Cent. Eur. J. Math. • 11(12) • 2013 • 2076-2088
DOI: 10.2478/s11533-013-0307-8

VERSITA

Central European Journal of **Mathematics**

The automorphism groups of foliations with transverse linear connection

Research Article

Nina I. Zhukova[1]*, Anna Yu. Dolgonosova[1]†

1 *Nizhny Novgorod State University, Department of Mechanics and Mathematics, Gagarin Ave. 23, Nizhny Novgorod, 603095, Russia*

Received 12 March 2012; accepted 17 January 2013

Abstract: The category of foliations is considered. In this category morphisms are differentiable maps sending leaves of one foliation into leaves of the other foliation. We prove that the automorphism group of a foliation with transverse linear connection is an infinite-dimensional Lie group modeled on LF-spaces. This result extends the corresponding result of Macias-Virgós and Sanmartín Carbón for Riemannian foliations. In particular, our result is valid for Lorentzian and pseudo-Riemannian foliations.

MSC: 53C12, 53-XX, 58BXX

Keywords: Foliation • Linear connection • Automorphism group • Foliated bundle • Infinite-dimensional Lie group

1. Introduction

Foliations with transverse linear connection are investigated. Works of Molino [9], Kamber and Tondeur [3], Bel'ko [1] are devoted to different aspects of this class of foliations.

Let $\mathcal{F}ol$ be the category of foliations whose objects are foliations and morphisms are smooth maps between foliated manifolds mapping leaves to leaves. By smoothness (manifolds, mappings, bundles) we shall mean the smoothness of the class C^∞. Let (M, F) be an arbitrary smooth foliation with transverse linear connection. We investigate the group $\mathcal{D}(M, F)$ of diffeomorphisms of the manifold M whose elements are the automorphisms of the foliation (M, F) in the category $\mathcal{F}ol$.

* *E-mail: n.i.zhukova@rambler.ru*
† *E-mail: dolgonosova@rambler.ru*

Palais [11] introduced a smooth structure on the set $C^\infty(M, M')$ of smooth maps $M \to M'$ of smooth compact manifolds M and M'. He applied as model spaces inductive limits of Hilbert spaces. The diffeomorphism group $\mathrm{Diff}(M)$ of a compact manifold M was studied by Leslie, Omori and other authors. Leslie and Omori introduced structures of infinite-dimensional manifolds on $\mathrm{Diff}(M)$ modeled on Fréchet spaces and on inductive limits of Hilbert spaces respectively.

When the manifold M is non-compact the application of FD-topology (see Section 4) allowed Michor [8] to introduce a smooth structure on $\mathrm{Diff}(M)$ modeled on LF-spaces, i.e., on inductive limits of Fréchet spaces.

The objective of this paper is to introduce a structure of a smooth infinite-dimensional manifold modeled on LF-spaces in the group of all automorphisms of a foliation with transverse linear connection in the category $\mathcal{F}o\ell$. In order to introduce the structure of a smooth infinite-dimensional manifolds modeled on LF-spaces in the group of diffeomorphisms of a smooth manifold, Michor [8] used the construction of a local addition. Macias-Virgós and Sanmartín Carbón [7] adapted this method to foliations and applied it to Riemannian foliations. The goal of our work is to extend the result of Macias-Virgós and Sanmartín Carbón for Riemannian foliations [7] to foliations with transverse linear connection. The following theorem is the main result of this work.

Theorem 1.1.
Let (M, F) be a foliation with transverse linear connection of an arbitrary codimension q on an n-dimensional manifold M. Then the automorphism group $\mathcal{D}(M, F)$ of this foliation in the category $\mathcal{F}o\ell$ admits a structure of an infinite-dimensional Lie group modeled on LF-spaces.

The results of Macias-Virgós and Sanmartín Carbón [7, Theorem 13, Corollary 14] allow us to reduce the proof of this theorem to the construction of an adapted local addition for (M, F). Let $\mathfrak{M}$ be a smooth q-dimensional distribution on M which is transverse to (M, F). For a foliation (M, F) with transverse linear connection we construct (Theorem 4.1) a special linear connection $\nabla^{\mathfrak{M}}$ on the foliated manifold M with respect to which $\mathfrak{M}$ and the tangent distribution TF of the foliation (M, F) are geodesic invariant in the sense of [6]. Due to the use of $\nabla^{\mathfrak{M}}$ our construction of an adapted local addition is simpler than Macias-Virgós and Sanmartín Carbón's one for Riemannian foliations [7].

As pseudo-Riemannian foliations and, in particular, Lorentzian foliations belong to the class of foliations with transverse linear connection, the following assertion is valid.

Corollary 1.2.
The statement of Theorem 1.1 is true for pseudo-Riemannian and Lorentzian foliations.

This article is organized as follows. First, we give basic concepts and notation (Section 2). Then we construct the foliated bundle of transverse frames and prove Propositions 3.1 and 3.4 about its properties (Section 3). The foliated bundle of transverse frames with the lifted foliation is applied for the construction of a special linear connection $\nabla^{\mathfrak{M}}$ for the foliation (M, F) (Section 4). We also recall the Macias-Virgós and Sanmartín Carbón results about the structure of the Lie group on the set of automorphisms of a foliation admitting an adapted local addition (Section 5). Section 6 contains the proof of our main Theorem 1.1. Here, the above mentioned special linear connection $\nabla^{\mathfrak{M}}$ is essentially used.

2. Basic concepts and notation

2.1. Notation

Let M be a Hausdorff, paracompact, connected smooth n-dimensional manifold. It is not necessarily compact. Algebra of smooth functions on M is denoted by $\mathfrak{F}(M)$, and $\mathfrak{F}(M)$-module of vector fields on M is designated by $\mathfrak{X}(M)$. Let $\mathfrak{M}$ be a smooth distribution on M. Then $\mathfrak{X}_{\mathfrak{M}}(M)$ is the set of vector fields which are sections of $\mathfrak{M}$. Let TF be the tangent distribution to the foliation (M, F), then $\mathfrak{X}_{TF}(M)$ is also denoted by $\mathfrak{X}_F(M)$.

2.2. Linear connections

Let $\mathfrak{M}$ be a k-dimensional distribution n-dimensional manifold M, where $0 < k \le n$. A *linear connection on the vector bundle* $\mathfrak{M}$ is the operator

$$\nabla\colon\ \mathfrak{X}(M)\times\mathfrak{X}_{\mathfrak{M}}(M)\ \to\ \mathfrak{X}_{\mathfrak{M}}(M), \qquad (X,Y)\mapsto \nabla_X Y,$$

enjoying the following properties for all $X, X_1, X_2 \in \mathfrak{X}(M)$, $Y, Z \in \mathfrak{X}_{\mathfrak{M}}(M)$ and $f, h \in \mathfrak{F}(M)$:

(C_1) $\nabla_{fX_1+hX_2}Z = f\nabla_{X_1}Z + h\nabla_{X_2}Z$;

(C_2) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$;

(C_3) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

If $\mathfrak{M} = TM$, then the operator ∇ is called a *linear connection on* M.

Further the pair (M,∇) is called the manifold with linear connection. The bilinear skew-symmetric tensor on M of the type $(1,2)$, which is defined by the equality

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad X, Y \in \mathfrak{X}(M),$$

is called the *torsion tensor* or the *torsion* of the linear connection ∇. A linear connection ∇ on M is said to be *symmetric*, if the torsion tensor T vanishes.

2.3. Foliations with transverse linear connection

A diffeomorphism $f\colon M^{(1)} \to M^{(2)}$ is said to be an isomorphism of connections $\nabla^{(1)}$ and $\nabla^{(2)}$ if

$$f_*\big(\nabla^{(1)}_X Y\big) = \nabla^{(2)}_{f_*X} f_* Y$$

for all vector fields $X, Y \in \mathfrak{X}\big(M^{(1)}\big)$, where f_* is the differential of f. Let N be a q-dimensional manifold and M be a smooth n-dimensional manifold, where $0 < q < n$. Unlike M the connectedness of the topological space N is not assumed. An N-cocycle is the set $\big\{U_i, f_i, \{k_{ij}\}\big\}_{i,j\in J}$ such that:

1. The family $\{U_i : i \in J\}$ forms an open cover of M.
2. The mappings $f_i\colon U_i \to N$ are submersions into N with connected fibers.
3. If $U_i \cap U_j \ne \emptyset$, $i, j \in J$, then a diffeomorphism $k_{ij}\colon f_j(U_i\cap U_j) \to f_i(U_i\cap U_j)$ is well defined and satisfies the equality $f_i = k_{ij}\circ f_j$.

