FINITE SUBGROUPS OF
DIFFEOMORPHISM GROUPS

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ABSTRACT. We prove (1) the existence of a noncompact smooth four-dimensional manifold M such that $\mathrm{Diff}(M)$ contains an isomorphic copy of *every* finitely presented group; (2) a finiteness theorem on finite simple subgroups of $\mathrm{Diff}(N)$ for compact manifolds N.

1. In [Po10, Sect. 2] was introduced the definition of abstract Jordan group and initiated exploration of the following two problems on automorphism groups and birational self-map groups of algebraic varieties:

Problem A. Describe algebraic varieties X for which $\mathrm{Aut}(X)$ is Jordan.
Problem B. The same with $\mathrm{Aut}(X)$ replaced by $\mathrm{Bir}(X)$.

Recall from [Po10, Def. 2.1] that Jordaness of a group is defined as follows:

Definition 1. A group G is called a *Jordan group* if there exists a positive integer d_G, depending on G only, such that every finite subgroup F of G contains a normal abelian subgroup whose index in F is at most d_G.

Informally Jordaness means that all finite subgroups of G are "almost" abelian in the sense that they are extensions of abelian groups by finite groups taken from a finite list.

Since the time of launching the exploration program of Problems A and B a rather extensive information on them has been obtained (see [Po13$_2$]), but, for instance, at this writing (October 2013) is still unknown whether there exists X with non-Jordan group $\mathrm{Aut}(X)$ (see Question 1 in [Po13$_2$]; it is interesting to juxtapose it with Corollary 2 below).

2. In this note we explore the counterpart of Problem A, in which algebraic varieties X are replaced by connected smooth topological manifolds M, and $\mathrm{Aut}(X)$ is replaced by $\mathrm{Diff}(M)$, the diffeomorphism group of M. We shall show that the situation for noncompact manifolds is quite different from that for compact ones.

3. First, we consider the case of noncompact manifolds.

Recall that a group is called finitely presented if it is presented by finitely many generators and relations.

Theorem 1. *There is a simply connected noncompact smooth oriented four-dimensional manifold C such that $\mathrm{Diff}(C)$ contains an isomorphic copy of every finitely presented group and this copy acts on C properly discontinuously.*

Remark 1. Since $\mathrm{Diff}(C)$ is a subgroup of $\mathrm{Diff}(C \times M)$ for every manifold M, one can replace "four" in Theorem 1 by any integer $n \geqslant 4$.

Proof of Theorem 1. As is known (see, e.g., [R10, Thm. 12.29]) Higman's Embedding Theorem [HNH61] (see also [R10, Thm. 12.18]) implies the existence of a universal finitely presented group, that is a finitely presented group $\mathcal{U}$ which contains as a subgroup an isomorphic copy of every finitely presented group. In turn, finite presentedness of $\mathcal{U}$ implies the existence of a connected compact smooth oriented four-dimensional manifold B whose fundamental group is isomorphic to $\mathcal{U}$ (see, e.g., [CZ93, Thm. 5.1.1]). Let $C \to B$ be the universal cover. Its deck transformation group is a subgroup of $\mathrm{Diff}(C)$ isomorphic to the fundamental group of B, i.e., to $\mathcal{U}$, and its action on C is properly discontinuous (see, e.g., [Ma67, Chap. V, Sect. 8]). This completes the proof. $\qquad\square$

4. Since every finite group is finitely presented, Theorem 1 yields

Corollary 1. *For every finite group,* $\mathrm{Diff}(C)$ *contains its isomorphic copy that acts on C freely.*

If G a Jordan group, then Definition 1 implies the existence of a constant d_G, depending on G only, such that the order of every nonabelian finite simple subgroup of G is at most d_G. Since there are nonabelian finite simple groups whose order is bigger than any given constant (for instance, the alternating group Alt_n is simple for $n \geqslant 5$ and $|\mathrm{Alt}_n| = n!/2 \xrightarrow[n \to \infty]{} \infty$), Theorem 1 yields

Corollary 2. *The group* $\mathrm{Diff}(C)$ *is non-Jordan.*

Corollary 2 answers the question raised in the correspondence with I. Mundet i Riera [Po13$_1$], [MiR13]. We note that there are connected noncompact manifolds M such that $\mathrm{Diff}(M)$ is Jordan. For instance, if $M = \mathbb{R}^n$, this is so for $n = 1, 2$, but for $n \geqslant 3$ the answer is unknown [MiR13]; it would be interesting to find it (here is worth noting that by [PS13, Cor. 1.9] the space Cremona group Cr_3 is Jordan).

5. If one focuses only on *finite* subgroups of diffeomorphism groups, it is possible to give somewhat more explicit constructions of manifolds M such that $\mathrm{Diff}(M)$ contains an isomorphic copy of *every* finite group. Below are three of them, each is based on the combination of the idea used in the proof of Theorem 1 with an appropriate result from group theory.

