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The Erdős–Vershik Problem for the Golden Ratio*

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Abstract. Properties of the Erdős measure and the invariant Erdős measure for the golden ratio and all values of the Bernoulli parameter are studied. It is proved that a shift on the two-sided Fibonacci compact set with invariant Erdős measure is isomorphic to the integral automorphism for a Bernoulli shift with countable alphabet. An effective algorithm for calculating the entropy of an invariant Erdős measure is proposed. It is shown that, for certain values of the Bernoulli parameter, this algorithm gives the Hausdorff dimension of an Erdős measure to 15 decimal places.

Key words: hidden Markov chain, Erdős measure, invariant Erdős measure, golden shift, integral automorphism, entropy, Hausdorff dimension of a measure.

Almost seventy years ago Erdős posed the following problem: What distribution function the random variable $\zeta = \zeta_1\rho + \zeta_2\rho^2 + \dots$, where $\zeta_1, \zeta_2, \dots$ are independent identically distributed random variables taking the values 0 and 1 with $P(\zeta_i = 0) = 1/2$ and $0 < \rho < 1$, can have?

We refer to the distribution of such a random variable ζ as an Erdős measure on the real line.

The problem of Erdős has been the subject of many papers. The authors of [3] defined an Erdős measure on the unit interval $[0, 1]$ and on the Fibonacci compactum, as well as an invariant Erdős measure on the Fibonacci compactum for the case $\rho = (\sqrt{5} - 1)/2$ (the reciprocal of the golden ratio $\beta = (\sqrt{5}+1)/2$). It was also proved in [3] that an Erdős measure is equivalent to an invariant Erdős measure on the Fibonacci compactum.

Vershik posed the problem about the ergodic properties of an invariant Erdős measure on the Fibonacci compactum. This problem was solved in [3].

In the previous paper [1], we discovered a connection between the Erdős–Vershik problem and the class of hidden Markov chains for the more general case of $0 < P(\zeta_i = 0) = q < 1$, $P(\zeta_i = 1) = p$, and $\rho = (\sqrt{5} - 1)/2$. In what follows we consider this case.

With the help of this connection, we shall prove an analog of one of the main results of [3]. Namely, we shall prove that a shift on the two-sided Fibonacci compactum with invariant Erdős measure is isomorphic to the integral automorphism for a Bernoulli shift with countable alphabet. We also obtain a formula for the entropy of an invariant Erdős measure.

The ratio of the entropy of an invariant Erdős measure to $\ln\beta$ is the Hausdorff dimension of this invariant Erdős measure on the Fibonacci compactum with metric $d(x, y) = \rho^{n(x,y)}$, where $n(x, y)$ is the length of the longest common prefix of the words x and y. This dimension is equal to the Hausdorff dimension of the corresponding Erdős measure on the real line. Recall that the Hausdorff dimension of a probability measure is the infimum of the Hausdorff dimensions of all sets of measure 1. The above statement follows from the equivalence of an Erdős measure and the corresponding ergodic invariant Erdős measure. (See [2], where a method for calculating the Hausdorff dimension of ergodic measures for symbolic dynamical systems and maps of the unit interval is described.)

A formula for the Hausdorff dimension of an Erdős measure on the real line was obtained by Feng in [6, Theorem 4.29]. Our formula coincides with Feng's formula. Thus, we have given a new derivation of Feng's formula. Hausdorff dimension cannot be calculated directly by Feng's formula because of the slow convergence of the series in this formula.

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In the case under consideration, Lalley [5] obtained yet another formula for the Hausdorff dimension of an Erdős measure on the real line. Using this formula and the Monte Carlo method, he obtained confidence intervals for the Hausdorff dimension of the Erdős measure at various values of p.

For the same values of p as in [5], we calculate the Hausdorff dimension of the Erdős measure to more decimal places, which, in particular, gives an idea of the accuracy of Lalley's statistical estimates. In our calculations we use the acceleration of convergence of the series in the formula for Hausdorff dimension. This acceleration is similar to that applied by Alexander and Zagier in [4].

Note that Lalley's calculations can be regarded as a calculation of the Lyapunov exponent for some sequence of random matrices (see [5]). The same can be said about our calculations, but we use a different sequence of random matrices.

