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Abstract

This article contributes to research dealing with the optimal dividend policy problem of a firm whose goal is to maximize the expected total discounted dividend payments before bankruptcy. We consider a model of firm whose cash surplus exhibits regime switching, but unlike the existing literature, we exclude diffusion from our model in order to overcome the well-known shortcoming of infinite money flows. Hence, we assume firm's cash surplus follows telegraph process, which leads to the problem of singular stochastic control. Surprisingly, this problem turns out to be more complicated than the ones arising in the models involving diffusion. We solve this problem using the method of variational inequalities and show that the optimal dividend policy is defined by two thresholds.

JEL classification: C61, G35.

Keywords: optimal dividend policy, regime switching, telegraph process.

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1 Introduction and the formulation of the model

The optimal dividend problem was first discussed in [9]. The key idea of this work was that the goal of a company is to maximize the expected present value of the flow of dividends before bankruptcy. In the simplest discrete framework it was shown that the optimal dividend strategy is of a threshold type — the surplus above certain level should be paid as dividends, and if the capital is less than this level, company should not pay any dividends. The renewed interest in the optimal dividend problems was stimulated by the articles [24], [15] and [1], which addressed the optimal dividend problems in continuous environment with firm's cash reserves following Brownian motion with drift. Numerous works which followed after them considered the optimal dividend problems for more complicated dynamics of cash reserves based on Brownian motion, see for example [4], [31], [10], [23], [28]. Another big strand of optimal dividend policy literature is based on compound Poisson process and presented in, for example, [11], [2] and [13] among many others.

Our point of interest is an optimal dividend problem in the model where firm's cash reserves follow telegraph process. Introduced in [14] and [18], telegraph process was extensively analyzed, for example, in [22], [12] and [3]. Notable generalizations of telegraph process include telegraph process with random velocities introduced in [30] and with alternating renewal process defining switching times in [33]. A jump-telegraph process with jumps occurring at the moments of switching is introduced in [25] and analyzed in [7], [19] and [6]. Telegraph process and its generalizations are widely used in finance as alternatives to the Brownian motion since it is free from the limitations of the Brownian motion such as infinite propagation velocities and independent log-returns increments on separated time intervals. In [21] telegraph process is used in the context of stochastic volatility. In [5] the very basic model of evolution of stock prices based on telegraph process is presented. In [8] a geometric telegraph process is used to describe the dynamics of the price of risky assets, and the analogue of the Black-Sholes equation is derived. In [25], [27] and [26], the jump-telegraph process is used to develop an arbitrage-free model of financial market. In [20] it is used in the option pricing model.

The models of optimal dividend policy with regime switching, such as [29], [34], [17], [32], [16] among others are closest to ours, but they involve diffusion, which is absent in our model. For example, our model may be considered as just a special case of [29] with two states of the world, first one with a positive drift coefficient, and the second one with a negative drift coefficient with diffusion coefficients set to zero. But, as we shall see, the absence of diffusion completely changes the nature of the problem and leads to substantially different results.

We now formulate our model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space of trajectories of changes of the state of the world and a filtration $\mathcal{F}(t)$ represents the information up to time t. We assume that cash reserves of a firm X_t follow the equation

$$X(t) = x + \int_0^t \mu_{\pi(u)} du - L(t), \tag{1}$$

where x is the initial level of reserves, $\pi(u) \in \{0, 1\}$ is the state of the world, $\mu_0 < 0$ and $\mu_1 > 0$ are the drift coefficients and $L(t) \in \mathcal{F}(t)$ is the total amount of dividends paid up to the time t, which is non-negative and non-decreasing and also assumed to be left-continuous with right limits. The switching between the states of the world is defined by the frequencies $\Lambda_0 > 0$ and $\Lambda_1 > 0$: if the state of the world is 0 then the probability of switching to the state 1 during the period of time Δt is $\Lambda_0 \Delta t$ and similarly for the state 1. The goal of firm is to maximize the expected total amount of dividends paid before bankruptcy time τ, which occurs when firm's level of reserves hits zero for the first time:

$$J(s, x, L(\cdot)) = \left[\int_0^\tau \mathrm{e}^{-ct} dL(t)|_{\pi(0)=s, X(0)=x} \right] \to \max_{L(\cdot)}. \tag{2}$$

We denote the admissible dividend policy, which maximizes $J(s, L(\cdot))$, by L^* and then denote

$$P(s, x) = EJ(s, x, L^*(\cdot)).$$

2 Analysis of the model

2.1 Variational inequalities

In this subsection we derive variational inequalities which the solution of the optimal dividend policy problem must satisfy. Consider a small interval $[0, \delta]$. Fix some ε and consider an admissible policy $L^{s,y}(\cdot)$ such that for any $y > 0$ and $s \in \{0, 1\}$

$$\mathbb{E}J(s, y, L^{s,y}(\cdot)) \geq P(s, y) - \varepsilon.$$

Let $W(t) = x + \mu_{\pi(t)} t$. Consider the following policy:

$$L_\varepsilon(t) = \begin{cases} 0, t < \delta, \\ L^{\pi(\delta), W(\delta)}(t-\delta), t \geq \delta. \end{cases}$$

This policy means that we pay no dividends before δ and then switch to suboptimal policy. We get

$$P(s,x) \geq \mathrm{e}^{c\delta}\mathbb{E}\left[P(\pi(\delta), W(\delta)) - \varepsilon\right]. \tag{3}$$

By the definition of the telegraph process

$$EP(\pi(\delta), W(\delta)) = (1-\Lambda_s\delta)P(s, x+\mu_s\delta) + \Lambda_s\delta P(1-s, x) + o(\delta). \tag{4}$$

Using (4) and the arbitrariness of ε, (3) may re rewritten as

$$P(s,x) \geq (1-c\delta)[(1-\Lambda_s\delta)P(s, x+\mu_s\delta) + \Lambda_s\delta P(1-s,x)].$$

Assuming $P(s,x)$ is continuously differentiable and using Taylor expansion, we get

$$P(s,x) \geq (1-c\delta)[(1-\Lambda_s\delta)(P(s,x) + \mu_s\delta\frac{\partial}{\partial x}P(s,x)) + \Lambda_s\delta P(1-s,x)].$$

Simplifying this expression and tending δ to zero we get the first variational inequality:

$$\mu_s\frac{\partial}{\partial x}P(s,x) - (\Lambda_s + c)P(s,x) + \Lambda_s P(1-s,x) \leq 0, s \in \{0,1\}. \tag{5}$$

