# A report on locally conformally Kähler manifolds

Liviu Ornea and Misha Verbitsky

To J. C. Wood at his sixtieth birthday

ABSTRACT. We present an overview of recent results in locally conformally Kähler geometry, with focus on the topological properties which obstruct the existence of such structures on compact manifolds.

## 1. Locally conformally Kähler manifolds

Locally conformally Kähler (LCK) geometry is concerned with complex manifolds of complex dimension at least two admitting a Kähler covering with deck transformations acting by holomorphic homotheties with respect to the Kähler metric.

We shall usually denote with M the LCK manifold, with (J,g) its Hermitian structure, with  $\Gamma \to \tilde{M} \to M$  the Kähler covering and with  $\tilde{\omega}$  the Kähler form on the covering.

Directly from the definition, one obtains the existence of an associated character

$$\chi: \Gamma \to \mathbb{R}^{>0}, \quad \chi(\gamma) = \frac{\gamma^* \tilde{\omega}}{\tilde{\omega}}.$$

This already puts some restrictions on  $\pi_1(M)$ . Others, more precise ones, will be obtained further.

Sometimes, the couple  $(\Gamma, \tilde{M})$  is called a **presentation** of the LCK manifold M. Here,  $\tilde{M}$  is understood as a Kähler manifold tgether with a group of holomorphic homotheties (called a homothetic Kähler manifold). The idea is that, as on M the metric can move in a conformal class, on the covering, the Kähler metric is not fixed but can be changed homothetically. Obviously, the same LCK manifold can admit many presentations and one can choose a minimal one and a maximal one (corresponding to the simply connected  $\tilde{M}$ ). However, the rank of the image of  $\Gamma$  in  $\mathbb{R}^{>0}$  is constant; it will be denoted  $\mathrm{rk}(M)$ . Clearly,  $\mathrm{rk}(M) \leqslant b_1(M)$  (see [GOPP]).

An equivalent definition - historically, the first one -, at the level of the manifold itself, requires the existence of an open covering  $\{U_{\alpha}\}$  with local Kähler metrics  $g_{\alpha}$  subject to the condition that on overlaps  $U_{\alpha} \cap U_{\beta}$ , these local Kähler metrics

<sup>2000</sup> Mathematics Subject Classification. Primary 53C55.

L.O. is partially supported by a CNCSIS PNII IDEI Grant nr. 529/2009.

are homothetic:  $g_{\alpha} = c_{\alpha\beta}g_{\beta}$ . The cocycle  $\{c_{UV}\}$  is represented by a closed one-form  $\theta$ . Locally,  $\theta\Big|_{U_{\alpha}} = df_{\alpha}$  and the metrics  $e^{f_{\alpha}}g_{\alpha}$  glue to a global metric whose associated two-form  $\omega$  satisfies the integrability condition  $d\omega = \theta \wedge \omega$ , thus being locally conformal with the Kähler metrics  $g_{\alpha}$ . Here  $\theta$  is a closed 1-form on M, called the Lee form. This gives another definition of an LCK structure (motivating also the name), which will be used in this paper.

**Definition 1.1.** Let  $(M, \omega)$  be a Hermitian manifold,  $\dim_{\mathbb{C}} M > 1$ , with  $d\omega = \theta \wedge \omega$ , where  $\theta$  is a closed 1-form. Then M is called a **locally conformally Kähler** (LCK) manifold.

Remark 1.2. i) Some authors include the Kähler manifolds as particular LCK manifolds. Although this is a legitimate choice, we prefer the dichotomy LCK versus Kähler, and hence we always assume that the LCK manifolds we work with are of non-Kähler type. Due to a result in [Va], namely: A compact locally conformally Kähler manifold which admits some Kähler metric, or, more generally, which satisfies the  $\partial \overline{\partial}$ -Lemma, is globally conformally Kähler, it is enough to assume  $[\theta] \neq 0 \in H^1(M, \mathbb{R})$ .

ii) The equation  $d\omega = \theta \wedge \omega$  makes sense also in absence of a complex structure, leading to the notion of **locally conformally symplectic manifold** (LCS). There is a great number of papers on this topic, among the authors of which we cite: A. Banyaga, G. Bande, S. Haller, D. Kotschick, A. Lichnerowicz, J.C. Marrero, I. Vaisman etc. Hence, any LCK structure underlies a LCS structure. Nothing is known on the converse. The corresponding question regarding the relation symplectic versus Kähler was since long solved by Thurston, [Th]. We still do not know if any (compact) LCS manifold admits an integrable, compatible complex structure which makes it a LCK manifold or not. The difficulty might come from the fact that the topology of a LCK manifold is not controlled. We believe the answer should be negative and hence we propose:

**Open Problem 1.** Construct a compact LCS manifold which admits no LCK metric.

The Lee form, which is the torsion of the Chern connection (see [G]), can also be interpreted in terms of presentations as follows. Abelianize the Serre sequence of  $\Gamma \to \tilde{M} \to M$  to get:

$$H_1(K,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to \Gamma^{ab} \to 1.$$

Then apply  $\operatorname{Hom}(\cdot, \mathbb{Z})$  to obtain:

$$0 \to \operatorname{Hom}(\Gamma^{ab}, \mathbb{Z}) \to H^1(M, \mathbb{Z}) \to H^1(K, \mathbb{Z}).$$

Tensoring with  $\otimes_{\mathbb{Z}}\mathbb{R}$ , exactness is conserved (as  $\mathbb{R}_{/\mathbb{Z}}$  is flat) and one arrives at:

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\Gamma^{ab}, \mathbb{R}) \xrightarrow{i} H^1_{DR}(M) \to H^1_{DR}(K).$$

Then  $i(\chi) = [\theta]$ , as proven in [PV].

We refer to  $[\mathbf{DO}]$  and  $[\mathbf{O}]$  for an overview of this geometry. Here we focus on our recent results and on related ones.