1. An Invariant Erdős Measure on the Fibonacci Compactum

Below we give the definition of an invariant Erdős measure on the Fibonacci compactum borrowed from [1]. In [1], the problem about the ergodic properties of an invariant Erdős measure was reduced to the study of the hidden Markov chain $\{\eta_i = f(\xi_i)\}$ generated by a Markov chain $\{\xi_i\}$ with 5 states 1, 2, 3, 4, and 5 and transition matrix P of the form

$$P = \begin{pmatrix} q & 0 & 0 & pq & p^2 \\ q & 0 & qp & 0 & p^2 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The initial distribution l is the stationary distribution, and the gluing function f is 0 for states 1, 2, and 3 and is 1 for states 4 and 5. The hidden Markov chain generates a probability distribution μ on its space of realizations. It is convenient to regard μ as the distribution of the infinite random binary word $\eta_1\eta_2\cdots\eta_n\cdots = f(\xi_1)f(\xi_2)\cdots f(\xi_n)\cdots$. Its support is the Fibonacci compactum consisting of all infinite binary Fibonacci words without subwords 11. This set is compact with respect to the metric $d(x,y) = \rho^{n(x,y)}$, where $n(x,y)$ is the length of the longest common prefix of the words x and y. The measure μ is an invariant Erdős measure on the Fibonacci compactum [1].

In the matrix P we take the blocks $P(00)$, $P(01)$, and $P(10)$ corresponding to the partition of the set $\{1,2,3,4,5\}$ into the two subsets $\{1,2,3\}$ and $\{4,5\}$:

$$P(00) = \begin{pmatrix} q & 0 & 0 \\ q & 0 & qp \\ 0 & 1 & 0 \end{pmatrix}, \qquad P(01) = \begin{pmatrix} pq & p^2 \\ 0 & p^2 \\ 0 & 0 \end{pmatrix}, \qquad P(10) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $l(0)$ denote the row whose entries are the first three entries of the row l. (Recall that l is the stationary distribution of the Markov chain with transition matrix P.) By $r(0)$ we denote the column $(1,1,1)^\top$ and by $r(1)$, the column $(1,1)^\top$.

Let $n \geqslant 2$, and let $a = a_1, \dots, a_n$ be a finite Fibonacci word. We set

$$P(a) = P(a_1a_2)\cdots P(a_{n-1}a_n).$$

Then

$$\mu(\{x : x_1\cdots x_n = a\}) = \mu(a) = l(a_1)P(a)r(a_n), \qquad \mu(a_1) = l(a_1)r(a_1).$$

Let $\widetilde{X}$ be the two-sided Fibonacci compactum consisting of all infinite two-sided binary Fibonacci words without subwords 11 and with fixed first position. Let T be a shift on the space $\widetilde{X}$. Consider the measure $\tilde{\mu}$ on the space $\widetilde{X}$ defined by

$$\tilde{\mu}(\{x : x_{1+j}x_{2+j}\cdots x_{n+j} = a_1\cdots a_n = a\}) = \mu(a) \qquad \text{for any} \quad j \in \mathbb{Z} \quad \text{and} \quad n \geqslant 1.$$

The measure $\tilde{\mu}$ is an *invariant* Erdős measure on the space $\widetilde{X}$.

2. The Golden Shift

A finite Fibonacci word is called an elementary word if it has the form $1(0)^{k+1}$, where $k = 0, 1, 2, \dots$. An elementary word is said to be *even* if k is even and *odd* if k is odd.

In [3], the subset of regular words was introduced. The definition of a regular word given in [3] is as follows. Let $\widetilde{X}_1$ be the subset of Fibonacci words $x = \dots x_{-1}x_0x_1\dots$ in the space $\widetilde{X}$ with $x_1 = 1$ containing infinitely many 1's both to the left and to the right of the first position. For each x in $\widetilde{X}_1$, we introduce numbers $y_i(x)$ for $i \in \mathbb{Z}$, where $y_i(x)+1$ is the number of 0's between the ith and the $(i+1)$st occurrence of 1 in the word x (the first 1 occupies the first position).

Definition. A word $x \in \widetilde{X}_1$ is said to be *regular* if in this word odd numbers $y_i(x)$ occur infinitely many times both to the left and to the right of the first position (there are infinitely many elementary Fibonacci words) and the number $y_0(x)$ is odd (the first elementary Fibonacci word to the left of the first position is odd).