To obtain another one, we fix x, $\delta > 0$ and denote $y = x - \delta$. Consider the policy $L_\varepsilon(t) = \delta + L^{x-\delta}(t)$, which prescribes to pay δ instantaneously and then use the policy $L^{x-\delta}$. We get

$$P(s,x) \geq \delta + P(s, x-\delta) + \varepsilon.$$

Again using Taylor expansion and arbitrariness of ε we get the second variational inequality

$$\frac{\partial}{\partial x}P(s,x) \geq 1. \tag{6}$$

Now combining (5), (6) and the obvious boundary condition $P(0,0) = 0$, we get the following

Theorem 1. Let the function P be continuously differentiable. Then it satisfies the following Hamilton-Jacobi-Bellman equation:

$$\max\{\mu_s \frac{\partial}{\partial x} P(s,x) - (\Lambda_s + c)P(s,x) + \Lambda_s P(1-s,x), 1 - \frac{\partial}{\partial x} P(s,x)\}, s \in \{0,1\}, \\ P(0,0) = 0. \tag{7}$$

2.2 Solution of Hamilton-Jacobi-Bellman equation

Standard arguments verify that $P(s,\cdot)$ is concave. This implies that there exist $m_s, s \in \{0,1\}$ such that $P(s,x) > 1$ for $x < m_s$ and $P(s,x) = 1$ for $x \geq m_s$. Denote $m = \min(m_0, m_1)$ and $M = \max(m_0, m_1)$. We now analyze two different cases.

Case 1. $m_0 \geq m_1$. In this case we have three domains. In the lower domain $[0, m]$ it follows from (5) that function P follows equations

$$\mu_s \frac{\partial}{\partial x} P(s,x) - (\Lambda_s + c)P(s,x) + \Lambda_s P(1-s,x) = 0 \tag{8}$$

for $s \in \{0,1\}$ with the boundary condition $P(0,0) = 0$. Applying Laplace transform to it, we get

$$\xi \,\mathcal{L}\,(s,\xi) - P\,(s,0) = -\frac{\Lambda_s \mathcal{L}\,(1-s,\xi)}{\mu_s} + \frac{\mathcal{L}\,(s,\xi)\,\Lambda_s}{\mu_s} + \frac{\mathcal{L}\,(s,\xi)\,c}{\mu_s},$$

where $\mathcal{L}(s,\xi)$ is the Laplace transform of $P(s,x)$. This leads to

$$\mathcal{L}\,(0,\xi) = -\frac{P\,(1,0)\,\Lambda_0 \mu_1}{\xi^2 \mu_0 \mu_1 - c\xi\,\mu_0 - c\xi\,\mu_1 - \xi\,\Lambda_0 \mu_1 - \xi\,\Lambda_1 \mu_0 + c^2 + c\Lambda_0 + c\Lambda_1}, \\ \mathcal{L}\,(1,\xi) = -\frac{-P\,(1,0)\,\xi\,\mu_0 \mu_1 + P\,(1,0)\,c\mu_1 + P\,(1,0)\,\Lambda_0 \mu_1}{\xi^2 \mu_0 \mu_1 - c\xi\,\mu_0 - c\xi\,\mu_1 - \xi\,\Lambda_0 \mu_1 - \xi\,\Lambda_1 \mu_0 + c^2 + c\Lambda_0 + c\Lambda_1}. \tag{9}$$

Considering denominator as the square polynomial on ξ, we may rewrite (9) as

$$\mathcal{L}\,(0,\xi) = -\frac{P\,(1,0)\,\Lambda_0}{\mu_0\,(\xi - a)\,(\xi - b)}, \mathcal{L}\,(1,\xi) = -\frac{-P\,(1,0)\,\xi\,\mu_0 \mu_1 + P\,(1,0)\,c\mu_1 + P\,(1,0)\,\Lambda_0 \mu_1}{\mu_1 \mu_0\,(\xi - a)\,(\xi - b)}, \tag{10}$$

where

$$a = \frac{c\mu_0 + c\mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 + \sqrt{\Omega}}{2\mu_1 \mu_0}, b = \frac{c\mu_0 + c\mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 - \sqrt{\Omega}}{2\mu_1 \mu_0} \tag{11}$$

and

$$\Omega = c^2{\mu_0}^2 - 2\,c^2\mu_0\mu_1 + c^2\mu_1^2 - 2\,c\Lambda_0\mu_0\mu_1 + 2\,c\Lambda_0\mu_1^2 + \\ + 2\,c\Lambda_1\mu_0^2 - 2\,c\Lambda_1\mu_0\mu_1 + \Lambda_0^2\mu_1^2 + 2\,\Lambda_0\Lambda_1\mu_0\mu_1 + \Lambda_1^2\mu_0^2. \tag{12}$$

We now derive some inequalities which will be used later.

Lemma 1. For any values of the parameters of the model the following inequalities hold.

1. $a < 0$ and $b > 0$.
2. $-bc\mu_0 - b\Lambda_0\mu_1 - b\Lambda_1\mu_0 + c^2 + c\Lambda_0 + c\Lambda_1 > 0$.
3. $bc\mu_0 + bc\mu_1 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1 < 0$.
4. $-bc^2\mu_0 - 2\,bc\Lambda_1\mu_0 - b\Lambda_0\Lambda_1\mu_1 - b\Lambda_1^2\mu_0 + c^3 + c^2\Lambda_0 + 2\,c^2\Lambda_1 + c\Lambda_0\Lambda_1 + c\Lambda_1^2$ ¿ 0.

Proof. 1. Inequality $a < 0$ may be rewritten as

$$\sqrt{\Omega} < -(c\mu_0 + c\mu_1 + \Lambda_0\mu_1 + \Lambda_1\mu_0).$$

If the expression in parentheses is positive, it is obviously true. If it is negative, we square both sides of inequality and after some simplifications get

$$4\,c^2\mu_0\mu_1 + 4\,c\Lambda_0\mu_0\mu_1 + 4\,c\Lambda_1\mu_0\mu_1 < 0, \tag{13}$$

which is always true. Inequality $b > 0$ is also reduced to (13).

2. Substituting (11) we get

$$(c\mu_0 + \Lambda_0\mu_1 + \Lambda_1\mu_0)\,\sqrt{\Omega} < c^2\mu_0^2 - c^2\mu_0\mu_1 + c\Lambda_0\mu_1^2 + 2c\Lambda_1\mu_0^2 - c\Lambda_1\mu_0\mu_1 + \Lambda_0^2{\mu_1}^2 + 2\,\Lambda_0\Lambda_1\mu_0\mu_1 + \Lambda_1^2\mu_0^2.$$

The expression on the right side is always positive. If the expression in parentheses on the left side is negative, the proof is concluded. If it is positive, we square both sides of inequality and get

$$-4\,c^3\Lambda_0\mu_0^2\mu_1^2 + 4\,c^3\Lambda_0\mu_0\mu_1^3 - 4\,c^2\Lambda_0^2\mu_0^2\mu_1^2 + 4\,c^2\Lambda_0^2\mu_0\mu_1^3 - 4\,c^2\Lambda_0\Lambda_1\mu_0^2\mu_1^2 + 4\,c^2\Lambda_0\Lambda_1\mu_0\mu_1^3 < 0,$$

which is always true.