Definition 2.1.
Let a foliation (M,F) be given by an N-cocycle $\big\{U_i, f_i, \{k_{ij}\}\big\}_{i,j\in J}$. If the manifold N admits a linear connection ∇^N such that every local diffeomorphism k_{ij} is an isomorphism of the linear connections induced by ∇^N on open subsets $f_i(U_i\cap U_j)$ and $f_j(U_i\cap U_j)$, then we refer to (M,F) as a *foliation with transverse linear connection* defined by the (N,∇^N)-cocycle $\big\{U_i, f_i, \{k_{ij}\}\big\}_{i,j\in J}$.

We emphasize that vanishing of the torsion tensor of a linear connection ∇^N on N is not assumed.

2.4. Inducted connection on a submanifold

Let M be any n-dimensional manifold provided by a linear connection ∇. Consider an arbitrary p-dimensional immersion submanifold L of M. The number $q = n - p$ is the codimension of L. Suppose that at each point $x \in L$ a q-dimensional subspace $\mathfrak{M}_x$ of the tangent vector space T_xM is given, and $\mathfrak{M}_x$ is transverse to T_xL and smoothly depends on x. Then L is said to be an $\mathfrak{M}$-*equipped submanifold*. As $T_yM = T_yL \oplus \mathfrak{M}_y$, $y \in L$, so each vector field $X \in \mathfrak{X}(M)\restriction_L$ may be represented in the form $X = X^F \oplus X^{\mathfrak{M}}$, where $X^F \in \mathfrak{X}(L)$, $X^{\mathfrak{M}} \in \Gamma\mathfrak{M}$ is a section of $\mathfrak{M}$. Denote the canonical projection $\mathfrak{X}(M)\restriction_L \to \mathfrak{X}(L)$, $X \mapsto X^F$ by pr^F.

Define a linear connection $\nabla^{L,\mathfrak{M}}$ on L in the following a way. Consider vector fields $X, Y \in \mathfrak{X}(L)$ and any point $x \in L$. As is well known [4], the covariant derivative of Y along X at x is defined by values of $X\restriction_x$ and Y at any small neighborhood of x in L. Let W and V be neighborhoods of x in L, with the closure $\mathrm{Cl}\,W$ belonging to V and $X_V = X\restriction_V$, $Y_V = Y\restriction_V$. There are prolongations $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ of vector fields $X_{\mathrm{Cl}\,W}$ and $Y_{\mathrm{Cl}\,W}$. We have $X\restriction_W = \widetilde{X}\restriction_W$ and $Y\restriction_W = \widetilde{Y}\restriction_W$. As is well known, the equality

$$\nabla^{L,\mathfrak{M}}_X Y\restriction_x = \mathrm{pr}^F(\nabla_{\widetilde{X}}\widetilde{Y})\restriction_x, \qquad x \in L,$$

defines a linear connection $\nabla^{L,\mathfrak{M}}$ on the $\mathfrak{M}$-equipped submanifold L of (M,∇). The connection $\nabla^{L,\mathfrak{M}}$ is named the $\mathfrak{M}$-*induced connection*.

Recall that a submanifold L of the manifold of linear connection (M,∇) is called *totally geodesic*, if for each $x \in L$ and every vector $X \in T_xL$ the geodesic line $\gamma_X(s)$ such that $\gamma_X(0) = x$ and $\dot{\gamma}_X(0) = X$ belongs to L.

Generally speaking, the connection $\nabla^{L,\mathfrak{M}}$ depends on the equipment $\mathfrak{M}$ of the submanifold L. As it is known [12, Lecture 3], if ∇ is a symmetry connection on M, then the induced connection $\nabla^{L,\mathfrak{M}}$ does not depend on the equipment $\mathfrak{M}$ of the submanifold L if and only if L is the totally geodesic submanifold of (M,∇).

2.5. Geodesic invariant distributions and totally geodesic foliations

A smooth distribution $\mathfrak{N}$ on the manifold of linear connection (M,∇) is called *geodesic invariant* [6], if for any $x \in M$ and each vector $X \in \mathfrak{N}_x$ the geodesic line $\gamma_X(s)$ such that $\gamma_X(0) = x$ and $\dot{\gamma}_X(0) = X$, is an integral curve of $\mathfrak{N}$. A foliation (M,F) of a manifold M with a linear connection ∇ is called *totally geodesic*, if its tangent distribution TF is geodesic invariant or equivalent, if all its leaves are totally geodesic submanifolds.

3. The foliated bundle of transverse frames

3.1. Projectable connections [10]

We keep notation from [4]. By $P(M,G)$ we mean a principal G-bundle. Let M be an n-dimensional smooth manifold and $\pi\colon P \to M$ be the projection of $P(M,G)$. A G-connection in $P(M,G)$ is a G-invariant n-dimensional distribution Q on P transverse to fibres of the submersion $\pi\colon P \to M$. As known [4], the connection may be defined by $\mathfrak{g}$-valued 1-form ω on M satisfying some conditions. The form ω is named the *connection form*. If $P = L(M)$ is the frame bundle of M, then the existence of the connection in $P(M,G)$ is equivalent to the existence of a linear connection ∇ on M.

Consider $P(M,G)$ with a G-invariant foliation $(P,\mathfrak{F})$ such that images of its leaves form a foliation (M,F) of the same dimension m, where $0 < m < n$. A connection Q on P is said to be *transverse*, if $T\mathfrak{F} \subset Q$ or equivalent, if $i_X\omega = 0$, $X \in \mathfrak{X}_{\mathfrak{F}}(P)$, where i_X is the interior product by X. A transverse connection Q on P is called *projectable with respect to* $(P,\mathfrak{F})$, if $i_X d\omega = 0$ for all $X \in \mathfrak{X}_{\mathfrak{F}}(P)$.

3.2. Principal foliated bundles

Let (M,F) be a smooth foliation of codimension q. A q-dimensional distribution $\mathfrak{M}$ on M is said to be *transverse to the foliation* (M,F) if $\mathfrak{M}$ satisfies the equality $T_xM = \mathfrak{M}_x \oplus T_xF$, $x \in M$, where $\oplus$ is the symbol of the direct sum of vector spaces.

Let (M, F) be a foliation of codimension q with transverse linear connection defined by an (N, ∇^N)-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$. For brevity let us denote $H = \mathrm{GL}(q, R)$. Let $\mathfrak{h}$ be the Lie algebra of a Lie group H. Denote the projection of the frame bundle of N by $p\colon P \to N$, then $P = P(N, H)$ is the principal H-bundle. Let $V_i = f_i(U_i)$, then $P_i = p^{-1}(V_i)$ is a sub-bundle of the H-bundle P. Let $\mathcal{R}_i = f_i^* P_i = \{(x, z) \in U_i \times P_i : f_i(x) = p(z)\}$ be the pullback bundle of P_i with respect to the submersion f_i. We define the projections $\widehat{p}_i\colon \mathcal{R}_i \to U_i$, $(x, z) \mapsto x$, and $\widehat{f}_i\colon \mathcal{R}_i \to P_i$, $(x, z) \mapsto z$, where $(x, z) \in \mathcal{R}_i$. Suppose that a q-dimensional distribution $\mathfrak{M}$ on the manifold M is transverse to the foliation (M, F). Identify the vector quotient bundle TM/TF with the distribution $\mathfrak{M}$. We consider a point $(x, z) \in \mathcal{R}_i$ as a basis $\{e_{n-q+\alpha}\}$ of the vector space $\mathfrak{M}_x$ such that $f_{i*x} e_{n-q+\alpha} = \epsilon_\alpha$, where $\alpha = 1, \dots, q$, $\{\epsilon_\alpha\} = z$ is the frame at $v = f_i(x) \in N$ and f_{i*x} is the differential of f_i at x. The pair (x, z) is named an *$\mathfrak{M}$-frame* at x.

Introduce the following binary relation S in the disjoint sum $Y = \bigsqcup_{i\in J} \mathcal{R}_i$. Let $(x, z) \in \mathcal{R}_i, (\widetilde{x}, \widetilde{z}) \in \mathcal{R}_j$. Let us assume $(x, z)\, S\, (\widetilde{x}, \widetilde{z})$ if the following conditions hold:

(i) $x = \widetilde{x} \in U_i \cap U_j$;

(ii) $\widetilde{z} = k_{ji*f_i(x)} \circ z$, where $k_{ji*f_i(x)}$ is the differential of the local diffeomorphism k_{ji} at the point $f_i(x)$.