Following [3], we denote the set of regular words by $\widetilde{X}_0$. On the space $\widetilde{X}_0$, a conditional Erdős measure $\tilde{\mu}_0$ is naturally defined. This measure is proportional to the measure $\tilde{\mu}$, and $\tilde{\mu}_0(\widetilde{X}_0) = 1$.

According to [3], a finite Fibonacci word b is called a *block* if it is an odd elementary word or the concatenation $c_1c_2\cdots c_{s-1}c_s$, where $s \geqslant 2$, the c_i with $i \leqslant s-1$ are even elementary words, and the elementary word c_s is odd.

Let B be the set of all blocks. We can identify this set with the set B' of finite words $b' = k_1, \dots, k_s$ such that if $s = 1$, then k_1 is an odd number, if $s > 1$, then $k_1, \dots, k_{s-1}$ are even numbers, and k_s is an odd number.

The length of a block b for which $b' = k_1, \dots, k_s$ is equal to $\phi(b) = k_1 + \dots + k_s + 2s$. The correspondence $b \to b'$ gives a parameterization of blocks important for our purposes. The authors of [3] used another parameterization of blocks.

Any regular word $x \in \widetilde{X}_0$ has a unique expansion into blocks $b_i(x)$, where $i \in \mathbb{Z}$. The block $b_1(x)$ starts at the first position. The block $b_2(x)$ starts after the first block, etc. The block $b_0(x)$ ends at position 0. The block $b_{-1}(x)$ ends before the block $b_0(x)$, etc.

Let $x \in \widetilde{X}_0$. The least positive integer j such that $T^j x \in \widetilde{X}_0$ is equal to the length of the block $b_1(x)$. Let us denote the length of the block $b_1(x)$ by $F(x) = \phi(b_1(x))$; then the derivative automorphism $S = T'\colon x \mapsto T^{F(x)}x$ is the left shift of the word x by $F(x)$.

This map of the set of regular words was called the two-sided golden shift in [3]. It is clear that $b_i(x) = b_1(S^{i-1}(x))$ for $i \in \mathbb{Z}$. Note that the measure $\tilde{\mu}_0$ on the space $\widetilde{X}_0$ is invariant with respect to the two-sided golden shift.

From the above construction, we see that the golden shift S can be identified with the shift $\widetilde{S}$ on the space $\widetilde{Z} = \{\tilde{z} = \cdots z_{-1}z_0z_1\cdots\}$ of two-sided words with distinguished first position over the alphabet B. The isomorphism is given by the rule

$$x \mapsto \cdots b_{-1}(x)b_0(x)b_1(x)\cdots .$$

We introduce the function $\widetilde{F}(\tilde{z}) = \phi(z_1(\tilde{z}))$, where $\phi(b)$ is the block length defined above.

We call $1(0)^{k+1}1$ an elementary cycle. An elementary cycle is odd if k is an odd number.

In [3], a subset $\widetilde{X}^{\mathrm{reg}} \subset \widetilde{X}$ was introduced.

Definition. The subset $\widetilde{X}^{\mathrm{reg}} \subset \widetilde{X}$ consists of all words $x = \cdots x_{-1}x_0x_1\cdots$ in which odd cycles occur infinitely many times both to the left and to the right of the first position.

The subset $\widetilde{X}^{\mathrm{reg}}$ is invariant with respect to T, and $\tilde{\mu}(\widetilde{X}^{\mathrm{reg}}) = 1$.

The integral automorphism $\widehat{T}$constructed from the shift $\widetilde{S}$ and the positive integer-valued function $\widetilde{F}(\tilde{z})$ is the transformation of the space $\widehat{Z}$ of pairs $(\tilde{z}, j)$, where $j = 0, \dots, F(\tilde{z})-1$ and $\tilde{z} \in \widetilde{Z}$, defined by the formulas

$$(\tilde{z}, j) \mapsto (\tilde{z}, j+1)$$

if $j < \widetilde{F}(\tilde{z})-1$ and $(\tilde{z}, \widetilde{F}(\tilde{z})-1) \mapsto (\widetilde{S}\tilde{z}, 0)$.

