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INTERACTION OF SHORT SINGLE-COMPONENT VECTOR SOLITONS

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We study interaction of different-polarization single-component vector solitons of the envelope function in anisotropic media within the framework of the system of two coupled third-order nonlinear Schrödinger equations which allow for the third-order linear dispersion, nonlinear dispersion, nonlinear cross-phase modulation, and cross-nonlinear dispersion. The regimes of mutual reflection, passage, and asymptotic approach of the solitons are obtained. It is shown that the character of interaction of such solitons is determined by the initial relationship of their amplitudes and phases. The stationary mutual locations of interacting solitons and their coupled, the so-called breather states are discussed. The roles of the cubic nonlinearity, cubic cross-nonlinearity, and cross-nonlinear dispersion during interaction of solitons are studied.

1. INTRODUCTION

The studies of the stationary nonlinear wave packets, i.e., solitons, attract significant interest at present. This interest is related to the fact that solitons can propagate to large distances without distortions in their shape and energy loss and may be used, in particular, as information carriers, e.g., in nonlinear fiber-optic communication lines. The soliton solutions exist for many nonlinear differential equations which appear in various parts of physics when studying the propagation of intense wave fields in nonlinear dispersive media, i.e., optical pulses in fiber optics, electromagnetic waves in plasmas, surface waves in deep water [1–4], etc.

2. SOLITON CLASSIFICATION

2.1. Scalar solitons

The dynamics of the high-frequency wave packets $U(\xi,t) = \exp(i\omega t - ik\xi)$ with a short envelope is described in the third order of the nonlinear dispersion theory with respect to the wave amplitude, which takes into account both the second- and the third-order infinitesimal terms, i.e., the third-order linear and nonlinear dispersion [5]. In this case, by a small quantity we mean $\Delta \sim 1/L \sim \partial/\partial\xi \sim U$, where L is the wave-packet length. Correspondingly, the quantities $1/L^2 \sim \partial^2/\partial\xi^2 \sim U^2 \sim \Delta^2$ and $U^2/L \sim U^2\partial/\partial\xi \sim 1/L^3 \sim \partial^3/\partial\xi^3 \sim \Delta^3$ are the second- and third-order infinitesimal terms, respectively. In this case, the basic model equation for describing the wave propagation in isotropic media is the third-order nonlinear Schrödinger equation [6–13]

$$2i\frac{\partial U}{\partial t} + 2i\beta\frac{\partial(|U|^2\,U)}{\partial\xi} + q\frac{\partial^2 U}{\partial\xi^2} + 2\alpha\,|U|^2\,U + i\gamma\frac{\partial^3 U}{\partial\xi^3} = 0, \qquad (1)$$

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where $q = -\partial^2\omega/\partial k^2$ and $\gamma = -\partial^3\omega/(3\,\partial k^3)$ are the parameters describing the second and third-order linear dispersion, $\alpha = \partial\omega/\partial(|U^2|)$ is the cubic-nonlinearity parameter, $\omega = \omega(k, |U^2|)$ is the nonlinear dispersion relation, and β is the nonlinear-dispersion coefficient.

For $\gamma = \beta = 0$, Eq. (1) is reduced to the classical nonlinear Schrödinger equation [14, 15], which has a solution in the form of a soliton [14] resulting from the balance of dispersive spreading and nonlinear compression of a wave packet if $\alpha q > 0$. The dynamics and interaction of such solitons were studied in detail in, e.g., [14, 16–18].

In the presence of all terms, Eq. (1) has a solution in the form of a short scalar soliton of the envelope function. Such a solution exists as a result of the balance of the third-order linear dispersion and nonlinear dispersion. It is shown in [12] that such short scalar solitons are stable for $\gamma\beta > 0$. Since their length is determined by the ratio β/γ, short solitons, which differ from the solitons of the classical Schrödinger equation, exist even for small γ and β. Interaction of such solitons is described in detail in [10–12].

As was shown in the full-scale experiments for deep-water waves [19, 20], a nonstationary wave packet with a small length of 2–5 wavelengths during its propagation at the distances exceeding five wave packet widths exhibits effects such as nonlinear aberration of the wave packet, which are well described by Eq. (1), but cannot be explained by the classical Schrödinger equation. Similar results, which confirmed the correctness of using Eq. (1) to describe short wave packets, were also obtained for electromagnetic waves in, e.g., [21, 22].

2.2. Vector solitons

Propagation of the vector wave field $\mathbf{E} = \mathbf{e}_1 U(\xi, t)\exp(i\omega t - ik_u\xi) + \mathbf{e}_2 W(\xi, t)\exp(i\omega t - ik_w\xi)$ with short envelope functions U and W and the mutually orthogonal polarization components in an anisotropic nonlinear dispersive medium is correctly described in the third-order (aberration) approximation of the theory of dispersion of nonlinear waves in anisotropic media [2]. Here, $\mathbf{e}_1 \perp \mathbf{e}_2$ are the mutually orthogonal unit-length vectors which determine the directions of the different-polarization wave-field components U and W. This approximation allows for the third-order nonlinear cross terms [23–25]. In this approximation, for a small difference between the wave numbers of the polarization components ($|k_u - k_w| \ll k_u$), the slowly varying envelope functions U and W are described by the model system of two coupled third-order Schrödinger equations [3]

$$2i\left[\frac{\partial U}{\partial t} + \beta\frac{\partial(|U|^2\,U + \sigma_\beta\,|W|^2\,U)}{\partial\xi} + \frac{\beta\sigma_\beta}{2}\frac{\partial(W^2U^*)}{\partial\xi}\right] + q\frac{\partial^2 U}{\partial\xi^2} + 2\alpha\,(|U|^2 + \sigma_\alpha\,|W|^2)\,U + \alpha\sigma_\alpha W^2U^* + i\gamma\frac{\partial^3 U}{\partial\xi^3} = 0, \quad (2)$$

$$2i\left[\frac{\partial W}{\partial t} + \beta\frac{\partial(|W|^2\,W + \sigma_\beta\,|U|^2\,W)}{\partial\xi} + \frac{\beta\sigma_\beta}{2}\frac{\partial(U^2W^*)}{\partial\xi}\right] + q\frac{\partial^2 W}{\partial\xi^2} + 2\alpha\,(|W|^2 + \sigma_\alpha\,|U|^2)\,W + \alpha\sigma_\alpha U^2W^* + i\gamma\frac{\partial^3 W}{\partial\xi^3} = 0, \quad (3)$$

where U^* and W^* are the quantities which are complex conjugate with respect to U and W, σ_α is the parameter of the nonlinear cross-phase modulation, and σ_β is the parameter of the cross-nonlinear dispersion. For $\gamma = \beta = 0$, this system is reduced to two coupled classical nonlinear Schrödinger equations which have a solution in the form of extended vector solitons [1, 16, 26–28].

