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PATTERN RECOGNITION AND IMAGE PROCESSING

An Axiomatic Approach to Measuring of Information of Sign-based Image Representations

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Abstract—A sign-based image representation is introduced and its main properties are investigated. In particular, the problem of measuring information of sign-based representations is discussed. For this purpose, axiomatic measures of information of images and their sign-based representations are introduced. It is shown that these measures are very close in properties to the Shannon entropy. The paper also contains results of calculating the values of the information measures introduced for synthetic and real images.

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INTRODUCTION

Many pattern recognition problems can be efficiently solved by algorithms using sign-based image representations. In [1], sign-based image representations were applied to face detection problem, which consists in finding image parts containing faces and not containing the background. Face detection is an important preliminary stage for face recognition, face identification, gender recognition, age estimation and emotions recognition, etc. In [1, 2], this approach was employed for identification of faces, when in response to a query face, the most similar faces belonging to the image base are found. The application of sign-based image representation in models of active contours [2] makes it possible to solve efficiently in computation the problem of localizing anthropometric face features such as eyebrows contours, the coordinates of eye corners and the centers of pupils, nose and lips contours, and the face boundary. Sign-based representation has also shown its advantage in near duplicates detection in large image collections [3]. This problem is topical, e.g., in search engines [4, 5], since one of the cost functions of data retrieval is the diversity of query results. In addition, detection of fuzzy duplicates is of great interest in the struggle against spam [6], distributed as graphical files. The idea of the transition from the source signal or image representation to signs of a certain functional is widely applied both in pattern recognition and analysis of stochastic processes.

One of the analogs of sign-based representation is the description of the shape of an object in the form of a chain code, first proposed by H. Freeman [7]. A chain code is a method for specifying a contour by a sequence of adjacent pixels, i.e., $(\mathbf{x}_i)_{i=1}^N$, where two-dimensional vectors $\mathbf{x}_i$ have integer coordinates, and if $\Delta\mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i = (l, m)$, where $i \in \{1, \ldots, N-1\}$, then $l, m \in \{-1, 0, 1\}$. Therefore in the chain code, the position of the next pixel relative to the previous one is coded by a pair of numbers (l, m) or, which is equivalent, by their signs). Thus the chain code can be considered as one of the examples of sign-based data representation.

The closest analog of sign-based representation is the well-known morphological approach proposed by Yu.P. Pyt'ev [8]. The Pyt'ev morphology is based on the idea of dividing the image into parts characterized by constant image intensities, and the image itself is represented in the form of orthogonal characteristic functions, which differ from zero only on the subsets corresponding to regions with constant intensity values. The set of images that can be obtained from the initial image under the action of a certain function of intensity values are called an image "shape" [9, 10]. In the proposed approach, complete and neighborhood sign-based representations are considered. The set of images corresponding to complete sign-based representation coincides with the Pyt'ev shape concept in the class of strictly increasing intensity transformations. However the set of images obtained based on neighborhood sign-based representation is wider that the Pyt'ev image shape.

Despite a wide range of problems solved using sign-based representations, its properties have not been investigated yet. Note that the most interesting is the problem on an information measure for this image

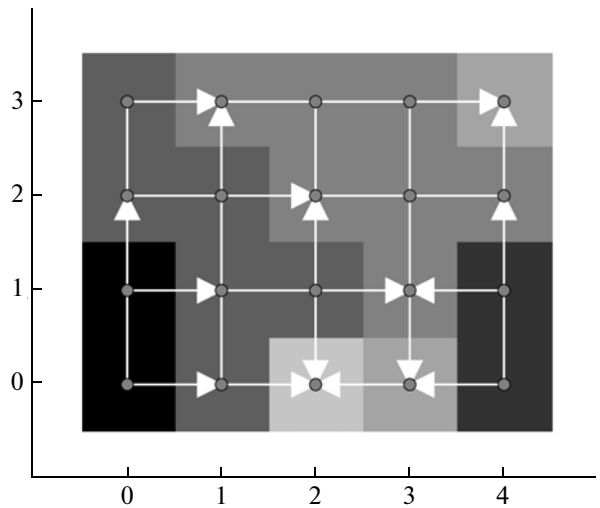

Fig. 1. Example of neighborhood sign-based representation (1.2), when the neighborhood of each pixel consists of directly adjacent pixels.

representation. Information measures of a special type were applied in image processing in [11–13]. Within the scope of this paper, we propose an axiomatic approach to introducing information measures in images and the sign-based representations corresponding to them, as well as an uncertainty measure of a sign-based representation, describing quantitatively information loss in the transition from an image to its sign-based representation.

1. SIGN-BASED IMAGE REPRESENTATION

By an image we mean a nonnegative integer-valued function $f = f(x_1, x_2)$ given at points of a integer grid $\Omega = I_N \times I_M = \{1, \ldots, N\} \times \{1, \ldots, M\}$, i.e., $f: \Omega \to \mathbb{Z}_+$. A pair of numbers $\mathbf{x} = (x_1, x_2)$, $x_1 \in I_N$, $x_2 \in I_M$ is called a *pixel*, and $f(\mathbf{x})$ is the intensity value of the image f at the pixel $\mathbf{x}$. The set of all images $f : \Omega \to \mathbb{Z}_+$ is denoted by $\mathcal{F}$.

D e f i n i t i o n 1. A relation $\tau \subseteq \Omega \times \Omega$ is a sign-based representation of the image $f \in \mathcal{F}$ if the following properties hold:

(1) if $(\mathbf{x}_i, \mathbf{x}_j) \in \tau$, then $f(\mathbf{x}_i) \le f(\mathbf{x}_j)$;

(2) if $(\mathbf{x}_i, \mathbf{x}_j) \in \tau$, $(\mathbf{x}_j, \mathbf{x}_i) \notin \tau$, then $f(\mathbf{x}_i) < f(\mathbf{x}_j)$.

As examples, consider certain methods of introducing a sign-based representation in images. By a *complete sign-based representation*, we mean a representation such that possesses the connectivity property, i.e., contains all pairs of image points such that

$$\tau_c = \left\{(\mathbf{x}, \mathbf{y}) \in \Omega^2 | f(\mathbf{x}) \le f(\mathbf{y})\right\}. \tag{1.1}$$

Note that a complete sign-based representation of the image f is uniquely determined by the connectivity property of the relation. Let us also introduce a *window sign-based relation*, which is a compact variant of sign-based representation, when only relations on adjacent pixels

$$\tau_\varepsilon = \left\{(\mathbf{x}, \mathbf{y}) \in \Omega^2 | f(\mathbf{x}) \le f(\mathbf{y}), \mathbf{y} \in O_\varepsilon(x)\right\}, \tag{1.2}$$

are considered, where $O_\varepsilon(\mathbf{x})$ is a neighborhood of the point $\mathbf{x}$, e.g.,

$$O_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \Omega \,|\, \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}, \tag{1.3}$$

where $\|\mathbf{x} - \mathbf{y}\| = |x_1 - y_1| + |x_2 - y_2|$.

Figure 1 presents an example of neighborhood sign-based representation. Note that for applied problem, it is the neighborhood sign-based representation (1.2) with neighborhood (1.3) that is of the greatest interest; therefore in what follows, we consider sign-based representations given by relations on adjacent pixels.

The class of images corresponding to a sign-based representation τ is denoted by $\mathcal{F}_\tau$. Let τ be a certain sign-based representation of image $f \in \mathcal{F}$, and let τ^{Tr} be the transitive closure of the relation τ. Then it is

obvious that τ^{Tr} is also a sign-based representation of f, and $\mathcal{F}_{\tau^{Tr}} = \mathcal{F}_\tau$. Keeping this in mind, we can restrict ourselves with reflexive and transitive relations (i.e., quasi-order relations). We denote by $\mathcal{T}$ the set of these relations.