Theorem 1 [3]**.** *The shift T on the space $\widetilde{X}^{\mathrm{reg}}$ is isomorphic to the integral automorphism $\widehat{T}$ on the space $\widehat{Z}$. The isomorphism is given by the formula $T^j x \mapsto (\cdots b_{-1}(x)b_0(x)b_1(x)\cdots, j)$, where $0 \leqslant j \leqslant F(x)-1$ and $x \in \widetilde{X}_0$.*

3. The Golden Shift and The Invariant Erdős Measure

Consider the matrices $M(k)$ defined by

$$M(k) = P(10)P^k(00)P(01) \qquad \text{for} \quad k = 0, 1, 2, \dots .$$

The following lemma is valid.

Lemma 1. *The matrix $M(k)$ has the form*

$$M(k) = \begin{cases} M_o(n), & k = 2n+1, \\ M_e(n), & k = 2n, \end{cases}$$

where

$$M_o(n) = \begin{pmatrix} pq^{2n+2} & p^2q^{2n+1} \\ pq^{n+2}\frac{p^{n+1}-q^{n+1}}{p-q} & p^2q^{n+1}\frac{p^{n+1}-q^{n+1}}{p-q} \end{pmatrix},$$
$$M_e(n) = \begin{pmatrix} pq^{2n+1} & p^2q^{2n} \\ pq^{n+2}\frac{p^{n}-q^{n}}{p-q} & p^2q^{n}\frac{p^{n+1}-q^{n+1}}{p-q} \end{pmatrix}.$$

Proof. The characteristic polynomial of the matrix $P(00)$ is equal to $x^3 - qx^2 - pqx + pq^2$. By the Cayley–Hamilton theorem, this gives the following recurrent relation for the sequence of matrices $M(k) = P(10)(P(00))^kP(01)$, where $k = 0, 1, 2, \dots$:

$$M(k+3) = qM(k+2) + pqM(k+1) - pq^2M(k), \qquad k = 0, 1, \dots .$$

A direct check shows that the sequence of matrices

$$\widehat{M}(k) = \begin{cases} M_o(n), & k = 2n+1, \\ M_e(n), & k = 2n, \end{cases}$$

satisfies the same recurrent relation. Moreover, a direct check gives $M(0) = \widehat{M}(0)$, $M(1) = \widehat{M}(1)$, and $M(2) = \widehat{M}(2)$. This means that $M(k) = \widehat{M}(k)$ for $k \geqslant 0$, which proves the lemma.

The matrices $M_e(n)$ are nonsingular, and the matrices $M_o(n)$ are singular. We can write the matrices $M_o(n)$ in the form $M_o(n) = u(n)v$, where

$$u(n) = \begin{pmatrix} q^{2n+1} \\ q^{n+1}\frac{p^{n+1}-q^{n+1}}{p-q} \end{pmatrix}$$

and $v = (pq, p^2)$.

Consider the set B' of finite words $b' = k_1 \cdots k_s$ such that if $s = 1$, then $k_1 = 2n_1 + 1$, if $s > 1$, then $k_j = 2n_j$ for $j \leqslant s-1$, and $k_s = 2n_s + 1$.

For any block $b \in B$ with $b' = k_1 \cdots k_s$ we define the matrix

$$M(b) = M(k_1 \cdots k_s) = M(k_1) \cdots M(k_s) = M_e(n_1) \cdots M_e(n_{s-1})M_o(n_s).$$

We set the notation $u(b) = u(n_s)$ and $M_e(b) = M_e(n_1) \cdots M_e(n_{s-1})$ for $s > 2$; for $s = 1$, $M_e(b)$ denotes the identity matrix. We have

$$M(b) = M_e(b)u(b)v.$$

We also set

$$p(b) = vM_e(b)u(b).$$

Now let us calculate the distribution of the random variable $b_1(x)$ on the set $\widetilde{X}_0$ with measure $\tilde{\mu}_0$, that is,

