

ATTRACTORS OF CONFORMAL FOLIATIONS

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Abstract. We investigated conformal foliations (M, F) of codimension $q \geq 3$ and proved a criterion for them to be Riemannian. In particular, the application of this criterion allowed us to proof the existence of an attractor that is a minimal set for each non-Riemannian conformal foliation. Moreover, if foliated manifold is compact then non-Riemannian conformal foliation (M, F) is $(Conf(S^q), S^q)$ -foliation with finitely many minimal sets. They are all attractors, and each leaf of the foliation belongs to the basin of at least one of them. The specificity of the proper conformal foliations is indicated. Special attention is given to complete conformal foliations.

1 Introduction

The goal of this work is to present recent results of the author [1, 2] on the investigation of the structure of conformal foliations.

1.1 The Lichnerowicz Conjecture

Remind that two Riemannian metrics h and g on a manifold M are called conformally equivalent if there exists a positive smooth function f on M with $h = fg$. Each class $[g]$ of conformally equivalent Riemannian metrics is named a conformal structure on M , and the pair $(M, [g])$ is said to be a conformal manifold.

Recall that a diffeomorphism $f : N_1 \rightarrow N_2$ between Riemannian manifolds (N_1, g_1) and (N_2, g_2) is named conformal if there exists a smooth function λ on N_1 with $f^*g_2 = \lambda g_1$. A conformal diffeomorphism f from a Riemannian manifold (N, g) to (N, g) is also said to be a conformal transformation.

The group of conformal transformations of a Riemannian manifold (M, g) is called *inessential* if it is a group of isometries of a Riemannian manifold (M, h) with $h \in [g]$. Otherwise, the group is called *essential*.

Lichnerowicz put forth the conjecture that for $n \geq 3$ every n -dimensional compact Riemannian manifold admitting an essential group of conformal transformations is the standard n -dimensional sphere S^n .

The articles by Obata [3], Alekseevskii [4, 5], Ferrand [6] and others are devoted to proving this conjecture.

It was established also that if the group of conformal transformations of a non-compact Riemannian manifold M is essential then M is the n -dimensional Euclidean space. In 1996 Ferrand [6] gave a complete proof of the Lichnerowicz conjecture including the case of noncompact manifolds.

1.2 Conformal Foliations

Vaisman [7] introduced the conformal foliations (M, F) as foliations admitting a transversal conformal structure.

Suppose that given are:

1) n -dimensional manifold M and a possibly disconnected q -dimensional manifold N , where $0 < q < n$;

2) an open cover $\{U_i | i \in J\}$ of M ;

3) submersions $f_i : U_i \rightarrow V_i$ with connected fibres, $V_i \subset N$,

and if $U_i \cap U_j \neq \emptyset$, then there exists conformal diffeomorphism $\gamma_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ such that $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$.

Maximal, with respect to inclusion, N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ enjoying these properties determines a new topology on M , whose base is the set of leaves of all submersions f_i . This topology is called the leaf topology, denoted by τ .

The path-connected components of the topological space (M, τ) form a partition $F = \{L_\alpha | \alpha \in A\}$ of M , and (M, F) is named the foliation with leaves L_α determined by the N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$.

Definition 1 A codimension $q \geq 3$ smooth foliation (M, F) is called conformal if (M, F) is determined by an N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$, and N admits a Riemannian metric g such that each γ_{ij} is the local conformal diffeomorphism of the corresponding open subsets.

If each γ_{ij} is an isometry, then (M, F) is named a Riemannian foliation.

Let $Conf(S^q)$ be the Lie group of all the conformal transformations of the q -dimensional sphere S^q .

Definition 2 If a foliation (M, F) is defined by N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$, where

- $N = S^q$ and
 - each γ_{ij} is a restriction of a transformation $f \in Conf(S^q)$,
- then refer to (M, F) as a $(Conf(S^q), S^q)$ -foliation.

1.3 The Tarquini – Frances Question

Tarquini [8] and then Frances and Tarquini [9] posed the following question about conformal foliations:

Is every codimension $q \geq 3$ conformal foliation on a compact manifold either a Riemannian foliation or a $(Conf(S^q), S^q)$ -foliation?

They refer to the positive answer to this question as the foliated analogous of the Lichnerowicz Conjecture.