Direct verification shows that S is an equivalence relation. Let $\mathcal{R} = Y/S$ be the quotient space and $\beta\colon Y \to \mathcal{R}$ be the quotient mapping. Note that for every $i \in J$ the restriction of $\beta\restriction_{\mathcal{R}_i}\colon \mathcal{R}_i \to \widetilde{U}_i = \beta(\mathcal{R}_i)$ is a bijection. By requirement that all restrictions $\beta\restriction_{\mathcal{R}_i}$ are diffeomorphisms we define the structure of a smooth manifold in $\mathcal{R}$.

We introduce the notation: $\widetilde{f}_i = \widehat{f}_i \circ \left(\beta\restriction_{\mathcal{R}_i}\right)^{-1}\colon \widetilde{U}_i \to P_i$ and $K_{ij}\colon \widetilde{f}_j(\widetilde{U}_i \cap \widetilde{U}_j) \to \widetilde{f}_i(\widetilde{U}_i \cap \widetilde{U}_j)$, $z \mapsto k_{ij*f_j(x)} \circ z$, where $z \in \widetilde{f}_j(\widetilde{U}_i \cap \widetilde{U}_j)$. It is not difficult to check that $\{\widetilde{U}_i, \widetilde{f}_i, \{K_{ij}\}\}_{i,j\in J}$ is the P-cocycle defining a foliation of $(\mathcal{R}, \mathcal{F})$ of the same dimension as the foliation (M, F). Remark, that for every $x \in M$ and $u \in \pi^{-1}(x)$ the restriction $\pi\restriction_{\mathcal{L}}$ of π on the leaf $\mathcal{L} = \mathcal{L}(u)$ of $(\mathcal{R}, \mathcal{F})$ is a covering map onto a leaf $L = L(x)$ of (M, F).

For any point $u \in \mathcal{R}$ there is a point $(x, z) \in \mathcal{R}_i$ such that $u = \beta((x, z))$. The equality $\pi(u) = x$ defines a submersion $\pi\colon \mathcal{R} \to M$. The relation $u \cdot a = \beta((x, z \cdot a))$, where $a \in H$, defines the right free smooth action of the Lie group H on $\mathcal{R}$. Thus, $\pi\colon \mathcal{R} \to M$ is the projection of the principal H-bundle $\mathcal{R}(M, H)$. The definitions of $(\mathcal{R}, \mathcal{F})$ and the action of H on $\mathcal{R}$ imply the H-invariance of this foliation. Thus we have the following statement.

Proposition 3.1.

Let (M, F) be a foliation of an arbitrary codimension q defined by N-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$ and $H = \mathrm{GL}(q, R)$. Then there are:

1) *the principal H-bundle with the projection $\pi\colon \mathcal{R} \to M$;*

2) *an H-invariant foliation $(\mathcal{R}, \mathcal{F})$, whose leaves cover the leaves of the foliation (M, F) via π.*

Definition 3.2.

The principal H-bundle $\pi\colon \mathcal{R} \to M$ satisfying Proposition 3.1 is called the *foliated bundle of transverse frames* (or *$\mathfrak{M}$-frames*) of the foliation (M, F). In this case $(\mathcal{R}, \mathcal{F})$ is named the *lifted foliation*.

Remark 3.3.

Originally foliated bundles appeared in the works of Molino [9] and of Kamber–Tondeur [3]. Foliated bundles were essentially used by the first author in [15, 16].

3.3. The projectable connection in the foliated bundle of transverse frames

Proposition 3.4.

Let (M, F) be a foliation of an arbitrary codimension q with transverse linear connection defined by (N, ∇^N)-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$. Let $\mathcal{R}(M, H)$ be the foliated bundle of transverse frames over (M, F) with lifted foliation $(\mathcal{R}, \mathcal{F})$ and Q_0 be the H-connection on P corresponding to ∇^N. Then there exist:

(i) *the unique H-connection $Q \supset T\mathcal{F}$ on $\mathcal{R}$ locally projectable onto H-connection Q_0 on P;*

(ii) *an $\mathbb{R}^q$-valued H-equivariant projectable 1-form θ on $\mathcal{R}$.*

Moreover, $(\mathcal{R}, \mathcal{F})$ *is an e-foliation.*

Proof. The linear connection ∇^N defines the principal H-connection Q_0 on the space P of the frame bundle $P(N, H)$. Let ω_0 be the $\mathfrak{h}$-valued 1-form of the connection Q_0 and θ_0 be the canonical $\mathbb{R}^q$-valued 1-form of Q_0 on the manifold P (see for example [4]).

The direct verification shows that the equalities $\omega\restriction_{\widetilde{U}_i} = (\widetilde{f}_i)^*\omega_0$ and $\theta\restriction_{\widetilde{U}_i} = (\widetilde{f}_i)^*(\theta_0)$, where $i \in J$, define the $\mathfrak{h}$-valued 1-form ω and $\mathbb{R}^q$-valued 1-form θ on the manifold $\mathcal{R}$. The H-equivariance of the 1-forms ω_0 and θ_0 on P implies the H-equivariance of the 1-forms ω and θ on $\mathcal{R}$. Hence ω defines the principal H-connection $Q = \ker\omega$ on $\mathcal{R}$.

It follows from the definitions that ω and θ are projectable with respect to the foliation $(\mathcal{R}, \mathcal{F})$ and $L_X\omega = 0$, $L_X\theta = 0$ for any vector field $X \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R})$. Thus, Q is a projectable H-connection on $\mathcal{R}$ with respect to the foliation $(\mathcal{R}, \mathcal{F})$. For each fixed $i \in J$ we have $\widetilde{f}_{i*u}Q_u = Q_0\restriction_{\widetilde{f}_i(u)}$, $u \in \widetilde{U}_i$. Hence H-connection $Q \supset T\mathcal{F}$ on $\mathcal{R}$ is locally projectable onto H-connection Q_0 on P.

Let $u \in \mathcal{R}$ and $x = \pi(u)$. Consider $\mathfrak{N}_u = \{Z_u \in T_u\mathcal{R} : \pi_{*u}(Z_u) \in \mathfrak{M}_x\}$ and $\mathfrak{V}_u = \{Z_u \in T_u\mathcal{R} : \pi_{u*}(Z_u) = 0\}$. Then $\mathfrak{N} = \{\mathfrak{N}_u : u \in \mathcal{R}\}$, $\mathfrak{V} = \{\mathfrak{V}_u : u \in \mathcal{R}\}$ and $\mathfrak{N} \cap Q$ are smooth distributions on $\mathcal{R}$, with $\mathfrak{N} = \mathfrak{V} \oplus (\mathfrak{N} \cap Q)$. Fix bases $\{E_\alpha\}$, $\alpha = 1, \dots, \dim\mathfrak{h}$, and $\{E_\beta\}$, $\beta = 1, \dots, q$, of the vector spaces $\mathfrak{h}$ and $\mathbb{R}^q$. Then at any point $u \in \mathcal{R}$ vectors $X_\alpha\restriction_u = \left(\omega\restriction_{\mathfrak{V}_u}\right)^{-1}(E_\alpha)$ and $X_\beta\restriction_u = \left(\theta\restriction_{\mathfrak{N}_u \cap Q_u}\right)^{-1}(E_\beta)$ are defined. The vector fields $\{X_\alpha, X_\beta\}$ form a transverse parallelization of the foliation $(\mathcal{R}, \mathcal{F})$. So $(\mathcal{R}, \mathcal{F})$ is an e-foliation. $\square$

4. The existence of special connections

Let (M, F) be a smooth foliation. Recall that a vector field $X \in \mathfrak{X}(M)$ is said to be *foliate* if for all $Y \in \mathfrak{X}_F(M)$ the Lie bracket $[X, Y]$ also belongs to $\mathfrak{X}_F(M)$ [10]. Foliated vector fields are also named basic. The function $h\colon M \to \mathbb{R}^1$ is named *basic* if it is constant on every leaf of the foliation.

A linear connection ∇ on M is said to be *projectable with respect to* (M, F), if each submersion $f\colon U \to V$ from (N, ∇^N)-cocycle $\left\{U_i, f_i, \{k_{ij}\}\right\}_{i,j\in J}$ determinating (M, F) satisfies the equality

$$f_*\left(\nabla_{X_U} Y_U\right) = \nabla^N_{f_*(X_U)} f_*(Y_U)$$

for any foliate vector fields X, Y on M. Remark that f_*X and f_*Y are also called f-connected vector fields with X and Y accordingly. It is not difficult to show that ∇ on M is projectable with respect to (M, F), iff every the above mentioned submersion $f\colon U \to V$ maps geodesics on (U, ∇) to geodesics on (V, ∇^N).