The set of images $\mathcal{F}_\tau$ is an analog of the concept of shape in the Pyt'ev morphology. Let Φ be a set of mappings φ: $\mathbb{Z}_+ \to \mathbb{Z}_+$ simulating the conditions of image registration. Then by shape in the Pyt'ev morphology, we mean the set of images $\mathcal{V}_f = \{\varphi \circ f | \varphi \in \Phi\}$. It can easily be seen that if we take as Φ the class of all strictly increasing images, then $\mathcal{V}_f \subseteq \mathcal{F}_\tau$. The inverse inclusion $\mathcal{F}_\tau \subseteq \mathcal{V}_f$ holds only for the case of the complete sign-based representation of the image f, and, in this situation, the concepts of the Pyt'ev shape and the sign-based representation coincides. In the other cases, in particular, for neighborhood sign-based representations, the set $\mathcal{F}_\tau$ is wider than the set $\mathcal{V}_f$.

We can attach to a sign-based representation of the image f an oriented graph $G_\tau = (\Omega, E_\tau)$, whose set of vertices is generated by the set Ω, and the set of edges E_τ coincides with the set τ, and if $(\mathbf{x}, \mathbf{y}) \in \tau$, then the corresponding graph edge is directed from the vertex $\mathbf{x}$ to the vertex $\mathbf{y}$. It is convenient to interpret a sign-based representation as a graph G_τ in order to apply it for studying the properties of the sign-based representation, as well in its visualization.

Let us formulate necessary and sufficient conditions to provide that an arbitrary relation τ given on Ω is a neighborhood sign-based representation of a certain image. For this purpose, we use the graph structure of the sign-based representation.

Denote by E the set of all edges (including loops) that join adjacent pixels, and also introduce the graph $\bar{G}_\tau = (\Omega, E \backslash E_\tau)$ (here strictly speaking, the graph $\bar{G}_\tau$ is not supplementary to G_τ). Consider the equivalence relation $\theta = (\tau \cap \tau^{-1})^{Tr}$, where τ^{-1} is the inverse relation to τ, i.e., $\tau^{-1} = \{(\mathbf{x}, \mathbf{y}) \in \Omega^2 | (\mathbf{y}, \mathbf{x}) \in \tau\}$, as well as the partition of the set of vertices of Ω into factor-sets v_i connected with the relation θ. Note that the given partition is uniquely determined. If τ is a neighborhood sign-based representation of an image, then equivalence classes correspond to connected sets of pixels having the same intensity. Then we introduce the following graphs on the set $V = \{v_1, \ldots, v_n\}$ of all equivalence classes of the relation θ: $G_\tau^\theta = (V, E_\tau^\theta)$ and $\bar{G}_\tau^\theta = (V, \bar{E}_\tau^\theta)$, where $(v_i, v_j) \in E_\tau^\theta$, if there exists a pair $(\mathbf{x}, \mathbf{y}) \in E_\tau$ such that $\mathbf{x} \in v_i$ and $\mathbf{y} \in v_j$; similarly if there $(v_i, v_j) \in \bar{E}_\tau^\theta$ is a pair $(\mathbf{x}, \mathbf{y}) \in E \backslash E_\tau$ such that $\mathbf{x} \in v_i$ and $\mathbf{y} \in v_j$. Thus the graph $G_\tau^\theta = (V, E_\tau^\theta)$ is the extension of the graph $G_\tau = (\Omega, E_\tau)$ to the set of equivalence classes V. The adjacency relation of pixels can also be extended to equivalence classes. We assume that equivalence classes v_i and v_j are adjacent if there are adjacent pixels $\mathbf{x} \in v_i$ and $\mathbf{y} \in v_j$. If we denote by E^θ the set of all edges (including loops) between adjacent equivalence classes, then it is clear that $\bar{G}_\tau^\theta = (V, E^\theta \backslash E_\tau^\theta)$. In what follows, we also suppose that the function f is also defined on equivalence classes, i.e., $f(v) = f(\mathbf{x})$ when $v \in V$ and $\mathbf{x} \in v$.

The following proposition gives necessary and sufficient conditions for an arbitrary given graph $G_\tau = (\Omega, E_\tau)$ to be a graph of a neighborhood sign-based representation of an image f.

P r o p o s i t i o n 1. A graph $G_\tau = (\Omega, E_\tau)$ is a neighborhood sign-based representation of an image f if and only if

(a) $E_\tau \subseteq E$, where E is the set of all pairs of adjacent vertices;

(b) $\bar{G}_\tau^\theta$ is an acyclic graph;

(c) $\bar{G}_\tau^\theta$ is an asymmetric graph on the set of edges E^θ; i.e., $(v_i, v_j) \in E_\tau^\theta$ for $v_i \neq v_j$ if and only if $(v_j, v_i) \notin E_\tau^\theta$.

P r o o f. *Necessity.* Let $G_\tau = (\Omega, E_\tau)$ be the graph of sign-based representation of the image f. Then $E_\tau \subseteq E$ by definition. Let us show that $\bar{G}_\tau^\theta$ is an acyclic graph. Indeed if in this graph there exists a loop consisting of the vertices $v_1, \ldots, v_k$, then the inequality holds

$$f(v_1) > f(v_2) > \ldots > f(v_k) > f(v_1),$$

which is impossible for obvious reasons. Thus, the necessity is proved.

Sufficiency. Let all hypotheses of the proposition hold. We show that in this case there is a function $f:\Omega\to\mathbb{Z}_+$ such that the graph G_τ is its sign-based representation. We prove this using an iterative procedure of construction of acyclic transitive graphs $G^{(k)}=(V,E^{(k)})$, $k=1,2,\ldots,n$, such that

(1) $E^{(1)}=(\bar{E}_\tau^\theta)^{Tr}$;

(2) $E^{(1)}\subset E^{(2)}\subset\ldots\subset E^{(n)}\subset E^\theta$.

We proceed with this procedure until, at a certain step n, the graph $G^{(n)}=\left(V,E^{(n)}\right)$ becomes asymmetric; i.e., for any pair $(v_i,v_j)\in V\times V$ of different vertices ($v_i\neq v_j$), we have the condition $(v_i,v_j)\in E^{(n)}$ if and only if $(v_j,v_i)\notin E^{(n)}$. Let us show how to construct the graph $G^{(k+1)}=\left(V,E^{(k+1)}\right)$ at the step k if we have the acyclic transitive graph $G^{(k)}=(V,E^{(k)})$. Suppose that the graph $G^{(k)}$ is not asymmetric (otherwise, the construction of the extension of graphs is not required). Then there exists a pair of vertices $(v_i,v_j)\in V\times V$ such that $(v_i,v_j)\notin E^{(k)}$ and $(v_j,v_i)\notin E^{(k)}$. We add to the graph $G^{(k)}$ the edge (v_i,v_j) and obtain as a result the graph $(V,E^{(k)}\cup\{(v_i,v_j)\})$. Note that this graph is acyclic, since otherwise there exists a cycle $v_1,\ldots,v_i,v_j,\ldots,v_k$ containing the edge (v_i,v_j). However this is impossible since because of the transitivity of the graph $G^{(k)}$, this implies that $(v_j,v_i)\in E^{(k)}$, which contradicts the choice of the pair of vertices (v_i,v_j). Then we find the transitive closure of this graph and choose $G^{(k+1)}=(V,(E^{(k)}\cup\{(v_i,v_j)\})^{Tr})$, which satisfies all necessary conditions. Note that since the set $V\times V$ is finite, by all means, at some step we arrive at an asymmetric, acyclic, transitive graph $G^{(n)}$. This graph is a graph of a certain strict-order relation; therefore there is a function $f:V\to\mathbb{Z}_+$ such that $(v_i,v_j)\in E^{(n)}$ if and only if $f(v_i)>f(v_j)$. Then we extend the function f from the equivalence classes to all pixels Ω by $f(\mathbf{x})=f(v)$ if $\mathbf{x}\in v$. It is obvious that the constructed function $f:\Omega\to\mathbb{Z}_+$ satisfies all necessary conditions; i.e., the sufficiency and the whole proposition are proved.

R e m a r k 1. In the proof of Proposition 1, we actually showed how to construct images corresponding to a given sign-based representation. However enumerating in this way all possible variants, we do not enumerate all possible images up to an increasing monotonous transformation. This is the case since all equivalence classes are arranged by the strict-order relation. Nevertheless, cases when the image f corresponds to the available sign-based representation are possible, and there are indices i,j such that for nonadjacent equivalence classes v_i and v_j, we have $f(v_i)=f(v_j)$.