Let us find the energy variation rate of the components U and W. Multiply Eq. (1) by U^* and add the obtained equation and its complex-conjugate analog. Then integrate the resulting equation over ξ from $-\infty$ to $+\infty$ with allowance for zero conditions at infinity $(U, W)_{\xi\to\pm\infty} \to 0$. As a result, we obtain the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty}|U|^2\,\mathrm{d}\xi=-\frac{\beta\sigma_\beta}{4}\int\limits_{-\infty}^{+\infty}\left[4\,|U|^2\frac{\partial|W|^2}{\partial\xi}+(U^*)^2\frac{\partial(W^2)}{\partial\xi}+U^2\frac{\partial(W^*)^2}{\partial\xi}\right]\mathrm{d}\xi$$

$$+\frac{i}{2}\alpha\sigma_\alpha\int\limits_{-\infty}^{+\infty}\left[(WU^*)^2-(W^*U)^2\right]\mathrm{d}\xi. \quad (4)$$

for the energy variation rate of the component U. By analogy, Eq. (3) for W yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty}|W|^2\,\mathrm{d}\xi=-\frac{\beta\sigma_\beta}{4}\int\limits_{-\infty}^{+\infty}\left[4\,|W|^2\frac{\partial|U|^2}{\partial\xi}+(W^*)^2\frac{\partial(U^2)}{\partial\xi}+W^2\frac{\partial(U^*)^2}{\partial\xi}\right]\mathrm{d}\xi$$

$$+\frac{i}{2}\alpha\sigma_\alpha\int\limits_{-\infty}^{+\infty}\left[(UW^*)^2-(U^*W)^2\right]\mathrm{d}\xi. \quad (5)$$

Equations (4) and (5) show that the different-polarization wave fields interact with each other. Adding Eqs. (4) and (5), we obtain the law of conservation for the total energy of the vector wave packet:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty}(|U|^2+|W|^2)\,\mathrm{d}\xi=0. \quad (6)$$

On the other hand, subtracting Eq. (4) from Eq. (5) and taking into account that $U=|U|\,\exp(i\varphi_u)$ and $W=|W|\,\exp(i\varphi_w)$, we obtain the following expression for the variation rate of the difference of the component energies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty}(|U|^2-|W|^2)\,\mathrm{d}\xi=2\beta\sigma_\beta\int\limits_{-\infty}^{+\infty}|U|^2\frac{\partial|W|^2}{\partial\xi}\mathrm{d}\xi-\beta\sigma_\beta\int\limits_{-\infty}^{+\infty}|U^2|\frac{\partial|W|^2}{\partial\xi}\cos[2\,(\varphi_w-\varphi_u)]\,\mathrm{d}\xi$$

$$+\beta\sigma_\beta\int\limits_{-\infty}^{+\infty}|U|^2\,|W|^2\left(\frac{\partial\varphi_w}{\partial\xi}-\frac{\partial\varphi_u}{\partial\xi}\right)\sin[2\,(\varphi_w-\varphi_u)]\,\mathrm{d}\xi-2\alpha\sigma_\alpha\int\limits_{-\infty}^{+\infty}|U|^2\,|W|^2\sin[2\,(\varphi_w-\varphi_u)]\,\mathrm{d}\xi. \quad (7)$$

The first three terms on the right-hand side of Eq. (7) correspond to the effects of cross-nonlinear dispersion, while the last term corresponds to the effect of cross-phase modulation. The first term on the right-hand side of Eq. (7) describes the amplitude effect of interaction between the polarization components of the wave field, while the other terms in this equation describe the phase effects of the interaction.

The system of Eqs. (2) and (3) has a solution in the form of a two-component soliton [3]

$$U(\xi,t)=\frac{A_0}{\cosh[A_0\sqrt{\beta\,(2+3\sigma_\beta)/(2\gamma)}\,(\xi-V_{u,w}t)]}\exp(i\Omega_{u,w}t+iK\xi),\qquad W=\pm U,$$

where

$$\Omega_{u,w}=\frac{\alpha\gamma}{2}\left(1+\frac{3}{2}\sigma_\alpha\right)A_0^2-\frac{\alpha\gamma}{2\beta}K^2$$

are the additional frequencies of the components of a short vector soliton, $K=(q\beta-\alpha\gamma)/(2\beta\gamma)$ is the soliton wave number which is the same for the components U and W, and

$$V_{u,w} = \frac{\beta}{2}\left(1 + \frac{3}{2}\sigma_\alpha\right) A_0^2 + Kq - \frac{3}{2}\gamma K^2$$

is the soliton velocity.

On the other hand, the system of Eqs. (2) and (3) has two solutions in the form of short single-component vector solitons with different polarizations [3]

$$U(\xi, t) = \frac{A_u}{\cosh[A_u\sqrt{\beta/\gamma}\,(\xi - V_u t)]}\exp(i\Omega_u t + iK\xi), \qquad W = 0;$$
$$W(\xi, t) = \frac{A_w}{\cosh[A_w\sqrt{\beta/\gamma}\,(\xi - V_w t)]}\exp(i\Omega_w t + iK\xi), \qquad U = 0,$$

where A_u and A_w are the amplitudes of the single-component vector solitons with different polarizations, $\Omega_x = A_x^2\alpha\gamma/2 - K^2\alpha\gamma/(2\beta)$ and $V_x = A_x^2\beta/2 + Kq - 3\gamma K^2/2$ are the additional frequencies and velocities of the solitons, and the subscript x denotes u or w. These soliton solutions taken separately are the exact solutions of the scalar uncoupled third-order nonlinear Schrödinger equation given by Eq. (1) [10–12]. It is shown in [12] that such single-component solitons are stable with respect to small perturbations under the condition $\gamma\beta > 0$. In our further studies, we will consider only the cases where this condition is satisfied.

In this work, the interaction of single-component solitons is studied for the arbitrary parameters σ_α and σ_β. The study is performed analytically in the adiabatic approximation and numerically. Some results for $\sigma_\alpha = \sigma_\beta$ were obtained in [29].

3. INTERACTION OF SHORT SINGLE-COMPONENT VECTOR SOLITONS

Let us consider the initial value problem of interaction between two short single-component different-polarization vector solitons of the envelope function. Let two different-amplitude solitons be present in anisotropic medium at the initial time $t = 0$ at the distance ξ_0 from each other:

$$U(\xi, t = 0) = \frac{A_1(0)\exp(iK\xi)}{\cosh[A_1(0)\varepsilon\,(\xi - \xi_0)]}, \qquad W(\xi, t = 0) = \frac{A_2(0)\exp(iK\xi)}{\cosh[A_2(0)\varepsilon\xi]}, \tag{8}$$

where the constant $\varepsilon = \sqrt{\beta/\gamma}$. Assume that the soliton parameters vary slowly during the interaction, so that their evolution for $t > 0$ can be described in the adiabatic approximation:

$$U(\xi, t) = A_1(t)\exp\left[i\int_0^t \Omega_u(\tilde t)\,\mathrm{d}\tilde t + iK\xi\right] \Big/ \cosh\left[A_1(t)\varepsilon\left(\xi - \xi_0 - \int_0^t V_u(\tilde t)\,\mathrm{d}\tilde t\right)\right],$$
$$W(\xi, t) = A_2(t)\exp\left[i\int_0^t \Omega_w(\tilde t)\,\mathrm{d}\tilde t + iK\xi\right] \Big/ \cosh\left[A_2(t)\varepsilon\left(\xi - \int_0^t V_w(\tilde t)\,\mathrm{d}\tilde t\right)\right], \tag{9}$$

where $\Omega_u(t) = A_1^2\alpha\gamma/2 - K^2\alpha\gamma/(2\beta)$ and $\Omega_w(t) = A_2^2\alpha\gamma/2 - K^2\alpha\gamma/(2\beta)$ are the additional frequencies of the interacting vector solitons, and $V_u(t) = A_1^2\beta/2 + Kq - 3\gamma K^2/2$ and $V_w(t) = A_2^2\beta/2 + Kq - 3\gamma K^2/2$ are their velocities. The distance between the centers of the interacting solitons vary by the law

$$\Delta\xi = \xi_0 + \int_0^t [V_u(\tilde t) - V_w(\tilde t)]\,\mathrm{d}\tilde t = \xi_0 + \frac{\beta C_0}{2}\int_0^t \Delta A(\tilde t)\,\mathrm{d}\tilde t, \tag{10}$$

where $\Delta A = A_1 - A_2$ is the difference between the amplitudes of the interacting single-component solitons and the notation C_0 is specified in Eq. (12).