Let $\mathfrak{M}$ be an arbitrary smooth q-dimensional distribution on a manifold M transverse to the foliation (M, F). Therefore any smooth vector field X on M can be written in the form $X = X^F + X^{\mathfrak{M}}$, where $X^F \in \mathfrak{X}_F(M)$, $X^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. Then the canonical projections are defined in the following way:

$$p^F\colon \mathfrak{X}(M) \to \mathfrak{X}_F(M), \qquad X \mapsto X^F, \qquad\qquad p^{\mathfrak{M}}\colon \mathfrak{X}(M) \to \mathfrak{X}_{\mathfrak{M}}(M), \qquad X \mapsto X^{\mathfrak{M}}.$$

Let ∇ be a linear connection on M. A geodesic γ on (M, ∇) is called $\mathfrak{M}$-*geodesic*, if it is an integral curve of a distribution $\mathfrak{M}$.

Theorem 4.1.
Let (M, F) *be a foliation with transverse linear connection of codimension* q *and* $\mathfrak{M}$ *be a* q*-dimensional distribution transverse to* (M, F)*. Then there is a linear connection* $\nabla^{\mathfrak{M}}$ *on* M *such that*

(i) *both* $\mathfrak{M}$ *and* TF *are geodesic invariant distributions on* $\left(M, \nabla^{\mathfrak{M}}\right)$*, and the connections induced via* $\mathfrak{M}$ *on leaves of* (M, F) *are symmetric;*

(ii) *the equality* $\nabla^0_X Y = p^{\mathfrak{M}}\left(\nabla^{\mathfrak{M}}_X Y\right)$*, where* $X \in \mathfrak{X}(M)$*,* $Y \in \mathfrak{X}_{\mathfrak{M}}(M)$*, defines a projectable connection on* $\mathfrak{M}$*;*

(iii) *any submersion* $f\colon U \to V$ *from* $\left(N, \nabla^N\right)$*-cocycle defining* (M, F) *has the following properties:*

a) *the projection* $\sigma = f\circ\gamma$ *of any geodesic* γ *from* U *is a geodesic on* V, *i.e.* $\nabla^{\mathfrak{M}}$ *is a projectable connection with respect to* (M,F);

b) *if* $\sigma\colon[0,1]\to V$ *is a geodesic in* V *and* $x\in f^{-1}(\sigma(0))$, *then there is* $\varepsilon\in(0,1]$ *and* $\mathfrak{M}$*-lift* $\gamma=\gamma(s)$, $s\in[0,\varepsilon]$, *of* $\sigma\restriction_{[0,\varepsilon]}$ *to the point* x, *and* γ *is* $\mathfrak{M}$*-geodesic.*

Proof. According to Proposition 3.1 there is the foliated bundle $\mathcal{R}(M,H)$ of $\mathfrak{M}$-frames over (M,F) with the projection $\pi\colon\mathcal{R}\to M$. There is an open covering $\{V_\alpha\}$ of manifold M with the set of transition functions $\psi_{\beta\alpha}\colon V_\alpha\cap V_\beta\to \mathrm{GL}(q,R)$ [4, Chapter 1, § 5]. Consider the embedding $j\colon \mathrm{GL}(q,R)\to \mathrm{GL}(n,R)$,

$$A\mapsto\begin{pmatrix}E_{n-q} & 0\\ 0 & A\end{pmatrix},$$

where $A\in\mathrm{GL}(q,R)$ and E_{n-q} is the unit $(n-q)$-dimensional matrix, of the Lie group $\mathrm{GL}(q,R)$ into the Lie group $\mathrm{GL}(n,R)$. Then there are maps

$$\varphi_{\beta\alpha}=j\circ\psi_{\beta\alpha}\colon\ V_\alpha\cap V_\beta\to\mathrm{GL}(n,R),$$

satisfying the equality $\varphi_{\gamma\alpha}(x)=\varphi_{\gamma\beta}(x)\cdot\varphi_{\beta\alpha}(x)$, $x\in V_\alpha\cap V_\beta\cap V_\gamma$, where the symbol $\cdot$ denotes the product of the elements in the group $\mathrm{GL}(n,R)$. According to [4, Chapter 1, Proposition 5.2] there is the principal $\mathrm{GL}(n,R)$-bundle $\widehat{\pi}\colon\widehat{\mathcal{R}}\to M$ with the transition functions $\varphi_{\beta\alpha}$. Identify $\mathrm{GL}(q,R)$ with the closed subgroup $j(\mathrm{GL}(q,R))$ of the Lie group $\mathrm{GL}(n,R)$, then $\mathcal{R}\subset\widehat{\mathcal{R}}$, moreover $\widehat{\pi}\restriction_{\mathcal{R}}=\pi$. Thus, we consider $\mathcal{R}$ as the reduced sub-bundle of $\mathrm{GL}(n,R)$-bundle $\widehat{\mathcal{R}}\big(M,\mathrm{GL}(n,R)\big)$. Let Q be the H-connection on $\mathcal{R}$ satisfying Proposition 3.4. Then

$$\widetilde{f}_{i*}Q_u=Q_0\restriction_{\widetilde{f}_i(u)},\qquad u\in\widetilde{U}_i\subset\mathcal{R},\quad i\in J. \tag{1}$$

Connection Q is extended to the $\mathrm{GL}(n,R)$-connection $\widehat{Q}$ on $\widehat{\mathcal{R}}$ in the following way. Consider arbitrary $z\in\widehat{\mathcal{R}}$. Let $x=\widehat{\pi}(z)$. Then there exist $u\in\widehat{\pi}^{-1}(x)\cap\mathcal{R}$ and a unique $a\in\mathrm{GL}(n,R)$ such that $z=u\cdot a$. Define $\widehat{Q}_z=R_{a*}(Q_u)$. Due to $\mathrm{GL}(q,R)$-invariance of Q the distribution $\widehat{Q}$ is really defined. Remark that $\widehat{Q}$ is $\mathrm{GL}(n,R)$-invariant distribution on $\widehat{R}$ and transverse to the fibers of the bundle $\widehat{\pi}\colon\widehat{\mathcal{R}}\to M$. Hence the connection $\widehat{Q}$ defines some linear connection ∇ on the manifold M. Let us show that $\mathfrak{M}$ is a totally geodesic distribution on (M,∇).

Since the property of $\mathfrak{M}$ to be totally geodesic is local, it is sufficient to prove that it holds in a neighborhood of a point $x\in M$. Let $x\in U$, where $f\colon U\to V$ is a submersion from the $\big(N,\nabla^N\big)$-cocycle defining the foliation (M,F). Consider an arbitrary vector $X\in\mathfrak{M}_x$, let $f_{*x}(X)=Y$, then $Y\in T_yN$, where $y=f(x)$. There is a geodesic $\sigma=\sigma(s)$, $s\in(-\varepsilon,\varepsilon)$, of $\big(N,\nabla^N\big)$ satisfying the conditions $\sigma(0)=y$, $\dot\sigma(0)=Y$. Due to the theorem about existence and uniqueness of solutions of ordinary differential equations, there is a number δ, $0<\delta\le\varepsilon$, and a local $\mathfrak{M}$-lift $\gamma=\gamma(s)$, $s\in(-\delta,\delta)$, of σ to the point x. It means that γ is a such integral curve of the distribution $\mathfrak{M}$ that $\gamma(0)=x$ and $f\circ\gamma=\sigma\restriction_{(-\delta,\delta)}$. As $f_{*x}\colon\mathfrak{M}_x\to T_xN$ is an isomorphism of vector spaces, so $\dot\gamma(0)=X$. Show that γ is geodesic on (M,∇).

