Moduli spaces of Gorenstein quasi-homogeneous surface singularities

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1. In this paper we describe the topology of the space of Gorenstein hyperbolic quasi-homogeneous surface singularities. We determine the number of connected components of this space and describe their topology. Quasi-homogeneous surface singularities and their affine coordinate rings were actively studied in the 1970s by Dolgachev, Milnor, Neumann, Pinkham, and others. We call a singularity hyperbolic if the corresponding orbifold is hyperbolic, namely, of the form $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the hyperbolic plane and Γ a Fuchsian group. A normal isolated surface singularity is Gorenstein if and only if there exists a nowhere vanishing holomorphic 2-form on a punctured neighbourhood of the singular point. For example, all isolated singularities of complete intersections are Gorenstein. Dolgachev [1] proved that Gorenstein hyperbolic quasi-homogeneous surface singularities of level m are in 1-to-1 correspondence with singular m-co-spin bundles on compact hyperbolic Riemann orbifolds, that is, line bundles whose mth tensor power is isomorphic to the tangent bundle.

There is a 1-to-1 correspondence between m-co-spin and m-spin bundles. Classical ($m = 2$) spin bundles on compact Riemann surfaces coincide with Riemann's Theta-characteristics. Their modern interpretation and classification was proposed by Atiyah [2] and Mumford [3]. A classification of 2-spin bundles and their moduli on non-compact surfaces was given in [4]. A description of the moduli spaces of m-spin bundles was given in [5] for compact surfaces with punctures and in [6], [7] for surfaces with arbitrary finitely generated fundamental group.

2. Definition. Let p be a point on a compact hyperbolic Riemann orbifold P. Let $\pi_1^0(P,p)$ be the set of all non-trivial elements of the orbifold fundamental group $\pi_1(P,p)$ that can be represented by simple contours. An *m-Arf function* is a map $\sigma\colon \pi_1^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$ such that for any $a, b \in \pi_1^0(P,p)$ the following conditions are satisfied: 1) $\sigma(bab^{-1}) = \sigma(a)$; 2) $\sigma(a^{-1}) = -\sigma(a)$ if the element a is not of order 2; 3) $\sigma(ab) = \sigma(a) + \sigma(b)$ if a and b can be represented by simple contours intersecting in exactly the one point p with the intersection index not equal to 0; 4) $\sigma(ab) = \sigma(a)+\sigma(b)-1$ if a) the intersection index of a and b is equal to 0, b) a and b can be represented by simple contours intersecting in exactly the one point p, and c) the oriented contours a, b and $(ab)^{-1}$ are freely homotopic to pairwise disjoint simple contours $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ whose orientations are opposite to those induced by the complex structure of the sphere with 3 holes which they cut out of P; 5) for any elliptic element c of order p the number $p\sigma(c) + 1$ is a multiple of m.

Theorem 1. *There exists a 1-to-1 correspondence between Gorenstein hyperbolic quasi-homogeneous surface singularities of level m and m-Arf functions on compact hyperbolic Riemann orbifolds.*

The proof is based on the correspondence between m-spin bundles on the orbifold $\mathbb{H}/\Gamma$ and lifts of the Fuchsian group $\Gamma \subset PSL(2,\mathbb{R})$ into the unique m-fold covering $\varphi\colon G_m \to PSL(2,\mathbb{R})$.

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3. Let us describe the simplest topological invariants of an m-Arf function σ. They are the genus g of the orbifold P and the orders $p_1,\dots,p_r$ of the singular points of P. It follows immediately from the definition of m-Arf functions that the group Mod of homotopy classes of orientation-preserving autohomeomorphisms of the orbifold P acts naturally on the set Σ of all m-Arf functions on P.

Theorem 2. *The set Σ of all m-Arf functions on a compact hyperbolic orbifold P of signature $(g:p_1,\dots,p_r)$ is not empty if and only if the orders $p_1,\dots,p_r$ are co-prime with m and $(p_1\cdots p_r)\biggl(\sum_{1\leqslant i\leqslant r}\dfrac{1}{p_i}-(2g-2)-r\biggr)$ is a multiple of m. In this case the number of m-Arf functions is equal to m^{2g}. The action of the group* Mod *on the non-empty set Σ is transitive if $g=0$ or if $g>1$ and m is odd, it has exactly two orbits if $g>1$ and m is even, and it has a number of orbits equal to the number of divisors of* $\gcd(m,p_1-1,\dots,p_r-1)$ *if $g=1$.*

The proof uses the explicit expressions for Dehn generators of the group Mod.

4. We shall now use m-Arf functions to study the space S^m of hyperbolic Gorenstein quasi-homogeneous surface singularities of level m. The space S^m is the disjoint union of subsets $S^m_{g:p_1,\dots,p_r}$ which correspond to the simplest topological invariants of the m-Arf functions. Theorem 2 implies the following theorem.

Theorem 3. *The space $S^m_{g:p_1,\dots,p_r}$ is not empty if and only if the orders $p_1,\dots,p_r$ are co-prime with m and $(p_1\cdots p_r)\biggl(\sum_{1\leqslant i\leqslant r}\dfrac{1}{p_i}-(2g-2)-r\biggr)$ is a multiple of m. The non-empty set $S^m_{g:p_1,\dots,p_r}$ is connected if $g=0$ or if $g>1$ and m is odd, it has exactly two connected components if $g>1$ and m is even, and it has a number of connected components equal to the number of divisors of* $\gcd(m,p_1-1,\dots,p_r-1)$ *if $g=1$.*

Theorems 1–3 and the Fricke–Klein theorem [8] in the form in [9] imply the next theorem.

Theorem 4. *Each connected component of the space $S^m_{g:p_1,\dots,p_r}$ is homeomorphic to $\mathbb{R}^d/G$, where $d=6g-6+2r$ and $G\subset$* Mod *is a discrete group.*

Bibliography

[1] I. V. Dolgachev, *Math. Ann.* **265**:4 (1983), 529–540.

[2] M. F. Atiyah, *Ann. Sci. École Norm. Sup.* (4) **4**:1 (1971), 47–62.

[3] D. Mumford, *Ann. Sci. École Norm. Sup.* (4) **4**:2 (1971), 181–192.

[4] С. М. Натанзон, *Матем. заметки* **45**:4 (1989), 111–116; English transl., S. M. Natanzon, *Math. Notes* **45**:4 (1989), 341–345.

[5] T. J. Jarvis, *Internat. J. Math.* **11**:5 (2000), 637–663.

[6] С. Натанзон, А. Пратусевич, *УМН* **60**:2(362) (2005), 169–170; English transl., S. M. Natanzon and A. M. Pratusevich, *Russian Math. Surveys* **60**:2 (2005), 363–364.

[7] S. Natanzon and A. Pratoussevitch (Pratusevich), *J. Lie Theory* **19**:1 (2009), 107–148.

[8] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen.* B. I, II, Johnson Reprint Corp., New York; Teubner, Stuttgart 1965.

[9] С. М. Натанзон, *Модули римановых поверхностей, вещественных алгебраических кривых и их супераналоги*, МЦНМО, М. 2003, 176 с.
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