The phase difference of these solitons is written as

$$\varphi_u - \varphi_w = \int_0^t [\Omega_u(\tilde{t}) - \Omega_w(\tilde{t})]\,\mathrm{d}\tilde{t} = \frac{\alpha\gamma C_0}{2}\int_0^t \Delta A(\tilde{t})\,\mathrm{d}\tilde{t} = \frac{\alpha\gamma}{\beta}(\Delta\xi - \xi_0). \tag{11}$$

Substituting Eqs. (8) into the energy-conservation law given by Eq. (6), we obtain the relationship for the amplitudes of the interacting solitons

$$A_1(t) + A_2(t) = A_1(0) + A_2(0) = C_0. \tag{12}$$

Differentiating Eq. (10) with respect to time, substituting the result into Eq. (7), and introducing the notation $D = A_1/A_2 = (C_0 + \Delta A)/(C_0 - \Delta A)$, we obtain the system

$$\frac{\mathrm{d}\Delta A}{\mathrm{d}t} = I_1 \frac{3\varepsilon\beta\sigma_\beta\,[C_0^2 - (\Delta A)^2]}{16\cosh^2[(C_0 + \Delta A)\,\varepsilon\,\Delta\xi/2]}\left[2 - \cos\left(\frac{2\alpha\gamma}{\beta}(\Delta\xi - \xi_0)\right)\right] + I_2\frac{\alpha\sigma_\alpha}{8}\left[C_0^2 - (\Delta A)^2\right]^2 \sin\left[\frac{2\alpha\gamma}{\beta}(\Delta\xi - \xi_0)\right], \tag{13}$$

$$\frac{\mathrm{d}\Delta\xi}{\mathrm{d}t} = \frac{\beta C_0}{2}\Delta A, \tag{14}$$

where

$$I_1 = \int_{-\infty}^{+\infty} \frac{\tanh(\eta)\,\mathrm{d}(\tanh\eta)}{\cosh^2(D\eta)\left\{1 - \tanh(D\eta)\tanh[(C_0 + \Delta A)\,\varepsilon\,\Delta\xi/2]\right\}^2},$$

$$I_2 = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{\cosh^2[(C_0 - \Delta A)\,\varepsilon\xi/2]\cosh^2[(C_0 + \Delta A)\,\varepsilon(\xi - \Delta\xi)/2]}.$$

In what follows, we consider the interaction of the short single-component solitons with a small amplitude difference $|\Delta A| \ll C_0$. In this case, assuming that the parameter $D \approx 1$ on the right-hand side of Eq. (13), we obtain the system of equations

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = \sigma_\beta \frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)}\left\{6 - 4\sin^2 p\,[\rho_0 - \rho)]\right\} - \sigma_\alpha p \frac{4\,(\rho - \tanh\rho)}{\tanh^3(\rho)\cosh^2(\rho)}\sin[2p\,(\rho_0 - \rho)], \tag{15}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = a, \tag{16}$$

where $\rho = A_0\varepsilon\,\Delta\xi$, $\rho_0 = A_0\varepsilon\xi_0$, $\tau = tA_0^3\varepsilon\beta$, $a = \Delta A/A_0$, and $p = \alpha\gamma/(A_0\varepsilon\beta)$.

3.1. Interaction of the vector solitons without allowance for the phase effects

Under the condition $p = 0$, the terms allowing for the soliton phases in Eq. (15) disappear. Then after the introduction of new variables $\tau' = \tau\sqrt{\sigma_\beta}$ and $a' = a/\sqrt{\sigma_\beta}$ (the primes are omitted in what follows), the system of Eqs. (15) and (16) takes the form

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = 6\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)}, \tag{17}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = a. \tag{18}$$

The phase trajectories of the system of Eqs. (17) and (18) can explicitly be described by the equation

$$a^2 + 12\frac{\rho - \tanh\rho}{\cosh^2(\rho)\tanh^3(\rho)} = a_{\pm\infty}^2, \tag{19}$$

where $a_{\pm\infty}$ is the initial difference between the amplitudes of the interacting single-component solitons at a sufficiently large distance $|\rho| \gg 1$. Figure 1 shows the phase plane of the system of Eqs. (17) and (18). It is evident that a single equilibrium state $a = 0$ exists for this system, while $\rho = 0$ is the saddle point.

Curves *1* correspond to the mutual passage of the interacting solitons through each other and curves *2* are the saddle separatrices and correspond to the infinitely long approach of the interacting solitons with their amplitudes equalized, which is realized for the critical value $a_{\mathrm{c}} = a_\infty$. Due to Eq. (19), the critical difference of the initial amplitudes $a_{\mathrm{c}} = a_\infty$, i.e., the mutual-reflection interval equals 2 (in terms of the absolute value). Curves *3* correspond to the mutual reflection of solitons from each other.

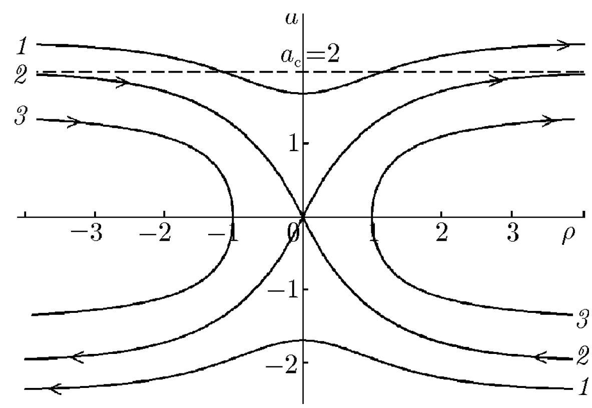

Fig. 1. The phase plane for the system of Eqs. (17) and (18).

Note that the inequality $|a| \ll 1$, which restricts the correct use of Eqs. (15) and (16), is assumed in this work. However, the numerical-simulation results shown below qualitatively coincide with the corresponding analytical results and also confirm the possibility of using analytical expressions for moderate values of $|a| \sim 1$. This remark is valid for all the analyzed cases.

In the case of mutual reflection, we can determine the minimum distance to which the solitons approach by analyzing Eq. (19). Taking into account that the difference of the soliton amplitudes is $a = 0$ for the minimum distance $\rho = \rho_{\min}$, we consider two limiting cases.

If $|a_{\pm\infty}| \sim a_{\mathrm{c}} = 2$, but $a_{\pm\infty} < a_{\mathrm{c}}$ (the case of mutual reflection), one can assume that the minimum distance $\rho_{\min} \ll 1$ and use the Taylor–Maclaurin asymptotic series expansion in Eq. (19). Then, using Eq. (19), we obtain $\rho_{\min}^2 = 5\,(4 - a_{\pm\infty}^2)/12$, e.g., $|\rho_{\min}| \approx 0.4$ for $|a_{\pm\infty}| = 1.9$.

For small $|a_{\pm\infty}| \ll 1$, we can assume that $|\rho_{\min}| \gg 1$ and use in Eq. (19) the limiting expressions which lead to the equality $48\,(\rho - 1)/\exp(2\rho) = a_{\pm\infty}^2$. Here, $|\rho_{\min}| \to 1$ for $|a_{\pm\infty}| \to 0$, e.g., $|\rho_{\min}| \approx 5$ for $|a_{\pm\infty}| = 0.1$ or $|\rho_{\min}| \approx 7.5$ for $|a_{\pm\infty}| = 0.01$.

3.2. Interaction of the vector solitons without the cross-phase modulation

If $\sigma_\beta = 0$, then after the introduction of the new variables $\tau' = \tau\sqrt{\sigma_\beta}$ and $a' = a/\sqrt{\sigma_\beta}$ (the primes are omitted in what follows), the system of Eqs. (15) and (16) is reduced to the form

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = 6\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)} - 4\sin^2[p\,(\rho_0 - \rho)]\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)}, \tag{20}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = a. \tag{21}$$

It has the only saddle-type equilibrium state at $a = 0$, and $\rho = 0$. Figure 2 shows the phase plane of Eqs. (20) and (21) for $p = 1$ and various values of the initial difference of the soliton phases: $p\rho_0 = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$.