The following notion, coming from conformal geometry, is crucial for the way we understand LCK geometry:

**Definition 1.3.** Let  $(M, \omega, \theta)$  be an LCK manifold, and L the trivial line bundle, associated to the representation  $GL(2n,\mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}$ , with flat connection defined as  $D := d + \theta$ . Then L is called **the weight bundle** of M.

Its holonomy coincides with the character  $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$  whose image is called the monodromy group of M.

We shall denote with the same letter, D, the corresponding covariant derivative on M. It is precisely the Weyl covariant derivative associated to  $\nabla = \nabla^g$  and  $\theta$ , uniquely defined by the conditions:

$$DJ = 0, \qquad Dg = \theta \otimes g.$$

The complexified weight bundle will also be denoted L. As  $d\theta = 0$ , L is flat and defines a local system and hence one can compute its cohomology.

On the other hand, in LCK geometry, one tries to work on the Kähler covering. But there, the interesting tensorial objects, in particular differential forms  $\alpha$ , are the ones satisfying:  $\gamma^*\alpha = \chi(\gamma)\alpha$  for every  $\gamma \in \Gamma$ . We call such forms **automorphic**.

The advantage of using the weight bundle is that automorphic objects on Mare regarded as objects on M with values in L.

#### 1.1. Examples.

- 1.1.1. Diagonal Hopf manifolds. ([GO], [KO], [Ve].) Let  $H_A := (\mathbb{C}^n \setminus \{0\})/\langle A \rangle$ with  $A = \operatorname{diag}(\alpha_i)$  endowed with:
  - Complex structure as the projection of the standard one of  $\mathbb{C}^n$ .
  - LCK metric constructed as follows: Let C > 1 be a constant and

$$\varphi(z_1,\ldots,z_n) = \sum |z_i|^{\beta_i}, \quad \beta_i = \log_{|\alpha_i|^{-1}} C$$

a potential on  $\mathbb{C}^n$ .

Then  $A^*\varphi = C^{-1}\varphi$  and hence:  $\Omega = \sqrt{-1}\partial\overline{\partial}\varphi$  is Kähler and  $\Gamma \cong \mathbb{Z}$ acts by holomorphic homotheties with respect to it.

Note that the Lee field:  $\theta^{\sharp} = -\sum z_i \log |\alpha_i| \partial z_i$  is parallel. It is also important to observe that the LCK metric here is constructed out of an automorphic potential. The construction will be extended to non-diagonal Hopf manifolds.

1.1.2. Compact complex surfaces. Belgun, [Be], gave the complete list of compact complex surfaces which admit metrics with parallel Lee form ( $\nabla \theta = 0$ ), being, in particular, LCK. Such metrics are called **Vaisman** and will be treated separately, in section 2.1 (see Theorems 2.6, 2.7).

Recently, Fujiki and Pontecorvo constructed LCK metrics on parabolic and hyperbolic Inoue surfaces. These examples are also bihermitian and hence related to generalized Kähler geometry. We also note that in [AD], the LCK metric of the diagonal Hopf surface  $q_{GO}$  found [GO] was deformed to a family of bihermitian metrics  $(g_t, J, J^t)$  with  $J^t = \varphi_t(J)$ , where  $\varphi_t$  is a path of diffeomorphisms; as  $t \to 0$ ,  $J^t \to J$  and  $g_t/t \to g_{GO}$ .

More generally, Brunella, [Br2], proved that all surfaces with global spherical shells, also known as Kato surfaces (as the previous mentioned parabolic and hyperbolic Inoue surfaces are) do admit LCK metrics. Previously he constructed families of LCK metrics only on Enoki surfaces, [Br1].

On the other hand, Belgun also proved in [Be] that a certain type of Inoue surfaces does not admit any LCK metric. As these surfaces are deformations of other Inoue surfaces with LCK metric (found in [Tr]), this proves, in particular, that, unlike the Kähler class, the LCK class is not stable at small deformations. By contrast, the LCK class share with the Kähler one the stability to blowing up points, [Tr], [Vu1].

1.1.3. Oeljeklaus-Toma manifolds, [OT]. Let K be an algebraic number field of degree  $n:=(K:\mathbb{Q})$ . Let then  $\sigma_1,\ldots,\sigma_s$  (resp.  $\sigma_{s+1},\ldots,\sigma_n$ ) be the real (resp. complex) embeddings of K into  $\mathbb{C}$ , with  $\sigma_{s+i}=\overline{\sigma}_{s+i+t}$ , for  $1\leqslant i\leqslant t$ . Let  $\mathcal{O}_K$  be the ring of algebraic integers of K. Note that for any  $s,t\in\mathbb{N}$ , there exist algebraic number fields with precisely s real and s0 complex embeddings.

Using the embeddings  $\sigma_i$ , K can be embedded in  $\mathbb{C}^m$ , m = s + t, by

$$\sigma: K \to \mathbb{C}^m, \quad \sigma(a) = (\sigma_1(a), \dots, \sigma_m(a)).$$

This embedding extends to  $\mathcal{O}_K$  and  $\sigma(\mathcal{O}_K)$  is a lattice of rank n in  $\mathbb{C}^m$ . This gives rise to a properly discontinuous action of  $\mathcal{O}_K$  on  $\mathbb{C}^m$ . On the other hand, K itself acts on  $\mathbb{C}^m$  by

$$(a,z) \mapsto (\sigma_1(a)z_1,\ldots,\sigma_m(a)z_m).$$

Note that if  $a \in \mathcal{O}_K$ ,  $a\sigma(\mathcal{O}_K) \subseteq \sigma(\mathcal{O}_K)$ . Let now  $\mathcal{O}_K^*$  be the group of units in  $\mathcal{O}_K$  and set

$$\mathcal{O}_K^{*,+} = \{ a \in \mathcal{O}_K^* \mid \sigma_i(a) > 0, \ 1 \leqslant i \leqslant s \}.$$