To enumerate all variants, we should change the procedure of generating acyclic transitive graphs considered in the proof of Proposition 1. In this case, the sequence of acyclic transitive graphs $G^{(k)}=(V^{(k)},E^{(k)})$, $k=1,2,\ldots,n$, is constructed as follows.

We set $G^{(1)}=(\bar{G}_\tau^\theta)^{Tr}$. Let $G^{(k)}=(V^{(k)},E^{(k)})$ be the acyclic transitive graph generated at the kth step, and let $G^{(k)}$ be not the graph of a strict-order relation. Then there exist a pair of vertices $(v_i,v_j)\in V^{(k)}\times V^{(k)}$ such that $(v_i,v_j)\notin E^{(k)}$ and $(v_j,v_i)\notin E^{(k)}$. Then we construct the graph $G^{(k+1)}$ either in Proposition 1, i.e., $G^{(k+1)}=(V^{(k)},(E^{(k)}\cup\{(v_i,v_j)\})^{Tr})$ or by merging the vertices v_i and v_j into a single vertex $v_i\cup v_j$. As a result, we obtain the graph $(V^{(k+1)},E^*)$, where $V^{(k+1)}=\left(V^{(k)}\cup\{v_i\cup v_j\}\right)\setminus\left(\{v_i\}\cup\{v_j\}\right)$, and

$$\begin{aligned}E^*=&\left\{(v_l,v_m)\in E^{(k)}\middle|\,v_l\notin\{v_i,v_j\},v_m\notin\{v_i,v_j\}\right\}\cup\\&\cup\left\{(v_i\cup v_j,v_m)\middle|(v_i,v_m)\in E^{(k)}\right\}\cup\\&\cup\left\{(v_i\cup v_j,v_m)\middle|(v_j,v_m)\in E^{(k)}\right\}\cup\\&\cup\left\{(v_m,v_i\cup v_j)\middle|(v_m,v_i)\in E^{(k)}\right\}\cup\\&\cup\left\{(v_m,v_i\cup v_j)\middle|(v_m,v_j)\in E^{(k)}\right\};\end{aligned}$$

after the operation of its transitive closure, we arrive at the desired graph $G^{(k+1)}=(V^{(k+1)},(E^*)^{Tr})$.

Using the same method as in Proposition 1, it is easy to show that at each step, we obtain an acyclic transitive graph $G^{(k)}$, and the procedure for constructing graphs will surely terminate at a certain graph $G^{(n)}$

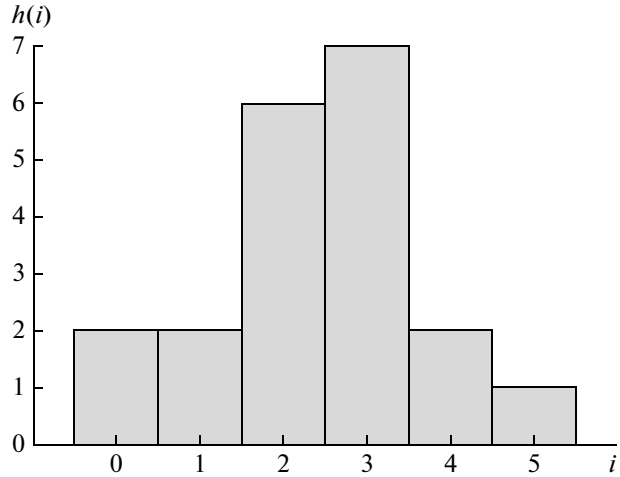

Fig. 2. Histogram of the model image presented in Fig. 1.

of a strict-order relation. It is easy to notice that applying this procedure, we can enumerate all possible images corresponding to a given sign-based representation.

2. AXIOMATIC INTRODUCTION OF AN INFORMATION IMAGE MEASURE

Since a certain set of images correspond a sign-based representation, there arises the problem of choice from this set of images an image such that preserves contour information containing in the sign-based representation and which, at the same time, is the most "typical" representative; i.e., it does not contain redundant information on intensity grade of pixels. For this purpose we analyze how to measure the amount of given information for images.

We will construct the image information measure within the framework of axiomatic approach, according to which it is necessary to define a certain finite number of axioms (desired properties of the information measure), which determine it uniquely. Taking into consideration the fact that we do not suppose to take into account the positional relationship of pixels in the information measure (only information connected with intensity grades is investigated), as the image we consider an arbitrary function $f : \Omega \to \mathbb{Z}_+$, which maps a finite set of pixels Ω in the set of nonnegative integers. It is necessary to note that the axioms introduced in what follows not completely reflect the properties of the sign-based image representation. The information measure taking into account the properties of the sign-based representation is introduced and investigated in the next section.

A x i o m 1. An information measure is a functional $U : \mathcal{F} \to [0, +\infty)$.

We assume that intensity values in themselves do not convey any information, and information is contained in intensity discontinuities. Thus, the amount of information in an image with intensity constant at each point is zero.

A x i o m 2. Let $f : \Omega \to \mathbb{Z}_+$, and let the set of values $f(\Omega) = \{f(\mathbf{x}) | \mathbf{x} \in \Omega\}$ of the function f be single-element, i.e., $|F(\Omega)| = 1$. Then $U(f) = 0$.

Then we formulate axioms that allow one to determine classes of image transformations that do not change their information. Since in the information measure, we take into account only intensity grades, the transformation that implements "mixture" of pixels in images does not affect the information.

A x i o m 3. Let $f : \Omega_1 \to \mathbb{Z}_+$, and let $\psi : \Omega_1 \to \Omega_2$ be a bijection. Then $U(\psi \circ f) = U(f)$.

The transformation that is connected with assignment of new intensity values to pixels by means of bijective mappings also does not affect the information of intensity grades, since the fact of distinction of intensity grades itself, rather than their particular values is important. Taking this into account, we formulate the following axiom.

A x i o m 4. Let $f : \Omega_1 \to \mathbb{Z}_+$, and let $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+$ be a bijection. Then $U(f \circ \varphi) = U(f)$.

Assume that $h_f : \mathbb{Z}_+ \to \mathbb{Z}_+$ is the histogram of the image f, whose value $h_f(i)$ gives the number of pixels in the image with intensity i (see Fig. 2). It is obvious that this function is not zero only on a certain finite set of integers. Axioms 3 and 4 imply the following important corollary.

C o r o l l a r y 1. Let images be given by functions $f : \Omega_1 \to \mathbb{Z}_+$ and $g : \Omega_2 \to \mathbb{Z}_+$. Then $U(f) = U(g)$ if there exists a bijection $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $h_g(i) = h_f(\varphi(i))$ for any $i \in \mathbb{Z}_+$.

P r o o f. Since $h_g(i) = h_f(\varphi(i))$ for any $i \in \mathbb{Z}_+$, we can construct the bijection $\psi : \Omega_1 \to \Omega_2$ between the sets Ω_1 and Ω_2 satisfying the following condition: if $f(\mathbf{x}) = i$, then $g(\psi(\mathbf{x})) = \varphi(i)$. Therefore the image g admits a representation in the form $g = \psi \circ f \circ \varphi$. Applying successively axioms 2 and 3, we find that $U(g) = U(\psi \circ f \circ \varphi) = U(f \circ \varphi) = U(f)$.

Note that the proved corollary makes it possible to simplify the problem since it is sufficient to determine the value of the functional U on all possible sequences of the form $(h_f(0), h_f(1), \ldots)$.

In what follows, we also consider the functional $\bar{U}(f) = U(f)/|\Omega|$ showing the mean value of the pixel information for the image $f: \Omega \to \mathbb{Z}_+$. Assume that the image g consists of k copies of the image f; in this case $h_g(i) = k\, h_f(i)$ for any $i \in \mathbb{Z}_+$. This condition can be expressed in terms of frequencies of occurrence of pixels in the images f and g

$$p_g(i) = \frac{h_g(i)}{\sum\limits_{j \in \mathbb{Z}_+} h_g(j)},$$

$$p_f(i) = \frac{h_f(i)}{\sum\limits_{j \in \mathbb{Z}_+} h_f(j)}$$

in the form $p_g(i) = p_f(i)$ for any $i \in \mathbb{Z}_+$. It is natural to suppose that $\bar{U}(f) = \bar{U}(g)$ for these images. Taking axiom 2 into account, we formulate the following axiom.