$$\tilde{\mu}_0(\{x \in \widetilde{X}_0 : b_1(x) = b\}).$$

It follows from the definition of the measure $\tilde{\mu}_0$ that

$$\begin{aligned}
\tilde{\mu}_0(\{x \in \widetilde{X}_0 : b_1(x) = b\}) &= \sum_{n=0}^{\infty} \frac{\tilde{\mu}(\{x \in \widetilde{X}_0 : y_0(x) = 2n+1,\, b_1(x) = b\})}{\tilde{\mu}(\widetilde{X}_0)} \\
&= \sum_{n=0}^{\infty} \frac{l(1)u(n)vM(b)r(1)}{\tilde{\mu}(\widetilde{X}_0)} = \sum_{n=0}^{\infty} \frac{l(1)u(n)vM_e(b)u(b)vr(1)}{\tilde{\mu}(\widetilde{X}_0)} \\
&= vM_e(b)u(b) \sum_{n=0}^{\infty} \frac{l(1)u(n)vr(1)}{\tilde{\mu}(\widetilde{X}_0)} \\
&= vM_e(b)u(b) \sum_{n=0}^{\infty} \frac{\tilde{\mu}(\{x \in \widetilde{X}_0 : y_0(x) = 2n+1\})}{\tilde{\mu}(\widetilde{X}_0)} = p(b).
\end{aligned}$$

Hence, in particular, $\sum_{b \in B} p(b) = 1$.

Since the golden shift S preserves the measure $\tilde{\mu}_0$ on the space $\widetilde{X}_0$, it follows that the random variables $b_i(x) = b_1(S^{i-1}x)$ with $i \in \mathbb{Z}$ are identically distributed and

$$\tilde{\mu}_0(\{x \in \widetilde{X}_0 : b_i(x) = b\}) = p(b).$$

In a similar way, we calculate the joint distribution of the random variables

$$b_1(x), \dots, b_m(x)$$

on the set $\widetilde{X}_0$ with measure $\tilde{\mu}_0$:

$$\begin{aligned}
&\tilde{\mu}_0(\{x \in \widetilde{X}_0 : b_1(x) = b^1, \dots, b_m(x) = b^m\}) \\
&\quad = \sum_{n=0}^{\infty} \tilde{\mu}_0(\{x \in \widetilde{X}_0 : y_0(x) = 2n+1,\, b_1(x) = b^1, \dots, b_m(x) = b^m\}) \\
&\quad = \sum_{n=0}^{\infty} \frac{l(1)u(n)vM_e(b^1)u(b^1)vM_e(b^2)u(b^2) \cdots vM_e(b^m)u(b^m)vr(1)}{\tilde{\mu}(\widetilde{X}_0)} \\
&\quad = p(b^1) \cdots p(b^m).
\end{aligned}$$

Thus the identically distributed random variables $b_j(x)$ (defined on the space $\widetilde{X}_0$ with measure $\tilde{\mu}_0$) are independent.

Consider the Bernoulli measure $\hat{\nu}$ on the space $\widetilde{Z}$ with one-dimensional distribution

$$p(b) = \hat{\nu}(\{\tilde{z} : z_j = b\}) = vM_e(b)u(b), \qquad \text{where} \quad b \in B \quad \text{and} \quad j \in \mathbb{Z}.$$

Recall that $F(x) = \phi(b_1(x))$ and $\widetilde{F}(\tilde{z}) = \phi(z_1(\tilde{z}))$.

The $\widehat{T}$-invariant measure $\hat{\mu}$ on the set of pairs $\{(\tilde{z}, j),\, \tilde{z} \in \widetilde{Z},\, 0 \leqslant j \leqslant \widetilde{F}(\tilde{z}) - 1\}$ is defined as follows. The set of pairs $(\tilde{z}, 0)$, where $\tilde{z} \in \widetilde{Z}$, can be identified with the set $\widetilde{Z}$, on which the measure $\hat{\nu}$ is defined; the measure $\hat{\mu}$ is given by the formula

$$\int f(\tilde{z}, j)\, d\hat{\mu}(\tilde{z}, j) = \frac{\int \sum_{j=0}^{\widetilde{F}(\tilde{z})-1} f(\tilde{z}, j)\, d\hat{\nu}(\tilde{z})}{\int \widetilde{F}(\tilde{z})\, d\hat{\nu}(\tilde{z})}.$$

Theorem 2. *The shift T on the subset $\widetilde{X}^{\mathrm{reg}}$ with invariant Erdős measure is isomorphic to the integral automorphism $\widehat{T}$ with measure $\hat{\mu}$. The isomorphism is given by the formula $T^j x \mapsto (\cdots b_{-1}(x) b_0(x) b_1(x) \cdots, j)$, where $0 \leqslant j \leqslant F(x) - 1$ and $x \in \widetilde{X}_0$.*

The proof of Theorem 2 follows easily from the above considerations.