Let $\widehat{\omega}$ be the connection form and $\widehat{\theta}$ be the canonical $\mathbb{R}^n$-value 1-form on $\widehat{\mathcal{R}}$ defined by the connection ∇. Remind, that $B_\xi\in\mathfrak{X}(\widehat{\mathcal{R}})$ is called the *standard horizontal vector field* if $\widehat{\omega}(B_\xi)=0$ and $\widehat{\theta}(B_\xi)=\xi=\mathrm{const}\in\mathbb{R}^n$. It is known [4, Chapter III, Proposition 6.3] that γ is geodesic in (M,∇) if and only if γ is the projection of an integral curve of some standard horizontal vector field. Since σ is geodesic, with $\sigma(0)=y$, then for $v\in p^{-1}(y)$ there is the Q_0-horizontal lift $\sigma_0=\sigma_0(s)$, $s\in(-\varepsilon,\varepsilon)$, of curve σ to the point v, moreover $\theta_0(\dot\sigma_0(s))=\xi=\mathrm{const}\in\mathbb{R}^q$. Let $\widetilde{f}\colon\widetilde{U}=\pi^{-1}(U)\to P$ be the submersion defined by f and satisfying the equality $p\circ\widetilde{f}=f\circ\pi$. Take $u\in\widetilde{f}_i^{-1}(v)\cap\pi^{-1}(x)\subset\mathcal{R}$. Then there is $\widehat{Q}$-horizontal lift $\widehat{\gamma}=\widehat{\gamma}(s)$, $s\in(-\delta,\delta)$, of curve γ to the point u. Therefore the equality $\pi\circ\widehat{\gamma}=\gamma$ implies that $\widehat{\gamma}$ is an integral curve of the distribution $\widehat{\mathfrak{M}}=\{\widehat{\mathfrak{M}}_u : u\in\widehat{\mathcal{R}}\}$, where $\widehat{\mathfrak{M}}_u=\big\{Z\in\widehat{Q}_u:\widehat{\pi}_{*u}(Z)\in\mathfrak{M}_x,\ x=\widehat{\pi}(u)\big\}$. As $\widehat{\theta}\restriction_{\widetilde{U}}=\widehat{j}\circ\theta$, where $\widehat{j}\colon\mathbb{R}^q\to\mathbb{R}^n\cong\mathbb{R}^{n-q}\times\mathbb{R}^q$, $\xi\mapsto(0_{n-q},\xi)$ and 0_{n-q} is zero in $\mathbb{R}^{n-q}$, then (1) implies the equality

$$\widehat{\theta}\big(\dot{\widehat{\gamma}}(s)\big)=\widehat{j}\circ\theta\big(\dot{\widehat{\gamma}}(s)\big)=\widehat{j}\circ\theta_0\big(\widetilde{f}_{i*}\dot{\widehat{\gamma}}(s)\big)=\widehat{j}\circ\theta_0(\dot\sigma_0(s)),\qquad s\in(-\delta,\delta),$$

whence $\widehat{\theta}\big(\dot{\widehat{\gamma}}(s)\big)=\widehat{j}\circ\theta(\dot{\widehat{\gamma}}(s))=(0_{n-q},\xi)=\eta\in\mathbb{R}^n$, if $\theta_0(\dot\sigma_0(s))=\xi\in\mathbb{R}^q$, $s\in(-\delta,\delta)$. Therefore, $\widehat{\gamma}(s)$, $s\in(-\delta,\delta)$, is an integral curve of the horizontal vector field B_η on $\widehat{\mathcal{R}}$. It means that $\gamma(s)=\pi(\widehat{\gamma}(s))$ is geodesic on (M,∇). Thus, the distribution $\mathfrak{M}$ is a geodesic invariant distribution on (M,∇).

Now let us show that the linear connection ∇ is projectable with respect to (M,F). Consider an arbitrary geodesic v on U. Let $v(0)=x$ and $\dot v(0)=X\in T_xM$. Take $u_0\in\mathcal{R}\cap\widehat\pi^{-1}(x)$ and $\eta=u_0^{-1}(X)\in\mathbb{R}^n$. Let $\mathrm{pr}\colon\mathbb{R}^n\cong\mathbb{R}^{n-q}\times\mathbb{R}^q\to\mathbb{R}^q$ be the canonical projection and $\zeta=\mathrm{pr}(\eta)\in\mathbb{R}^q$. There is an integral curve $\widehat v$ of the standard vector field B_η through $\widehat v(0)=u_0$ on $\mathcal{R}$. Then $v=\widehat\pi\circ\widehat v$ and $\widehat\sigma=\widetilde f\circ\widehat v$ is the integral curve of the standard vector field B_ζ through $v_0=\widetilde f(u_0)$ on P. Since $f\circ\widehat\pi=p\circ\widetilde f$, it is easy to see that $\sigma=p\circ\widehat\sigma$ is a geodesic on V, and $f\circ v=\sigma$. This proves that ∇ is a projectable connection with respect to (M,F).

According to results of Willmore [14] and Walker [13], there is a unique linear connection $\nabla^{(1)}$ without torsion on M such that the foliation (M,F) is parallel with respect to the connection $\nabla^{(1)}$. Since each parallel distribution is geodesic invariant, the foliation (M,F) is totally geodesic on $\big(M,\nabla^{(1)}\big)$.

Define a new connection $\nabla^{\mathfrak{M}}$ on the manifold M by the equality

$$\nabla^{\mathfrak{M}}_X Y=\nabla^{(1)}_X Y^F+\nabla_X Y^{\mathfrak{M}},\qquad X,Y\in\mathfrak{X}(M),\tag{2}$$

where the linear connection ∇ is given above and $Y^F=p^F(Y)$, $Y^{\mathfrak{M}}=p^{\mathfrak{M}}(Y)$. The direct verification shows that $\nabla^{\mathfrak{M}}$ is really a linear connection on M. In accordance with (2), $p^{\mathfrak{M}}(\nabla^{\mathfrak{M}}_X Y)=p^{\mathfrak{M}}(\nabla_X Y)$ for all $X\in\mathfrak{X}(M)$, $Y\in\mathfrak{X}_{\mathfrak{M}}(M)$. Taking into consideration that ∇ is a projectable connection with respect to (M,F), we get the assertion (ii) of Theorem 4.1.

The property of the distribution $\mathfrak{M}$ to be geodesic invariant with respect to the connection ∇ and the definition of the connection $\nabla^{\mathfrak{M}}$ by the equality (2) imply the property of $\mathfrak{M}$ to be geodesic invariant on $(M,\nabla^{\mathfrak{M}})$. Similarly, the property of TF to be geodesic invariant on $(M,\nabla^{\mathfrak{M}})$ follows from the same property of TF with respect to the connection $\nabla^{(1)}$ and the definition of $\nabla^{\mathfrak{M}}$.

The distribution $\mathfrak{M}$ plays the role of the equipment of a leaf L of the foliation (M,F). Hence the connections $\nabla^{\mathfrak{M}}$ and $\nabla^{(1)}$ induce through $\mathfrak{M}$ the same connection without torsion on L. It finishes the proof of the statement (i) of Theorem 4.1. The properties a) and b) follow from the corresponding properties of the connection ∇ proved above. $\square$

Remark 4.2.
Using the proof of Theorem 4.1 it is easy to see that a linear connection ∇ on M is projectable with respect to the foliation (M,F) if and only if the respective induced $\mathrm{GL}(q,R)$-connection Q in the foliated bundle of transverse frames $\mathcal{R}\big(M,\mathrm{GL}(q,R)\big)$ is projectable with respect to the lifted foliation $(\mathcal{R},\mathcal{F})$ in the sense of subsection 3.1.

5. The Lie group of automorphisms of foliations with an adapted local addition

The goal of this section is to recall the results of Macias-Virgós and Sanmartín Carbón [7].

5.1. Michor's topology

Let M and M' be two smooth manifolds. Consider the set $C^\infty(M,M')$ of smooth maps from M to M'. Recall some notions and notation of different topologies on the set $C^\infty(M,M')$ from [8] (see also [5]).

Let $0\le r\le\infty$ and $J^r(M,M')$ be the space of r-jets of smooth maps from $C^\infty(M,M')$. Let CO^r be the compact C^r-topology on $C^\infty(M,M')$. It is the topology induced by the embedding $j^r\colon C^\infty(M,M')\to C^0(M,J^r(M,M'))$ from the compact open topology.

Let C^r-topology of Whitney or WO^r-topology on the set $C^\infty(M,M')$ be defined as the topology having the basis

$$W^r(\Omega)=\big\{f\in C^\infty(M,M'):(j^rf)(M)\subseteq\Omega\big\},$$

where Ω is any open set in $J^r(M,M')$. The basis of D-topology on $C^\infty(M,M')$ is given by the sets

$$D(L,\Omega)=\big\{f\in C^\infty(M,M'):(j^rf)(L_m)\subseteq\Omega_m\ \text{for all } m\big\},$$

where $L = \{L_m\}$ is a locally finite countable collection of closed sets in M and $\Omega = \{\Omega_m\}$ is a family of open sets in $\mathrm{J}^\infty(M, M')$. If the manifold M is not compact, the space $\mathrm{C}^\infty(M, M')$ with D-topology is not locally path connected. This problem was solved by the addition of new open sets that are the equivalence classes of the relation: $f \sim g$ if g coincides with f on the complement to some compact set in M'. The resulting topology is called *FD-topology* or *Michor's topology*.