3. Substituting (11) we get

$$(c\mu_0 + c\mu_1 + \Lambda_0\mu_1 + \Lambda_1\mu_0)\,\sqrt{\Omega} < c^2\mu_0^2 + c^2\mu_1^2 + 2\,c\Lambda_0\mu_1^2 + 2\,c\Lambda_1{\mu_0}^2 + \Lambda_0^2\mu_1^2 + 2\,\Lambda_0\Lambda_1\mu_0\mu_1 + \Lambda_1^2\mu_0^2$$

If the expression in parentheses on the left side is negative, the proof is concluded. If it is positive, we square both sides of inequality and get

$$0 < 4\,c^4\mu_0^2\mu_1^2 + 8\,c^3\Lambda_0\mu_0^2\mu_1^2 + 8\,c^3\Lambda_1\mu_0^2\mu_1^2 + 4\,c^2\Lambda_0^2\mu_0^2{\mu_1}^2 + 8\,c^2\Lambda_0\Lambda_1\mu_0^2\mu_1^2 + 4\,c^2\Lambda_1^2\mu_0^2\mu_1^2,$$

which is always true.

4. Substituting (11) in it, we get

$$\left(c^2\mu_0 + 2\,c\Lambda_1\mu_0 + \Lambda_0\Lambda_1\mu_1 + \Lambda_1^2\mu_0\right)\sqrt{\Omega} < c^3\mu_0^2 - c^3\mu_0\mu_1 - c^2\Lambda_0\mu_0\mu_1 + 3\,c^2\Lambda_1\mu_0^2 -$$
$$-\,2\,c^2\Lambda_1\mu_0\mu_1 + c\Lambda_0\Lambda_1\mu_0\mu_1 + c\Lambda_0\Lambda_1\mu_1^2 + 3\,c\Lambda_1^2{\mu_0}^2 - c\Lambda_1^2\mu_0\mu_1 + \Lambda_0^2\Lambda_1{\mu_1}^2 + 2\,\Lambda_0\Lambda_1^2\mu_0\mu_1 + \Lambda_1^3\mu_0^2.$$

The expression on the right side is always positive. Indeed, it may be rewritten as

$$c\Lambda_0\Lambda_1\mu_1^2 + \Lambda_0^2\Lambda_1\mu_1^2 + \left(3\,c{\Lambda_1}^2 - 3\,c^2\Lambda_1 + c^3 + \Lambda_1^3\right)\mu_0^2 - \mu_1\left(c^3 + c{\Lambda_1}^2 - 2\,\Lambda_0\Lambda_1^2 + 2\,c^2\Lambda_1\right)\mu_0. \quad (14)$$

The necessary condition for this inequality to hold is that the coefficient of μ_0^2 is positive:

$$3c\Lambda_1^2 - 3c^2\Lambda_1 + c^3 + \Lambda_1^3 > 0.$$

The cubic polynomial in the left side of inequality has one zero $\Lambda_1 - \sqrt[3]{2}\Lambda_1$ with respect to c, which is negative, and this polynomial is positive for big values of c, hence it is always positive and the necessary condition is satisfied. Also note that (14) is positive for $\mu_0 = 0$. Hence, if the derivative of (14) in $\mu_0 = 0$ is non-positive, inequality is proven. Assume it is positive:

$$-\mu_1 c^3 - c\Lambda_1^2\mu_1 + 2\Lambda_0\Lambda_1^2\mu_1 - 2c^2\Lambda_1\mu_1 > 0. \quad (15)$$

The maximum of (14) is achieved at

$$\tilde{\mu}_0 = \frac{1}{2}\frac{\mu_1\left(c^3 + c{\Lambda_1}^2 - 2\,\Lambda_0{\Lambda_1}^2 + 2\,c^2\Lambda_1\right)}{3\,c{\Lambda_1}^2 - 3\,c^2\Lambda_1 + c^3 + {\Lambda_1}^3}.$$

The value of (14) in this point is

$$-\frac{1}{4}\frac{\mu_1^2 c\Upsilon}{3\,c\Lambda_1^2 - 3\,c^2\Lambda_1 + c^3 + \Lambda_1^3}, \quad (16)$$

where

$$\Upsilon = c^5 + 4\,\Lambda_1 c^4 + 6\,c^3\Lambda_1^2 + 4\,c^2\Lambda_1^3 + c\Lambda_1^4 - 4\,\Lambda_1\left(-3\,c\Lambda_1 + c^2 + 3\,\Lambda_1^2\right)\Lambda_0^2 -$$
$$4\,\Lambda_1\left(c^3 - 2\,c^2\Lambda_1 + 2\Lambda_1^3 + 5c\Lambda_1^2\right)\Lambda_0.$$

For (16) to be positive, Υ has to be negative, because the expression in denominator is shown earlier to be positive. The coefficient of Λ_0^2 in Υ is negative, hence Υ is negative for big values of Λ_0. Now consider the minimal value of Λ_0 defined by (15):

$$\tilde{\Lambda}_0 = \frac{1}{2}\frac{\mu_1 c^3 + c\Lambda_1^2\mu_1 + 2\,c^2\Lambda_1\mu_1}{\mu_1\Lambda_1^2}.$$

The value of Υ in this point is

$$\Upsilon(\tilde{\Lambda}_0) = -\frac{c\,(3\,\Lambda_1^2 + 2\,c\Lambda_1 + c^2)\,(3\,c\Lambda_1^2 - 3c^2\Lambda_1 + c^3 + \Lambda_1^3)\,(\Lambda_1 + c)^2}{\Lambda_1^3} < 0.$$

It is left to show that derivative of Υ in the point Λ_0^* is negative. Indeed,

$$\frac{\partial}{\partial\Lambda_0}\Upsilon|_{\Lambda_0=\Lambda_0^*} = -4\,\frac{(2\Lambda_1^2 + 2\,c\Lambda_1 + c^2)\,(3c\Lambda_1^2 - 3\,c^2\Lambda_1 + c^3 + \Lambda_1^3)}{\Lambda_1} < 0.$$

