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GEOMETRY OF TOTALLY REAL GALOIS FIELDS OF DEGREE 4

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Abstract. We will consider a totally real Galois field K of degree 4 as the linear coordinate space $\mathbb{Q}^4 \subset \mathbb{R}^4$. An element $k \in K$ is called strictly positive, if all its conjugates are positive. The set of strictly positive elements is a convex cone in K. The convex hull of strictly positive integral elements is a convex subset of this cone and its boundary Γ is an infinite union of 3-dimensional polyhedrons. The group U of strictly positive units acts on Γ: the action of a strictly positive unit permutes polyhedrons. Fundamental domains of this action are the object of study in this work. We mainly present some interesting examples.

1. Introduction

Let K be a cubic totally real Galois field, defined by polynomial $p = x^3+ax+b \in \mathbb{Q}[x]$. Elements of K will be written in the form $kx^2 + lx + m$, $k, l, m \in \mathbb{Q}$. This notation allows us to identify K with 3-dimensional space $\mathbb{Q}^3 \subset \mathbb{R}^3$. . An element $k \in K$ is called strictly positive, if all its conjugates are positive, i.e. if x_1, x_2, x_3 are (real) roots of p then an element $kx^2 + lx + m$ is strictly positive, if

$$\begin{cases} kx_1^2 + lx_1 + m > 0 \\ kx_2^2 + lx_2 + m > 0 \\ kx_3^2 + lx_3 + m > 0 \end{cases}$$

These three conditions define a convex cone C in $\mathbb{R}^3$. Let $O \subset C$ be the set of strictly positive integral elements of K, $\overline{O}$ — its convex hull (an infinite polyhedral subset of C) and Γ be the boundary of this convex hull — an infinite polygonal 2-dimensional complex in $\mathbb{R}^3$. Let U be the group of strictly positive units (a free Abelian group of rank 2). The action of a positive unit on Γ induces a permutation of its faces. The fundamental domain of the action of U on Γ is a finite union of polygons with pairwise identified sides — a torus. In [1] the structure of this polygonal complex was studied.

In this work we will study a 4-dimensional analogue of this problem.

2. Totally real Galois fields of degree 4

Let K be a totally real Galois field of degree 4 defined by a polynomial $p \in \mathbb{Q}[x]$. Elements of the field will be written in the form $kx^3 + lx^2 + mx + n$, $k, l, m, n \in \mathbb{Q}$, and the field will be considered as 4-dimensional space $\mathbb{Q}^4 \subset \mathbb{R}^4$ with coordinates k, l, m, n. By O will be denoted the set of integral strictly positive elements from

K: an element kx^3+lx^2+mx+n is called strictly positive if

$$\begin{cases} kx_1^3+lx_1^2+mx_1+n>0 \\ kx_2^3+lx_2^2+mx_2+n>0 \\ kx_3^3+lx_3^2+mx_3+n>0 \\ kx_4^3+lx_4^2+mx_4+n>0 \end{cases}$$

where x_1, x_2, x_3, x_4 are (real) roots of p. These four conditions define a convex cone $C \subset \mathbb{R}^4$. The convex hull $\overline{O}$ of the set O is an infinite polyhedral 4-dimensional subset of C. Its boundary Γ is an infinite 3-dimensional polyhedral complex. Let $U \subset O$ be the group of strictly positive units — a free Abelian group of rank 3. The action of a positive unit on Γ induces a permutation of its 3-dimensional polyhedra. We will study fundamental domain of action of U on Γ. This domain — a finite union of 3-dimensional polyhedrons with pairwise identified faces is homeomorphic to 3-dimensional torus [2].

Galois group G of our field K is either $\mathbb{Z}_4$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In both cases G contains an invariant subgroup of order 2, so K contains a subfield L of degree 2. L is a decomposition field of a second degree polynomial with rational coefficients and positive discriminant d. Hence, K is a second degree algebraic extension of L, defined by polynomial $r \in \mathbb{Q}(\sqrt{d})[x]$.