The only torsion elements in the ring  $\mathcal{O}_K^*$  are  $\pm 1$ , hence the Dirichlet units theorem asserts the existence of a free Abelian group G of rank m-1 such that  $\mathcal{O}_K^* = G \cup (-G)$ . Choose G in such a away that it contains  $\mathcal{O}_K^{*,+}$  (with finite index). Now  $\mathcal{O}_K^{*,+}$  acts multiplicatively on  $\mathbb{C}^m$  and, taking into account also the above additive action, one obtains a free action of the semi-direct product  $\mathcal{O}_K^{*,+} \ltimes \mathcal{O}_K^*$  on  $\mathbb{C}^m$  which leaves invariant  $H^s \times \mathbb{C}^t$  (as above, H is the open upper half-plane in  $\mathbb{C}$ ). The authors then show that it is possible to choose a subgroup U of  $\mathcal{O}_K^{*,+}$  such that the action of  $U \ltimes \mathcal{O}_K$  on  $H^s \times \mathbb{C}^t$  be properly discontinuous and co-compact. Such a subgroup U is called admissible for K. The quotient

$$X(K,U) := (H^s \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K)$$

is then shown to be a m-dimensional compact complex (affine) manifold, differentiably a fiber bundle over  $(S^1)^s$  with fiber  $(S^1)^n$ .

For t = 1, X(K, U) admits LCK metrics. Indeed,

$$\varphi: H^s \times \mathbb{C} \to \mathbb{R}, \quad \varphi = \prod_{j=1}^s \frac{i}{z_j - \overline{z}_j} + |z_m|^2$$

is a Kähler potential on whose associated 2-form  $i\partial \overline{\partial} \varphi$  the deck group acts by linear holomorphic homotheties. On the other hand, one sees that the potential itself is not automorphic (in particular, these manifolds cannot be Vaisman, see §2.1).

A particular class of manifolds X(K,U) is that of *simple type*, when U is not contained in  $\mathbb{Z}$  and its action on  $\mathcal{O}_K$  does not admit a proper non-trivial invariant submodule of lower rank (which, as the authors show, is equivalent to the assumption that there is no proper intermediate field extension  $\mathbb{Q} \subset K' \subset K$  with  $U \subset \mathcal{O}_{K'}$ ). If X(K,U) is of simple type, then  $b_1(X(K,U)) = s$  (a more direct proof

than the original one appears in  $[\mathbf{PV}]$ ,  $b_2(X(K,U)) = \binom{s}{2}$ . Moreover, the tangent bundle TX(K,U) is flat and  $\dim H^1(X(K,U),\mathcal{O}_{X(K,U)}) \geqslant s$ . In particular, X(K,U) are non-Kähler.

Observe that for s = t = 1 and  $U = \mathcal{O}_K^{*,+}$ , X(K,U) reduces to an Inoue surface  $S_M$  with the metric given in [**Tr**].

Now, for s=2 and t=1, the six-dimensional X(K,U) is of simple type, hence has the following Betti numbers:  $b_0=b_6=1$ ,  $b_1=b_5=2$ ,  $b_2=b_4=1$ ,  $b_3=0$ . This disproves Vaisman's conjecture claiming that a compact LCK, non-Kähler, manifold must have an odd odd Betti number.

These manifolds can be used to obtain examples of LCK structures with arbitrary rank (recall that  $\operatorname{rk}(M)$  is the rank of  $\chi(\Gamma)$  in  $\mathbb{R}^{>0}$ ). Specifically:

**Theorem 1.4.** [PV] Let the number field K admit exactly two non-real embeddings and M = X(K, U). Then:

- i) If n is odd (hence if  $\dim_{\mathbb{C}}(M)$  is even), then  $\mathrm{rk}(M) = b_1(M)$  (i.e. the rank is maximal).
- ii) If n is even, then either  $\operatorname{rk} M = b_1(M)$  or  $\operatorname{rk}(M) = \frac{b_1(M)}{2}$ ; this last situation occurs if and only if K is a quadratic extension of a totally real number field.

Concrete examples of number fields which lead to ii) above are also constructed in [PV].

## 2. Locally conformally Kähler manifolds with potential

**Definition 2.1.** [OV7] (M, J, g) is a LCK manifold with (automorphic) potential if M admits a Kähler covering with automorphic potential.

Remark 2.2. The definition we gave in [OV3] was slightly more restrictive: we asked the potential to be a proper function (*i.e.* to have compact levels). The properness of the potential is equivalent to the weight bundle having monodromy  $\mathbb{Z}$ . Later on, we proved in [OV6] that on any compact LCK manifold with automorphic potential, there exists another LCK metric with automorphic potential and monodromy  $\mathbb{Z}$ . The proof amounts to a deformation of the weight bundle together with its connection form.

However, we have strong reasons to believe that the deformation is not necessary:

Conjecture 1. Any compact LCK manifold with automorphic potential has monodromy  $\mathbb{Z}$ .

The existence of a potential for the Kähler metric of the covering can be shown to be equivalent with the equation  $(\nabla \theta)^{1,1} = 0$ , introduced in [K] under the name of pluricanonical Kähler-Weyl and studied also in [KK].

**Proposition 2.1.** [OV7] M admits a Kähler covering with automorphic potential if and only if  $(\nabla \theta)^{1,1} = 0$ .