■

Now we can invert Laplace transforms in (10) and get

$$\begin{aligned} P\,(0,x) &= \frac{P\,(1,0)\,\Lambda_0\left(\mathrm{e}^{bx} - \mathrm{e}^{ax}\right)}{\mu_0\,(a-b)}, x \in [0,m],\\ P\,(1,x) &= \frac{P\,(1,0)\left(\mathrm{e}^{ax}\,(a\mu_0 - c - \Lambda_0) + (-b\mu_0 + c + \Lambda_0)\,\mathrm{e}^{bx}\right)}{\mu_0\,(a-b)}, x \in [0,m]. \end{aligned} \tag{17}$$

The threshold level m is defined by the condition $\frac{\partial}{\partial x}P(1,x) = 1$:

$$\frac{P\,(1,0)\left(a\mathrm{e}^{am}\,(a\mu_0 - c - \Lambda_0) + (-b\mu_0 + c + \Lambda_0)\,b\mathrm{e}^{bm}\right)}{\mu_0\,(a-b)} = 1. \tag{18}$$

Substituting (18) into (17), we get

$$\begin{aligned} P\,(0,x) &= \frac{\Lambda_0\left(\mathrm{e}^{bx} - \mathrm{e}^{ax}\right)}{a\mathrm{e}^{am}\,(a\mu_0 - c - \Lambda_0) + (-b\mu_0 + c + \Lambda_0)\,b\mathrm{e}^{bm}}, x \in [0,m],\\ P\,(1,x) &= \frac{\mathrm{e}^{ax}\,(a\mu_0 - c - \Lambda_0) + (-b\mu_0 + c + \Lambda_0)\,\mathrm{e}^{bx}}{a\mathrm{e}^{am}\,(a\mu_0 - c - \Lambda_0) + (-b\mu_0 + c + \Lambda_0)\,b\mathrm{e}^{bm}}, x \in [0,m]. \end{aligned} \tag{19}$$

We now consider the middle domain $[m, M]$. In this domain the function $P(1, \cdot)$ follows the equation $\frac{\partial}{\partial x}P(1, x) = 1$. Integrating it and using obvious boundary condition, we get

$$P(1, x) = x - m + P(1, m), x \in [m, M]. \tag{20}$$

Function $P(0, \cdot)$ follows (8). Substituting (20) into (8) and solving the differential equation, we get

$$P(0, x) = \frac{((x - m + P(1, m))(c + \Lambda_0) + \mu_0)\,\Lambda_0}{(\Lambda_0 + c)^2} + C\,\mathrm{e}^{\frac{(\Lambda_0 + c)x}{\mu_0}}, x \in [m, M]. \tag{21}$$

Since $P(0, \cdot)$ is assumed to be continuously differentiable, we impose two conditions: $P(0, m_-) = P(0, m_+)$ and $\frac{\partial}{\partial x}P(0, m_-) = \frac{\partial}{\partial x}P(0, m_+)$ but they turn out to be identical:

$$C = -\frac{\Lambda_0\left(P(1, 0)(\Lambda_0 + c)\left(a\mathrm{e}^{am} - b\mathrm{e}^{bm}\right) + \mu_0(a - b)\right)}{(a - b)(\Lambda_0 + c)^2}\mathrm{e}^{-\frac{(\Lambda_0 + c)m}{\mu_0}}. \tag{22}$$

Substituting (22) into (21) and also substituting $P(1, m)$ found from (19), we get

$$\begin{gathered} P(0, x) = \frac{\Lambda_0}{\Lambda_0 + c}\left(\frac{A\mathrm{e}^{am} - B\mathrm{e}^{bm}}{aA\mathrm{e}^{am} - bB\mathrm{e}^{bm}} - cm + cx - \Lambda_0 m + \Lambda_0 x + \mu_0\right) + \\ \frac{\Lambda_0\mu_0^2\left(a^2\mathrm{e}^{am} - b^2\mathrm{e}^{bm}\right)}{(\Lambda_0 + c)^2\left(aA\mathrm{e}^{am} + bB\mathrm{e}^{bm}\right) + \left(aA\mathrm{e}^{am} + bB\mathrm{e}^{bm}\right)}\mathrm{e}^{\frac{\Lambda_0(x - m) + c(x - m)}{\mu_0}}, x \in [m, M], \end{gathered} \tag{23}$$

where $B = -b\mu_0 + c + \Lambda_0$, $A = -a\mu_0 + c + \Lambda_0$.

We now consider conditions the solution must satisfy.

Condition 1. $\frac{\partial}{\partial x}P(0, x) \geq 1$ for $x \in [0, m]$. To guarantee the fulfilment of this inequality, we can demand $\frac{\partial}{\partial x}P(0, x)|_{x=m} \geq 1$ and $\frac{\partial^2}{\partial x^2}P(0, x) \leq 0$ for $x \in [0, m]$. First inequality may re rewritten as

$$\frac{\Lambda_0\left(b\mathrm{e}^{bm} - a\mathrm{e}^{am}\right)}{a(a\mu_0 - c - \Lambda_0)\mathrm{e}^{am} + (-b\mu_0 + c + \Lambda_0)b\mathrm{e}^{bm}} \geq 1. \tag{24}$$

After some simplifications, denominator may be represented as $\mu_0(c + \Lambda_1)\left(a\mathrm{e}^{am} - b\mathrm{e}^{bm}\right) + c(\Lambda_0 + c + \Lambda_1)\left(\mathrm{e}^{bm} - \mathrm{e}^{am}\right)$ and hence is obviously positive. Inequality (24) is thus equivalent to

$$\left(-bc\mu_0 - b\Lambda_0\mu_1 - b\Lambda_1\mu_0 + c^2 + c\Lambda_0 + c\Lambda_1\right)\mathrm{e}^{-am+bm} + ac\mu_0 + a\Lambda_0\mu_1 + a\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1 \leq 0.$$

Part 2 of Lemma 1 states that the coefficient of the exponent is always positive. Hence, this inequality may be rewritten as

$$\mathrm{e}^{-am+bm} \leq K_1^* = \frac{ac\mu_0 + a\Lambda_0\mu_1 + a\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1}{bc\mu_0 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1}. \tag{25}$$

Now consider the inequality $\frac{\partial^2}{\partial x^2}P(0,x) \leq 0$ for $x \in [0,m]$. It can be rewritten as

$$\begin{aligned} &\left(bc\mu_0 + bc\mu_1 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1\right)\mathrm{e}^{-ax+bx} - \\ &- ac\mu_0 - ac\mu_1 - a\Lambda_0\mu_1 - a\Lambda_1\mu_0 + c^2 + c\Lambda_0 + c\Lambda_1 \geq 0. \end{aligned} \tag{26}$$