Let $r = x^2+2sx+t$, $s,t \in \mathbb{Q}(\sqrt{d})$, then roots x_1, x_2 of r are of the form

$$x_{1,2} = -s \pm \sqrt{s^2-t} \in K \Rightarrow \sqrt{s^2-t} \in K.$$

If $s^2-t = m+n\sqrt{d}$, $m,n \in \mathbb{Q}$, then $\sqrt{m+n\sqrt{d}} \in K$, i.e. all conjugates $\pm\sqrt{m \pm n\sqrt{d}}$ belong to K. Thus, the field K is the decomposition field of the biquadratic polynomial

$$q = (x-\sqrt{m+n\sqrt{d}})(x+\sqrt{m+n\sqrt{d}})(x-\sqrt{m-n\sqrt{d}})(x+\sqrt{m-n\sqrt{d}}) = \\ = x^4-2mx^2+m^2-n^2d = x^4-2ax^2+b.$$

Such biquadratic polynomial defines a totally real Galois field if: 1) all its roots are real, i.e. if $a>0$, $b>0$ and $d=a^2-b>0$; 2) a root $y_1 = \sqrt{a+\sqrt{d}}$ of q polynomially generates all other roots of q, in particular, the root $y_2 = \sqrt{a-\sqrt{d}}$, which can represented only in the form $y_2 = ky_1^3+ly_1$. This gives us two possibilities:

- either $ky_2^3+ly_2 = -y_1$, then $G=\mathbb{Z}_4$, $a^2b-b^2 = bd = c^2$, $l = (2a^2-b)/c = (a^2+d)/c$ and $k = -a/c$;
- or $ky_2^3+ly_2 = y_1$, then $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $b = c^2$, $l = 2a/c$ and $k = -1/c$.

Remark 1. If b is a full square and a^2b-b^2 is also a full square, then polynomial x^4-2ax^2+b is reducible.

In what follows we'll assume that $2a$ and b are positive integers. Let us describe at first the structure of the fundamental domain for fields with smallest a and b: for the field $K_1 = \mathbb{Q}[x]/(x^4-4x^2+1)$ and for the field $K_2 = \mathbb{Q}[x]/(x^4-4x^2+2)$ (polynomial x^4-3x^2+1 is reducible).

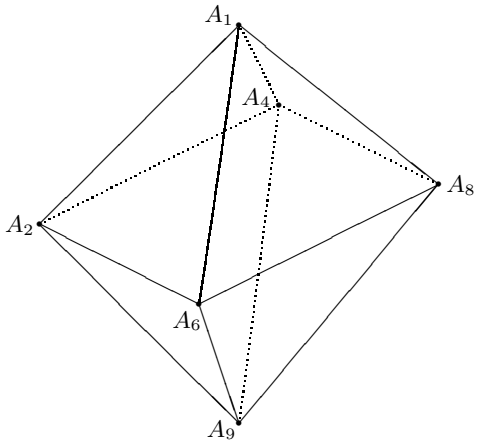
3. The field $K_1 = \mathbb{Q}[x]/(x^4-4x^2+1)$

Galois group here is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Transformations $x \mapsto x^3-4x$ and $x \mapsto -x$ are generators of Galois group. Elements kx^3+lx^2+mx+n, $k,l,m,n \in \mathbb{Z}$ are integral.

Hyperplane $k+l+n-1=0$ is a support hyperplane for the set O. It contains 9 integral strictly positive elements:

$$\begin{array}{lll} A_1=(-2,0,6,3), & A_2=(-2,3,2,0), & A_3=(-1,0,3,2),\\ A_4=(-1,0,4,2), & A_5=(-1,1,2,1), & A_6=(-1,2,0,0),\\ A_7=(-1,2,1,0), & A_8=(0,0,0,1), & A_9=(0,1,0,0). \end{array}$$

Elements A_1, A_2, A_4, A_6, A_8 and A_9 are units. Element A_3 is a midpoint of segment $[A_1, A_8]$ and has norm 4; element A_7 is a midpoint of segment $[A_2, A_9]$ and has norm 4; element A_5 is a midpoint of segment $[A_4, A_6]$ and has norm 9. The convex hull of these points in our hyperplane is an octahedron:



Faces of this octahedron are not identified by the action of group U. Hence, the construction of fundamental domain is not finished.