A x i o m 5. Let $f, g \in \mathcal{F}$, and let $p_g(i) = p_f(i)$ for all $i \in \mathbb{Z}_+$. Then $\bar{U}(f) = \bar{U}(g)$.

Let $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ be an arbitrary mapping (not necessarily injective), and let $f \circ \phi$ be the image obtained as a result of action of the transformation ϕ on the image $f : \Omega \to \mathbb{Z}_+$. If ϕ is not an injection, then we lose a part of information about the intensity grades in the initial image f, namely, in this case, the sets $\phi^{-1}(b) = \{a \in \phi(\Omega) | \phi(a) = b\}$ for $b \in \phi(f(\Omega))$ are not singletons necessarily. This means that the initial image is "coarsened" by assigning to pixels that are "close in intensity" the same value. Note that the given transformation is widespread in image processing, when it is necessary to reduce the number of intensity grades saving the most typical cuts of the image function.

We assume that $\phi(f(\Omega)) = \{b_1, \ldots, b_n\}$; then the sets $\Omega_k = \{\mathbf{x} \in \Omega | \phi(f(\mathbf{x})) = b_k\}$, $k = 1, \ldots, n$ obviously specify a partition of the set Ω. If the mapping ϕ is injective, then the images $f_k : \Omega_k \to \mathbb{Z}_+$ that are the restrictions of the function f on the sets Ω_k have zero information according to axiom 2 since $|f_k(\Omega_k)| = 1$. When the mapping ϕ is injective, $|f_k(\Omega_k)| \neq 1$. Therefore the quantities $\sum\limits_{k=1}^{n} U(f_k)$ characterize the total information loss under the mapping. Thus assuming this additive nature of accumulation of uncertainty, we can introduce the following additivity axiom.

A x i o m 6. Let $f : \Omega \to \mathbb{Z}_+$, $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ and $\phi(f(\Omega)) = \{b_1, \ldots, b_n\}$, and let $\Omega_k = \{\mathbf{x} \in \Omega | \phi(f(\mathbf{x})) = b_k\}$. Consider the set $f_k : \Omega_k \to \mathbb{Z}_+$, as well as the restrictions of the functions f to the sets Ω_k. Then $\sum\limits_{k=1}^{n} U(f_k) + U(f \circ \phi) = U(f)$.

Let us express the axioms presented above in terms of the functional $\bar{U}$. By axiom 5, it is sufficient to determine this functional for the number sequence $P = (p(i))_{i \in \mathbb{Z}_+}$ such that $p(i) \geq 0$ and $\sum\limits_{i \in \mathbb{Z}_+} p(i) = 1$. Note that the value $p(i)$ can be interpreted as the probability of occurrence in the image of a pixel with intensity i; therefore P can be treated as a probability measure. Then we can define the probability $P(A)$ of any subset $A \subseteq \mathbb{Z}_+$ by the expression

$$P(A) = \sum_{i \in A} p(i).$$

In what follows, we use standard notation of probability theory; in particular, let $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$, then P^ϕ is the probability measure given by the equality $P^\phi(A) = P\{i \in \mathbb{Z}_+ | \phi(i) \in A\}$. Note that within the scope of the posed problem, it is not necessary to consider all possible probability measures. By construction, all $p(i)$ are rational numbers and only a finite set of these numbers are different from zero. We denote by M_{pr}, the set of these probability measures on the algebra of subsets of $\mathbb{Z}_+$.

Corollary 2. The functional $\bar{U}$ on M_{pr} has the following properties:

(1) $\bar{U}(P) \geq 0$ for all $P \in M_{pr}$.

(2) $\bar{U}(P) = 0$ if there exists $i \in \mathbb{Z}_+$ such that $P(i) = 1$.

(3) Assume that the mapping $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ is injective, then $\bar{U}\left(P^\phi\right) = \bar{U}(P)$ for all $P \in M_{pr}$.

(4) Let $P \in M_{pr}$, $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$, and $A = \{i \in \mathbb{Z}_+ | p(i) > 0\}$. Consider the partition of the set A into subsets representing the preimages of elements of the set $B = \{b_1, \ldots, b_n\} = \phi(A)$; i.e., the partition consists of the sets $A_k = \{i \in A | \phi(i) = b_k\}$. Then

$$\bar{U}(P) = \sum_{k=1}^{n} P(A_k)\bar{U}(P_{A_k}) + \bar{U}\left(P^\phi\right),$$

where the conditional probability measures P_{A_k} are given by the expression $P_{A_k}(C) = P(C \cap A_k)/P(A_k)$, $C \subseteq \Omega$.

Proof. Properties (1)–(3) directly follows from axioms 1–4. Let us show that property (4) follows from axiom 6. Let $P \in M_{pr}$, and let the notation of Corollary 2 be used. Let us choose a function $f : \Omega \to \mathbb{Z}_+$ so that $p(i) = h_f(i)/N$, where $h_f(A) = |\Omega|$. We write the equality from axiom 6 using the functional $\bar{U}$

$$\sum_{k=1}^{n} h_f(A_k)\bar{U}(f_k) + h_f(A)\bar{U}(f \circ \phi) = h_f(A)\bar{U}(f),$$

where $h_f(A) = \sum_{i \in A_k} h_f(i)$. Note that $P(A_k) = h_f(A_k)/h_f(A)$ and the probability measures P_{A_k}, P^ϕ correspond to the images $f_k, f \circ \phi$. Thus we arrive at the conclusion that property (4) also holds.

The properties listed in Corollary 2 are well-known properties of the Shannon entropy. For example, property (2) accumulates properties of symmetry and extension. Property (4) is an additivity property, which is formulated as follows. Let ξ be a random variable with values in $\mathbb{Z}_+$ and $\eta = \phi(\xi)$. Then for the Shannon entropy S, we have $S(\xi, \eta) = S(\xi) = S(\xi|\eta) + S(\eta)$. In this case, $S(\xi|\eta) = S(\xi)$ since values of η completely depend on values of ξ.

It can easily be shown that properties (2) and (3) can be obtained from properties (1) and (4). Indeed if in property (4) we take the bijection $\phi(i) = i$ as ϕ for all $i \in \mathbb{Z}_+$, then $\bar{U}(P) = \bar{U}\left(P^\phi\right)$, i.e.,

$$\sum_{k=1}^{n} P(A_k)\bar{U}(P_{A_k}) = 0.$$

This equality is possible by property (1) only in the case when $\bar{U}(P_{A_k}) = 0$ for all k. Therefore it remains to notice that in this case the probability measures P_{A_k} are the Dirac measures, i.e., they are such as in property (2). Thus property (2) follows from properties (1) and (4).

Assume that the mapping $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ is injective. Then applying the formula from property (4), we find out that

$$\sum_{k=1}^{n} P(A_k)\bar{U}(P_{A_k}) = 0,$$

since all probability measures P_{A_k} are Dirac measures; i.e., property (3) also follows from properties (1)–(4).

P r o p o s i t i o n 2. Let the functional $\bar{U}$ on M_{pr} satisfy the properties listed in Corollary 2. Then $\bar{U}$ is the Shannon entropy; i.e.,

$$\bar{U}(P) = -c\sum_{i \in A} p(i)\ln p(i), \tag{2.1}$$

where $P \in M_{pr}$, $A = \{i \in \mathbb{Z}_+ | p(i) > 0\}$, and $c \geq 0$.