Part 3 of Lemma 1 states that the coefficient of the exponent is always negative. We now show that if (25) holds, (26) also holds. Indeed $\mathrm{e}^{-ax+bx} \leq \mathrm{e}^{-am+bm} leq K_1^*$, hence

$$\begin{aligned} &\left(bc\mu_0 + bc\mu_1 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1\right)\mathrm{e}^{-ax+bx} - \\ &- ac\mu_0 - ac\mu_1 - a\Lambda_0\mu_1 - a\Lambda_1\mu_0 + c^2 + c\Lambda_0 + c\Lambda_1 > \\ &\frac{\left(bc\mu_0 + bc\mu_1 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1\right)\left(ac\mu_0 + a\Lambda_0\mu_1 + a\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1\right)}{-bc\mu_0 - b\Lambda_0\mu_1 - b\Lambda_1\mu_0 + c^2 + c\Lambda_0 + c\Lambda_1} - \\ &- ac\mu_0 - ac\mu_1 - a\Lambda_0\mu_1 - a\Lambda_1\mu_0 + c^2 + c\Lambda_0 + c\Lambda_1 = \\ &-\frac{c^2\left(c + \Lambda_0 + \Lambda_1\right)\left(c\mu_0 - c\mu_1 + 2\,\Lambda_0\mu_0 - \Lambda_0\mu_1 + \Lambda_1\mu_0\right)}{\mu_0\left(bc\mu_0 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1\right)} > 0. \end{aligned}$$

Condition 2. $\frac{\partial}{\partial x}P(1,x) \geq 1$ for $x \in [0,m]$. This inequality may be rewritten as

$$\frac{a\mathrm{e}^{ax}\left(a\mu_0 - c - \Lambda_0\right) + \left(-b\mu_0 + c + \Lambda_0\right)b\mathrm{e}^{bx}}{a\mathrm{e}^{am}\left(a\mu_0 - c - \Lambda_0\right) + \left(-b\mu_0 + c + \Lambda_0\right)b\mathrm{e}^{bm}}, x \in [0,m] \geq 1.$$

which is obviously equivalent to the condition that the function in numerator has negative derivative for any $x \in [0,m]$. This condition after some simplifications may be rewritten as

$$\begin{aligned} &\left(-bc^2\mu_0 - 2\,bc\Lambda_1\mu_0 - b\Lambda_0\Lambda_1\mu_1 - b\Lambda_1^2\mu_0 + c^3 + c^2\Lambda_0 + 2\,c^2\Lambda_1 + c\Lambda_0\Lambda_1 + c\Lambda_1^2\right)\mathrm{e}^{-ax+bx} \leq \\ &- ac^2\mu_0 - 2\,ac\Lambda_1\mu_0 - a\Lambda_0\Lambda_1\mu_1 - a\Lambda_1^2\mu_0 + c^3 + c^2\Lambda_0 + 2\,c^2\Lambda_1 + c\Lambda_0\Lambda_1 + c\Lambda_1^2, x \in [0,m]. \end{aligned} \tag{27}$$

Part 4 of Lemma 1 states that the coefficient of the exponent is always positive. If the expression in parentheses on the left side is negative, inequality is proven. If it is positive, we square both sides of inequality and after some simplifications get

$$0 < -4c^4\Lambda_0\Lambda_1\mu_0\mu_1^3 - 8c^3\Lambda_0^2\Lambda_1\mu_0\mu_1^3 - 8c^3\Lambda_0\Lambda_1^2\mu_0\mu_1^3 - 4c^2\Lambda_0^3\Lambda_1\mu_0\mu_1^3 - 8c^2\Lambda_0^2\Lambda_1^2\mu_0\mu_1^3 - 4c^2\Lambda_0\Lambda_1^3\mu_0\mu_1^3,$$

which is always true. Hence, (28) may be rewritten as

$$\mathrm{e}^{x(b-a)} \leq K_2^* = \frac{a(c^2\mu_0 + 2c\Lambda_1\mu_0 + \Lambda_0\Lambda_1\mu_1 + \Lambda_1^2\mu_0) - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c\Lambda_1^2}{b(c^2\mu_0 + 2c\Lambda_1\mu_0 + \Lambda_0\Lambda_1\mu_1 + \Lambda_1^2\mu_0) - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c\Lambda_1^2}. \quad (28)$$

Condition 3. Inequality (5) in the middle domain for $s = 1$ has the following form:

$$\mu_1 - (c + \Lambda_1)(x + P(1, m) - m) + \Lambda_1 P(0, x) \leq 0.$$

It holds as equality for $x = m$. To guarantee that it holds for $x \in [m, m']$ for some $m' > m$ we impose the following condition

$$\frac{\partial}{\partial x} P(0, x)\,|_{x=m} \leq \frac{c + \Lambda_1}{\Lambda_1}.$$

Substituting (23) in this inequality, we get

$$0 \leq \frac{-b\,(bc\mu_0 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1)\,\mathrm{e}^{bm} + a\,(ac\mu_0 + a\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1)\,\mathrm{e}^{am}}{\Lambda_1\,(a^2\mathrm{e}^{am}\mu_0 - ac\mathrm{e}^{am} - a\mathrm{e}^{am}\Lambda_0 - b^2\mathrm{e}^{bm}\mu_0 + b\mathrm{e}^{bm}c + b\mathrm{e}^{bm}\Lambda_0)}. \quad (29)$$

Denominator may be rewritten as

$$\frac{\Lambda_1}{\mu_1}\left(\mu_0(c + \Lambda_1)\left(a\mathrm{e}^{am} - b\mathrm{e}^{bm}\right) + c\,(c + \Lambda_0 + \Lambda_1)\left(\mathrm{e}^{bm} - \mathrm{e}^{am}\right)\right) \quad (30)$$

and hence is always positive. The condition that numerator is non-negative can be rewritten as

$$\begin{aligned}&\left(-bc^2\mu_0 - 2\,bc\Lambda_1\mu_0 - b\Lambda_0\Lambda_1\mu_1 - b\Lambda_1^2\mu_0 + c^3 + c^2\Lambda_0 + 2\,c^2\Lambda_1 + c\Lambda_0\Lambda_1 + c\Lambda_1^2\right)\mathrm{e}^{-am+bm} + \\ &+ ac^2\mu_0 + 2\,ac\Lambda_1\mu_0 + a\Lambda_0\Lambda_1\mu_1 + a{\Lambda_1}^2\mu_0 - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c{\Lambda_1}^2 \geq 0,\end{aligned} \quad (31)$$

which is exactly

$$\mathrm{e}^{-am+bm} \geq K_2^*. \quad (32)$$

Hence, the only possibility of both (28) and (32) to be satisfied is

$$\mathrm{e}^{(b-a)m} = K_2^* = \frac{a(c^2\mu_0 + 2\,c\Lambda_1\mu_0 + \Lambda_0\Lambda_1\mu_1 + \Lambda_1^2\mu_0) - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c\Lambda_1^2}{b(c^2\mu_0 + 2\,c\Lambda_1\mu_0 + \Lambda_0\Lambda_1\mu_1 + \Lambda_1^2\mu_0) - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c\Lambda_1^2}. \quad (33)$$