As known, for $q \geq 3$ a conformal foliation is a $(Conf(S^q), S^q)$ -foliation if and only if it is transversally conformally flat.

Frances and Tarquini [9] gave a positive answer to this question under some additional assumptions.

1.4 Attractors and Minimal Sets of Foliations

Let (M, F) be a foliation. A *saturated set* is a union of leaves.

Definition 3 *A nonempty closed saturated subset \mathcal{M} of M is said to be an attractor of a foliation (M, F) if there exists a saturated open neighbourhood $Attr(\mathcal{M})$ of \mathcal{M} such that the closure of every leaf from $Attr(\mathcal{M})$ includes \mathcal{M} . The set $Attr(\mathcal{M})$ is named a basin of this attractor. If in addition $Attr(\mathcal{M}) = M$ then \mathcal{M} is called a global attractor.*

Definition 4 *A minimal set of a foliation (M, F) is a nonempty closed saturated subset of M without proper subsets enjoying these properties.*

In mathematical encyclopedia Anosov [10] said that the study of minimal sets is one of the fundamental problems in the topological dynamics, hence also in the qualitative theory of foliations.

1.5 Results

We consider conformal foliations as Cartan foliations and use the construction of principal foliated bundle. Thanks this we apply the results of our previous works [11, 12]. In the foliation theory the germ holonomy groups are usually used. By analogy to [12] we gave different interpretations for the germ holonomy group of a leaf of a conformal foliation (Theorem 1).

We proved a criterion for conformal foliation to be Riemannian (Theorem 2). Application of this criterion and some results on local conformal geometry allowed us

to prove the existence of an attractor for every non-Riemannian conformal foliation of codimension $q > 2$ (Theorem 4).

As known ([13, 14]), there are smooth foliations on non-compact manifolds without minimal sets. We proved that any non-Riemannian conformal foliation of codimension $q > 2$ on noncompact manifold has the minimal set that is an attractor of this foliation (Corollary 1).

We also described the structure of conformal foliations (M, F) in the cases when:

- 1) the foliated manifold M is compact (Theorem 5);
- 2) these foliations are complete (Theorem 6);
- 3) foliations (M, F) are proper (Theorem 7).

1.6 The Positive Answer to the Frances — Tarquini Question and a Proof of the Conjecture of Ghys

- Theorem 5 implies the positive answer to the Frances — Tarquini question about conformal foliations on compact manifolds.
- Deroin and Kleptsyn [15] indicated that all known examples of a transversely conformal foliation having a diffuse transversely invariant measure have a transverse metric which is transversely invariant. According [15], for transversely conformal foliations this has been conjectured by Ghys. A proof of this conjecture for conformal foliations of codimension $q \geq 3$ (in the conforming class of the smoothness) follows from Theorem 5.

Notations

- Let $G = Conf(S^q)$ be the Lie group of all conformal transformations of the q -dimensional sphere S^q and H be the stabilizer in G of an arbitrary point of S^q . Then H is a semidirect product of a conformal group $CO(q) = R^+ \cdot O(q)$ and the group R^q .
- The Lie group H is isomorphic to the the Lie group $Sim(E^q)$ of all conformal transformations of the q -dimensional Euclidean space E^q .
- We assume here that $q \geq 3$.

2 A criterion for a conformal foliation to be Riemannian

2.1 The Foliated Bundle of a Conformal Foliation

The construction of a foliated bundle was essentially used. Foliated bundles were introduced in works of Molino and Kamper – Tondeur.

Remark that any conformal foliation may be considered as Cartan foliation of the type (G, H) , where $G = \text{Conf}(S^q)$ is the Lie group of all conformal transformations of S^q and H be the stabilizer in G of an arbitrary point of S^q .

Let $\mathfrak{g}, \mathfrak{h}$ are the Lie algebras of the Lie groups G and H , respectively. Then we have the following statement [11].