Indeed, hyperplane $4k+5l/2+m+n-1=0$ is also a support hyperplane for the set O. It contains 5 integral strictly positive elements:

$$A_1,\ A_3,\ A_6,\ A_8 \text{ and also an element } A_{10}=(0,0,-1,2),$$

which is a unit. The convex hull of these points is a tetrahedron, which is glued to our octahedron (by the triangular face $A_1A_6A_8$). In thus obtained polyhedron some faces (not all!) are identified:

(1) multiplication by the unit A_9 identifies the face $A_1A_8A_{10}$ with the face $A_2A_6A_9$: $A_1 \to A_2$, $A_8 \to A_9$, $A_{10} \to A_6$;
(2) multiplication by the unit A_4 identifies the face $A_6A_8A_{10}$ with the face $A_1A_2A_4$: $A_6 \to A_2$, $A_8 \to A_4$, $A_{10} \to A_1$;
(3) multiplication by the unit A_{10} identifies the face $A_4A_8A_9$ with the face $A_1A_6A_{10}$: $A_4 \to A_1$, $A_8 \to A_{10}$, $A_9 \to A_6$.

Faces $A_1A_4A_8$, $A_1A_2A_6$, $A_6A_8A_9$ $A_2A_4A_9$ are "free". Hence, the construction of fundamental domain is not finished.

Indeed, hyperplane $k+l/2+n-1=0$ is also a support hyperplane for the set O. It contains 5 integral strictly positive elements:

$$A_1,\ A_3,\ A_4,\ A_8 \text{ and also an element } A_{11}=(-4,-2,15,8),$$

which is a unit. The convex hull of these points is a tetrahedron, which is glued to our octahedron (by the triangular face $A_1A_4A_8$). In thus obtained polyhedron all its 2-dimensional faces are pairwise identified:

(1) multiplication by the unit A_9 identifies the face $A_1A_8A_{11}$ with the face $A_2A_4A_9$: $A_1 \to A_2$, $A_8 \to A_9$, $A_{11} \to A_4$;
(2) multiplication by the unit A_6 identifies the face $A_4A_8A_{11}$ with the face $A_1A_2A_6$: $A_4 \to A_2$, $A_8 \to A_6$, $A_{11} \to A_1$;
(3) multiplication by the unit A_{11} identifies the face $A_6A_8A_9$ with the face $A_1A_4A_{11}$: $A_6 \to A_1$, $A_8 \to A_{11}$, $A_9 \to A_4$.

The construction of the fundamental domain is finished.

4. The field $K_2 = \mathbb{Q}[x]/(x^4 - 4x^2 + 2)$

Galois group here is $\mathbb{Z}_4$. Transformation $x \mapsto x^3 - 3x$ is a generator of Galois group. Elements $kx^3 + lx^2 + mx + n$, $k, l, m, n \in \mathbb{Z}$ are integral.

Hyperplanes

$$\begin{array}{ll} \Pi_1 : k + 2l + m + 2n - 2 = 0, & \Pi_2 : 3k + 4l + 2m + 4n - 4 = 0, \\ \Pi_3 : k + 2l + 2n - 2 = 0, & \Pi_4 : k + 4l - m + 4n - 4 = 0, \\ \Pi_5 : k - 2l + m - 2n + 2 = 0, & \Pi_6 : 3k - 4l + 2m - 4n + 4 = 0, \\ \Pi_7 : k - 2l - 2n + 2 = 0, & \Pi_8 : k - 4l - m - 4n + 4 = 0 \end{array}$$

are support hyperplanes for the set O. Each hyperplane contains 5 integral elements. Three elements $A = (0, 0, 0, 1)$, $B = (0, -2, 0, -1)$ and $(A + B)/2$ belong to each hyperplane. Thus, the convex hull of the set $O \cap \Pi_i$ is an tetrahedron.

Besides A, B and $(A + B)/2$, hyperplane Π_1 contains elements $C = (1, 0, -3, 2$ and $D = (0, 1, -2, 1)$, hyperplane Π_2 — D and $E = (-2, 4, 1, -2)$, hyperplane Π_3 — E and $F = (-2, 3, 2, -1)$, hyperplane Π_4 — F and $G = (-1, 0, 3, 2)$, hyperplane Π_5 — G and $H = (0, 1, 2, 1)$, hyperplane Π_6 — H and $K = (2, 4, -1, -2)$, hyperplane Π_7 — K and $L = (2, 3, -2, -1)$, hyperplane Π_8 — L and C. The union of these tetrahedrons is something like two octagonal pyramids $ACDEFGHKL$ and $BCDEFGHKL$ with common base $CDEFGHKL$. Here elements A, B, D, F, H and L are units and elements C, E, G and K have norm 2.