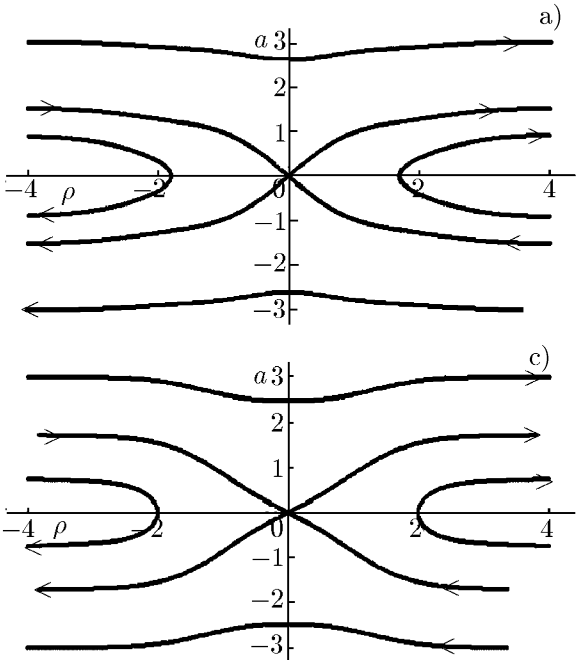

Fig. 2. The phase plane for the system of Eqs. (20) and (21) for $p = 1$ and various values of the parameter $p\rho_0 = 0$ (*a*), $\pi/4$ (*b*), $\pi/2$ (*c*), and $3\pi/4$ (*d*).

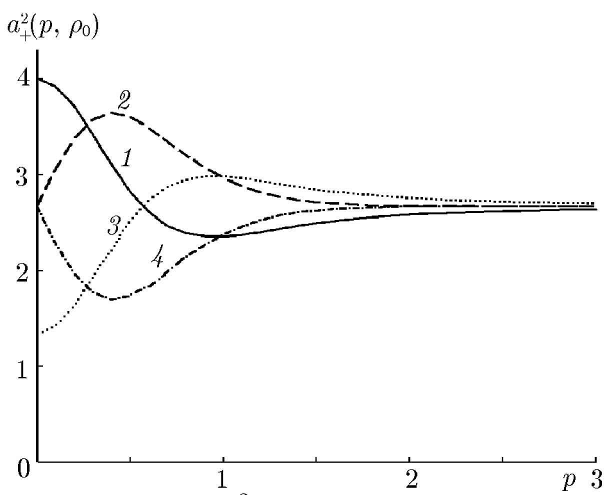

Fig. 3. Function $a_+^2(p, \rho_0)$ for various values of $2p\rho_0$: curves *1*, *2*,*3*, and *4* correspond to $2p\rho_0 = 0$, $2p\rho_0 = \pi/2$, $2p\rho_0 = \pi$, and $2p\rho_0 = 3\pi/2$, respectively. The quantities a_-^2 are equal to a_+^2 for the following replacements of the parameter $2p\rho_0$: $0 \to 0$, $\pi/2 \to 3\pi/2$, $\pi \to \pi$, and $3\pi/2 \to \pi/2$.

Variation in $p\rho_0$ leads only to variation in the "reflection intervals," i.e., the maximum distance from the saddle separatrices to the horizontal axis.

Depending on the initial conditions, the cases of the mutual reflection of solitons, soliton passage through each other, and the infinitely long approach of the solitons are possible.

Integrating the system of Eqs. (20) and (21) under the condition of equality of the soliton amplitudes in the equilibrium state, i.e., $a(\rho = 0) = 0$, we determine the "reflection intervals," i.e., the distances from the separatrices (separately for $\rho \to +\infty$ and $\rho \to -\infty$) to the horizontal axis:

$$a_{\pm\infty}^2 = 4 - 8 \int_0^{\pm\infty} \sin^2[p\,(\rho_0 - \rho)]\, \frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)} \mathrm{d}\rho.$$

The dependences $a_+^2(p, \rho_0)$ are plotted in Fig. 3 for various values of the parameters $2p\rho_0 = 0$, $\pi/2$, π, and $3\pi/2$.

The values of $a_\pm^2(2p\rho_0)$ are limited and approximately tend to 2.66 for $p \to +\infty$. Therefore, the inequality $|a_{\pm\infty}| < 2$ always holds, i.e., the reflection interval is smaller than that described in Sec. 3.1 for $p = 0$ and approximately tends to 1.63 for large p. In particular, in the cases shown in Fig. 2 we have $|a_{\pm\infty}| \approx 1.53$ for $2p\rho_0 = 0$, $|a_{+\infty}| \approx 1.72$ for $2p\rho_0 = \pi/2$, $|a_{\pm\infty}| \approx 1.73$ for $2p\rho_0 = \pi$, and $|a_{+\infty}| \approx 1.54$ for $2p\rho_0 = 3\pi/2$.

In the case of mutual reflection, integrating Eqs. (20) and (21), one can determine the minimum distance $\rho_{\min}$ of the soliton approach. If the soliton amplitudes are identical for the minimum approach

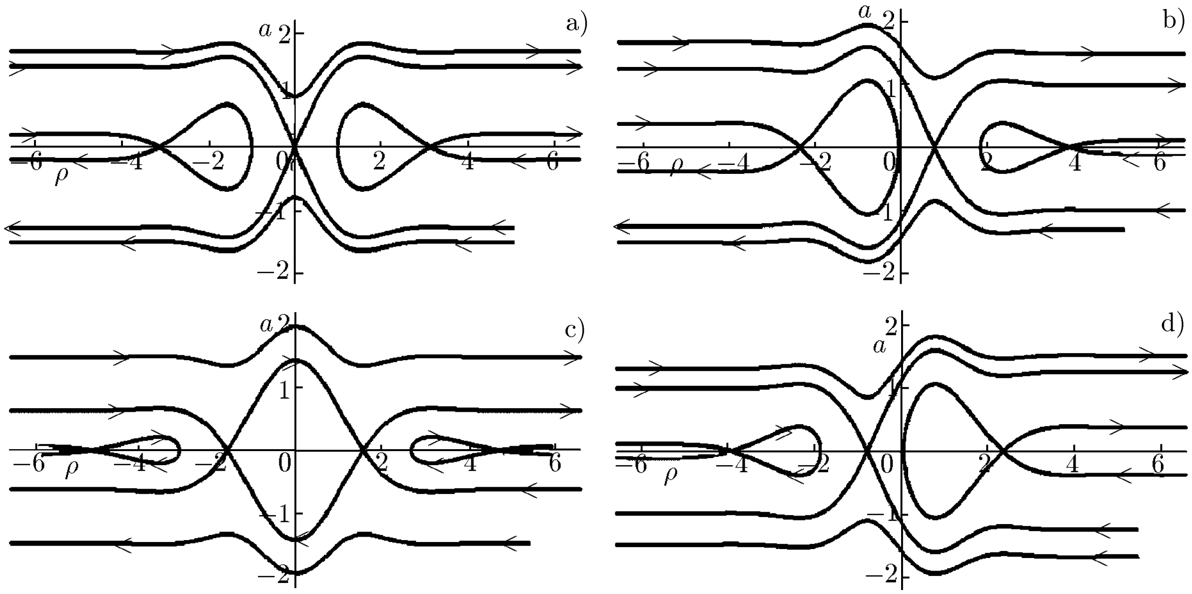

Fig. 4. The phase plane of the system of Eqs. (22) and (23) for $p = 1$ and various values of $p\rho_0 = 0$ (*a*), $\pi/2$ (*b*), π (*c*), and $3\pi/2$ (*d*). Out of the equilibrium states, only the saddles which are close to the origin of coordinates are shown.