Proposition 1 *Let (M, F) be a conformal foliation of codimension $q \geq 3$. Then the following objects are defined:*

- 1) a principal H -bundle $\pi : \mathcal{R} \rightarrow M$;
- 2) an H -invariant foliation $(\mathcal{R}, \mathcal{F})$ which π transforms into (M, F) ;
- 3) a \mathfrak{g} -valued 1-form ω on \mathcal{R} with the properties:
 - (i) $\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field corresponding to A ;
 - (ii) $R_a^* \omega = \text{Ad}_G(a^{-1}) \omega$ for any $a \in H$, where Ad_G is the adjoint representation of the Lie group G in its Lie algebra \mathfrak{g} ;
 - (iii) the Lie derivative $L_X \omega$ vanishes for every vector field X tangent to the leaves of $(\mathcal{R}, \mathcal{F})$.

The H -bundle $\pi : \mathcal{R} \rightarrow M$ is said to be *foliated*. The foliation $(\mathcal{R}, \mathcal{F})$ is named a *lifted* foliation, and $(\mathcal{R}, \mathcal{F})$ is a transversally parallelizable foliation, i.e., an e -foliation.

Definition 5 *A conformal foliation (M, F) of codimension $q \geq 3$ is said to be complete, if so is the associated lifted e -foliation $(\mathcal{R}, \mathcal{F})$, i.e., if complete is every vector field X on \mathcal{R} such that $\omega(X) = \text{constant}$.*

2.2 Interpretations of the Holonomy Groups of a Conformal Foliation

An application of the foliated H -bundle over a conformal foliation (M, F) allowed us gave the following important interpretations of the holonomy groups of leaves.

Theorem 1 *Let (M, F) be an arbitrary conformal foliation of codimension $q \geq 3$ and $\pi : \mathcal{R} \rightarrow M$ be the projection of the foliated H -bundle over (M, F) with lifted foliation $(\mathcal{R}, \mathcal{F})$. For each leaf $L = L(x)$ of (M, F) consider the leaf $\mathcal{L} = \mathcal{L}(u)$.*

where $u \in \mathcal{R}$, $\pi(u) = x$, of the lifted foliation $(\mathcal{R}, \mathcal{F})$. Then the germ holonomy group $\Gamma(L, x)$ of L is isomorphic to the following groups:

- the subgroup $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of H ;
- the group of deck transformations of the regular covering $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow L$.

If in conditions of the Theorem 1 we consider an other point $u' \in \pi^{-1}(x)$ and the leaf $\mathcal{L}' = \mathcal{L}'(u')$, then the group $H(\mathcal{L}')$ must be conjugated to $H(\mathcal{L})$ in H . Therefore, the following definition makes sense.

Definition 6 *Refer to the holonomy group of a leaf L of a conformal foliation as relatively compact or inessential if the corresponding subgroup $H(\mathcal{L})$ of the Lie group H is relatively compact. Otherwise the holonomy group of a leaf is called essential.*

2.3 When a Conformal Foliation is a Riemannian one?

Thanks to well-known works of Reinhart, Molino, Haefliger, Salem, Carriere and other authors, now Riemannian foliations form the most investigated class of foliations with transverse geometric structure. Hence it is very significant to know when a smooth foliation is Riemannian.

We established the following criterion for a conformal foliation to be Riemannian.

Theorem 2 *If (M, F) is a codimension $q \geq 3$ conformal foliation modeled on a conformal geometry $(N, [g])$, then there exists of a Riemannian metric $d \in [g]$ such that (M, F) is a Riemannian foliation modeled on (N, d) if and only if every holonomy group of this foliation be relatively compact.*

Corollary 1 *If a conformal foliation (M, F) is not Riemannian then it has a leaf with essential holonomy group.*

3 The Existence of Attractors of Conformal Foliations

3.1 A closure of a Leaf with Essential Holonomy Group

The following two theorems were proved without the assumptions of compactness of foliated manifolds and completeness of conformal foliations. In proof of Theorem 3 we considered and applied a conformal geometry on non-Hausdorff manifolds.

Theorem 3 *If (M, F) is a non-Riemannian conformal foliation of codimension $q \geq 3$, then:*

(i) for each leaf L with essential holonomy group the closure $\bar{L} = \mathcal{M}$ is an attractor, while

- *either \mathcal{M} is a minimal set*
- *or \mathcal{M} includes a closed leaf that is also an attractor;*

(ii) the union K of the closures of all leaves with essential holonomy group is a closed saturated subset of M , and $(M \setminus K, F_{M \setminus K})$ is a Riemannian foliation.