In thus constructed polyhedron some faces (not all!) are identified:

- the multiplication by unit H identifies the face ACD with the face BHK: $A \to H$, $C \to K$, $D \to B$;
- the multiplication by unit L identifies the face AFG with the face BKL: $A \to L$, $F \to B$, $G \to K$;
- the multiplication by unit D identifies the face AGH with the face BDE: $A \to D$, $G \to E$, $H \to B$;
- the multiplication by unit F identifies the face ALC with the face BEF: $A \to F$, $C \to E$, $L \to B$.

Faces ADE, AEF, AHK, AKL, BCD, BFG, BGH and BLC are "free". The construction of the fundamental domain is not finished.

$3k + 2l + m + n - 1 = 0$ also is a support hyperplane for the set O. It contains 7 integral strictly positive elements: A, D, E, F, $M = (0, 0, -1, 2)$, $N = (-2, 4, 0, -1)$ and $(-1, 2, 0, 0) = (A + N)/2 = (D + F)/2 = (E + M)/2$. The convex hull of these points is an octahedron, where (A, N), (D, F) and (E, M) — pairs of opposite vertices. Here element N is a unit and element M has norm 2.

In thus obtained polyhedron — two octagonal pyramids with a common base and octahedron glued to them by the faces ADE and AFE, the following faces are identified:

- the multiplication by unit N identifies the face AHK with the face EFN: $A \to N$, $H \to F$, $K \to E$;
- the multiplication by unit N identifies the face AKL with the face NDE: $A \to N$, $K \to E$, $L \to D$;
- the multiplication by unit H identifies the face ADM with the face BGH: $A \to H$, $D \to B$, $M \to G$;
- the multiplication by unit L identifies the face AFM with the face BCL: $A \to L$, $F \to B$, $M \to C$;
- the multiplication by unit $B^{-1}F$ identifies the face BCD with the face FMN: $B \to F$, $C \to M$, $D \to N$;
- the multiplication by unit $B^{-1}D$ identifies the face BFG with the face DMN: $B \to D$, $F \to N$, $G \to M$.

The construction of the fundamental domain is finished.

5. The simplest fundamental domains in the case of cyclic Galois group

The combinatorics of a fundamental domain can be quite complex. The natural question here is: how simple it can be? It turns out that there exists an one parameter family of fields with a simple domains and also one field apart. At first we will study the special field.

5.1. **The field $K = \mathbb{Q}[x]/(x^4 - 15x^2 + 45)$.** The transformation $x \mapsto x^3/3 - 3x$ is a generator of Galois group. Elements of the form

$$(k,l,m,n) + (\frac{i}{6},0,0,\frac{i}{2}) + (0,\frac{j}{6},\frac{j}{2},\frac{j}{2}), \quad k,l,m,n \in \mathbb{Z}, \quad i,j = 0,\dots,5,$$

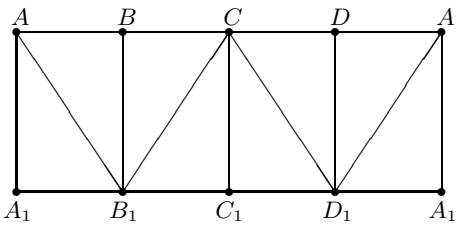
are integral.

Hyperplane $\Pi : 3k + 6l + m + n = 1$ is a support hyperplane for the set O. It contains 8 integral strictly positive elements — all of them are units:

$$A = (0,0,0,1),\ B = \Big(\frac{1}{6},-\frac{1}{3},-2,\frac{9}{2}\Big),\ C = \Big(\frac{1}{3},-\frac{1}{2},-\frac{7}{2},\frac{13}{2}\Big),\ D = \Big(\frac{1}{6},-\frac{1}{6},-\frac{3}{2},3\Big),$$

$$A_1 = \Big(0,\frac{1}{3},0,-1\Big),\ B_1 = \Big(0,\frac{1}{6},-\frac{1}{2},\frac{1}{2}\Big),\ C_1 = \Big(\frac{1}{6},\frac{1}{6},-\frac{3}{2},1\Big),\ D_1 = \Big(\frac{1}{6},\frac{1}{3},-1,-\frac{1}{2}\Big).$$

The convex hull of the set $O \cap \Pi$ is a decahedron with 2 quadrangular faces $ABCD$ and $A_1B_1C_1D_1$ — bases, and 8 triangular faces. Bases are parallelograms, that are parallel to each other. In the picture below is presented the pattern of triangular faces of this "prism":



Faces of $ABCDA_1B_1C_1D_1$ are identified in the following way:

- the multiplication by unit A_1 identifies the base $ABCD$ with the face $A_1B_1C_1D_1$: $A \to A_1$, $B \to B_1$, $C \to C_1$, $D \to D_1$;
- the multiplication by unit D identifies the face AA_1B_1 with the face CDD_1: $A \to D$, $A_1 \to D_1$, $B_1 \to C$;
- the multiplication by unit D_1 identifies the face ABB_1 with the face CC_1D_1: $A \to D_1$, $B \to C$, $B_1 \to C_1$;
- the multiplication by unit B identifies the face AA_1D_1 with the face BCB_1: $A \to B$, $A_1 \to B_1$, $D_1 \to C$;
- the multiplication by unit B_1 identifies the face ADD_1 with the face CB_1C_1: $A \to B_1$, $D \to C$, $D_1 \to C_1$.

The construction of the fundamental domain is finished.

5.2. **Fields K_n.** Let $p_n = x^4 - (n^2+4)x^2 + n^2 + 4$, where n is an odd positive integer, and let K_n be the field, defined by p_n.

Lemma 1. *Polynomials p_n are irreducible.*

Proof. If p_n is reducible, then either it has a positive integral root, or it has an irreducible factor of degree 2.

Let a be a positive integral root of p_n: $a^4 - (n^2+4)a^2 + n^2 + 4 = 0$. Each prime factor of a is a factor of the number n^2+4 and vice versa. Let q be such prime factor, $\alpha > 0$ be an exponent of a and $\beta > 0$ be an exponent of n^2+4. In the set of 3 numbers $\{4\alpha, 2\alpha+\beta, \beta\}$ cannot be exactly one minimal, thus, $4\alpha = \beta$. As this is true for each prime factor of a, then n^2+4 is a full square — a contradiction.

Let p_n be a product of two irreducible polynomials of degree 2 with integral coefficients, then these factors are $x^2 - ax + b$ and $x^2 + ax + b$:

$$(x^2 - ax + b)(x^2 + ax + b) = x^4 - (n^2+4)x^2 + n^2 + 4.$$

But then again n^2+4 is a full square. □

Remark 2. Fields K_n are not pairwise different. Indeed, fields K_1 and K_{11} coincide: if $y = -3x^3 + 5x \in K_1$, then $y^4 - 125y^2 + 125 = 0$.

5.3. **The field $K_1 \simeq \mathbb{Q}[x]/(x^4 - 5x^2 + 5)$.** The transformation $x \mapsto x^3 - 3x$ is a generator of Galois group. Elements of the form $kx^3 + lx^2 + mx + n$, $k, l, m, n \in \mathbb{Z}$ are integral.

Hyperplane $\Pi : 2k + 2l + m + n = 1$ is a support hyperplane for the set O. Π contains 10 integral strictly positive elements:

$$\begin{array}{ll} A = (0,0,0,1), & A_1 = (0,1,0,-1), \\ B = (0,0,-1,2), & B_1 = (-1,2,1,-2), \\ C = (2,-2,-8,9), & C_1 = (0,1,-2,1), \\ D = (2,-2,-7,8), & D_1 = (1,0,-3,2), \\ E = (1,-1,-4,5), & E_1 = (0,1,-1,0). \end{array}$$

The convex hull of this points in Π is a decahedron, combinatorially the same, as in subsection 5.1. The only difference is that centers E and E_1 of bases $ABCD$ and $A_1B_1C_1D_1$, respectively, also are integral and strictly positive.

5.4. **The field** $K_3 \simeq \mathbb{Q}[x]/(x^4-13x^2+13)$**.** The transformation $x \mapsto x^3/3-11x/3$ is a generator of Galois group. Elements of the form

$$(k,l,m,n)+\left(\frac{i}{3},0,\frac{i}{3},0\right)+\left(0,\frac{j}{3},0,\frac{j}{3}\right), \quad k,l,m,n \in \mathbb{Z}, \quad i,j=0,1,2,$$

are integral.

Hyperplane $\Pi : 2k+2l+m+n=1$ is a support hyperplane for the set O. Π contains 36 integral strictly positive elements. The convex hull of this points in Π is a decahedron, combinatorially the same, as in subsection 5.1. Besides vertices A,B,C,D,A_1,B_1,C_1,D_1 of decahedron the intersection $\Pi \cap O$ contains:

(1) centers of bases $ABCD$ and $A_1B_1C_1D_1$;
(2) two points on each segment $[A,A_1]$, $[B,B_1]$, $[C,C_1]$ and $[D,D_1]$, which divide each segment into 3 equal parts;
(3) 18 points inside the decahedron.