For the proof, one first proves by direct computation that  $(\nabla \theta)^{1,1} = 0$  is equivalent with the equation:

$$d(J\theta) = \omega - \theta \wedge J\theta.$$

This can also be put in terms of Weyl connection as:

$$(D\theta)^{1,1} = (\theta \otimes \theta)^{1,1} - \frac{1}{2}g.$$

Now, let  $\tilde{M}$  be a covering of M on which the pull-back of  $\theta$  is exact. Denote, for convenience, with the same letters the pull-backs to  $\tilde{M}$  of  $\theta$ ,  $\omega$  and D. As locally D is the Levi-Civita connection of the local Kähler metrics, its pull-back on  $\tilde{M}$  is the Levi-Civita connection of the Kähler metric on  $\tilde{M}$  globally conformal with  $\omega$ . Then let  $\psi := e^{-\nu}$ , where  $d\nu = \theta$ . We have

$$dd^c\psi = -e^{-\nu}dd^c\nu + e^{-\nu}d\nu \wedge d^c\nu = e^{-\nu}(d^c\theta + \theta \wedge J\theta) = \psi\omega,$$

and hence the pluricanonical condition implies that  $\psi$  is an automorphic potential for the Kähler metric  $\psi\omega$ . The converse is true by a similar argument.

A second characterization can be given in terms of Bott-Chern cohomology. Let  $\Lambda^{1,1}_{\chi,d}(\tilde{M})$  be the space of closed, automorphic (1, 1)-forms on  $\tilde{M}$ , and  $C^{\infty}_{\chi}(\tilde{M})$  the space of automorphic functions on  $\tilde{M}$ . Then

$$H^{1,1}_{BC}(M,L) := \frac{\Lambda^{1,1}_{\chi,d}(\tilde{M})}{dd^c(C^{\infty}_{\chi}(\tilde{M}))}$$

is the **Bott-Chern group** of the LCK manifold (it is finite-dimensional and does not depend on the choice of the presentation). It is now clear that

**Lemma 2.3.** [OV6] 
$$M$$
 is LCK with potential if  $[\Omega] = 0 \in H^{1,1}_{BC}(M,L)$ .

The main properties of LCK manifolds with automorphic potential are listed in the following:

**Theorem 2.4.** [OV3] i) The class of compact LCK manifolds with potential is stable to small deformations.

ii) Compact LCK manifolds with potential, of complex dimension at least 3, can be holomorphically embedded in a (non-diagonal), Hopf manifold.

From i), it follows that the Hopf manifold  $(\mathbb{C}^N \setminus 0)/\Gamma$ , with  $\Gamma$  cyclic, generated by a non-diagonal linear operator, is LCK with potential. This is the appropriate generalization of the (non-Vaisman) non-diagonal Hopf surface. Then ii) says that the Hopf manifold plays in LCK geometry the rôle of the projective space in Kähler geometry.

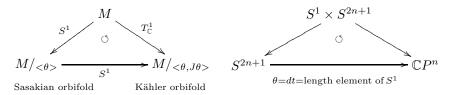
**2.1. Vaisman manifolds.** Among the LCK manifolds with potential, a most interesting class is the Vaisman one. A Vaisman metric is a Hermitian metric with parallel Lee form. It can be easily seen that the Kähler metric of the covering has global automorphic potential  $\varphi = \tilde{\omega}(\pi^*\theta, \pi^*\theta)$ .

The Lee field of a Vaisman manifold is Killing and, being parallel, it has constant length. Conversely, a LCK metric with Killing Lee field of constant length is Vaisman (see, e.g. [DO, Proposition 4.2]). On the other hand, it was proven in [Ve, Proposition 6.5] that a complex compact submanifold of a compact Vaisman manifold must be tangent to the Lee field. In particular, the submanifold enherits a LCK metric whose Lee field is again Killing and of constant length. Hence:

**Proposition 2.2.** Complex compact submanifolds of a compact Vaisman manifold are again Vaisman.

As the LCK metric of the diagonal Hopf manifold is Vaisman, this provides a wide class of examples. On the other hand, on surfaces there exists the complete list of compact examples, see above.

On Vaisman manifolds, the vector field  $\theta^{\sharp}$  is holomorphic and Killing, and hence it generates a totally geodesic, Riemannian, holomorphic foliation  $\mathcal{F}$ . When this is quasi-regular, one may consider the leaf space and obtain a fibration in elliptic curves over a Kähler orbifold. Similarly, when  $\theta^{\sharp}$  has compact orbits, the leaf space is a Sasakian orbifold, [**B1**], over which M fibers in circles. The two principal fibrations are connected by the Boothby-Wang fibration in a commutative diagram whose model is the classical Hopf fibering:



This is, in fact, the generic situation, because we proved in [OV2] that the Vaisman structure of a compact manifold can always be deformed to a quasi-regular one.

From the above, it is clear that Vaisman structures may exist on the total space of some elliptic fibrations on compact Kähler manifolds. The precise statement is:

**Theorem 2.5.** [Vu2] Let X, B be compact complex manifolds,  $X \to B$  an elliptic principal bundle with fiber E. If the Chern classes of this bundle are linearly independent in  $H^2(B, \mathbb{R})$ , then X carries no locally conformally Kähler structure.

This contrasts with the case of an induced Hopf fibration over a projective manifold B, when one of the Chern classes vanishes.

For surfaces, we have a complete list of those who admit Vaisman metrics:

**Theorem 2.6.** [Be] Let M be a compact complex surface with odd  $b_1$ . Then M admits a Vaisman metric if and only if M is an elliptic surface (a properly elliptic surface, a - primary or secondary - Kodaira surface, or an elliptic Hopf surface) or a diagonal Hopf surface.

Using the "if" part of this result we can prove:

**Theorem 2.7.** Let be a minimal, non-Kähler compact surface, which is not of class VII. Then M is a Vaisman elliptic surface.

Indeed, recall that a compact complex surface is called **class VII** if it has Kodaira dimension  $-\infty$  and  $b_1(M) = 1$ . It is called **minimal** if it has no rational curves with self-intersection -1. Now, from Kodaira's classification of surfaces, it follows that the algebraic dimension of M is 1 (see e.g. [T, Theorem 5]). Also from Kodaira's classification it follows that M is elliptic [T, Theorem 3]. On the other hand, a non-Kähler compact complex surface has odd  $b_1$  ([Bu] and [L]) It only remains to apply Belgun's result.