P r o o f. Usually, when numbers $p(i)$ are real, the uniqueness of the Shannon entropy is proved by the continuity axiom. In the proposed proof, the continuity axiom is unnecessary. First we find out what the values of the functional $\bar{U}$ are on the probability measures P_n, for which $p_n(i) = 1/n$ if $i \in \{1,...,n\}$, and $p_n(i) = 0$ otherwise. Let us introduce the function $u(n) = \bar{U}(P_n)$. Assume that $\{A_1,... A_m\}$ is a partition of the set $\{1,...,km\}$, where k and m are natural numbers possessing the property $|A_i| = k$, $i = 1, \ldots, m$. Consider the mapping $\phi(1, \ldots, km) \to \{1,...,m\}$ such that $\phi(i) = j$ if $i \in A_j$. Applying the formula from property (4) of Corollary 2 to the probability measure P_{km}, we find that

$$\bar{U}(P_{km}) = \sum_{i=1}^{m} P_{km}(A_i)\bar{U}(P_A) + \bar{U}(P_m).$$

It is clear that $P_{km}(A_i) = 1/m$, $\bar{U}(P_{A_i}) = \bar{U}(P_k)$; therefore $\bar{U}(P_{km}) = \bar{U}(P_k) + \bar{U}(P_m)$ or $u(km) = u(k) + u(m)$. It is known from the theory of functional equations [14] that if a function $u : \mathbb{N} \to [0,+\infty)$ has the properties $u(1) = 0$ and $u(km) = u(k) + u(m)$ for any k, $m \in \mathbb{N}$, then it can be represented in the form $u(n) = c\ln n$, where $c \geq 0$ and $n \in \mathbb{N}$. Therefore $\bar{U}(P_n) = c\ln n$, $n \in \mathbb{N}$.

Then we consider the probability measure $P \in M_{pr}$ for which $p(i) = k_i/N$, where $k_i \in \mathbb{N}$, $i = 1, \ldots, m$, $\sum_{i=1}^{m} k_i = N$. Let P_N be a probability measure and let the partition $\{A_1,..., A_m\}$ of the set $\{1,..., N\}$ be such that $|A_i| = k_i$, $i = 1,...,m$. We assume that the mapping $\phi : \{1,..., N\} \to \{1,...,m\}$ is given by $\phi(i) = j$ if $j = A_j$. Then using the formula of property (4) of the Corollary 2 for the probability measure P_N, we have

$$\bar{U}(P_N) = \sum_{i=1}^{m} P_N(A_i)\bar{U}(P_{A_i}) + \bar{U}(P).$$

Note that in this formula $\bar{U}(P_N) = c\ln N$, $P_N(A_i) = k_i/N$, $\bar{U}(P_{A_i}) = c\ln k_i$, $i = 1,...,m$. Therefore we have

$$\bar{U}(P) = c\ln N - c\sum_{i=1}^{m} p(i)\ln k_i = c\sum_{i=1}^{m} p(i)\ln N -$$

$$- c\sum_{i=1}^{m} p(i)\ln k_i = -c\sum_{i \in A} p(i)\ln p(i).$$

We can conditionally assume that $p(i)\ln p(i) = 0$ if $p(i) = 0$. Then the expression for information takes the form

$$\bar{U}(P) = -c\sum_{i \in \mathbb{Z}_+} p(i)\ln p(i).$$

It is clear that for an arbitrary image $f : \Omega \to \mathbb{Z}$, the information measure is

$$U(f) = -cN\sum_{i \in \mathbb{Z}_+} p_f(i)\ln p_f(i),$$

where $N = |\Omega|$ and $p_f(i) = h_f(i)/N$. The information measure is defined uniquely by the normalization condition. For example, if we assume that the most informative image of n pixels has the information value equal to 1, then $c = 1/(n\ln n)$.

Note that there is the following probability interpretation of the mean information $\bar{U}(f)$ of a pixel of the image $f : \Omega \to \mathbb{Z}_+$. Let P be a probability measure on the algebra of all subsets of Ω given by the equality $P(A) = |A|/|\Omega|$. Then the mapping $f : \Omega \to \mathbb{Z}_+$ can be considered as a random variable, and obviously the mean information of the image is the Shannon entropy $\bar{U}(f) = S(f)$.

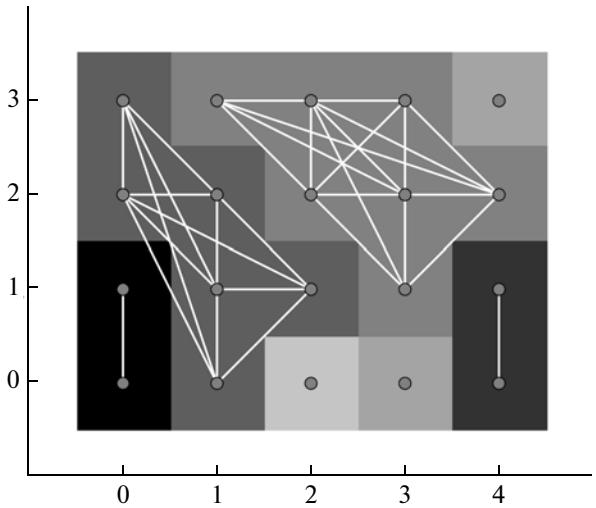

Fig. 3. Graph of the equivalence relation θ of the model image. Equivalence classes v_i correspond to connected components of the graph.

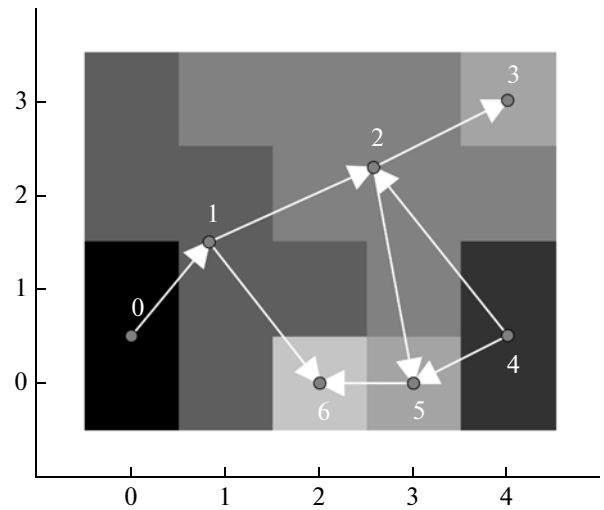

Fig. 4, Graph of extension of the relation τ to the equivalence classes of the model image.

Then we introduce a vector random variable $\xi = (\xi_1, \ldots, \xi_N)$, where $N = |\Omega|$ and random variables are independent and have the same probability distribution as the variable f. Then

$$S(\xi) = \sum_{i=1}^{N} S(\xi_i) = N\bar{U}(f) = U(f).$$

Note also that the entropy is widely applied in estimating the information amount of an image in the coding theory [15], and to determine the entropy, the probabilities of appearance of an image with fixed intensity values at each pixel are considered. The estimation or simulation of these probabilities is a quite complex problem, as a result of this, to calculate entropy, as a rule, we should pass to analysis of conditional probabilities and make the assumption that the image is a Markovian process of order no more than k, simplifying in this way the procedure of estimating the conditional probabilities. In the proposed approach, we take into account the probability of a pixel separately taken, rather than the whole image, and, based on the introduced axioms, prove that the information measure obtained as a result has the properties of the Shannon entropy.

3. INFORMATION AND UNCERTAINTY MEASURES FOR A SIGN-BASED REPRESENTATION

Let us discuss the problem of measuring the uncertainty of information about the intensity grades of the image if we only know its sign-based representation. We describe a sign-based representation using quasi-order relations on Ω, i.e., reflexive and transitive relations. The graph of this relation is obtained as a transitive closure of the graph $\bar{G}_\tau$. In what follows, we do not take into account the method of generating this relation. Only the fact that a class of images $\mathcal{F}_\tau \subseteq \mathcal{F}$ corresponds to the sign-based representation τ is the only important for us. Thus it is necessary to construct a functional on the set of quasi-order relations that measure the uncertainty of image description quantitatively using the sign-based representation. We denote by $\hat{U}$ this functional and by $\mathcal{T}$, the set of all quasi-orders on the set Ω.

As above, this functional is defined in terms of a set of its desired properties; however the functional $\hat{U}$ should interact in a certain way with the functional U on $\mathcal{F}$ characterizing the information amount of the image. Note that each sign-based representation also has its information amount, which is measured using the functional, also denoted by U. The interaction of the given functional is described by the following axiom.