We emphasize that WO^r-topology is finer than both CO^r-topology and $\mathrm{WO}^{r'}$-topology for $r \geq r'$. The D-topology is finer than WO^∞-topology. By the definition, Michor's topology is finer than D-topology.

5.2. The adapted local addition

Denote the projection of the tangent bundle to the manifold M by $\tau\colon TM \to M$. Let $s_0\colon M \to TM$, $x \mapsto 0_x$, be the zero section of τ assigning to an arbitrary point x of M the zero vector 0_x of T_xM. Thus $M_0 = s_0(M)$ is a submanifold of TM diffeomorphic to M. A map $E\colon S \to M$ of some open neighborhood S of the submanifold M_0 of TM is said to be a *local addition* [8], if the following two conditions are satisfied:

(D_1) $E(0_x) = x$ for every $x \in M$.

(D_2) The map $(\tau, E)\colon S \to M \times M$, $X_x \mapsto (x, E(X_x))$, $X_x \in S$, is a diffeomorphism onto some neighborhood W of the diagonal $\Delta = \{(y, y) : y \in M\}$ in the product of manifolds $M \times M$.

Suppose that a foliation (M, F) of codimension q is given. There is a foliated chart (U, φ) at every point $x \in M$. Let $F_U = F\restriction_U$ be the restriction of the foliation (M, F) onto U. Then $(U, F\restriction_U)$ is a simple foliation, which is isomorphic (in the category $\mathcal{F}o\ell$) to the standard foliation $(\mathbb{R}^n, F_{\mathrm{st}})$. Therefore the leaf space $\widehat{U} = U/F_U$ is a smooth q-dimensional manifold diffeomorphic to $\mathbb{R}^q$, and the canonical projection $\pi_U\colon U \to \widehat{U}$ is a submersion.

A local addition E on a foliated manifold M is called *adapted* to the foliation (M, F) [7], if for any foliated chart (U, φ) and the canonical projection $\pi_U\colon U \to \widehat{U}$ there is an open neighborhood $S_U \subset S$ of submanifold $U_0 = s_0(U)$ in TU and a local addition $\overline{E}_U\colon \overline{S}_U = (\pi_U)_*(S_U) \to \widehat{U}$, moreover $E(S_U) \subset U$, and the following diagram is commutative:

$$\begin{tikzcd} S_U \arrow[r, "E"] \arrow[d, "(\pi_U)_*"] & U \arrow[d, "\pi_U"] \\ \overline{S}_U \arrow[r, "\overline{E}_U"] & \widehat{U}. \end{tikzcd}$$

Macias-Virgós and Sanmartín Carbón proved that for any Riemannian foliation there is an adapted local addition [7].

5.3. The Lie group of automorphisms of foliations

Denote the group of all automorphisms of a foliation (M, F) in the category of foliations $\mathcal{F}o\ell$ by $\mathcal{D}(M, F)$. Let f be an arbitrary element of the group $\mathcal{D}(M, F)$. Consider the pullback f^*TM of the tangent bundle $\tau\colon TM \to M$. A smooth map $X\colon M \to TM$ satisfying the equality $\tau \circ X = f$ is named a *vector field X along f*. Let V be an open set in M. A vector field X along f is said to be *foliated*, if for any basic function h defined on V, the function Xh given by $(Xh)_x = X_xh$, $x \in f^{-1}(V)$, is basic. The set of foliated vector fields along f is denoted by $\Gamma^F(f^*TM)$. Let $\Gamma^F_c(f^*TM)$ be the set of foliated vector fields along f with compact support.

Suppose that a foliation (M, F) admits an adapted local addition. The application of Michor's results [8] allowed Macias-Virgós and Sanmartín Carbón [7] to prove that $\Gamma^F_c(f^*TM)$ with FD-topology is an LF-space, i.e., it is a complete locally path connected vector space, which is an inductive limit of Fréchet spaces.

Recall of the construction of an atlas on the automorphism group $\mathcal{D}(M, F)$ in the category $\mathcal{F}o\ell$ of (M, F) [7]. Two elements $g, f \in \mathcal{D}(M, F)$ are said to be equivalent $f \sim g$ if g coincides with f on the complement to some compact set.

According to our assumption, for (M, F) there is an adapted local addition $E\colon S \to M$. Then $W = (\tau, E)(S)$ is an open neighborhood of the diagonal of $M \times M$. Let f be any automorphism from $\mathcal{D}(M, F)$. The topology in $\mathcal{D}(M, F)$ is defined as the induced topology from the LF-manifold $\mathrm{Diff}(M)$ introduced by Michor [8] (see also [5]). Then an open neighborhood U_f of f in $\mathcal{D}(M,F)$ is given by

$$U_f = \big\{h \in \mathcal{D}(M,F) : h \sim f,\ (f(x), h(x)) \in W,\ x \in M\big\}.$$

The map $\varphi_f \colon U_f \to \Gamma^F_c(f^*TM)$ is defined by $\varphi_f(h)(x) = X_x$, where X_x is the unique vector in $S \cap T_{f(x)}M$ such that $(\tau, E)(X_x) = (f(x), h(x))$. It is proved in [7] that the map $h \in U_f$ is a morphism of $\mathcal{F}ol$ if the vector field $X = \varphi(h)$ along f is foliated. Moreover, for every $f \in \mathcal{D}(M,F)$ the map $\varphi_f \colon U_f \to \Gamma^F_c(f^*TM)$ is a homeomorphism onto an open neighborhood of the zero section in the vector space $\Gamma^F_c(f^*TM)$, and the last space is isomorphic to the vector space of the Lie algebra $\mathfrak{X}_c(M,F)$ of foliate vector fields with compact support.

Thus the following statement is a corollary of [7, Theorem 13 and Corollary 14].

Theorem 5.1.
Let (M,F) be a foliation that admits an adapted local addition. Then the group $\mathcal{D}(M,F)$ of foliation preserving diffeomorphisms is an infinite-dimensional Lie group modeled on the Lie algebra $\mathfrak{X}_c(M,F)$, which is an LF-space.

6. Proof of Theorem 1.1

Let (M,F) be a foliation of an arbitrary codimension q with transverse linear connection defined by $\big(N,\nabla^N\big)$-cocycle $\big\{U_i, f_i, \{k_{ij}\}\big\}_{i,j\in J}$. Fix a q-dimension distribution $\mathfrak{M}$ transverse to (M,F). Denote the linear connection on M satisfying Theorem 4.1 by $\nabla^{\mathfrak{M}}$. Further we consider open subsets of M and N with the linear connections inducted by $\nabla^{\mathfrak{M}}$ and by ∇^N, respectively.

6.1. Neighborhoods $\Omega(W,\varepsilon)$

Consider a submersion $f \colon U \to V$ from the maximal $\big(N,\nabla^N\big)$-cocycle determining (M,F). Let g^U and g^V be Riemannian metrics on U and V, respectively. Define a new Riemannian metric g on U by the following formula:

$$g(X,Y) = g^U\big(X^F, Y^F\big) + g^V\big(f_*\big(X^{\mathfrak{M}}\big), f_*\big(Y^{\mathfrak{M}}\big)\big)$$

for $X = X^F + X^{\mathfrak{M}}$ and $Y = Y^F + Y^{\mathfrak{M}}$ in $\mathfrak{X}(U)$.

Remark that $f \colon (U,g) \to (V,g^V)$ is a Riemannian submersion, and $\mathfrak{M}_U$ is a horizontal distribution for f [2]. It is well known that for a Riemannian submersion f the length of a smooth horizontal curve μ is equal to the length of its projection $f\circ\mu$. By the norms of vectors $X \in T_xU$ and $Y \in T_wV$ we shall mean the numbers $\|X\|_x = \sqrt{g_x(X,X)}$ and $\|Y\|^V_w = \sqrt{g^V_w(Y,Y)}$.

Let W and W^V be open relatively compact subsets in U and V properly. Introduce for $\varepsilon > 0$ the following notation:

$$\Omega(W,\varepsilon) = \{X \in T_xU : x \in W,\ \|X\|_x < \varepsilon\}, \qquad \Omega\big(W^V,\varepsilon\big) = \big\{Y \in T_wV : w \in W^V,\ \|X\|^V_w < \varepsilon\big\}.$$

Put $W_\varepsilon = \exp(\Omega(W,\varepsilon))$, $W^V_\varepsilon = \exp^V\big(\Omega(W^V,\varepsilon)\big)$, where $\exp$ and $\exp^V$ are exponential mappings with respect to the connections $\nabla^{\mathfrak{M}}$ and ∇^N correspondingly, if these sets are defined. As was proved by Whitehead (for example, see [4, Chapter III, § 8], at every point of the manifold provided with a linear connection there is a normal convex neighborhood. Therefore without loss of generality we may assume ε to be so small that the restrictions $\exp\restriction_{\Omega(W,\varepsilon)\cap T_xU}$, $x \in W$, and $\exp^V\restriction_{\Omega(W^V,\varepsilon)\cap T_wV}$, $w \in W^V$, are diffeomorphisms onto images belonging to U and V, respectively.