distance, i.e., for $a(\rho_{\min})$, then integrating Eqs. (20) and (21) from $\rho_{\pm\min}$ to $\pm\infty$, we obtain

$$a^2_{\pm\infty,\mathrm{lim}} = 12\frac{\rho_{\pm\min} - \tanh(\rho_{\pm\min})}{\tanh^3(\rho_{\pm\min})\cosh^2(\rho_{\pm\min})} - 8\int\limits_{\rho_{\pm\min}}^{\pm\infty} \sin^2[p\,(\rho_0 - \rho)]\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)}\mathrm{d}\rho.$$

Herefrom, e.g., for the cases shown in Fig. 2 we obtain for $|\rho_{\pm\min}| = 0.1$ that $|a_{\pm\infty,\mathrm{lim}}| \approx 1.53$ for $2p\rho_0 = 0$, $|a_{+\infty,\mathrm{lim}}| \approx 1.71$ and $|a_{-\infty,\mathrm{lim}}| \approx 1.54$, for $2p\rho_0 = \pi/2$ $|a_{\pm\infty,\mathrm{lim}}| \approx 1.72$ for $2p\rho_0 = \pi$, and $|a_{+\infty,\mathrm{lim}}| \approx 1.54$ and $|a_{-\infty,\mathrm{lim}}| \approx 1.71$ for $2p\rho_0 = 3\pi/2$.

3.3. Interaction of vector solitons without cross-nonlinear dispersion

If $\sigma_\beta = 0$, then after introducing the new variables $\tau' = \tau\sqrt{\sigma_\alpha}$ and $a' = a/\sqrt{\sigma_\alpha}$ (the primes are omitted in what follows), the system of Eqs. (15) and (16) takes the form

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -4p\sin[2p\,(\rho_0 - \rho)]\frac{\rho - \tanh\rho}{\tanh^3(\rho)\cosh^2(\rho)}, \tag{22}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = a. \tag{23}$$

It has infinitely many equidistant equilibrium states at $a = 0$ and $\rho = 2k\pi/(2p) + \rho_0$ of the saddle type and at $a = 0$ and $\rho = (2k+1)\,\pi/(2p) + \rho_0$ of the center type (here, $k \in Z$ is any integer). Figure 4 shows the phase plane for the system of Eqs. (22) and (23) for $p = 1$ and various values $2p\rho_0 = 0$, $\pi/2$ of the initial difference of the soliton phases, π, and $3\pi/2$. The center-type equilibrium states exist inside the existing separatrix loops or bundles.

Variation in p proportionally changes only the equilibrium-state coordinates and the "reflection interval" value without a qualitative changer in the system-trajectory behavior.

As is evident from the above phase portraits, only one qualitatively distinguished equilibrium state exists, namely, the so-called B-saddle which is the closest to the origin of coordinates and has no separatrix loops. Only for $2p\rho_0 = \pi$ (Fig. 4*c*), there are two such saddles and having common coupled separatrices.

If the initial conditions are between the corresponding separatrices, then all saddles determine the limiting distance of the soliton approach. However, the distinguished saddle determines the least limiting approach distance. In the cases where the B-saddle deviates from the origin of coordinates, it is possible that this distance has the other sign compared with the initial distance. Therefore, during mutual reflection, the solitons can approach each other and interact, pass through each other, separate to a certain distance, and again start approaching, pass through each other in the opposite direction, and finally reflect. Such behavior is possible if, e.g., $2p\rho_0 = \pi/2$ (see Fig. 4*b*) during the motion along the phase trajectories between the B-saddle separatrices from the region of negative ρ.

If the initial difference in the amplitudes of interacting solitons exceeds the corresponding value for the B-saddle separatrix, the solitons pass through each other.

For the initial conditions at the separatrix of any saddle, the solitons approach each other for an infinitely long time. In this case, after the solitons are separated by a distance which is equal to the saddle coordinate, they proceed to move without changing their mutual location. However, such a "separatrix-type" interaction mechanism is unstable with respect to small perturbations. Such perturbations due to external influence are evidently possible since the soliton approach time is infinitely long. Therefore, the interaction mechanism is not structurally stable.

If the initial conditions are specified exactly at a saddle, the solitons remain at a distance equal to the saddle coordinate and then continue to move with the same amplitudes without changing the mutual location. Therefore, it is possible to realize the coupled, but nonperiodic (non-breather) state of the vector single-component solitons interacting at some distance. However, it is expedient to consider such a "saddle-type" mechanism of soliton interaction in actual media only for finite times since it is not structurally stable for the above-mentioned reasons.

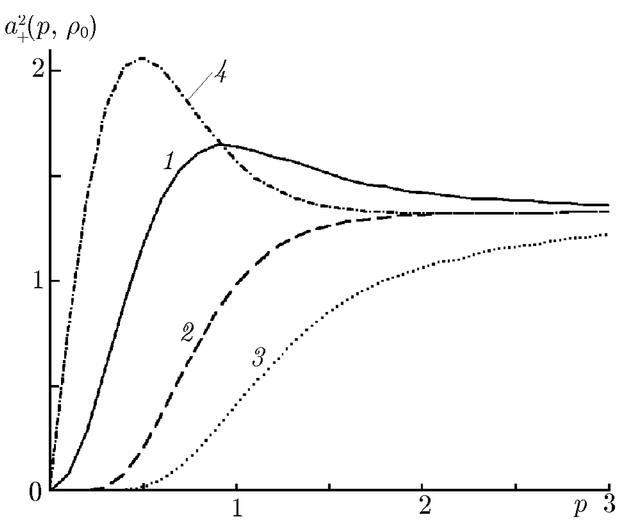

Fig. 5. Function $a_+^2(p,\rho_0)$ for various values of $2p\rho_0$: curves *1*, *2*, *3*, and *4* correspond to $2p\rho_0 = 0$, $2p\rho_0 = \pi/2$, $2p\rho_0 = \pi$, and $2p\rho_0 = 3\pi/2$, respectively. The quantities a_-^2 are equal to a_+^2 for the following replacements of the parameter $2p\rho_0$: $0 \to 0$, $\pi/2 \to 3\pi/2$, $\pi \to \pi$, and $3\pi/2 \to \pi/2$.

It is worthy of note that the initial conditions are specified by the quantity ρ_0 or, more exactly, by $\rho_0 + \pi n/p$ due to the periodicity of the function in Eq. (22). Therefore, the initial distance always coincides with the coordinates of one of the saddles in the phase plane. Thus, specifying the initial conditions, it is impossible to choose the trajectory located on or inside some separatrix loop or inside the separatrix bundle. If a choice of such a closed trajectory were possible, this would correspond to mutual periodic motion of interacting solitons at a certain distance from each other. This would result in realization of a breather of different-polarization vector solitons located at a certain distance from each other, i.e., solitons which never overlap. Such breathers can appear only due to random external actions for long times in the case of the "saddle" coupled state of solitons.