5.5. **General case.** Let assume that a field K_n is not isomorphic to any field K_m, $m<n$. Put $n=2a+1$, then transformation

$$x \mapsto \frac{x^3}{2a+1}-\frac{4a^2+4a+3}{2a+1}\,x$$

is a generator of Galois group. Elements of the form

$$(k,l,m,n)+i\left(\frac{1}{2a+1},0,\frac{2a-1}{2a+1},0\right)+j\left(0,\frac{1}{2a+1},0,\frac{2a-1}{2a+1}\right),$$

$$k,l,m,n \in \mathbb{Z}, \quad i,j=0,\dots,2a,$$

are integral.

Hyperplane $\Pi : 2k+2l+m+n=1$ is a support hyperplane for the set O. The convex hull of the set $\Pi \cap O$ in Π is the same decahedron with vertices A,B,C,D,A_1,B_1,C_1,D_1 and bases — parallelograms $ABCD$ and $A_1B_1C_1D_1$. In internal coordinates k,l,m of the hyperplane Π bases lie in parallel planes $k+l=0$ and $k+l=1$. All vertices are units: $A=1$,

$$B=\frac{2(a+1)x^3-2(a+1)x^2-(8a^3+16a^2+16a+7)x+8a^3+16a^2+18a+8}{2a+1},$$

$$C=2x^3-2x^2-(8a^2+8a+8)x+8a^2+8a+9,$$

$$D=\frac{2ax^3-2ax^2-(8a^3+8a^2+8a+1)x+8a^3+8a^2+10a+2}{2a+1},$$

$$A_1=x^2-1, \quad B_1=\frac{x^3+2ax^2-(2a+3)x+2}{2a+1},$$

$$C_1=x^2-2x+1, \quad D_1=\frac{-x^3+(2a+2)x^2-(2a-1)x-2}{2a+1}.$$

The multiplication by unit A_1 identifies bases $ABCD$ and $A_1B_1C_1D_1$. Other 8 triangular faces of the "prism" are pairwise identified in the following way:

$$\triangle AA_1B_1 \sim \triangle CDD_1, \quad \triangle ABB_1 \sim \triangle CC_1D_1,$$
$$\triangle BB_1C \sim \triangle D_1AA_1, \quad \triangle B_1CC_1 \sim \triangle DD_1A.$$

Besides vertices, the intersection $\Pi \cap O$ contains centers of bases, $2a$ points on each segment $[A, A_1]$, $[B, B_1]$, $[C, C_1]$ and $[D, D_1]$, which divide each segment into $2a+1$ equal parts, and points inside the decahedron.

Remark 3. Integers of the field K_{11} are of the form

$$(k, l, m, n) + i\left(\frac{1}{275}, 0, \frac{3}{11}, 0\right) + j\left(0, \frac{1}{55}, 0, \frac{4}{11}\right).$$

Therefore, the described above "prism" contains *several* fundamental decahedrons of the field K_{11}.

6. Simple fundamental domains in the case of Galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

In the case of Klein group we don't know fields, where fundamental domain consists of one 3-dimensional face of Γ, but there are examples, where this domain consists of two.

6.1. **The field $K = \mathbb{Q}[x]/(x^4 - 9x^2 + 9)$.** Transformations $x \mapsto x^3/3 - 3x$ and $x \mapsto -x$ are generators of Galois group. An element $kx^3 + lx^2 + mx + n \in K$ is integral, if $3k, 3l, m, n \in \mathbb{Z}$.

Hyperplane $\Pi_1 : 6k + 3l + m + n - 1 = 0$ is a support hyperplane for the set O. It contains 14 integral strictly positive elements. This elements belong to 2 parallel 2-dimensional planes in Π_1: $P_0 : 9k+3l+m = 0$ and $P_1 : 9k+3l+m = 1$ (seven in each). Elements $A = (0, 0, 0, 1)$, $B = (0, 1/3, -1, 1)$, $C = (-1/3, 4/3, -1, 0)$, $D = (-2/3, 2, 0, 1)$, $E = (-2/3, 5/3, 1, -1)$, $F = (-1/3, 2/3, 1, 0)$ and $G = (-1/3, 1, 0, 0$ belong to the plane P_0. Points A, B, C, D, E and F are vertices of centrally-symmetric hexagon; A, C, D and F are units; B and E have norm 4; G is the center of hexagon and has norm 9.