The transversal Kählerian foliation  $\mathcal{F}$  permits the use of transversal foliations techniques (basic operators etc.) The following result concerning unicity of Vaisman structures was obtained this way:

**Theorem 2.8.** [OV5] Let (M, J) be a compact complex manifold admitting a Vaisman structure, and  $V \in \Lambda^{n,n}(M)$  a nowhere degenerate, positive volume form. Then M admits at most one Vaisman structure with the same Lee class, such that the volume form of the corresponding Gauduchon metric is equal to V.

Another recent application of this technique is the following:

**Theorem 2.9.** [**OP**] Let  $(M^{2m}, g, J)$  be a compact Vaisman manifold. The metric g is geometrically formal (i.e. the product of every harmonic forms is again harmonic) if and only if  $b_p(M) = 0$  for  $2 \le p \le 2m - 2$  and  $b_1(M) = b_{2m-1}(M) = 1$ , hence M has the real homology of a Hopf manifold.

The connection between Vaisman and Sasakian geometries is clearly seen in:

**Theorem 2.10.** [OV1] Compact Vaisman manifolds are mapping tori over  $S^1$  with Sasakian fibre. More precisely: the universal cover  $\tilde{M}$  is a metric cone  $N \times \mathbb{R}^{>0}$ , with N compact Sasakian manifold and the deck group is isomorphic with  $\mathbb{Z}$ , generated by  $(x,t) \mapsto (\lambda(x), t+q)$  for some  $\lambda \in \operatorname{Aut}(N)$ ,  $q \in \mathbb{R}^{>0}$ .

This result was recently used to prove the following:

**Theorem 2.11.** [MO] On compact Vaisman manifolds whose Weyl connection does not have holonomy in Sp(n) and which are not diagonal Hopf manifolds, conf(M, [g]) = aut(M).

Indeed, the statement follows from the fact that Killing fields with respect to the Gauduchon metric (and a Vaisman metric is Gauduchon) are holomorphic, [MO], and from the more general, referring to Riemannian cones:

**Theorem 2.12.** [MO] Let  $(M, g) := (W, h) \times \mathbb{R}/_{\{(x,t) \sim (\psi(x), t+1)\}}$ , with  $\psi \in \text{Iso}(W, h)$ , W compact. Then conformal vector fields on (M, g) are Killing.

For Vaisman manifolds, the conclusion of ii) in Theorem 2.4 can be sharpened:

**Theorem 2.13.** [OV3] A compact complex manifold of dimension of least 3 admits a Vaisman metric if and only if it admits a holomorphic embedding into a diagonal Hopf manifold.

Taking into account also the relation between Sasaki and Vaisman geometries, a first application of this Kodaira-Nakano type theorem was a corresponding embedding result in Sasakian geometry:

**Theorem 2.14.** [OV4] A compact Sasakian manifold M admits a CR-embedding into a Sasakian manifold diffeomorphic to a sphere, and this embedding is compatible with the respective Reeb fields.

Moreover, we showed that this is the best result one may hope: assuming the existence of a model manifold in Sasakian geometry, analogue of the projective space in complex geometry, leads to a contradiction. A key point in the proof of the theorem was showing that if Z is a closed complex submanifold of a compact Kähler manifold  $(M, \omega)$ ,  $[\omega] \in H^2(M)$  is the Kähler class of M, and  $\omega_0$  is a Kähler form on Z such that its Kähler class coincides with the restriction  $[\omega]|_Z$ , then there exists a Kähler form  $\overline{\omega} \in [\omega]$  on M such that  $\overline{\omega}|_Z = \omega_0$ . Recently, using a same type of argument, van Coevering gave a more direct proof of the embedding in  $[\mathbf{C}]$ .

We also used Theorem 2.13 to prove that, diffeomorphically, LCK with automorphic potential and Vaisman manifolds are the same:

**Theorem 2.15.** [OV7] Let  $(M, \omega, \theta)$  be an LCK manifold with potential with complex dimension at least 3. Then there exists a deformation of M which admits a Vaisman metric.

For the proof, one considers a holomorphic embedding of M in a Hopf manifold  $H = (\mathbb{C}^N \setminus \{0\})\langle A \rangle$ , then observes that M corresponds to a complex subvariety Z of  $\mathbb{C}^N$ , smooth outside of  $\{0\}$  and fixed by A. The operator A admits a Jordan-Chevalley decomposition A := SU, with S diagonal and U unipotent and one can show that S preserves Z. Then  $M_1 := (Z \setminus \{0\})/\langle S \rangle$  is a deformation of M (as S is contained in a  $\mathrm{GL}(\mathbb{C}^n)$ -orbit of A) and is Vaisman as contained in the Hopf manifold  $H_S := (\mathbb{C}^n \setminus \{0\})/\langle S \rangle$ .

The above result shows that all known topological obstructions to the existence of a Vaisman metric on a compact complex manifold (see e.g. [DO]) apply to LCK manifolds with potential. It allows, in particular, to determine the fundamental group of compact LCK manifolds with potential. Indeed, one first deforms the structure to a Vaisman one, then deforms this one to a quasi-regular one (see above) which fibers in elliptic curves over a Kähler basis X. At this point, one considers the homotopy sequence of the fibering:

$$\pi_2(X) \stackrel{\delta}{\longrightarrow} \pi_1(T^2) \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 0$$

and observes that  $\operatorname{rk}(\operatorname{Im}(\delta)) \leq 1$  in  $\pi_1(T^2)$ , as the Chern classes of the  $S^1 \times S^1$ -fibration are: one trivial (as M fibers on  $S^1$ ), the other one non-trivial, as M is non-Kähler, and the total space of an elliptic fibration with trivial Chern classes is Kähler. Hence:

Corollary 2.1. [OV7] The fundamental group of a compact LCK manifold M with an automorphic potential admits an exact sequence

$$0 \longrightarrow G \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 0$$

where  $\pi_1(X)$  is the fundamental group of a Kähler orbifold, and G is a quotient of  $\mathbb{Z}^2$  by a subgroup of rank 1.