Applying this theorem to $p = 1/2$, we immediately obtain a new proof of one of the main results in the paper [3].

4. Probabilities of Blocks and the Passage to Binary Words

For the probabilities of blocks b with $b' = k_1 \cdots k_s$, where $k_j = 2n_j$, $j < s$, and $k_s = 2n_s + 1$, we have the formula

$$p(b) = vM_e(b)u(n_s).$$

If $s = 1$, then

$$M_e(b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and if $s \geqslant 2$, then

$$M_e(b) = M_e(n_1) \cdots M_e(n_{s-1}).$$

Consider the function $[n]_\alpha = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}$, where $\alpha = p/q$, and the matrix

$$A(n) = \begin{pmatrix} \alpha[n+1]_\alpha & [n]_\alpha \\ \alpha & 1 \end{pmatrix}.$$

Evidently,

$$A(n) = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} \alpha & 0 \\ \alpha & 1 \end{pmatrix}.$$

We shall use this relation in what follows.

Let us rewrite the formula for $p(b)$ in another form by using the relation

$$CM_e(n)C = pq^{2n+1}A(n), \qquad \text{where} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $s \geqslant 2$, then

$$p(b) = vM_e(n_1) \cdots M_e(n_{s-1})u(n_s) = p^{s-1}q^{k_1+\cdots+k_{s-1}+s-1}vCA(n_1) \cdots A(n_{s-1})Cu(n_s).$$

It is clear that $Cu(n_s) = q^{2n_s+1}(1/\alpha)A(n_s)(1,0)^\top$. Hence

$$p(b) = p(n_1 \cdots n_s) = \alpha^{s-1}q^{\phi(b)}(\alpha, 1)A(n_1) \cdots A(n_s)(1,0)^\top,$$
$$\phi(b) = k_1 + \cdots + k_s + 2s.$$

Now let us obtain a new formula for $p(n_1 \cdots n_s)$ in terms of binary words.

Consider the matrices

$$\widetilde{M}(0) = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \widetilde{M}(1) = \begin{pmatrix} \alpha^2 & 0 \\ \alpha^2 & \alpha \end{pmatrix}.$$

We have

$$\alpha A(n) = (\widetilde{M}(0))^n \widetilde{M}(1).$$

Let $u = (\alpha, \alpha)^\top$. Recall that by definition $q = 1/(1+\alpha)$. Therefore

$$\begin{aligned}(1+\alpha)^{\phi(b)}p(b) &= \alpha^{s-1}(\alpha, 1)A(n_1) \cdots A(n_s)(1,0)^\top \\ &= (\alpha, 1)\widetilde{M}(0)^{n_1}\widetilde{M}(1) \cdots \widetilde{M}(0)^{n_{s-1}}\widetilde{M}(1)\widetilde{M}(0)^{n_s}\widetilde{M}(1)(1,0)^\top/\alpha \\ &= (\alpha, 1)\widetilde{M}(0)^{n_1}\widetilde{M}(1) \cdots \widetilde{M}(0)^{n_{s-1}}\widetilde{M}(1)\widetilde{M}(0)^{n_s}u = (\alpha, 1)\widetilde{M}(b)u.\end{aligned}$$

In this formula,

$$\widetilde{M}(b) = \widetilde{M}(0)^{n_1}\widetilde{M}(1) \cdots \widetilde{M}(0)^{n_{s-1}}\widetilde{M}(1)\widetilde{M}(0)^{n_s}.$$

Let D_{n-1}, where $n = 1, 2, \dots$, denote the set of all binary words of length $n-1$. The set D_1 contains only the empty word. Let us associate the block b for which $b' = (2n_1) \cdots (2n_s + 1)$, $n_1 + \cdots + n_s + s = n$, and $n \geqslant 2$ (the length of the block b equals $2n+1$) with a binary word $d \in D_{n-1}$ by the rule $d = i_1 \cdots i_{n-1} = (0)^{n_1}1 \cdots (0)^{n_{s-1}}1(0)^{n_s}$. In this word, if $n_i = 0$, then $(0)^{n_i}$ is the empty word.