Lemma 6.1.
Let $f\colon U \to V$ be any submersion from $\left(N, \nabla^N\right)$-cocycle determining (M, F). Then at each point $z \in U$ there is an open neighborhood $W = W(z) \subset U$ and a number $\delta = \delta(z) > 0$ satisfying the following condition (in the notation introduced above):

$$\text{for any geodesic } \sigma \text{ of } \nabla^N \text{ from } V(v) = f(W_\delta),\ v = f(z), \text{ such that } \|\dot\sigma(0)\|^V_{\sigma(0)} < \delta,$$
$$\text{and for each point } x \in f^{-1}(\sigma(0)) \cap W_\delta \text{ there is } \mathfrak{M}\text{-lift } \gamma \text{ of } \sigma \text{ to } x, \text{ and } \gamma \text{ is } \mathfrak{M}\text{-geodesic of } \nabla^{\mathfrak{M}} \text{ in } U.$$

Proof. Take an arbitrary point $z \in U$. Let ε be a small positive number and W be an open relatively compact neighborhood of z in U such that $W_\varepsilon \subset U$. Any submersion is an open map, hence $W^V = f(W)$ is an open relatively compact neighborhood of $v = f(z)$ in V.

In conformity with (iii) of Theorem 4.1 any geodesic μ from $\left(U, \nabla^{\mathfrak{M}}\right)$ projects onto a geodesic $f \circ \mu$ from $\left(V, \nabla^N\right)$. As $f\colon U \to V$ is a Riemannian submersion, so $\|(\dot\mu(0))\| \geq \|(f \dot\circ \mu)(0)\|^V$, with $\|(\dot\mu(0))\| = \|(f \dot\circ \mu)(0)\|^V$ for any $\mathfrak{M}$-geodesic μ on $\left(U, \nabla^{\mathfrak{M}}\right)$. Therefore the set W^V_ε is defined, and the inclusions $W^V_\varepsilon \subset f(W_\varepsilon) \subset V$ are valid. Let $\gamma_X(s) = \exp sX$, where $s \in [0,1]$. Thanks to the following property of geodesics $\gamma_{\lambda X}(s) = \gamma_X(\lambda s)$, $\lambda \in [0,1]$, 1-parametric family of neighborhoods $\{W_{\lambda\varepsilon}\}$ exists in U and continuously depends on λ. This family may be considered as a compression from W_ε, $\lambda = 1$, to W, $\lambda = 0$. Due to the continuity of the exponential mapping it implies the existence of a number $\delta_0 > 0$ such that for every δ, $0 < \delta < \delta_0$, we have $\exp(\Omega(W_\delta, \delta)) \subset W_\varepsilon$ and $f(W_\delta) \subset W^V_\varepsilon$. Take one of such δ and show that $W(z)$ and δ satisfy Lemma 6.1.

Consider an arbitrary geodesic σ from $V(v) = f(W_\delta)$ of (V, ∇^N) such that $\|\dot Y\|^V_w < \delta$, where $w = \sigma(0)$ and $Y = \dot\sigma(0)$. For every $x \in f^{-1}(w) \cap W_\delta$ the restriction $f_*\restriction_{\mathfrak{M}_x}\colon \mathfrak{M}_x \to T_w V$ is an isometry of Euclidean vector spaces $(\mathfrak{M}_x, g_x)$ and $\left(T_w V, g^V_w\right)$. Hence there is a unique vector $X \in \mathfrak{M}_x$ such that $f_{*x}(X) = Y$, with $\|X\|_x = \|Y\|^V_w < \delta$. Therefore $X \in \Omega(W, \delta)$ and there exists a geodesic $\gamma_X(s) = \exp_x sX$, $s \in [0,1]$, moreover $\gamma_X(s) \in W_\delta \subset U$. Because $\mathfrak{M}$ is a geodesic invariant distribution by (i) of Theorem 4.1, γ_X is an $\mathfrak{M}$-geodesic on $\left(U, \nabla^{\mathfrak{M}}\right)$. In accordance with the statement (iii) of Theorem 4.1 the projection $\nu = f \circ \gamma_X$ is a geodesic in $\left(V, \nabla^N\right)$. Note that $\nu = \sigma$ as geodesics in $\left(V, \nabla^N\right)$ having the same tangent vector Y at common point w. Thus γ_X is $\mathfrak{M}$-lift of σ into point x. □

6.2. The map $E_W\colon \Omega(W, \delta) \to U$

Suppose that the submersion $f\colon U \to V$, the number δ and $\Omega(W, \delta)$ satisfy Lemma 6.1. Define a map $E_W\colon \Omega(W, \delta) \to U$ in the following way. For $X \in \Omega(W, \delta) \cap T_x M$ and $y = \exp_x p^F(X)$ let us put

$$E_W(X) = \exp_y \circ \left(f_{*y}\restriction_{\mathfrak{M}_y}\right)^{-1} \circ f_{*x} \circ p^{\mathfrak{M}}(X).$$

In other words, for any $X \in \Omega(W, \delta) \cap T_x M$ and $y = \exp_x X^F$ the geodesic $\sigma(s) = f \circ \exp sX^{\mathfrak{M}}$, $s \in [0,1]$, is defined, and $v = \sigma(0) = f(x) \in V$. Then $E_W(X) = \gamma(1)$, where γ is the $\mathfrak{M}$-lift of σ to the point y.

The definition of the Riemannian metric g implies the inequality $\|X^F\|_x \leq \|X\|_x$, hence for $X \in \Omega(W, \delta) \cap T_x M$ it is necessary $X^F \in \Omega(W, \delta) \cap T_x M$ and $y = \exp_x X^F \in W_\delta$. According to Lemma 6.1 the lift γ exists and belongs to U. Therefore, the map $E_W\colon \Omega(W, \delta) \to U$ is really defined. It is clear that the map $E_W\colon \Omega(W, \delta) \to U$ is smooth. It is not difficult to show the validity of the following assertion.

Lemma 6.2.
At each $u \in W$ there is such neighborhood $D \subset \Omega(W, \delta)$ of 0_u in $T_u M$ that the restriction $E_W\restriction_D\colon D \to U$ is a diffeomorphism onto the image $E_W(D)$.

6.3. The set S_U and the map $E_U\colon S_U \to U$

Take any $z \in U$. Consider a neighborhood $W = W(z)$ and $\Omega(W, \delta)$, $\delta = \delta(z)$, satisfying Lemma 6.1. Remark that $U = \bigcup_{z \in U} W(z)$. Let $S_U = \bigcup_{z \in U} \Omega\big(W(z), \delta(z)\big)$ and $X \in S_U$. Then there is a set $\Omega(W, \delta)$ containing X. Let

$E_W\colon \Omega(W,\delta) \to U$ be the map given above. The equality $E_U(X) = E_W(X)$ defines a map $E_U\colon S_U \to U$. Indeed, for $X \in S_U \cap T_x W$ and $y = \exp X^F$ we have $E_W(X) = \gamma(1)$, where γ is the $\mathfrak{M}$-lift of the geodesic $\sigma(s) = f \circ \exp sX^{\mathfrak{M}}$, $s \in [0,1]$, to the point y. Hence $E_W(X)$ does not depend on the choice of z, $W = W(z)$ and $\delta = \delta(z)$. Therefore the map $E_U\colon S_U \to U$ is well defined and smooth.

Each submersion is an open map, hence $f(W(z)) = V(v)$ is an open neighborhood of $v = f(z)$ in V, and $V = \bigcup_{v\in V} V(v)$. For the set $S_V = \{f_{*x}(X) : X \in S_U \cap T_x U,\ x \in U\}$ the following diagram is commutative:

$$\begin{tikzcd} S_U \arrow[r, "E_U"] \arrow[d, "f_*"] & U \arrow[d, "f"] \\ S_V \arrow[r, "\exp"] & V. \end{tikzcd} \tag{3}$$

6.4. The map $(\tau, E_U)\colon S_U \to U \times U$

Let $\tau\colon TM \to M$ be the projection of the tangent bundle. Remark that the proof of the following lemma is similar to the proof of [7, Proposition 19] and it is given here for the completeness of the description. Let $n = \dim M$.