**Remark 2.16.** In fact, in [OV7] we only proved that the rank of the subgroup must be  $\leq 1$ , but the recent Theorem 2.5 above ([Vu2]) shows that rk(M) = 0 would imply M is Kähler (see also Remark 1.2).

Corollary 2.2. [OV7] A non-Abelian free group cannot be the fundamental group of a compact LCK manifold with potential.

This corollary, as well as other topological restrictions, was first obtained by Kokarev and Kotschick using harmonic forms and a LCK version of Siu-Beauville result:

**Theorem 2.17.** [KK] Let M be a closed complex manifold admitting a LCK structure with potential (pluricanonical Kähler-Weyl). Then the following statements are equivalent:

- i) M admits a surjective holomorphic map with connected fibers to a closed Riemann surface of genus  $\geq 2$ ;
- ii)  $\pi_1(M)$  admits a surjective homomorphism to the fundamental group of a closed Riemann surface of genus  $\geq 2$ ;
  - iii)  $\pi_1(M)$  admits a surjective homomorphism to a non-Abelian free group.

The above can be generalized to:

**Theorem 2.18.** [KK] Let M be a closed complex manifold admitting a LCK structure with potential, and N a closed Riemannian manifold of constant negative curvature. If  $\varphi : \pi_1(M) \to \pi_1(N)$  is a representation with non-cyclic image, then there exists a compact Riemann surface S and a holomorphic map  $h : M \to S$  with connected fibers such that  $\varphi$  factors through  $h_*$ .

In particular, if N is a closed real hyperbolic manifold, dim  $N \ge 4$ , then any map  $f: M \to N$  has degree zero.

Other topological obstructions to the existence of a LCK structure with potential were obtained by Kokarev in [K] using harmonic maps techniques. For example, one of his results is:

**Theorem 2.19.** [K] Let M be a compact LCK manifold of the same homotopy type as a locally Hermitian symmetric space of non-compact type whose universal cover does not contain the hyperbolic plane as a factor. If M admits a LCK metric with potential, then it admits a global Kähler metric.

On the other hand, on compact Vaisman manifolds the cohomology of L (which is the Morse-Novikov cohomology of the operator  $d-\theta \wedge$ ) is simple:  $H^*(M, L_\theta) = 0$  follows easily from the Structure theorem 2.10 (here the subscript  $\theta$  makes precise the structure of local system of L).

**Theorem 2.20.** [OV6] Let (MJ) be a compact complex manifold, of complex dimension at least 3, endowed with a Vaisman structure with 2-form  $\omega$  and Lee form  $\theta$ . Let  $\omega_1$  be another LCK-form (not necessarily Vaisman) on (MJ), and let  $\theta_1$  be its Lee form. Then  $\theta_1$  is cohomologous with the Lee form of a Vaisman metric, and  $[\omega_1] = 0 \in H^2(M, L_{\theta_1})$ .

By contrast, on an Inoue surface, which does not admit any Vaisman metric, there exists a LCK metric, compatible with the solvmanifold structure, with non-vanishing Morse-Novikov class of the LCK two-form, [Ba].

We end this section with a result which determines all compact nilmanifolds admitting an invariant LCK structure (generalizing a result of L. Ugarte in dimension 4):

**Theorem 2.21.** [S] Let (M, J) be a non-toral compact nilmanifold with a left-invariant complex structure. If (M, J) has a locally conformally Kähler structure, then (M, J) is biholomorphic to a quotient of  $(H(n) \times \mathbb{R}, J_0)$ , where H(n) is the generalized Heisenberg group and  $J_0$  is the natural complex structure on the product.

The author mentions that he does not know if the biholomorphism he finds passes to the quotient; in other words, he does not know if the compact LCK nilmanifold is isomorphic or biholomorphic with  $H(n) \times S^1$ . On the other hand, one sees that, in particular, left invariant LCK structures on compact nilmanifolds are of Vaisman type. We tend to believe that the result is true in more general setting, namely without the assumption of left (or right) invariance. It is tempting to state:

Conjecture 2. Every LCK compact nilmanifod is, up to covering, the product of the generalized Heisenberg group with  $S^1$ .

## 3. Transformation groups of LCK manifolds

The study of this topic went in two directions. The first one is characterizing the various groups appearing in LCK geometry (conformalities, isometries, affinities with respect to the Levi-Civita or the Weyl connection, holomorphicities) and determination of their interrelations. The second one is characterizing different subclasses of LCK manifolds by the existence of a particular subgroup of one of these groups.

In the first direction, we mention the above Theorem 2.11 and the following local result:

**Theorem 3.1.** [MO] On any LCK manifold,  $\mathfrak{aff}(M, \nabla) = \mathfrak{aut}(M)$ , provided that  $\operatorname{Hol}_0(D)$  is irreducible and  $\operatorname{Hol}_0(D)$  is not contained in  $\operatorname{Sp}(n)$ .

For the proof, one first shows that  $\mathfrak{aff}(M,\nabla)\subseteq\mathfrak{h}(M,J)$  (thus generalizing the analogue result for Kähler manifolds). Indeed, let  $f\in \mathrm{Aff}(M,D)$ . We show that it is  $\pm$  - holomorphic.

Define  $J'_x := (d_x f)^{-1} \circ J_{f(x)} \circ (d_x f)$ . Then J' is D-parallel. To show that  $J' = \pm J$ , we decompose JJ' = S (symm.) + A (antisymm.). Then S is  $\nabla$ -parallel and hence it has constant eigenvalues; thus the corresponding eigenbundles are D-parallel.

By  $\operatorname{Hol}_0(D)$  irreducible,  $S = k \operatorname{id}, k \in \mathbb{R}$ . Similarly,  $A^2 = p \operatorname{id}, p \in \mathbb{R}$ .