The product of matrices corresponding to this binary word has the form

$$\widetilde{M}(d) = \widetilde{M}(i_1)\cdots\widetilde{M}(i_{n-1}) = \widetilde{M}(b).$$

The empty word with $n = 1$ gives the identity matrix.

Thus, for any word b with $b' = (2n_1)\cdots(2n_s + 1)$ of length $\phi(b) = 2n + 1$, we have

$$p(b) = (\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top q^{2n+1}.$$

5. The Generating Function of the Block Length

The generating function of the block length $\phi(b)$ is defined by

$$\begin{aligned}\Phi(z) = \sum_b p(b) z^{\phi(b)} &= \sum_b (\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top q^{2n+1} z^{\phi(b)} \\ &= \sum_{n=1} \sum_{d \in D_{n-1}} (\alpha, 1)\widetilde{M}(d)(\alpha, \alpha)^\top q^{2n+1} z^{2n+1} \\ &= q^3 z^3 (\alpha, 1)(\mathrm{Id} - q^2 z^2 \widetilde{M})^{-1}(\alpha, \alpha)^\top,\end{aligned}$$

where $\widetilde{M} = \widetilde{M}(0) + \widetilde{M}(1)$ and Id is the identity matrix.

Hence we obtain

$$\Phi(z) = \frac{pqz^3}{1 - (1 - pq)z^2}\,.$$

Knowing the generating function of the block length, we calculate the mean value of the block length:

$$\mathrm{E}\,\phi(\,\cdot\,) = \Phi'(1) = 1 + \frac{2}{pq}\,.$$

Let us expand $\Phi(z)$:

$$\Phi(z) = \sum_{n=0}^{\infty} pq(1 - pq)^{n-1} z^{2n+1}.$$

Thus, the probability that a block has length $2n + 1$ is equal to $pq(1 - pq)^{n-1}$.

6. A Formula for Calculating the Entropy

In this section, we determine the entropy by using binary logarithms. It was shown that

$$p(b) = \frac{1}{(1 + \alpha)^{\phi(b)}}(\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top.$$

Hence

$$\log_2 p(b) = -\phi(b)\log_2(1 + \alpha) + \log_2(\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top.$$

Therefore,

$$\mathrm{E}(-\log_2 p(\,\cdot\,)) = \log_2(1 + \alpha)\,\mathrm{E}\,\phi(\,\cdot\,) - \mathrm{E}(\log_2(\alpha, 1)\widetilde{M}(\,\cdot\,)(\alpha, \alpha)^\top).$$

It follows from Theorem 2 and the Abramov formula [7] for the entropy of an integral automorphism that the entropy of the invariant Erdős measure is equal to

$$\begin{aligned}H &= \frac{\mathrm{E}(-\log_2 p(\,\cdot\,))}{\mathrm{E}\,\phi(\,\cdot\,)} \\ &= \log_2(1 + \alpha) - \frac{1}{\mathrm{E}\,\phi(\,\cdot\,)} \sum_{n=1}^{\infty} \left[\sum_{b \in B_n} \log_2((\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^T)(\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top\right] \frac{1}{(1 + \alpha)^{2n+1}}\,.\end{aligned}$$

We know that

$$\mathrm{E}\,\phi(\,\cdot\,) = 1 + \frac{2}{pq} = 1 + \frac{2(1 + \alpha)^2}{\alpha}\,.$$

Let B_n be the set of words b for which $b' = (2n_1) \cdots (2n_s + 1)$, where $n_1 + \cdots + n_s + s = n$. We introduce the notation

$$k_n = \sum_{b \in B_n} \log_2((\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top)((\alpha, 1)\widetilde{M}(b)(\alpha, \alpha)^\top),$$

or

$$k_n = \sum_{d \in D_{n-1}} \log_2((\alpha, 1)\widetilde{M}(d)(\alpha, \alpha)^\top)((\alpha, 1)\widetilde{M}(d)(\alpha, \alpha)^\top).$$

We have

$$H = \log_2(1+\alpha) - \frac{1}{1 + \frac{2(1+\alpha)^2}{\alpha}} \sum_{n=1}^{\infty} k_n \left(\frac{1}{1+\alpha}\right)^{2n+1}.$$