This condition defines the optimal lower threshold level m. Now we need to show that it satisfies Condition 1. To do so, we show that $K_1^* > K_2^*$. Indeed,

$$\begin{aligned}
&K_1^* - K_2^* = -c^2\Lambda_0\mu_1\left(ac + a\Lambda_0 + a\Lambda_1 - bc - b\Lambda_0 - b\Lambda_1\right) \times \\
&\times \left(bc\mu_0 + b\Lambda_0\mu_1 + b\Lambda_1\mu_0 - c^2 - c\Lambda_0 - c\Lambda_1\right)^{-1} \times \\
&\times \left(bc^2\mu_0 + 2\,bc\Lambda_1\mu_0 + b\Lambda_0\Lambda_1\mu_1 + b{\Lambda_1}^2\mu_0 - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c\Lambda_1^2\right)^{-1} > 0.
\end{aligned}$$

We now that that upper threshold level is finite. Indeed, $\frac{\partial}{\partial x}P\left(0,x\right)|_{x=m} = \frac{c+\Lambda_1}{\Lambda_1} > 1$ and $\lim_{x\to\infty}\frac{\partial}{\partial x}P\left(0,x\right) = \frac{\Lambda_0}{c+\Lambda_0} < 1$. Hence, there exists a point $M \in (m,+\infty)$ such that $\frac{\partial}{\partial x}P\left(0,x\right)|_{x=M} = 1$. Equalizing $\frac{\partial}{\partial x}P\left(0,x\right)$ to one, we get the equation defining M.

$$\mathrm{e}^{\frac{(\Lambda_0+c)M}{\mu_0}} = -\frac{c\left(a^2\mathrm{e}^{am}\mu_0 - ac\mathrm{e}^{am} - a\mathrm{e}^{am}\Lambda_0 - b^2\mathrm{e}^{bm}\mu_0 + b\mathrm{e}^{bm}c + b\mathrm{e}^{bm}\Lambda_0\right)}{\Lambda_0\mu_0\left(a^2\mathrm{e}^{\frac{m(a\mu_0-c-\Lambda_0)}{\mu_0}} - b^2\mathrm{e}^{\frac{m(b\mu_0-c-\Lambda_0)}{\mu_0}}\right)}. \quad (34)$$

Hence, there exists the unique solution for the optimal dividend problem in the Case 1, and this solution is defined by threshold levels (33) and (34). Now we need to analyze the signs of thresholds. Condition $m > 0$ is obviously equivalent to $K_2^* > 1$, which may be rewritten as

$$\frac{\left(c^2\mu_0 + 2\,c\Lambda_1\mu_0 + \Lambda_0\Lambda_1\mu_1 + {\Lambda_1}^2\mu_0\right)\sqrt{\Omega}}{\mu_0\mu_1} < 0.$$

Thus we arrive to the condition of the positivity of thresholds:

$$(c+\Lambda_1)^2\mu_0 + \Lambda_0\Lambda_1\mu_1 > 0. \quad (35)$$

Case 2. $m_0 < m_1$. Again, we should consider three domains. In the lower domain $[0,m]$, as in the previous case, function P follows (8), which leads to (17) but the boundary condition is now $\frac{\partial}{\partial x}P(0,x) = 1$:

$$\frac{P\left(1,0\right)\Lambda_0\left(-a\mathrm{e}^{am} + b\mathrm{e}^{bm}\right)}{\mu_0\left(a-b\right)} = 1.$$

Substituting it to (17), we get

$$P\left(0,x\right)=\frac{\mathrm{e}^{bx}-\mathrm{e}^{ax}}{-a\mathrm{e}^{am}+b\mathrm{e}^{bm}},P\left(1,x\right)=\frac{\mathrm{e}^{ax}\left(a\mu_0-c-\Lambda_0\right)+\left(-b\mu_0+c+\Lambda_0\right)\mathrm{e}^{bx}}{\Lambda_0\left(-a\mathrm{e}^{am}+b\mathrm{e}^{bm}\right)}. \tag{36}$$

For the middle domain, similarly to the Case 1, we get

$$P(0,x)=x-m+P(0,m),x\in[m,M]. \tag{37}$$

Function $P(1,\cdot)$ follows (8). Substituting (37) into (8) and solving the differential equation, we get

$$P\left(1,x\right)=\frac{\left(cP\left(0,m\right)+\Lambda_1 P\left(0,m\right)-cm+cx-\Lambda_1 m+\Lambda_1 x+\mu_1\right)\Lambda_1}{\left(c+\Lambda_1\right)^2}+C\mathrm{e}^{\frac{(c+\Lambda_1)x}{\mu_1}}. \tag{38}$$

Again, we impose two conditions: $P(1,m_-)=P(1,m_+)$ and $\frac{\partial}{\partial x}P(1,m_-)=\frac{\partial}{\partial x}P(1,m_+)$ but they turn out to be identical:

$$C=-\frac{\Lambda_1\mu_1}{\left(c+\Lambda_1\right)^2}\mathrm{e}^{-\frac{(c+\Lambda_1)m}{\mu_1}}+\frac{P\left(1,0\right)\left(\tilde{B}\mathrm{e}^{bm}-\tilde{A}\mathrm{e}^{am}\right)}{\left(a-b\right)\mu_0\left(c+\Lambda_1\right)}\mathrm{e}^{-\frac{(c+\Lambda_1)m}{\mu_1}},$$

where $\tilde{B}=-bc\mu_0-b\Lambda_1\mu_0+c^2+c\Lambda_0+c\Lambda_1$ and $\tilde{A}=-ac\mu_0-a\Lambda_1\mu_0+c^2+c\Lambda_0+c\Lambda_1$. Substituting it to (38), we get

$$\begin{gathered}P\left(1,x\right)=\frac{\Lambda_1}{\left(c+\Lambda_1\right)^2}\left(\frac{\left(c+\Lambda_1\right)\left(\mathrm{e}^{bm}-\mathrm{e}^{am}\right)}{-a\mathrm{e}^{am}+b\mathrm{e}^{bm}}-cm+cx-\Lambda_1 m+\Lambda_1 x+\mu_1\right)+\\ \frac{\mu_1\left(-b\tilde{B}\mathrm{e}^{bm}+a\tilde{A}\mathrm{e}^{am}\right)}{\left(c+\Lambda_1\right)^2\Lambda_0\left(a\mathrm{e}^{am}-b\mathrm{e}^{bm}\right)}\mathrm{e}^{\frac{(c+\Lambda_1)(x-m)}{\mu_1}}.\end{gathered} \tag{39}$$

We now consider some conditions the solution must satisfy and show they are inconsistent.