According to Corollary 1 any conformal non-Riemannian foliation has a leaf L with essential holonomy group. Therefore by Theorem 3 the closure $\mathcal{M} := \bar{L}$ is an attractor of conformal foliation (M, F) . Thus we have the following assertion.

Theorem 4 *Each codimension $q \geq 3$ conformal foliation (M, F)*

- *either is Riemannian*
- *or has an attractor \mathcal{M} that is the closure $\mathcal{M} = \bar{L}$ of a leaf L with essential holonomy group, and the restriction of the foliation to the attraction basin $(\text{Attr}(\mathcal{M}), F)$ is a $(\text{Conf}(S^q), S^q)$ -foliation.*

Theorems 3 and 4 imply the following statement.

Corollary 2 *Each codimension $q \geq 3$ non-Riemannian conformal foliation has a minimal set that is an attractor of this foliation.*

3.2 Conformal Foliations on Compact Manifolds

The notion of Ehresmann connection was introduced by Blumenthal and Hebda [16]. It belongs to differential topology. Using an Ehresmann connection we constructed a "trap for the leaves" in the proof of the following assertion for conformal foliations.

Proposition 2 *Let \mathcal{M} be a compact minimal set of a conformal foliation (M, F) , and all leaves from \mathcal{M} have inessential holonomy group. Then each open neighbourhood \mathcal{W} of \mathcal{M} includes a saturated open neighbourhood V consisting of leaves with inessential holonomy group.*

This proposition was essentially used in the proof of Theorem 5.

Theorem 5 *Every codimension $q \geq 3$ conformal foliation (M, F) on a compact manifold M is*

- *either a complete Riemannian foliation, and the closure of every leaf is a minimal set which is an embedded submanifold of M ,*
- *or a $(\text{Conf}(S^q), S^q)$ -foliation with finitely many minimal sets. They are all attractors formed by the closures of leaves with essential holonomy group, and each leaf of the foliation belongs to the basin of at least one of them.*

4 Global Attractors of Complete Conformal Foliations

Denote by $\text{Sim}(E^q)$ the Lie group of all conformal transformations of the Euclidean space E^q . The group $\text{Sim}(E^q)$ is equal to a semidirect product of the conformal Lie group $CO(q)$ and the normal subgroup R^q .

Definition 7 *A foliation (M, F) defined by N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ is said to be transversally similar or a $(\text{Sim}(E^q), E^q)$ -foliation if $N = E^q$ and each γ_{ij} is a restriction of a similar transformation from the group $\text{Sim}(E^q)$.*

Theorem 6 *Let (M, F) be a complete conformal foliation of codimension $q \geq 3$. Then one of the following three possibilities is realized:*

- 1) *the foliation (M, F) is Riemannian, and the closure of each its leaf forms a minimal set that is an embedded submanifold of M ;*
- 2) *(M, F) is a transversally similar foliation. It has a global attractor \mathcal{M} that is a minimal set containing all leaves with essential holonomy group;*
- 3) *(M, F) is a $(\text{Conf}(S^q), S^q)$ -foliation with a global attractor \mathcal{M} , and \mathcal{M} is either one or two leaves of this foliation or else \mathcal{M} is nontrivial minimal set coincided with the closure of every leaf having essential holonomy group.*

Moreover, in the cases 2) and 3) the restriction (M_0, F_0) of F onto $M_0 := M \setminus \mathcal{M}$ is a Riemannian foliation, and the closure of any leaf $L \subset M_0$ in M is equal to the union $\mathcal{M} \cup \mathcal{L}$ of \mathcal{M} and a closed submanifold \mathcal{L} coincided with the closure of L in M_0 .

Corollary 3 *If a complete conformal non-Riemannian foliation has a minimal set \mathcal{M} different from a closed leaf, then \mathcal{M} is a global attractor of this foliation.*

Remark 1 *Minimal sets of complete transversally similar foliations were investigated by the author in [11]. In particular, there we found conditions guaranteed for the global attractor of a complete transversally similar foliation (M, F) to be a smooth submanifold of M .*

5 Specificity of Proper Conformal Foliations

Definition 8 *A foliation (M, F) is called proper if all its leaves are embedded submanifolds of M . A leaf L is called closed if L is a closed subset of M .*

Emphasize that every minimal set of a proper foliation coincides with a closed leaf. Hence as application of Theorem 3 we have the following assertion.