The following statement holds: the entropy H does not change under the replacement of α by $1/\alpha$. This follows from the formula for the entropy, because under the replacement of α by $1/\alpha$ the matrices $\widetilde{M}(0)$ and $\widetilde{M}(1)$ are transformed accordingly into the matrices

$$\frac{1}{\alpha^2} C\widetilde{M}(1)C \qquad \text{and} \qquad \frac{1}{\alpha^2} C\widetilde{M}(0)C, \qquad \text{where} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Another proof is based on the fact that the replacement of α by $1/\alpha$ corresponds to the passage from the random variable $\rho\zeta = \zeta_1\rho^2 + \zeta_2\rho^3 + \cdots$ to the random variable $1 - \rho\zeta$. The Hausdorff dimension of a set is an invariant of the isometry $x \mapsto 1-x$, where $x \in [0,1]$. Hence, the definition of the Hausdorff dimension of a measure implies the required assertion. Of course, for $a > 1$ the series converges more rapidly. Below we use the above statement to calculate the Hausdorff dimension.

If $q = 1/2$, then

$$H = 1 - \frac{1}{9} \sum_{n=1}^{\infty} k_n \frac{1}{2^{2n+1}}.$$

Note that the formula $H/\log_2 \beta$ for the Hausdorff dimension of an invariant Erdős measure on the Fibonacci compactum coincides with the Alexander–Zagier formula for the Hausdorff dimension of an Erdős measure on the real line. The Alexander–Zagier formula was obtained in [4] with the help of the combinatorics of the Euclidean tree. Possibly, our formula corresponds to the combinatorics of the α-Euclidean tree.

The main difficulty in the calculation of the entropy H is the slow convergence of the corresponding series. The series for H converges too slowly for effective computation. Following the approach of Alexander and Zagier [4], we use some rearrangement of the series for H in order to accelerate convergence.

Let

$$\mu_n = k_n - [3]_\alpha k_{n-1}.$$

Then

$$(1 - [3]_\alpha x) \sum_{n=1}^{\infty} k_n x^n = \sum_{n=1}^{\infty} \mu_n x^n.$$

Consider

$$\lambda_n = 2\lambda_{n-1} - \lambda_{n-2} + \mu_n - [3]_\alpha \mu_{n-1}.$$

It is clear that

$$\sum_{n=1}^{\infty} k_n x^n = \frac{1}{1 - [3]_\alpha x} \sum_{n=1}^{\infty} \mu_n x^n,$$

$$(1-x)^2 \sum_{n=1}^{\infty} \lambda_n x^n = (1 - [3]_\alpha x) \sum_{n=1}^{\infty} \mu_n x^n,$$

$$\sum_{n=1}^{\infty} k_n x^n = \frac{(1-x)^2}{(1 - [3]_\alpha x)^2} \sum_{n=1}^{\infty} \lambda_n x^n.$$

Using the last relation and setting $x = 1/(1+\alpha)^2$, we obtain

$$H = \log_2(1+\alpha) - \frac{\alpha(2+\alpha)}{(1+2\alpha)} \sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{1+\alpha}\right)^{2n+1}.$$

This series converges more rapidly than the original series.

7. Results of Calculations

In the following table, we give the values of the Hausdorff dimension $H_{\text{dim}} = H/\log_2 \beta$ of invariant Erdős measures on the Fibonacci compactum.

The second column in the table contains the values of the Hausdorff dimension of Erdős measures for different probabilities p. In the third column, it is shown how many terms of the series are used in the formula for the Hausdorff dimension of the corresponding Erdős measure. The fourth column shows the results of Lalley's calculations.

p	H_{dim}	n	Lalley
0.05	0.392167680782199076	15	0.3877 ± 0.03
0.05	0.392167680782199076	14	
0.1	0.6101383374950678578	20	0.6085 ± 0.008
0.1	0.6101383374950678578	19	
0.2	0.849903398027151976	23	0.8499 ± 0.004
0.2	0.849903398027151972	22	
0.3	0.9513889802259870	24	0.9501 ± 0.002
0.3	0.9513889802259869	23	
0.4	0.9875456832532938	25	0.9868 ± 0.001
0.4	0.9875456832532931	24	
0.5	0.995713126685555526	24	0.9954 ± 0.0008
0.5	0.9957131266855555560	23	

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