A x i o m 7. Let $\tau \in \mathcal{T}$, $U_{\max}(\tau) = \sup\{U(f) | f \in \mathcal{F}_\tau\}$, then

$$U(\tau) + \hat{U}(\tau) = U_{\max}(\tau). \tag{3.1}$$

We stress that axiom 7 expresses the known principle in information theory proposed by G.J. Klir [16], according to which the property of being informative and uncertainty are connected with each other and their sum is a certain constant value. Note that the right side of formula (3.1) corresponds to information of the most informative image with the sign-based representation τ. Therefore, we deduce from formula (3.1) that $\hat{U}(\tau) = U_{\max}(\tau) - U(\tau)$. Thus the uncertainty amount of the sign-based representation τ of the image f is the difference of the information amount of the image and the information amount of its sign-based representation, and the image f is chosen from the principle of maximum of uncertainty.

We accept that the images $f_1, f_2 \in \mathcal{F}$ are equivalent if the there is a monotonously increasing bijection $\varphi : f_1(\Omega) \to f_2(\Omega)$ such that $f_2 = \varphi \circ f_1$. Assume that equivalent images contain the same information. Then the following axiom has to hold.

A x i o m 8. $\hat{U}(\tau) = 0$ if the relation $\tau \in \mathcal{T}$ is connected; i.e., any two elements $\omega_1, \omega_2 \in \Omega$ are comparable with each other.

If the relation $\tau \in \mathcal{T}$ is connected, then the class $f \in \mathcal{F}_\tau$ consists of images that are equivalent, and by the assumption equivalent images contain the same information. Therefore in this case, the sign-based representation τ preserves all necessary information about the image. Therefore, we can assume that $\hat{U}(\tau) = 0$. Note that $U(\tau) = U_{\max}(\tau)$ according to axiom 7.

A x i o m 9. Let $\tau_1 \subseteq \tau_2$ for $\tau_1, \tau_2 \in \mathcal{T}$, then $\hat{U}(\tau_1) \geq \hat{U}(\tau_2)$.

In the case when $\tau_1 \subseteq \tau_2$, we have more information describing the image with the help of the sign-based representation τ_2 compared with the sign-based representation τ_1. Therefore axiom 9 should hold.

A x i o m 10. Let $G_\tau = (\Omega, \tau)$ be the graph of a sign-based representation $\tau \in \mathcal{T}$, and let the sets $\Omega_1, \ldots, \Omega_m$ determine the connected components of the graph G_τ. Then $\sum_{k=1}^{m} U(\tau_{\Omega_k}) = U(\tau)$, where $\tau_{\Omega_k} = \tau \cap \Omega_k \times \Omega_k$ is a restriction of the relation τ to the set Ω_k, $k = 1, \ldots, m$.

The meaning of Axiom 10 is that the connected components of the graph G_τ are fragments of independent information; therefore the information amount of the whole representation has to be equal to the sum of information amounts of the given independent components.

The next problem is to study theoretically the properties of the functional U, $\hat{U}$, $U_{\max}$ on $\mathcal{T}$ and to consider the methods of determining U and $\hat{U}$. Let $\tau \in \mathcal{T}$, then the relation $\theta = \tau \cap \tau^{-1}$ is an equivalence relation. Figure 3 presents the graph of the equivalence relation θ for a model image. Let $V = \{v_1, \ldots, v_n\}$ be the set of all equivalence classes defined by the relation θ on Ω, and let τ^θ be the extension of the relation to the equivalence classes. We assume that $(v_i, v_j) \in \tau^\theta$ if there is a pair $(\omega_l, \omega_k) \in \tau$ such that $\omega_l \in v_i$ and $\omega_k \in v_j$. Figure 4 presents the graph of the relation τ^θ for the model image. It is known that the relation τ^θ on V obtained in this way is reflexive, antisymmetric, and transitive relation; i.e, a partial order relation, and we can always construct a relation of non-strict linear order ρ so that $\rho \supseteq \tau^\theta$. Then a class of images in which all v_i have different intensity grades correspond to the relation ρ. This implies the following proposition.

P r o p o s i t i o n 3. Let $\tau \in \mathcal{T}$, $\theta = \tau \cap \tau^{-1}$, and τ^θ be the extension of the relation τ on the set V of the equivalence classes generated by θ, then

$$U_{\max}(\tau) = -cN \sum_{i=1}^{n} p(i) \ln p(i), \tag{3.2}$$

where $p(i) = |v_i|/N$ and $N = |\Omega|$.

Definition 1 implies that any quasi-order relation is a sign-based representation of an image, in particular, the quasi-order relation $\tau = \tau \cap \tau^{-1} = \theta$. Proposition 3 implies Corollary 3.

C o r o l l a r y 3. Let $\theta = \tau$, i.e., τ is the equivalence relation. Then

$$\hat{U}(\tau) = -cN \sum_{i=1}^{n} p(i) \ln p(i),$$

where $p(i) = |v_i|/N$ and $N = |\Omega|$.

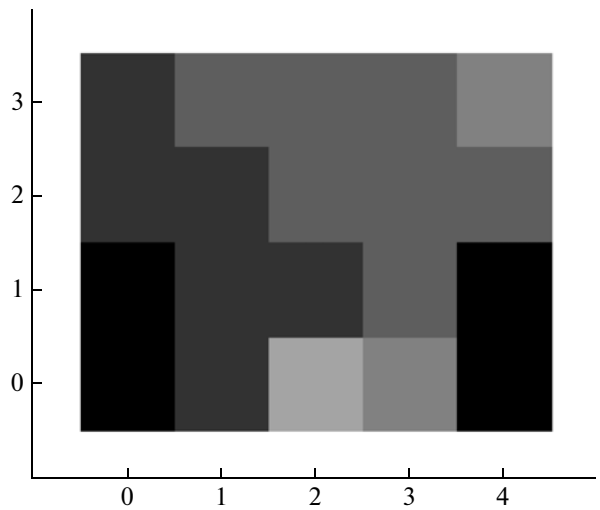

Fig. 5. Example of restored image based on the sign-based representation of the model image (Fig. 1).

P r o o f. In this case the sets $v_1, \ldots, v_n$ are connected components of the graph G_τ, and $\sum_{k=1}^{n} U(\tau_{v_k}) = U(\tau)$ by axiom 9. Since τ_{v_k} are connected relations, $U(\tau_{v_k}) = 0$, i.e., $U(\tau) = 0$ and $\hat{U}(\tau) = U_{\max}(\tau)$. This implies the required proof.

C o r o l l a r y 4. Assume that G_τ is the graph of a relation $\tau \in \mathcal{T}$ and its connected components are determined by the sets $\Omega_1, \ldots, \Omega_m$, and τ_{Ω_i}, $i = 1, \ldots, m$ are connected relations. Then

$$\hat{U}(\tau) = -cN\sum_{i=1}^{m} p(i)\ln p(i),$$

where $p(i) = |\Omega_i|/N$ and $N = |\Omega|$.

P r o o f. By the definition of $U_{\max}(\tau)$, there exists $f \in \mathcal{F}_\tau$ such that $U_{\max}(\tau) = U(f)$. Since each relation τ_{Ω_i} is connected, any two functions from the class $\mathcal{F}_{\tau_{\Omega_i}}$ are equivalent. This means that $U_{\max}(\tau_{\Omega_i}) = U_{\max}(f_i)$, where f_i is the restriction of the function f to the set Ω_i. Thus by axioms 7–10, we have

$$U(\tau) = \sum_{i=1}^{m} U(\tau_{\Omega_i}) = \sum_{i=1}^{m} U(f_i) \text{ and}$$

$$\hat{U}(\tau) = U_{\max}(\tau) - U(\tau) = U(f) - \sum_{i=1}^{m} U(f_i).$$

It is easy to test that the partition $V = \{v_1, \ldots, v_n\}$ is smaller than the partition $\{\Omega_1, \ldots, \Omega_m\}$; i.e., each set Ω_k can be represented as a union of sets v_i. Therefore we can choose the mapping $\phi : f(\Omega) \to \{1, \ldots, m\}$ so that $\Omega_k = \{\omega \in \Omega | \phi(f(\omega)) = k\}$. This is satisfied if $\phi(f(v_i)) = k$ for $v_i \subseteq \Omega_k$. Then by axiom 6 (additivity of the information measure) for the information measure U on $\mathcal{F}$, we obtain