Condition 1. $\frac{\partial}{\partial x}P(1,x)|_{x=m}\geq 1$ may be rewritten as

$$\frac{a\left(a\mu_0-c-\Lambda_0\right)\mathrm{e}^{am}+\left(-b\mu_0+c+\Lambda_0\right)b\mathrm{e}^{bm}}{\Lambda_0\left(-a\mathrm{e}^{am}+b\mathrm{e}^{bm}\right)}\geq 1.$$

Denominator is always positive and thus inequality may be rewritten as

$$\left(-bc\mu_0-b\Lambda_0\mu_1-b\Lambda_1\mu_0+c^2+c\Lambda_0+c\Lambda_1\right)\mathrm{e}^{-am+bm}+ac\mu_0+a\Lambda_0\mu_1+a\Lambda_1\mu_0-c^2-c\Lambda_0-c\Lambda_1\geq 0,$$

which is exactly $\mathrm{e}^{-am+bm}\geq K_1^*$.

Condition 2. Inequality (5) in the middle domain for $s = 0$, which may be rewritten as

$$P(1,x) \le \frac{\Lambda_0 + c}{\Lambda_0} \left(x - m + P(0,m)\right) - \frac{\mu_0}{\Lambda_0}. \tag{40}$$

Condition 3. Finiteness on the middle domain.

$$\frac{\partial}{\partial x} P(1,x) = -\frac{\Theta}{\mu_1 \left(c + \Lambda_1\right) \Lambda_0 \left(a\mathrm{e}^{am} - b\mathrm{e}^{bm}\right)} \mathrm{e}^{\frac{-cm + cx - m\Lambda_1 + x\Lambda_1}{\mu_1}} + \frac{\Lambda_1}{c + \Lambda_1},$$

where

$$\Theta = \mathrm{e}^{am} ac^2 \mu_0 + 2\,\mathrm{e}^{am} ac\Lambda_1 \mu_0 + \mathrm{e}^{am} a\Lambda_0 \Lambda_1 \mu_1 + \mathrm{e}^{am} a{\Lambda_1}^2 \mu_0 - \mathrm{e}^{bm} bc^2 \mu_0 - 2\mathrm{e}^{bm} bc\Lambda_1 \mu_0 -$$
$$- \mathrm{e}^{bm} b\Lambda_0 \Lambda_1 \mu_1 - \mathrm{e}^{bm} b\Lambda_1^2 \mu_0 - \mathrm{e}^{am} c^3 - \mathrm{e}^{am} c^2 \Lambda_0 - 2\,\mathrm{e}^{am} c^2 \Lambda_1 - \mathrm{e}^{am} c\Lambda_0 \Lambda_1 - \mathrm{e}^{am} c\Lambda_1^2 +$$
$$+ \mathrm{e}^{bm} c^3 + \mathrm{e}^{bm} c^2 \Lambda_0 + 2\,\mathrm{e}^{bm} c^2 \Lambda_1 + \mathrm{e}^{bm} c\Lambda_0 \Lambda_1 + \mathrm{e}^{bm} c\Lambda_1^2.$$

If we want the middle domain to be finite, we must demand the coefficient of exponent to be negative, which implies $\Theta < 0$, which may be rewritten as

$$\left(-bc^2\mu_0 - 2\,bc\Lambda_1\mu_0 - b\Lambda_0\Lambda_1\mu_1 - b\Lambda_1^2\mu_0 + c^3 + c^2\Lambda_0 + 2\,c^2\Lambda_1 + c\Lambda_0\Lambda_1 + c\Lambda_1^2\right) \mathrm{e}^{-am+bm} +$$
$$+ ac^2\mu_0 + 2\,ac\Lambda_1\mu_0 + a\Lambda_0\Lambda_1\mu_1 + a\Lambda_1^2\mu_0 - c^3 - c^2\Lambda_0 - 2\,c^2\Lambda_1 - c\Lambda_0\Lambda_1 - c\Lambda_1^2 \le 0.$$

This is exactly $\mathrm{e}^{-am+bm} \le K_2^*$. But it is already shown that $K_1^* > K_2^*$, so Conditions 1 and 3 cannot be satisfied at the same time. Condition 1 cannot violated for the solution of HJB equation, so suppose Condition 3 is violated, which means there is no upper threshold. But if Θ is positive, $P(1,x)$ increases exponentially, so Condition 2 is violated for big x. Condition 2 cannot violated for the solution of HJB equation, hence we arrive to the conclusion that there are no solutions of HJB equation in the Case 2. Thus we arrive to the following

Theorem 2. Let the parameters of the model are such that (35) is satisfied. Than thresholds m and M, defined by (33) and (34) respectively, are positive and the solution of HJB equation (7) is given by

$$P(0,x)=\begin{cases}\frac{\Lambda_0\left(e^{bx}-e^{ax}\right)}{-ae^{am}A+Bbe^{bm}}, x\le m,\\ \frac{\Lambda_0}{\Lambda_0+c}\left(\frac{Ae^{am}-Be^{bm}}{aAe^{am}-bBe^{bm}}-cm+cx-\Lambda_0 m+\Lambda_0 x+\mu_0\right)+\\ +\frac{\Lambda_0\mu_0^2\left(a^2e^{am}-b^2e^{bm}\right)}{(\Lambda_0+c)^2\left(aAe^{am}+bBe^{bm}\right)+\left(aAe^{am}+bBe^{bm}\right)}e^{\frac{(\Lambda_0+c)(x-m)}{\mu_0}}, x\in(m,M],\\ x-M+\frac{\Lambda_0}{\Lambda_0+c}\left(\frac{Ae^{am}-Be^{bm}}{aAe^{am}-bBe^{bm}}-cm+cM-\Lambda_0 m+\Lambda_0 M+\mu_0\right)+\\ +\frac{\Lambda_0\mu_0^2\left(a^2e^{am}-b^2e^{bm}\right)}{(\Lambda_0+c)^2\left(aAe^{am}+bBe^{bm}\right)+\left(aAe^{am}+bBe^{bm}\right)}e^{\frac{(\Lambda_0+c)(M-m)}{\mu_0}}, x>M,\end{cases} \tag{41}$$

$$P(1,x)=\begin{cases}\frac{-Ae^{ax}+Be^{bx}}{-aAe^{am}+bBe^{bm}}, x\in[0,m],\\ x-m+\frac{-Ae^{am}+Be^{bm}}{-aAe^{am}+bBe^{bm}}, x>m,\end{cases} \tag{42}$$

where $B=-b\mu_0+c+\Lambda_0$, $A=-a\mu_0+c+\Lambda_0$ and the associated dividend policy is

$$L^*(t)=(x-m)^+\mathbb{1}_{\{s=1\}}+(x-M)^+\mathbb{1}_{\{s=0\}}+\int_0^t\mu_1\mathbb{1}_{\{\pi(u)=1,X(u)=m\}}du.$$