Now, if  $A \neq 0$ , then  $A(X) \neq 0$  for some  $X \in TM$ , so  $0 > -g(AX, AX) = g(A^2X, X) = pg(X, X)$ , whence p < 0. Then  $K := A/\sqrt{-p}$  is D-parallel,  $K^2 = -\operatorname{id}$ , KJ = -JK, so (J, K) defines a D-parallel quaternionic structure structure on M, contradiction.

Hence  $A=0,\ JJ'=k\,\mathrm{id},\ \mathrm{so}\ J'=-kJ.$  But  ${J'}^2=-\mathrm{id},\ \mathrm{thus}\ k=\pm 1$  and so  $J'=\pm J.$ 

With similar arguments one proves that  $\mathfrak{aff}(M,D) \subseteq \mathfrak{conf}(M,[g])$ .

In the second direction, we first recall the following characterization of Vaisman manifolds:

**Theorem 3.2.** [KO] A compact LCK manifold admits a LCK metric with parallel Lee form if its Lie group of holomorphic conformalities has a complex one-dimensional Lie subgroup, acting non-isometrically on its Kähler covering.

We note that the above criterion assures the existence of a Vaisman metric in the conformal class of the given LCK one. We recently extended this result to obtain the existence of a LCK metric with automorphic potential, not necessarily conformal to the starting LCK one:

**Theorem 3.3.** [OV8] Let M be a compact LCK manifold, equipped with a holomorphic  $S^1$ -action. Suppose that the holonomy of the weight bundle L restricted to a general orbit of this  $S^1$ -action is non-trivial. Then  $\tilde{M}$  admits a global automorphic potential.

For the proof, a first step is to show that one can assume from the beginning that  $\omega$ , and hence, as J remains unchanged, g, is  $S^1$ -invariant (*i.e.* the action is isometric). Note that a similar argument was used in the proof of Theorem 2.20.

Indeed, we average  $\theta$  on  $S^1$  and obtain  $\theta' = \theta + df$  which is  $S^1$ -invariant. The cohomology class is conserved:  $[\theta] = [\theta']$ . Now we let  $\omega' = e^{-f}\omega$ : it is LCK, conformal to  $\omega$  and with Lee form  $\theta'$ .

Hence, we may assume from the beginning that  $\theta$  (corresponding to  $\omega$ ) is  $S^1$ -invariant.

We now average  $\omega$  over  $S^1$ , taking into account that:

(3.1) 
$$d(a^*\omega) = a^*\omega \wedge a^*\theta = a^*\omega \wedge \theta, \qquad a \in S^1$$

We thus find an  $\omega'$  which is  $S^1$ -invariant, with

$$d\omega' = \theta \wedge \omega'$$
.

As the monodromy of L along an orbit S of the  $S^1$ -action is precisely  $\int_S \theta$ , it is not changed by this averaging procedure.

This means that it is enough to make the proof assuming  $\omega$  is  $S^1$ -invariant.

On the other hand, the lift of  $S^1$  to  $\tilde{M}$  acts on  $\tilde{\omega}$  by homotheties, and the corresponding conformal constant is equal to the monodromy of L along the orbits of  $S^1$ . Thus, the image of the restriction of the character  $\chi$  to the lifted subgroup cannot be compact in  $\mathbb{R}^{>0}$  unless it is trivial, hence the  $S^1$  action lifts to an  $\mathbb{R}$  action.

In conclusion, we may assume from the beginning that  $S^1$  is lifted to an  $\mathbb{R}$  acting on  $\tilde{M}$  by non-trivial homotheties.

Now, let A be the vector field on  $\tilde{M}$  generated by the  $\mathbb{R}$ -action. A is holomorphic and homothetic (Lie<sub>A</sub>  $\Omega = \lambda \Omega$ ).

Let  $A^c = JA$ . Then:

$$dd^c |A|^2 = \lambda^2 \Omega + \operatorname{Lie}_{A^c}^2 \Omega$$

Read in Bott-Chern cohomology, this implies:

$$\operatorname{Lie}_{A^c}^2[\Omega] = -\lambda^2[\Omega] \in H^2_{BC}(M, L).$$

Hence  $V:=\operatorname{span}\{[\Omega],\operatorname{Lie}_{A^c}[\Omega]\}\subset H^2_{BC}(M,L)$  is 2-dimensional.

As  $\operatorname{Lie}_{A^c}$  acts on V with two 1-dimensional eigenspaces, corresponding to  $\sqrt{-1}\lambda$  and  $-\sqrt{-1}\lambda$ , it is essentially a rotation with  $\lambda\pi/2$ , and hence the flow of  $A^c$  satisfies:

$$e^{tA^c}[\Omega] = [\Omega], \text{ for } t = 2n\pi\lambda^{-1}, n \in \mathbb{Z}.$$

But also

$$\int_0^{2\pi\lambda^{-1}} e^{tA^c} [\Omega] dt = 0.$$

and hence

$$\Omega_1 := \int_0^{2\pi\lambda^{-1}} e^{tA^c} \Omega dt$$

is a Kähler form, whose Bott-Chern class vanishes,  $[\Omega_1] = 0 \in H^2_{BC}(M, L)$ , thus defining a LCK metric with automorphic potential by Lemma 2.3.

#### References

- [AD] V. Apostolov, G. Dloussky, Bihermitian metrics on Hopf surfaces, Math. Res. Lett. 15 (2008), 827–839.
- [Ba] A. Banyaga, Examples of non  $d_{\omega}$ -exact locally conformal symplectic forms, J. Geom. 87 (2007) 1–13.
- [Be] F.A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1–40.
- [Bl] D.E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Math. 203, Birkhäuser, Boston, Basel 2002.
- [Br1] M. Brunella, Locally conformally Kähler metrics on some non-Kählerian surfaces, Math. Ann. 346 (2010), 629–639.