Construction (i): Consider the direct product of symmetric groups

$$G := \prod_{n \geqslant 2} \mathrm{Sym}_n.$$

Since Coxeter groups have a solvable word problem [T69], G has a presentation with a recursively enumerable set of relations. By [HNH49], this and countability of G imply that G can be embedded in a group G^{II} that has two generators and a recursively enumerable set of relations. By Higman's Embedding Theorem, G^{II} is embeddable in a finitely presented group

$\widetilde{G}$. Thus we may identify G with a subgroup of $\widetilde{G}$. As above, finite presentedness of $\widetilde{G}$ implies the existence of a connected compact smooth oriented four-dimensional manifold whose fundamental group is isomorphic to $\widetilde{G}$. Since G is a subgroup of $\widetilde{G}$, this manifold is covered by a connected smooth oriented four-dimensional manifold whose fundamental group is isomorphic to G (see, e.g., [Ma67, Thm. 10.2]). The deck transformation group of the universal covering manifold S of the latter manifold is a subgroup of $\mathrm{Diff}(S)$ isomorphic to G and acting on S properly discontinuously. Since every finite group can be embedded in Sym_n for a suitable n, we conclude that Corollaries 1 and 2 hold true with C replaced by S.

Construction (ii): Let $\mathcal{N}$ be the subgroup of the group of all permutations of $\mathbf{Z}$ generated by the transposition $\sigma := (1,2)$ and the "translation" τ defined by the condition $\tau(i) = i+1$ for every $i \in \mathbf{Z}$; see [Po13$_2$, Example 4]. Since $\tau^m \sigma \tau^{-m} = (m+1, m+2)$ for every m, and the set of transpositions $(1,2), (2,3), \ldots, (n-1,n)$ generates the symmetric group Sym_n, the group $\mathcal{N}$ contains an isomorphic copy of every finite group. One can explicitly describe a set of defining relations of $\mathcal{N}$; this set is recursively enumerable. Arguing then as in Construction (i), one proves the existence of a simply connected noncompact smooth oriented four-dimensional manifold N such that $\mathrm{Diff}(N)$ contains an isomorphic copy of $\mathcal{N}$ acting on N properly discontinuously. Thus Corollaries 1 and 2 hold true with C replaced by N.

Remark 2. The group Sym_∞ of all permutations of $\mathbf{Z}$ that move only finitely many elements is a subgroup of $\mathcal{N}$. Therefore, there is a simply connected noncompact smooth oriented four-dimensional manifold whose diffeomorphism group contains an isomorphic copy of Sym_∞ that acts on this manifold properly discontinuously. Representation theory of Sym_∞ is the subject of many publications. It would be interesting to explore which part of representations of Sym_∞ is realizable in the cohomology of such a manifold.

The existence of an embedding of $\mathcal{N}$ in $\mathrm{Diff}(N)$ is worth to compare with the following conjecture formulated in [Po13$_2$, Sect. 3.3, Conj. 1]:

Conjecture 1. $\mathcal{N}$ is not embeddable in $\mathrm{Bir}(X)$ for every irreducible algebraic variety X.

Since $\mathcal{N}$ contains an isomorphic copy of every finite group, the following conjecture (cf. [Po13$_2$, Sect. 3.3, Question 8]) implies Conjecture 1:

Conjecture 2. For every irreducible algebraic variety X, there are only finitely many pairwise nonisomorphic nonabelian simple finite groups whose isomorphic copy is contained in $\mathrm{Bir}(X)$.

Construction (iii): Consider Richard J. Thompson's group V (concerning its definition and properties used below see, e.g. [CFP96, §6]). As V is finitely presented, the same argument as in the proof of Theorem 1 yields the existence of a simply connected noncompact smooth oriented four-dimensional manifold T whose diffeomorphism group $\mathrm{Diff}(T)$ contains

an isomorphic copy of V acting on T properly discontinuously. Since V contains a subgroup isomorphic to Sym_n for every n, the above Corollaries 1 and 2 hold true with C replaced by T.

Remark 3. Since the explicit presentation of V by 4 generators and 14 relations is known (see [CFP96, Lemma 6.1]), the proof of [CZ93, Thm. 5.1.1] yields the explicit way to construct T from the connected sum of four copies of $S^1 \times S^3$ successively glueing to it fourteen "handles" $D^2 \times S^2$.

6. Now we consider the case of compact manifolds.

According to [MiR13], the following conjecture is attributed to É. Ghys (we reformulate it using Definition 1):

Conjecture 3. *For every connected compact smooth manifold M the group* $\mathrm{Diff}(M)$ *is Jordan.*

There are several evidences in favor of Conjecture 3. For instance, it is true in either of the following cases:

(i) M is oriented and $\dim(M) \leqslant 2$ (see, e.g., [MiR10, Thm. 1.3]);
(ii) $\dim(M) = n$ and M admits an unramified covering $\widetilde{M} \to M$ such that $H^1(\widetilde{M}, \mathbf{Z})$ contains the cohomology classes $\alpha_1, \ldots, \alpha_n$ satisfying $\alpha_1 \cup \ldots \cup \alpha_n \neq 0$ (see [MiR10, Thm. 1.4(1)]).