$$\hat{U}(\tau) = U(f) - \sum_{i=1}^{m} U(f_i) = U(\phi \circ f).$$

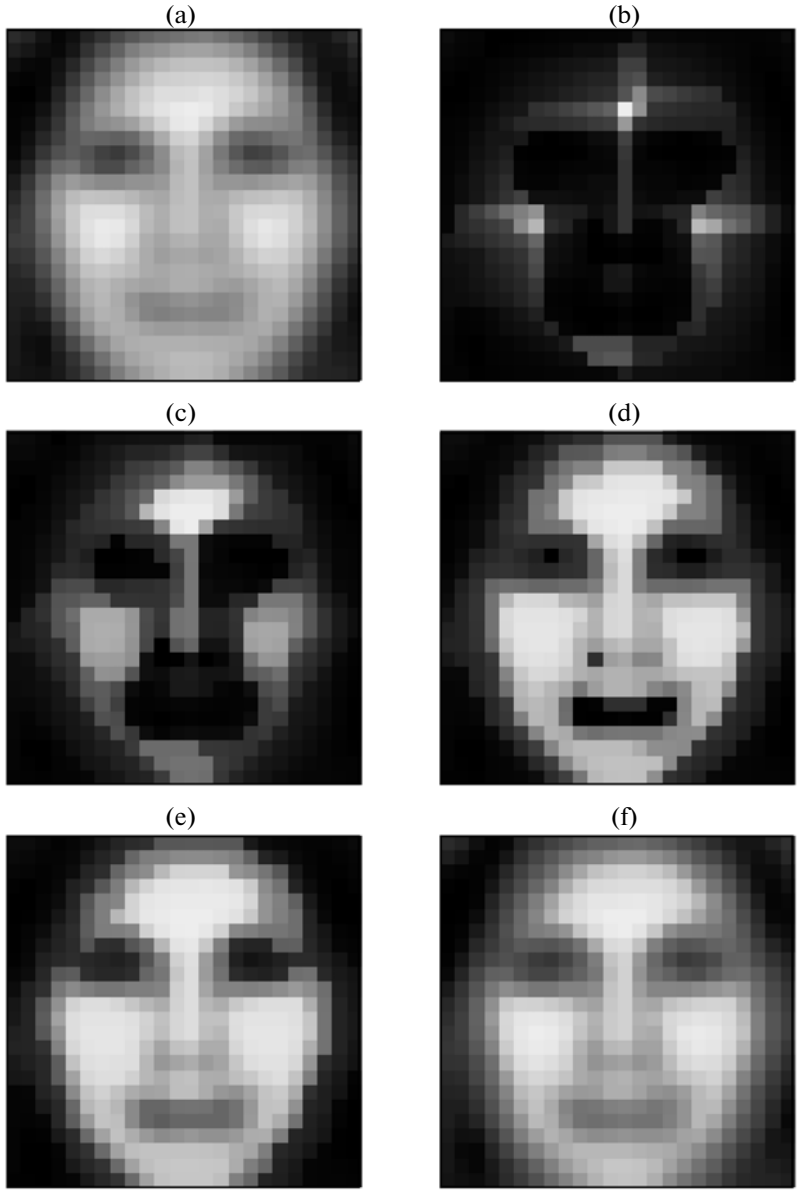

Fig. 6. Example of restoration of the image f based on neighborhood sign-based representations $\tau_f(\varepsilon)$: (a) for original image; (b)–(e) for images restored based on neighborhood sign-based representations $\tau_f(\varepsilon)$ with different parameters of the neighborhood ε ((b) $\varepsilon = 1$, (c) $\varepsilon = 2$, (d) $\varepsilon = 4$, (e) $\varepsilon = 6$); (f) for the image restored based on complete sign-based representation.

Note that the image $\phi \circ f$ is defined by the equality $\phi(f(\omega)) = k$ if $\omega \in \Omega_k$ and its information can be calculated by the formula

$$U(\phi \circ f) = -cN\sum_{i=1}^{m} p(i)\ln p(i),$$

where $p(i) = |\Omega_i|/N$ and $N = |\Omega|$.

R e m a r k 2. The result formulating in Corollary 3 can be represented in a simpler form. It is easy to prove that if the condition of Corollary 4 holds, then the relation $\tau \cup \tau^{-1}$ is an equivalence relation, and the partition $\{\Omega_1, \ldots \Omega_m\}$ is connected with this relation and $\hat{U}\left(\tau \cup \tau^{-1}\right) = \hat{U}(\tau)$ by Corollary 3.

P r o p o s i t i o n 4. Let $\tau \in \mathcal{T}$, and let $\alpha \subseteq \tau \cup \tau^{-1}$ be an equivalence relation. Then $\hat{U}(\tau) \le \hat{U}(\alpha)$.

P r o o f. Let $\tau_1 = \alpha \cap \tau$. It is clear that it is a reflexive, antisymmetric, and transitive relation, i.e., $\tau_1 \in \mathcal{T}$. Then we recall that $\tau_1 \cup \tau_1^{-1} = (\alpha \cap \tau) \cup (\alpha \cap \tau)^{-1} = \alpha \cap (\tau \cup \tau^{-1}) = \alpha$. Therefore, taking into account Remark 2, we have $\hat{U}(\tau_1) = \hat{U}(\alpha)$. Since $\tau_1 \subseteq \tau$ and by axiom 9, we have $\hat{U}(\tau) \le \hat{U}(\tau_1)$, i.e., $\hat{U}(\tau) \le \hat{U}(\alpha)$.

Proposition 4 makes it possible to introduce the upper estimate $\hat{U}_{up}$ of the uncertainty measure $\hat{U}$. Let $Eq(\tau \cup \tau^{-1})$ be a set of all equivalence relations which are involved in the relation $\tau \cup \tau^{-1}$. Then the functional $\hat{U}_{up}$ is defined as follows:

$$\hat{U}_{up}(\tau) = \min\left\{\hat{U}(\alpha) | \alpha \in Eq\left(\tau \cup \tau^{-1}\right)\right\}.$$

by Proposition 4, $\hat{U}(\tau) \le \hat{U}_{up}(\tau)$ for all $\tau \in \mathcal{T}$.

P r o p o s i t i o n 5. The functional $\hat{U}_{up}$ as an uncertainty measure of the sign-based representation and the functional $U = U_{\max} - \hat{U}_{up}$ as an information measure on the set of sign-based representations $\mathcal{T}$ satisfy axioms 7–10.

P r o o f. It is necessary to prove that axioms 7–10 are satisfied. Let $\tau \in \mathcal{T}$ and $\hat{U}_{up}(\tau) = \hat{U}(\alpha)$. It is clear that $\hat{U}_{\max}(\tau) = \hat{U}(\tau \cap \tau^{-1})$ and $\tau \cap \tau^{-1} \in Eq(\tau \cup \tau^{-1})$. Therefore $U_{\max}(\tau) \ge \hat{U}_{up}(\tau)$. Then we can determine the information amount of the sign-based representation by $U(\tau) = U_{\max}(\tau) - \hat{U}_{up}(\tau)$; i.e., Axiom 7 holds. Axiom 8 also holds since $\tau \cup \tau^{-1}$ is an equivalence relation determining the trivial partition $\{\Omega\}$, and obviously $\hat{U}_{up}(\tau) = \hat{U}\left(\tau \cup \tau^{-1}\right) = 0$. Axiom 9 takes place since $Eq\left(\tau_1 \cup \tau_1^{-1}\right) \subseteq Eq\left(\tau_2 \cup \tau_2^{-1}\right)$ if $\tau_1 \subseteq \tau_2$ and $\tau_1, \tau_2 \in \mathcal{T}$.