In the cases under consideration, integrating the system of Eqs. (22) and (23) under the condition $a(\mathrm{B-saddle}) = 0$, one can determine the "reflection intervals," i.e., the maximum distances from the specified-saddle separatrices (separately for $\rho \to +\infty$ and $\rho \to -\infty$) to the horizontal axis. These distances are described by the expression

$$a_{\pm\infty}^2 = -8p \int\limits_{\rho(\mathrm{B-saddle})}^{\pm\infty} \sin^2[2p\,(\rho_0 - \rho)] \frac{\rho - \tanh\rho}{\tanh^3(\rho)\cosh^2(\rho)} \mathrm{d}\rho.$$

The plots of $a_+^2(p,\rho_0)$ for $2p\rho_0 = 0$, $\pi/2$, π, and $3\pi/2$ are shown in Fig. 5. The value of $a_+^2(p,\rho_0)$ is always

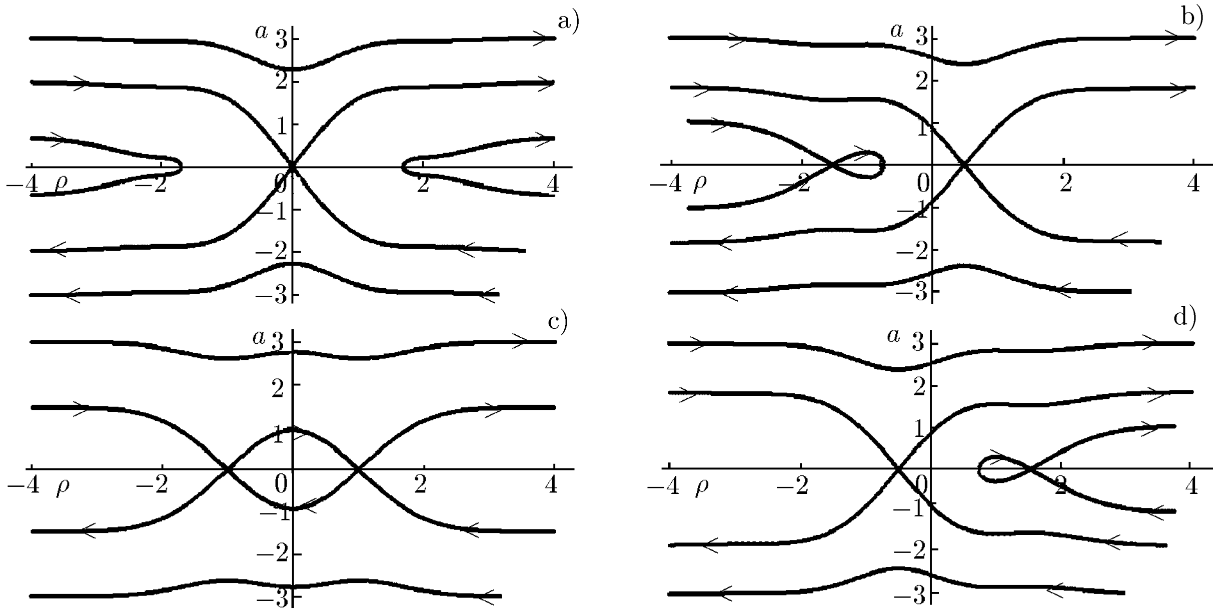

Fig. 6. The phase plane for the system of Eqs. (24) and (25) for $\delta = 1$, $p = 1$, and various values of $2p\rho_0 = 0$ (*a*), $\pi/2$ (*b*), π (*c*), and $3\pi/2$ (*d*). Out of the equilibrium state, only the saddles which are close to the origin of coordinates are shown.

limited and approximately tends to 1.33 for $p \to +\infty$. Namely, in the cases shown in Fig. 4 we have $|a_{\pm\infty}| \approx 1.28$ for $2p\rho_0 = 0$, $|a_{+\infty}| \approx 0.99$ for $2p\rho_0 = \pi/2$, $|a_{\pm\infty}| \approx 0.64$ for $2p\rho_0 = \pi$, and $|a_{+\infty}| \approx 1.26$ for $2p\rho_0 = 3\pi/2$. By analogy, we can also determine the "reflection intervals" for any saddle.

3.4. General case of allowance for all nonlinear effects of interaction

If all nonlinear coupling parameters are simultaneously present in the model equations, then, using the new variables $\tau' = \tau\sqrt{\sigma_\beta}$ and $a' = a/\sqrt{\sigma_\beta}$ and the notation $\sigma_\alpha/\sigma_\beta = \delta$ (the primes are omitted in what follows), the system of Eqs. (15) and (16) can be reduced to the form

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = 6\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)} - 4\sin^2[p\,(\rho_0 - \rho)]\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)} - \delta\frac{4p\,(\rho - \tanh\rho)}{\tanh^3(\rho)\cosh^2(\rho)}\sin[2p\,(\rho_0 - \rho)], \quad (24)$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = a. \quad (25)$$

This system of equations has at least one equilibrium state. Figure 6 shows the phase plane for the system of Eqs. (24) and (25) for $p = 1$ and $\delta = 1$ and various values of $2p\rho_0 = 0$, $\pi/2$, π, and $3\pi/2$.

Comparing the phase portraits for different values of the parameter $\delta = \sigma_\alpha/\sigma_\beta$, we see that its increase leads to an increase in the number of saddles and the corresponding number of centers inside the separatrix loops, as well as an increase in both the distance between the saddles and the "reflection interval." An increase in p leads to an increase in the number of saddles and the "reflection interval," but to a decrease in the distance between the existing saddles.

As is evident from the above phase portraits, only one qualitatively distinguished equilibrium state (B-saddle) exists. This saddle is the closest to the origin of coordinates and has no separatrix loops. Only for $2p\rho_0 = \pi$ (Fig. 6*c*), we have two such saddles which are coupled by the separatrix bundles.

The mutual passages of the vector solitons through each other, the "separatrix-type" interaction with

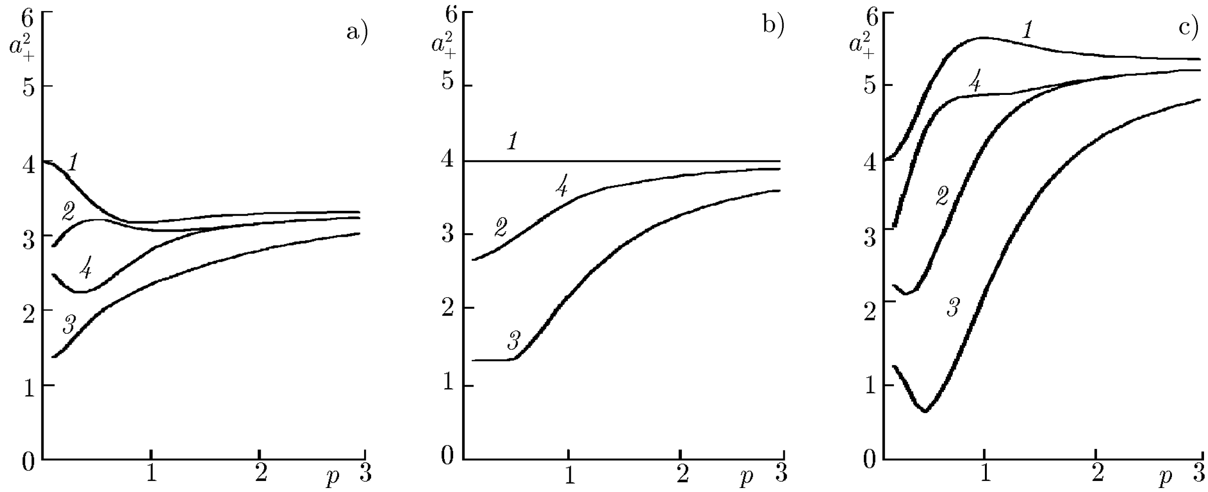

Fig. 7. The quantity a_+^2 for various $\delta = 1/2$ (*a*), 1 (*b*) and 2 (*c*) and $2p\rho_0$. Curves *1*,*2*, *3*, and *4* correspond to $2p\rho_0 = 0$, $2p\rho_0 = \pi/2$, $2p\rho_0 = \pi$, and $2p\rho_0 = 3\pi/2$, respectively. The values of a_-^2 are equal to the values of a_+^2 for the following replacements of the parameter $2p\rho_0$: $0 \to 0$, $\pi/2 \to 3\pi/2$, $\pi \to \pi$, and $3\pi/2 \to \pi/2$.

an infinitely slow equalization of the soliton amplitudes, and the mutual reflection of solitons from each other are the main interactions of the vector solitons. If the B-saddle is displaced off the origin of the phase-space coordinates, then the mutual reflection with double passage of solitons through each other is possible by analogy with the case described in Sec. 3.2.