Definition 1 implies that in every Jordan group every set of pairwise nonisomorphic nonabelian finite simple subgroups is finite. Theorem 2 below (based on the classification of finite simple groups) shows that a certain finiteness property of nonabelian finite simple subgroups indeed holds for diffeomorphism groups of connected compact manifolds. So Theorem 2 may be considered as another evidence in favor of Conjecture 3. It shows that noncompactness of the above-considered "highly symmetric" manifolds C, S, N, and T admitting a free action of every finite group is an inalienable property (the peculiarity here is considering actions of all finite groups on the *same* manifold: it is known [A57] that every finite group acts freely on some *compact* two-dimensional manifold, which, however, depends on this group).

7. In order to formulate Theorem 2, recall (see, e.g., [W09, 1.2]) that by the adverted classification the complete list of nonabelian finite simple groups (considered up to isomorphism) consists of 26 sporadic groups and the following infinite series depending on parameters:

- series $\mathcal{S}_1$ depending on one parameter $n \in \mathbb{N} := \{1, 2, \ldots\}$:

$$\mathrm{Alt}_n, n \geqslant 5; {}^2B_2(2^{2n+1}); {}^2G_2(3^{2n+1}); {}^2F_4(2^{2n+1}); \tag{1}$$

- series $\mathcal{S}_2$ depending on two parameters $p, a \in \mathbb{N}$, where p is a prime:

$$G_2(p^a), (p, a) \neq (2, 1); F_4(p^a); E_6(p^a); {}^2E_6(p^a); {}^3D_4(p^a); E_7(p^a); E_8(p^a); \tag{2}$$

- series $\mathcal{S}_3$ depending on three parameters $p, a, n \in \mathbb{N}$, where p is a prime:

$$
\begin{aligned}
&\mathrm{PSL}_n(p^a), n \geqslant 2, (n, p, a) \neq (2, 2, 1), (2, 3, 1); \\
&\mathrm{PSU}_n(p^a), n \geqslant 3, (n, p, a) \neq (3, 2, 1); \\
&\mathrm{PSp}_n(p^a), n \geqslant 2, (n, p, a) \neq (4, 2, 1); \\
&\mathrm{P\Omega}_{2n+1}(p^a), n \geqslant 3, p \neq 2; \\
&\mathrm{P\Omega}_{2n}^+(p^a), n \geqslant 4; \\
&\mathrm{P\Omega}_{2n}^-(p^a), n \geqslant 4.
\end{aligned}
\tag{3}
$$

Theorem 2. *Let M be a connected compact smooth manifold. There exists a real number b_M, depending on M only, such that every nonabelian finite simple subgroup F of $\mathrm{Diff}(M)$, belonging to one of the series $\mathcal{S}_1$, $\mathcal{S}_2$, or $\mathcal{S}_3$, has the following property:*

(i) *if $F \in \mathcal{S}_1$, then $n \leqslant b_M$;*
(ii) *if $F \in \mathcal{S}_2$, then $a \leqslant b_M$;*
(iii) *if $F \in \mathcal{S}_3$, then $a \leqslant b_M$ and $n \leqslant b_M$.*

Sketch of proof. One deduces from [MS63, Thm. 2.5] that there exists an integer e_M, depending on M only, such that the order of every elementary abelian subgroup of $\mathrm{Diff}(M)$ does not exceed e_M. This property is then applied to the suitable elementary abelian subgroups of the groups of each type listed in (1), (2), and (3); finding these elementary subgroups is performed by case by case considerations. The upper bounds of possible values of parameters for each of these types are obtained as the result of these considerations. The maximum of these upper bounds in then the sought-for bound b_M.

Below are several examples.

Let $F = \mathrm{Alt}_n$, $n \geqslant 5$. Let d be the quotient of dividing n by 3. The subgroup of Alt_n generated by the 3-cycles $(1, 2, 3), \ldots, (3d - 2, 3d - 1, 3d)$ is an elementary abelian group of order 3^d. Therefore, $3^d \leqslant e_M$; whence $n < 3(\log_3 e_M + 1)$.

Let $F = {}^2G_2(3^{2n+1})$, $n \geqslant 1$. Since F contains an elementary abelian subgroup of order 3^{6n+3} (see [W09, Thm. 4.2(i)]), we have $3^{6n+3} \leqslant e_M$; whence $n \leqslant (\log_3 e_M - 3)/6$.

Let $F = {}^3D_4(p^a)$. Since F contains an elementary abelian subgroup of order p^{9a} (see [W09, Thm. 4.3(i)]), we have $p^{9a} \leqslant e_M$; whence $a \leqslant (\log_p e_M)/9$.

Let $F = \mathrm{PSL}_n(p^a), n \geqslant 2, (n, p, a) \neq (2, 2, 1), (2, 3, 1)$. Since F contains an elementary abelian subgroup of order $p^{a(n-1)}$ (see [W09, Sect. 3.3.3]), we have $p^{a(n-1)} \leqslant e_M$; whence $a(n - 1) \leqslant \log_p e_M$.

In the other cases considerations are similar. We skip the details. $\qquad\square$

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