Let us prove that Axiom 10 holds. Let $G_\tau = (\Omega, \tau)$ be a graph of sign-based representation $\tau \in \mathcal{T}$ and let the sets $\Omega_1, \ldots, \Omega_m$ determine connected components of the graph G_τ. Assume that $\hat{U}_{up}(\tau) = \hat{U}(\alpha)$. Since the optimization problem of finding α in this case is decomposed into m independent optimization problems for each set Ω_i, we can assume that $\hat{U}_{up}\left(\tau_{\Omega_i}\right) = \hat{U}\left(\alpha_{\Omega_i}\right) = U_{\max}\left(\alpha_{\Omega_i}\right)$. Taking this into account, we have $U\left(\tau_{\Omega_i}\right) = U_{\max}\left(\tau_{\Omega_i}\right) - U_{\max}\left(\alpha_{\Omega_i}\right)$, $i = 1, \ldots, m$, and $U(\tau) = U_{\max}(\tau) - U_{\max}(\alpha)$. We should show that

$$\sum_{i=1}^{m} U\left(\tau_{\Omega_i}\right) = U(\tau)$$

or

$$\sum_{i=1}^{m} \left(U_{\max}\left(\tau_{\Omega_i}\right) - U_{\max}\left(\alpha_{\Omega_i}\right)\right) = U_{\max}(\tau) - U_{\max}(\alpha). \tag{3.3}$$

Equality (3.3) is transformed to the form

$$U_{\max}(\alpha) - \sum_{i=1}^{m} U_{\max}\left(\alpha_{\Omega_i}\right) = U_{\max}(\tau) - \sum_{i=1}^{m} U_{\max}\left(\tau_{\Omega_i}\right). \tag{3.4}$$

We transform the right and left sides of this equality. By definition, there is a function $f \in \mathcal{F}_\tau$ such that $U(f) = U_{\max}(\tau)$. Note that the choice of f is reduced to solving independent optimization problems on the sets Ω_i; therefore $U(f_i) = U_{\max}\left(\tau_{\Omega_i}\right)$, where f_i is the restriction of the function f to the set Ω_i, $i = 1, \ldots, m$. Thus the right side of (3.4) takes the form

$$U_{\max}(\tau) - \sum_{i=1}^{m} U_{\max}\left(\tau_{\Omega_i}\right) = U(f) - \sum_{i=1}^{m} U(f_i).$$

In accordance with the proposition, the function f has different values on the partition $V = \{v_1, \ldots, v_n\}$, which is induced by the equivalence relation $\tau \cap \tau^{-1}$. Since the partition V is smaller than the partition

Results of estimating the introduced information and uncertainty measures by an example of a facial image, $c = (NM)^{-1}$

ε	$U(f)$	$U_{\max}(\tau_f)$	$\hat{U}_{up}(\tau_f)$
1	4.683	6.264	5.230
2	4.683	6.151	4.232
4	4.683	5.919	2.988
6	4.683	5.662	2.184
∞	4.683	4.683	0.0

$\{\Omega_1,\ldots,\Omega_m\}$, we can choose the function $\varphi : f(\Omega) \to \{1,\ldots,m\}$ so that $\Omega_k = \{\omega \in \Omega | \varphi(f(\omega)) = k\}$. This is the case when $\varphi(f(v_i)) = k$ for $v_i \subseteq \Omega_k$. Then by Axiom 6 for the information measure U on $\mathcal{F}$, we obtain

$$U(f) - \sum_{i=1}^{m} U(f_i) = U(\varphi \circ f).$$

Here the function $g = \varphi \circ f$ is calculated by the rule $g(\omega) = k$ if $\omega \in \Omega_k$. Proceeding in this way, we can find out that

$$U_{\max}(\alpha) - \sum_{i=1}^{m} U_{\max}(\alpha_{\Omega_i}) = U(g),$$

i.e., Equality (3.4) holds and Axiom 10 also holds.

Thus the proved proposition allows us to use the functional $\hat{U}_{up}$ as the uncertainty measure of a sign-based representation. In addition Proposition 5 shows that axioms 7–10 are not contradictive.

The calculation of the value $U(f)$, $f \in \mathcal{F}$ and $U_{\max}(\tau)$, $\tau \in \mathcal{T}$, is not difficult, while the calculation of values of $\hat{U}_{up}(\tau)$, $\tau \in \mathcal{T}$, is a rather cumbersome problem. Consider the calculation of $U(f)$, $U_{\max}(\tau_f)$, and $\hat{U}_{up}(\tau_f)$, by the example of a model image. For finding $U(f)$, it is necessary to calculate the number of pixels in the image for each intensity level. Figure 2 presents the histogram of the model image, which contains six intensity levels with the frequencies 2/20, 2/20, 6/20, 7/20, 2/20, 1/20. Substituting the values of frequencies in expression (2.1) and taking into account the fact that $U(f) = N\bar{U}(f)$, we have $U(f) = 31.38c$, where c is the coefficient in expansion (3.2).

The calculation of maximum information amount of a sign-based representation is based on the analysis of equivalence classes of the relation $\tau \cap \tau^{-1}$. Figure 3 presents the graph of equivalence relation for the model image, consisting of seven connected components, containing 2, 6, 7, 1, 2, 1, and 1 elements, respectively. Substituting these values into expression (3.2), we obtain $U_{\max}(\tau_f) = 32.77c$. In this case, $U(f) < U_{\max}(\tau_f)$, since the same intensity values in the image correspond to different equivalence classes of the relation $\tau \cap \tau^{-1}$. Figure 4 shows the graph of extension of the relation τ to the equivalence classes v_i generated by the equivalence relation $\tau \cap \tau^{-1}$.

Figure 5 shows the image obtained by the restoration algorithm, which is a modified procedure of generating vertices of the graph (see the proof of Proposition 1 and Remark 1), in which at each step the next vertex is chosen based on the criterion of minimum information amount. For this purpose, we find in the graph $G^{(i)}$ a path with the greatest cost $\sum_{k=1}^{l} |v_{i_k}|$. We construct the graph $G^{(i+1)}$ by eliminating vertices $(v_{i_1},\ldots,v_{i_l})$ from the graph $G^{(i)}$. Let us generate the set Ω^{i+1} as follows:

$$\Omega_{i+1} = \bigcup_{k=1}^{l} v_{i_k}.$$

The result of operation of this algorithm is the partition $\{\Omega_1, ..., \Omega_m\}$ of the set of pixels Ω. Let us calculate the upper estimate $\hat{U}(\tau)$ by the following formula:

$$\hat{U}_{up}(\tau) \le -cN\sum_{i=1}^{m} p(i)\ln p(i), \tag{3.5}$$

where $p(i) = |\Omega_i|/N$ and $N = |\Omega|$. In this example, the path in the graph $G^{(0)}$ with the greatest cost is $\Omega_1 = \{0, 1, 2, 5, 6\}$, $|\Omega_1| = 17$. Let us construct the graph $G^{(1)}$ by eliminating from the graph $G^{(0)}$ the vertices of Ω_1; finally we obtain the graph consisting of the components $\Omega_2 = \{4\}$, $|\Omega_2| = 2$, $\Omega_3 = \{3\}$, $|\Omega_3| = 1$. Thus in accordance with (3.5), the value of the maximum uncertainty measure of the sign-based representation is $\hat{U}_{up}(\tau_f) \le 10.36c$. In other words, in the transition from the initial image to the corresponding sign-based representation, a significant part of information contained in the image is preserved. The information loss can easily be seen in the image restored based on the sign-based representation using the minimum information principle. Figure 6 shows that the restored image contains a minimum number of grades required for saving the order relation on adjacent pixels (see for comparison Fig. 4, in which it is clear that equivalence classes 0 and 4 have different intensities).

Figure 6 presents the results of restoring a facial image based on neighborhood sign-based representations constructed based on the initial image, as well as for different values of the parameter ε in expression (1.3). The table shows the results of calculating the functional $U(f)$, $U_{\max}(\tau_f)$, and $\hat{U}_{up}(\tau_f)$ for a facial image (Fig. 6) and the neighborhood sign-based representations corresponding to it. As can be seen from the table, with the growth of the size of the neighborhood within which relations on pixels in the neighborhood sign-based representation are considered, the upper estimate of the uncertainty measure reduces and becomes zero for the complete sign-based representation. In addition, the maximum information measure of the sign-based representation also reduces, and for the complete sign-based representation it coincides with the image information amount.

CONCLUSIONS

In this paper, a sign-based image representation was introduced and its most important properties were investigated. In view of the fact that a set of images correspond to a single sign-based representation, there arises the problem of choosing an image that characterizes the class of equivalent images in the best way. As the criterion of choice of the best image, we propose to use the minimum information principle, whose application is based on information and uncertainty measures of sign-based image representations. The axiomatic approach to information and uncertainty measures developed in the paper results in measures possessing the well-known properties of the Shannon entropy. In the concluding part of the paper, an example of calculating the information and uncertainty measures of sign-based representations was considered.

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