In the cases studied, integrating the system of Eqs. (24) and (25) under the condition of equality of the soliton amplitudes in the distinguished saddle equilibrium state $a(\mathrm{B-saddle}) = 0$, one can determine the "reflection intervals," i.e., the maximum distances from the B-saddle separatrices (separately for $\rho \to +\infty$ and for $\rho \to -\infty$) to the horizontal axis. These distances are described by the expression

$$a_{\pm\infty}^2 = \frac{12\,[\rho(\mathrm{B-saddle}) - \tanh[\rho(\mathrm{B-saddle})]}{\tanh^3[\rho(\mathrm{B-saddle})]\,\cosh^2[\rho(\mathrm{B-saddle})]} - 8\delta\rho \int\limits_{\rho(\mathrm{B-saddle})}^{\pm\infty} \sin[2p\,(\rho_0 - \rho)]\frac{\rho - \tanh\rho}{\tanh^3(\rho)\cosh^2(\rho)}\mathrm{d}\rho$$
$$- 8 \int\limits_{\rho(\mathrm{B-saddle})}^{\pm\infty} \sin^2[p\,(\rho_0 - \rho)]\frac{3\rho - 3\tanh\rho - \rho\tanh^2\rho}{\tanh^4(\rho)\cosh^2(\rho)}\mathrm{d}\rho.$$

The plots of the functions a_+^2 for $\rho \to +\infty$ are shown in Fig. 7 for various values of $2p\rho_0 = 0$, $\pi/2$, π, and $3\pi/2$ and various values of $\delta = 1/2$, 1 and 2. In any of these cases, the values of a_+^2 become constant (dependent on δ, but independent of $2p\rho_0$) for $p \to +\infty$.

Note that location of any saddle is determined by the quantities δ, $2p\rho_0$, and p. When analyzing the coordinates of various saddles, it was obtained that the dependences of these coordinates on the parameter p for its large values asymptotically tend to hyperbolic ones from below for any values of δ and $2p\rho_0$. However, the initial conditions for the distance between the solitons are given by the quantity ρ_0 or more exact, by the quantity $\rho_0 + \pi n/p$ due to the periodicity of the functions in Eqs. (24) and (25), while the initial coordinate in the phase plane depends on the parameter p according to the hyperbolic law. Therefore, the initial point in phase space is always located at a somewhat longer distance from the origin of coordinates on the horizontal direction than the distinguished or undistinguished saddle. As a result, by specifying the initial conditions, we cannot choose the trajectory which is located on or inside the separatrix loop (bundle) or directly at a saddle. Therefore, apart from the cases of the soliton reflection and passage, only the "separatrix" (structurally stable) cases of their interaction are possible, As a result, the breather or at least the stationary "saddle" state are not realized in the case under consideration.

3.5. Numerical simulation

To check the correctness of the above-obtained results, we consider numerically the dynamics of the wave packets described by Eq. (8) within the framework of the system of Eqs. (2) and (3) under the condition $\alpha = q = \beta = \gamma = 1$ for different values of the nonlinear-coupling parameters $\sigma_\alpha = \sigma_\beta$, and different initial amplitudes $A_1(0)$ and $A_2(0)$ of interacting single-component solitons.

As an example, Fig. 8 shows the distribution of the wave fields $|U|$ and $|W|$ of the interacting solitons at various times for the initial distance $\xi = -\pi$ between them and $\sigma_\alpha = \sigma_\beta = 1/16$, $A_1(0) = 1.7$, and $A_2(0) = 1$. As is evident from this distribution, the single-component mutually orthogonal vector wave packets retain the soliton-like shape during the interaction, although the interaction is accompanied by weak linear radiation of a part of the wave fields. This result confirms the correctness of the adiabatic approximation when describing the interaction of the considered solitons.

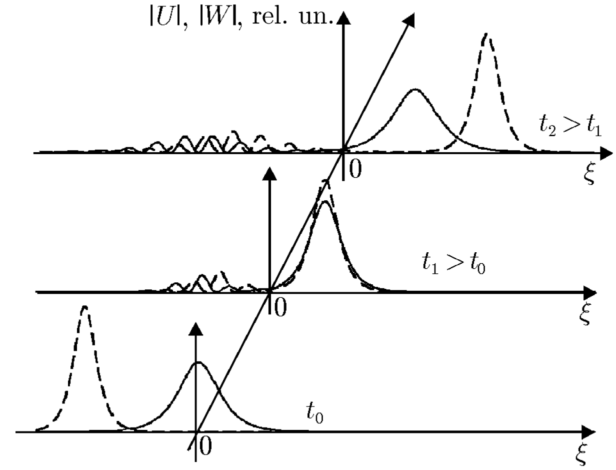

Fig. 8. Distribution of the wave fields of interacting single-component solitons at various times. Continuous curves correspond to the distribution of $|U|$, while the dash curves correspond to the distribution of $|W|$.

Figure 9 shows the results of numerical simulation of interaction of the single-component solitons. In particular, we present the relative difference

$$a_{\text{num}} = \frac{2}{\sqrt{\sigma}} \frac{\max(|U|) - \max(|W|)}{\max(|U|) + \max(|W|)} \approx \frac{2\,(A_1 - A_2)}{\sqrt{\sigma}\,(A_1 + A_2)}$$

of the maximum amplitudes of the interacting polarization components as a function of the distance $\Delta\xi = \xi_{\max|u|} - \xi_{\max|w|}$ in space between the maxima under the condition $A_2(0) = 1$ for different values of $A_1(0)$ and different initial distances ξ_0 between the solitons.

Curve *1* in Fig. 9*a* describes the passage of interacting solitons through each other for $A_1(0) = 1.75$ ($a_{\text{num}}(-\pi) = 2.18$), curve *2* corresponds to the "separatrix" type of interaction for $A_1(0) = 1.65$ ($a_{\text{num}}(-\pi) = 1.93$), and curve *3* describes mutual reflection of the solitons for $A_1(0) = 1.55$ ($a_{\text{num}}(-\pi) = 1.73$). Figure 9*a* corresponds to the phase plane for the system of Eqs. (24) and (25) for $\rho_0 = \pi k$ (i.e., an analog of Fig. 6*a*). The interval of mutual reflection for $\rho_0 = \pi k$ is numerically obtained equal to $(a_{\text{num}})_{\text{c}} \approx 1.93$ (the distance from curve *2* to the horizontal axis for $\Delta\xi = -\pi$), which differs only slightly from the corresponding value in the adiabatic approximation. The difference between the analytical and numerical results appears due to radiation of a part of the wave field during the interaction of solitons, which was not allowed for in the analytical consideration.

Curve *1* in Fig. 9*b* describes the passage of solitons through each other for $A_1(0) = 1.55$ ($a_{\text{num}}(-\pi) = 1.73$), curve *2* describes the "separatrix" type of interaction for $A_1(0) = 1.45$ ($a_{\text{num}}(-\pi) = 1.43$), and curve *3* describes mutual reflection of the solitons for $A_1(0) = 1.35$ ($a_{\text{num}}(-\pi) = 1.19$). Figure 9*b* corresponds to the phase plane for the system of Eqs. (24) and (25) for $\rho_0 = \pi k + \pi/4$ (see Fig. 6*c*). The interval of mutual reflection for $\rho_0 = \pi k + \pi/4$ is numerically obtained equal to $(a_{\text{num}})_{\text{c}} \approx 1.45$, which differs only slightly from a similar value obtained analytically.