Lemma 6.3.

The map $(\tau, E_U)\colon \Omega(W,\delta) \to U \times U$, $X \mapsto \big(\tau(X), E_U(X)\big)$, is a diffeomorphism onto an open neighborhood of the diagonal $\Delta_W = \{(x,x) : x \in W\}$ in the product $U \times U$.

Proof. It is sufficient to show that for each point $x \in W$ there is an open neighborhood $W' \subset W$ and a positive number $r < \delta$ such that (τ, E_U) is a diffeomorphism between $\Omega(W',r)$ and an open neighborhood $(\tau, E_U)(\Omega(W',r))$ of (x,x) in $U \times U$. It is a trivializing open set of the tangent bundle, because U is a contractible set. Therefore

$$(\tau, E_U)_{*0_x}\colon\ T_x W \times \mathbb{R}^n \to T_x U \times T_x U, \qquad n(X,v) \mapsto \big(X,\, X + (E_x)_{*0_x}(v)\big).$$

It follows from Lemma 6.2 that the map $(E_x)_{*0_x}\colon \mathbb{R}^n \cong T_{0_x} M \to T_x U$ is also an isomorphism of the indicated vector space, so

$$(\tau, E_U)_{*0_x}\colon\ T_{0_x}\Omega(W,\delta) \to T_{(x,x)}(U \times U)$$

is an isomorphism. Hence there is an open neighborhood Ω' of 0_x in $\Omega(W,\delta)$ such that $(\tau, E_U)\colon \Omega' \to U \times U$ is a diffeomorphism onto some open neighborhood W' of (x,x) in $U \times U$. Since Ω' contains some open neighborhood $\Omega(W',r)$ satisfying Lemma 6.2, the statement is proved. $\square$

6.5. Proof of Theorem 1.1

Let S be the union of S_U for the maximal $\big(N, \nabla^N\big)$-cocycle determining the foliation (M,F). Let $E\colon S \to M$ be defined by the equality $E(X) = E_U(X)$, $X \in S_U$. Show that this map $E\colon S \to M$ is an adapted local addition to the foliation (M,F).

At first, we check that $E\colon S \to M$ is well defined, i.e. $E_U(X) = E_{\widetilde{U}}(X)$ when $X \in S_U \cap S_{\widetilde{U}}$. Denote the corresponding submersions by $f\colon U \to V$ and $\widetilde{f}\colon \widetilde{U} \to \widetilde{V}$. Let $\overline{\gamma}(s) = \exp sX^{\mathfrak{M}}$, $s \in [0,1]$. Let $X = X^F + X^{\mathfrak{M}} \in S \cap T_u M$, then $y = \exp_x X^F \in U \cap \widetilde{U}$. According to the definition, $S_U(X) = \gamma(1)$, where γ is the $\mathfrak{M}$-lift to y of the geodesic $\sigma = f \circ \overline{\gamma}$ from V. By analogy, $S_{\widetilde{U}}(X) = \widetilde{\gamma}(1)$, where $\widetilde{\gamma}$ is the $\mathfrak{M}$-lift to y of the geodesic $\widetilde{\sigma} = \widetilde{f} \circ \overline{\gamma}$ from $\widetilde{V}$. Note that by Theorem 4.1, γ and $\widetilde{\gamma}$ are $\mathfrak{M}$-geodesics in M. Since (M,F) is a foliation with transverse linear connection, there is a local isomorphism of the induced linear connections $k\colon \widetilde{f}(U \cap \widetilde{U}) \to f(U \cap \widetilde{U})$ satisfying the equality $f = k \circ \widetilde{f}$. Hence $\sigma = f \circ \overline{\gamma} = (k \circ \widetilde{f}) \circ \overline{\gamma} = k \circ (\widetilde{f} \circ \overline{\gamma}) = k \circ \widetilde{\sigma}$ and $\dot{\sigma}(0) = k_{*\widetilde{v}}\big(\dot{\widetilde{\sigma}}(0)\big)$, where $\widetilde{v} = \widetilde{f}(u)$. Therefore $\dot{\sigma}(0) = k_{*\widetilde{v}}\big(\dot{\widetilde{\sigma}}(0)\big) = k_{*\widetilde{v}}\big(\widetilde{f}_{*y}\big(\dot{\widetilde{\gamma}}(0)\big)\big) = \big(k_{*\widetilde{v}} \circ \widetilde{f}_{*y}\big)\big(\dot{\widetilde{\gamma}}(0)\big) = f_{*y}\big(\dot{\widetilde{\gamma}}(0)\big)$. On the other hand, $\dot{\sigma}(0) = f_{*y}(\dot{\gamma}(0))$ and we have $\dot{\sigma}(0) = f_{*y}(\dot{\gamma}(0)) = f_{*y}\big(\dot{\widetilde{\gamma}}(0)\big)$.

As the map $f_{*y}\colon \mathfrak{M}_y \to T_vV$, $v = f(y)$, is an isomorphism of the corresponding vector spaces, it is necessary that $\dot\gamma(0) = \dot{\widetilde{\gamma}}(0)$. Thus, $\mathfrak{M}$-geodesics γ and $\widetilde{\gamma}$ have the properties: $\gamma(0) = \widetilde{\gamma}(0) = y$ and $\dot\gamma(0) = \dot{\widetilde{\gamma}}(0)$. Therefore $\gamma = \widetilde{\gamma}$ and $E_U(X) = E_{\widetilde{U}}(X)$.

The commutative diagram (3) and Lemma 6.3 imply that the map $E\colon S \to M$ is an adapted local addition to the foliation (M, F). Therefore, as it was shown by Macias-Virgós and Sanmartín Carbón (Theorem 5.1), the full automorphism group $\mathcal{D}(M, F)$ of this foliation (M, F) is an infinite-dimensional Lie group whose manifold is modeled on the LF-spaces. $\square$

Acknowledgements

This work was supported by the Federal Target Program *Scientific and Scientific-Pedagogical Personnel*, Project No. 14.B37.21.0361, and the Russian Federation Ministry of Education and Science, Project No. 1.1907.2011.

References

[1] Bel'ko I.V., Affine transformations of a transversal projectable connection of a manifold with a foliation, Math. USSR-Sb., 1983, 45, 191–204
[2] Besse A.L., Einstein Manifolds, Classics Math., Springer, Berlin, 2008
[3] Kamber F.W., Tondeur P., *G*-foliations and their characteristic classes, Bull. Amer. Math. Soc., 1978, 84(6), 1086–1124
[4] Kobayashi S., Nomizu K., Foundations of Differential Geometry I, Interscience, New York–London, 1963
[5] Kriegl A., Michor P.W., Aspects of the theory of infinite-dimensional manifolds, Differential Geom. Appl., 1991, 1(2), 159–176
[6] Lewis A.D., Affine connections and distributions with applications to nonholonomic mechanics, In: Pacific Institute of Mathematical Sciences Workshop on Nonholonomic Constraints in Dynamics, Calgary, August 26–30, 1997, Rep. Math. Phys., 1998, 42(1-2), 135–164
[7] Macias-Virgós E., Sanmartín Carbón E., Manifolds of maps in Riemannian foliations, Geom. Dedicata, 2000, 79(2), 143–156
[8] Michor P.W., Manifolds of Differentiable Mappings, Shiva Math. Ser., 3, Shiva, Nantwich, 1980
[9] Molino P., Propriétés cohomologiques et propriétés topologiques des feuilletages à connexion transverse projetable, Topology, 1973, 12, 317–325
[10] Molino P., Riemannian Foliations, Progr. Math., 73, Birkhäuser, Boston, 1988
[11] Palais R.S., Foundations of Global Non-Linear Analysis, Benjamin, New York–Amsterdam, 1968
[12] Postnikov M.M., Lectures in Geometry V, Factorial, Moscow, 1998 (in Russian)
[13] Walker A.G., Connexions for parallel distributions in the large, Quart. J. Math. Oxford Ser., 1955, 6, 301–308
[14] Willmore T.J., Connexions for systems of parallel distributions, Quart. J. Math. Oxford Ser., 1956, 7, 269–276
[15] Zhukova N.I., Minimal sets of Cartan foliations, Proc. Steklov Inst. Math., 2007, 256(1), 105–135
[16] Zhukova N.I., Global attractors of complete conformal foliations, Sb. Math., 2012, 203(3-4), 380–405