- [Br2] M. Brunella, Locally conformally Kaehler metrics on Kato surfaces, arXiv:1001.0530.
- [Bu] N. Buchdahl, On compact Kähler surfaces, Ann.Inst. Fourier, 49 (1999), 287–302
- [C] C. van Coevering, Examples of asymptotically conical Ricci-flat Kähler manifolds, Math. Z. DOI 10.1007/s00209-009-0631-7.
- [DO] S. Dragomir and L. Ornea, Locally conformal Kähler geometry, Progress in Math. 155, Birkhäuser, Boston, Basel, 1998.
- [FP] A. Fujiki and M. Pontecorvo, Anti-self-dual bihermitian structures on Inoue surfaces, arXiv:0903.1320.
- [G] P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte Math. Ann. 267 (1984), 495–518.
- [GO] P. Gauduchon and L. Ornea, Locally conformally Kähler metrics on Hopf surfaces, Ann. Inst. Fourier 48 (1998), 1107–1127.
- [GOPP] R. Gini, L. Ornea, M. Parton, P. Piccinni, Reduction of Vaisman structures in complex and quaternionic geometry, J. Geom. Phys. 56 (2006), no. 12, 2501–2522.
- [K] G. Kokarev, On pseudo-harmonic maps in conformal geometry, Proc. London Math. Soc. 99 (2009) 168–194.
- [KO] Y. Kamishima, L. Ornea, Geometric flow on compact locally conformally Kähler manifolds, Tohoku Math. J. 57 (2005), no. 2, 201–222.
- [KK] G. Kokarev, D. Kotschick, Fibrations and fundamental groups of Kähler-Weyl manifolds, Proc. Amer. Math. Soc., 138 (2010), 997–1010.
- [L] A. Lamari, Courrants kählériens et surfaces compactes, Ann. Inst. Fourier 49 (1999), 263–285
- [MO] A. Moroianu, L. Ornea, Transformations of locally conformally Kähler manifolds, Manuscripta Math. 130 (2009), 93–100.
- [OT] K. Oeljeklaus, M. Toma, Non-Kähler compact complex manifolds associated to number fields, Ann. Inst. Fourier 55 (2005), 1291–1300.
- [O] L. Ornea, Locally conformally Khler manifolds. A selection of results, Lecture notes of Seminario Interdisciplinare di Matematica. Vol. IV, 121–152, Lect. Notes Semin. Interdiscip. Mat., IV, S.I.M. Dep. Mat. Univ. Basilicata, Potenza, 2005.
- [OP] L. Ornea, M.V. Pilca, Remarks on the product of harmonic forms, . arXiv:1001.2129.
- [OV1] L. Ornea, M. Verbitsky, Structure theorem for compact Vaisman manifolds, Math. Res. Lett. 10 (2003), 799–805.
- [OV2] L. Ornea, M. Verbitsky, An immersion theorem for Vaisman manifolds, Math. Ann. 332 (2005), 121–143.
- [OV3] L. Ornea and M. Verbitsky, Locally conformal K\u00e4hler manifolds with potential, Math. Ann. DOI: 10.1007/s00208-009-0463-0. arXiv:math/0407231
- [OV4] L. Ornea and M. Verbitsky, Embeddings of compact Sasakian manifolds, Math. Res. Lett. 14 (2007), 703–710.
- [OV5] L. Ornea and M. Verbitsky, Einstein-Weyl structures on complex manifolds and conformal version of Monge-Ampère equation Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 51(99) (2008), 339–353.
- [OV6] L. Ornea and M. Verbitsky, Morse-Novikov cohomology of locally conformally Kähler manifolds, J. Geom. Phys. 59, (2009), 295–305, arXiv:0712.0107
- [OV7] L. Ornea and M. Verbitsky, Topology of locally conformally Kähler manifolds with potential, Int. Math. Res. Notices doi:10.1093/imrn/rnp144. arXiv:0904.3362.
- [OV8] L. Ornea and M. Verbitsky, Automorphisms of locally conformally Kähler manifolds, arXiv:0906.2836.
- [PV] M. Parton, V. Vuletescu, Examples of non-trivial rank in locally conformal Kähler geometry, arXiv:1001.4891.
- [S] H. Sawai, Locally conformal Kähler structures on compact nilmanifolds with left-invariant complex structures Geom. Dedicata 125 (2007), 93–101.
- [Th] W.P. Thurston, Some simple examples of symplectic manifolds, Proc. Amer. Math. Soc. 55 (1976), 467–468.
- [Tr] F. Tricerri, Some examples of locally conformal Kähler manifolds, Rend. Sem. Mat. Univ. Politec. Torino 40 (1982), 81–92.
- [Va] I. Vaisman, On locally and globally conformal Kähler manifolds, Trans. Amer. Math. Soc. 262 (1980), 533–542.

- [T] M. Toma, Holomorphic vector bundles on non-algebraic surfaces Dissertation, Bayreuth 1992 http://www.mathematik.uni-osnabrueck.de/staff/phpages/tomam/preprints.html
- [Ve] I. Verbitsky, Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds, (Russian) Tr. Mat. Inst. Steklova 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 64–91; translation in Proc. Steklov Inst. Math. 246 (2004), 54–78.
- [Vu1] V. Vuletescu, Blowing-up points on locally conformally K\u00e4hler manifolds, Bull. Math. Soc. Sci. Math. Roumanie 52(100) (2009) 387–390.
- [Vu2] V. Vuletescu, LCK metrics on elliptic principal bundles, arXiv:1001.0936.

University of Bucharest, Faculty of Mathematics, 14 Academiei str., 70109 Bucharest, Romania and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21, Calea Grivitei str. 010702-Bucharest, Romania

 $E\text{-}mail\ address: \texttt{lornea@gta.math.unibuc.ro,\ liviu.ornea@imar.ro}$ 

Institute of Theoretical and Experimental Physics, B. Cheremushkinskaya, 25, Moscow, 117259, Russia

 $E\text{-}mail\ address: \verb|verbit@verbit.ru||$