Corollary 4 *Each proper of codimension $q \geq 3$ non-Riemannian conformal foliation has a closed leaf with essential holonomy group that is an attractor of this foliation.*

Theorem 7 *Any complete proper conformal foliation (M, F) of codimension $q \geq 3$ has a structure of one of the following types:*

- *(M, F) is a transversally complete proper Riemannian foliation with closed leaves and its leaf space is a smooth q -dimensional orbifold;*
- *(M, F) is a complete proper non-Riemannian transversally similar foliation. There exists a unique closed leaf L_0 , and L_0 is a global attractor and has an essential holonomy group;*
- *the foliation (M, F) is not transversally similar. There exists a global attractor \mathcal{M} that coincides with one or two leaves of (M, F) . The restriction (M_0, F_0) of F onto the dense open subset $M_0 := M \setminus \mathcal{M}$ is a Riemannian foliation, and the leaf space M_0/F_0 admits a structure of q -dimensional smooth orbifold. The closure of any leaf $L \subset M_0$ is equal to the union $\mathcal{M} \cup L$.*

Remark 2 *As Weil foliations form a subclass of conformal foliations, so in the case of codimension $q \geq 3$ the main results for Weil foliations [17] follow from Theorems 6 and 7.*

Examples of conformal foliations with different kinds of attractors are constructed by the method of suspension of a group homomorphism.

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References

1. Zhukova N.I. *Attractors and an analog of the Lichnerowicz conjecture for conformal foliations*. Siberian Math. J. **52**, 3, 2011, Pp. 436-450.
2. Zhukova N.I. *Global attractors of complete conformal foliations*. Math. Sb. **203**, 2012.
3. M. Obata *The conjectures on conformal transformations of Riemannian manifolds*. J. Differential Geom. **6**, 2, 1971. Pp. 247-258.
4. D.V. Alekseevskii *Groups of conformal transformations of Riemannian spaces*. Math. Sb. **89**, 2, 1972. Pp. 280-286.
5. D.V. Alekseevskii *S^n and E^n are the only Riemannian spaces that admit an essential conformal transformation*. Uspekhi Mat. Nauk. **28**, 5, 1973. Pp. 225-226.
6. J. Ferrand *The action of conformal transformations on a Riemannian manifold*. Math. Ann. **304**, 2, 1996. Pp. 277-291.
7. I. Vaisman *Conformal foliations*, Kodai Math. J. **2**, 1, 1979, Pp. 26-37.
8. C. Tarquini *Feuilletages conformes*. Ann. Inst. Fourier. **52**, 2, 2004. Pp. 453-480.
9. Frances C. and Tarquini C. *Autour du theoreme de Ferrand.Obata*. Ann. Global Anal. Geom. **21**, 1, 2007. Pp. 51-62.
10. D.V. Anosov *Minima set* in Mathematical Encyclopedia. **3**, 1982. Pp. 690-691.
11. Zhukova N. *Minimal sets of Cartan foliations*. Proc. Steklov Inst. of Math. **256**, 2007. Pp. 105-135.
12. Zhukova N.I. *Complete foliations with transversal rigid geometries and their basis automorphisms*. Vestnik RUDN Ser. Mat. Fiz.Inform. **2**, 2009, Pp. 14-35.
13. Beniere J.-C., Meigniez G. *Flows without minimal set*. Erg. Theory and Dyn. Syst. **19**, 1, 1999. Pp. 21-30.
14. Inaba T. *An example of a flow on a non-compact surface without minimal set*. Erg. Theory and Dyn. Syst. **19**, 1, 1999. Pp. 31-33.
15. B.Deroin and V. Kleptsyn *Random conformal dynamical systems*. Geom. Funct. Anal. **17**, 4, 2007. Pp. 1043-1105.
16. R. A. Blumenthal and J. J. Hebda *Ehresmann connections for foliations*. Indiana Univ. Math. J. **33**, 4, 1984. Pp. 597-611.
17. Zhukova N.I. *Weil foliations*. Nonlinear Dynam. **6**, 1, 2010. Pp. 219-231.

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