Note that in the analytical part of this work, we assumed the inequality $|a| \ll 1$, which limited the correct use of analytical solutions. However, the results of numerical simulation qualitatively coincide with the corresponding analytical results and confirm the possibility of using analytical expressions for moderate values $|a| \sim 1$.

It shall also be noted that similarity of the analytical and numerical results is influenced by the values

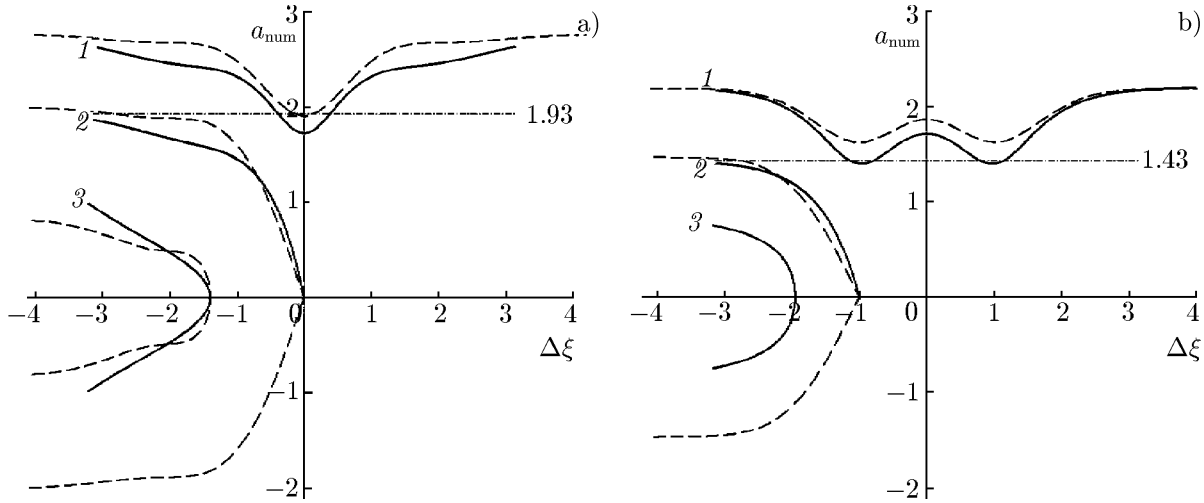

Fig. 9. Results of numerical calculation (solid curves) that are analogs of Figs. 6*a* and 6c. Panels *a* and*b* correspond to the conditions $\xi_0 = -\pi$ and $\xi_0 = -5\pi/4$, respectively. The dash curve show the corresponding phase trajectories obtained analytically.

of the nonlinear-coupling parameters σ_α and σ_β. With increasing these parameters, the character of the soliton interaction is retained, but their amplitude variation becomes more pronounced as a result of the interaction. This leads to a larger difference of analytical results from numerical simulation results.

4. CONCLUSIONS

In this work, we have analyzed the phase effects of interaction of short vector single-component solitons. Interaction of solitons was considered within the framework of a pair of coupled third-order nonlinear Schrödinger equations in the absence of stimulated Raman's scattering. The analytical and numerical results, which were obtained in the adiabatic approximation and numerical results are in good agreement for small parameters $\{\sigma_\alpha, \sigma_\beta\} \ll 1$ of mutual nonlinear coupling of the different-polarization components of the vector wave packet. The regimes of the passage solitons through each other, their mutual repulsion, and asymptotically slow approach are described. We have discussed the possibility of existence of the quasi-stationary states of interacting solitons known as breathers, i.e., solitons which do not completely overlap and are always located at a certain (variable) distance from each other, as well as the "saddle" interaction during which the coupled solitons are always located at a certain (constant) distance from each other. We have also described the regime of mutual reflection of interacting solitons with the double passage through each other. With increasing parameters σ_α and σ_β, the character of interaction of the single-component vector solitons is retained, but the soliton amplitude changes to a greater extent during the interaction. If $p = 0$, i.e., in the absence of cubic nonlinearity, the trajectories of the relative motion of solitons are described explicitly. For $p \neq 0$, the interaction character of solitons depends on their initial phase difference.

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REFERENCES

1. Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, New York (2001).
2. L. A. Dickey, *Soliton Equation and Hamiltonian Systems*, World Scientific, New York (2005).

3. Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: from Fibers to Photonic Crystals*, Academic Press, San Diego (2003).
4. G. P. Agrawal, *Fiber Optic Communication Systems*, Wiley, Hoboken, N.J. (2002).
5. J. R. Oliviera and M. A. Moura, *Phys. Rev. E*, **57**, 4751 (1998).
6. Y. J. Kodama, *Stat. Phys.*, **39**, 597 (1985).
7. C. E. Zaspel, *Phys. Rev. Lett.*, **82**, 723 (1999).
8. B. Hong and D. Lu, *Int. J. Nonlin. Sci.*, **7**, 360 (2009).
9. V. I. Karpman, *Eur. Phys. J. B*, **39**, 341 (2004).
10. E. M. Gromov and V. I. Talanov, *JETP*, **83**, No. 1, 73 (1996).
11. E. M. Gromov and V. I. Talanov, *Chaos*, **10**, 551 (2000).
12. E. M. Gromov, L. V. Piskunova, and V. V. Tyutin, *Phys. Lett. A*, **256**, 153 (1999).
13. N. L. Tsitsas, N. Rompolis, I. Kourakis, et al., *Phys. Rev. E.*, **79**, 037601 (2009).
14. V. E. Zakharov and A. B. Shabat, *Sov. Phys. JETP*, **34**, No. 1, 62 (1972).
15. A. Hasegava and F. Tappert, *Appl. Phys. Lett.*, **23**, 142 (1973).
16. S. V. Manakov, *Sov. Phys. JETP*, **38**, No. 2, 248 (1974).
17. J. Satsuma and N. Yajima, *Suppl. Progress Theor. Phys.*, **55**, 284 (1974).
18. S. N. Vlasov and V. I. Talanov, *Self-Focusing of Waves* [in Russian], Inst. Appl. Phys., Nizhny Novgorod (1997).
19. Su Ming-Yang, *Phys. Fluids*, **12**, 25 (1982).
20. E. M. Gromov and V. I. Talanov, in: *Int. School in Nonlinear Science, Nonlinear Waves, Synchronization, and Pattern*, Nizhny Novgorod State Univ., Nizhny Novgorod (1995), p. 23.
21. G. P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, San Diego (1989).
22. D. N. Christodoulides and R. I. Joseph, *Electron. Lett.*, **20**, 659 (1984).
23. J. Kim, *ETRI J.*, **23**, 9 (2001).
24. F. Lu, W. H. Lin, W. H. Knox, and G. P. Agrawal, *Phys. Rev. Lett.*, **93**, 183901 (2004).
25. E. M. Gromov, V. V. Tyutin, and D. E. Vorontzov, *Phys. Lett. A*, **287**, 233 (2001).
26. A. P. Fordy and P. P. Kullish, *Commun. Math. Phys.*, **89**, 427 (1983).
27. C. R. Menyuk, *J. Opt. Soc. Am. B*, **5**, 392 (1988).
28. V. I. Nayanov, *Multi-Field Solitons* [in Russian], Fizmztlit, Moscow (2006).
29. N. V. Aseeva, E. M. Gromov, and V. V. Tyutin, *Phys. Lett. A*, **376**, 718 (2012).