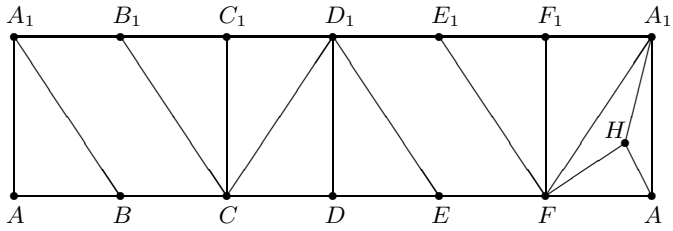
Elements $A_1 = (0, 1/3, 0, 0)$, $B_1 = (-1/3, 4/3, 0, -1)$, $C_1 = (-4/3, 4, 1, -4)$, $D_1 = (-2, 17/3, 2, -6)$, $E_1 = (-5/3, 14/3, 2, -5)$, $F_1 = (-2/3, 2, 1, -2)$ and $G = (-1, 3, 1, -3$ belong to the plane P_1. Points A_1, B_1, C_1, D_1, E_1 and F_1 are vertices of centrally-symmetric hexagon; A_1, C_1, D_1 and F_1 are units; B_1 and E_1 have norm 4; G_1 is the center of hexagon and has norm 9.

Polyhedron M_1 — the convex hull in Π_1 of the set $O \cap \Pi_1$, is a "prism" with two parallel hexagon bases and ten "side" faces: ABA_1, A_1B_1CB, B_1C_1C, C_1D_1C, CDD_1, DED_1, D_1E_1FE, E_1F_1F, FAA_1, F_1A_1F. The multiplication by unit A_1 identifies bases. Also

- the multiplication by unit F identifies the face ABA_1 with the face E_1F_1F: $A \to F$, $B \to E_1$, $A_1 \to F_1$;
- the multiplication by unit $A_1^{-1}F$ identifies the face A_1B_1CB with the face D_1E_1FE: $A_1 \to F$, $B_1 \to E_1$, $C \to D_1$, $B \to E$;
- the multiplication by unit $C^{-1}D$ identifies the face B_1C_1C with the face DED_1: $B_1 \to E$, $C_1 \to D_1$, $C \to D$.

Hyperplane $\Pi_2 : 3k+3l+n-1 = 0$ also is a support hyperplane for the set O. The intersection $\Pi_2 \cap O$ contains 4 elements: A, A_1, F and a unit $H = (-1/3, 0, 2, 2)$.

Polyhedron M_2 — the convex hull in Π_2 of the set $O \cap \Pi_2$, is a tetrahedron, glued to triangular "side" face of M_1. The "side" surface of $M_1 \cup M_2$ is presented below:



Faces C_1D_1C, CDD_1, F_1A_1F, AA_1H, A_1FH and FAH are pairwise identified in the following way:

- the multiplication by unit $C^{-1}A$ identifies the face C_1D_1C with the face AA_1H: $C_1 \to A_1$, $D_1 \to H$, $C \to A$;
- the multiplication by unit $C^{-1}A_1$ identifies the face CDD_1 with the face A_1FH: $C \to A_1$, $D \to H$, $D_1 \to F$;
- the multiplication by unit $A_1^{-1}A$ identifies the face F_1A_1F with the face FAH: $F_1 \to F$, $A_1 \to A$, $F \to H$.

The construction of the fundamental domain is finished.

6.2. **The field $K = \mathbb{Q}[x]/(x^4 - 25x^2 + 25)$.** Transformations $x \mapsto x^3/5 - 5x$ and $x \mapsto -x$ are generators of Galois group. An element $kx^3 + lx^2 + mx + n \in K$ is integral, if $5k, 5l, m, n \in \mathbb{Z}$. Hyperplanes $\Pi_1 : 5k + 5l + m + n = 1$ and $\Pi_2 : -75k + 20l - 3m + n = 1$ are support hyperplanes for the set O. The intersection $\Pi_1 \cap O$ contains 14 elements and the intersection $\Pi_2 \cap O$ — 4. The combinatorial structure of the fundamental polyhedron here is the same as in the previous example.

Remark 4. The case of the field $K = \mathbb{Q}[x]/(x^4 - 49x^2 + 49)$ is significantly more complex.

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