The optimal dividend strategy is is thus defined by two thresholds m and M. In the region lower than m firm should not pay any dividends in both states of the world. In the region between m and M firm should immediately pay an excess above m as dividends, if the state of the world is 1 and don't pay anything if the state of world is 0. This may look a bit counter-intuitive — in the state 0 firm loses money and then the state of the world switches to 1, it pays the excess above m. Why don't pay before switching? The answer is that in the case of paying before switching, firm then suffers losses, because the state of the world is 0, and n the case of paying at switching, it finds itself on the threshold in the state of the world 1 and makes more money. Finally, if firm has more money than M, in both states of the world it should immediately pay the excess above M as dividends (and then also the excess above m if the state of the world is 1).

2.3 Verification of solution

In this subsection we show that the solution of HJB equation described in Theorem 2 indeed defines the solution of the optimal dividend problem.

Theorem 3. Let G be a solution of HJB equation (7). Than it is the value function for problem (2) and the associated dividend policy is the optimal dividend policy.

Proof. Let $L(\cdot)$ be some admissible control. Denote the set of its discontinuities by Φ and let $L^d(t)=\sum_{u\in\Phi,s\le t}(L(u_+)-L(u))$ and $L^c(t)=L(t)-L^d(t)$ be discontinuous and continuous parts of L respectively. Denote also $f(t,s,x)=\mathrm{e}^{-ct}G(s,x)$. We have

$$\mathbb{E}_{s,x}d_t f(t,\pi(t),X(t))=$$
$$\left[\mu_{\pi(t)}\frac{\partial}{\partial x}f(t,\pi(t),X(t))+\frac{\partial}{\partial t}f(t,\pi(t),X(t))\right]dt-\frac{\partial}{\partial x}f(t,\pi(t),X(t))dZ^c(t)+$$
$$[f(t,\pi(t),X(t_+))-f(t,\pi(t),X(t))]I_{t\in\Phi}+[-\Lambda_{\pi(t)}f(t,\pi(t),X(t))+\Lambda_{\pi(t)}f(t,1-\pi(t),X(t))]dt=$$
$$\mathrm{e}^{-ct}[\mu_{\pi(t)}G(\pi(t),X(t))-cG(\pi(t),X(t))-\Lambda_{\pi(t)}G(\pi(t),X(t))+\Lambda_{\pi(t)}G(1-\pi(t),X(t))]dt-$$
$$-\mathrm{e}^{-ct}\frac{\partial}{\partial x}G(\pi(t),X(t))dZ^c(t)+\mathrm{e}^{-ct}[G(\pi(t),X(t_+))-G(\pi(t),X(t))]I_{t\in\Phi}.$$

Integrating this expression, we get

$$\mathrm{e}^{-c(t\wedge\tau)}G(\pi(t\wedge\tau),X(t\wedge\tau))=G(s,x)+\int_0^{t\wedge\tau}\mathrm{e}^{-cy}R(y)dy-$$
$$\int_0^{t\wedge\tau}\mathrm{e}^{-cy}\frac{\partial}{\partial x}G(\pi(y),X(y))dL^c(y)+\sum_{0\le y\le t\wedge\tau,y\in\Phi}\mathrm{e}^{-cy}\left(G(\pi(y),X(y_+))-G(\pi(y),X(y))\right),$$

where $R(y)=\mu_{\pi(y)}G(\pi(y),X(y))-cG(\pi(y),X(y))-\Lambda_{\pi(y)}G(\pi(y),X(y))+\Lambda_{\pi(y)}G(1-\pi(y),X(y))$. Taking conditional expectations, we get

$$\mathbb{E}_{s,x}\left[\mathrm{e}^{-c(t\wedge\tau)}G(\pi(t\wedge\tau),X(t\wedge\tau))\right]$$
$$=G(s,x)+\mathbb{E}_{s,x}\left[\int_0^{t\wedge\tau}\mathrm{e}^{-cy}R(y)dy\right]-\mathbb{E}_{s,x}\left[\int_0^{t\wedge\tau}\mathrm{e}^{-cy}\frac{\partial}{\partial x}G(\pi(t),X(t))dL^c(t)\right]+$$
$$\mathbb{E}_{s,x}\left[\sum_{0\le y\le t\wedge\tau,y\in\Phi}\mathrm{e}^{-cy}\left(G(\pi(t),X(t_+))-G(\pi(t),X(t))\right)\right].$$

Inequality (5) guarantees the integrand for the first integral is non-positive, and inequality (6) guarantees that for every $t\in\Phi$ $G(\pi(t),X(t_+))-G(\pi(t),X(t))\le X(t_+)-X(t)=L(t)-L(t_+)$. It also follows from (6) that $\mathrm{e}^{-cy}\frac{\partial}{\partial x}G(\pi(t),X(t))\ge\mathrm{e}^{-cy}$. Hence

$$\mathbb{E}_{s,x}\left[\mathrm{e}^{-c(t\wedge\tau)}G(\pi(t\wedge\tau),X(t\wedge\tau))\right]\le G(s,x)-\mathbb{E}_{s,x}\left[\int_0^{t\wedge\tau}\mathrm{e}^{-cy}dL(y)\right].$$

Note that for the dividend policy L^G, associated with the solution of HJB equation , this inequality turns into equality. Indeed, under this policy $R(y)=0$ almost everywhere, hence the first integral equals zero. Continuous flow of dividends corresponds to $X(t)=m$ and $s=1$, and we know that $\frac{\partial}{\partial x}G(1,X(t))|_{X(t)=m}=1$. Finally, in the points of discontinuity $G(\pi(t),X(t_+))-G(\pi(t),X(t))=X(t_+)-X(t)$. Hence, taking $t\to+\infty$, we get

$$G(s,x) \geq \mathbb{E}_{s,x}\left[\int_0^{t\wedge\tau} \mathrm{e}^{-cy} dL(y)\right]$$

for the arbitrary dividend policy with equality for the dividend policy associated with the solution of HJB equation (7).

■

3 Conclusion

It is shown that the optimal dividend policy in the model of firm's cash surplus following telegraph process is of a threshold type, which is in line with results for models with diffusion and Markov regime switching. However, we had to perform rather tricky analysis of variational inequalities to find these thresholds. Further research may involve generalization of our results for the arbitrary number of regimes and the analysis of links between our model and the models with diffusion.

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