LIE ELEMENTS IN THE GROUP ALGEBRA

YURII M. BURMAN

ABSTRACT. Given a representation V of a group G, there are two natural ways of defining a representation of the group algebra $k[G]$ in the external power $V^{\wedge m}$. The set $\mathcal{L}(V)$ of elements of $k[G]$ for which these two ways give the same result is a Lie algebra and a representation of G. For the case when G is a symmetric group and $V = \mathbb{C}^n$, a permutation representation, these spaces $\mathcal{L}(\mathbb{C}^n)$ are naturally embedded into one another. We describe $\mathcal{L}(\mathbb{C}^n)$ for small n and formulate questions and conjectures for future research.

1. SETTING AND MOTIVATION

Let V be a finite-dimensional representation of a group G over a field k. For every $g \in G$ and every m define linear operators $A_m(g), B_m(g) : V^{\wedge m} \to V^{\wedge m}$ as follows:

$$A_m(g)(v_1 \wedge \dots \wedge v_m) = g(v_1) \wedge \dots \wedge g(v_m)$$

$$B_m(g)(v_1 \wedge \dots \wedge v_m) = \sum_{p=1}^{m} v_1 \wedge \dots \wedge g(v_p) \wedge \dots \wedge v_m.$$

(here and below $v_1, \dots, v_m$ are arbitrary vectors in V). Then extend the operators $A_m, B_m : G \to \mathrm{End}(V^{\wedge m})$ to the group algebra $k[G]$ by linearity. Also take by definition $A_0(g) = 1$ (an operator $k \to k$) and $B_0(g) = 0$ for every $g \in G$.

Definition. An element $x \in k[G]$ satisfying $A_m(g) = B_m(g)$ for all $m = 0, 1, \dots, \dim V$ is called a Lie element of $k[G]$ (with respect to the representation V). The set of Lie elements is denoted by $\mathcal{L}(V)$.

Besides the associative algebra structure in $k[G]$ and $\mathrm{End}(V^{\wedge m})$ consider an associated Lie algebra structure in them, taking $[p, q] = pq - qp$.

Proposition 1. *Maps $A_m, B_m : k[G] \to \mathrm{End}(V^{\wedge m})$ are Lie algebra homomorphisms.*

Proof. It is clear that $A_m : G \to \mathrm{End}(V^{\wedge m})$ is an associative algebra homomorphism ($A_m(xy) = A_m(x)A_m(y)$ for all $x, y \in G$), hence a Lie algebra homomorphism. For B_m take $x = \sum_{g\in G} a_g g$, $y = \sum_{h\in G} b_h h$, to obtain

$$\begin{aligned} B_m(x)B_m(y)v_1 \wedge \dots \wedge v_m &= \sum_{h\in G, 1\le p\le m} b_h B_m(x) v_1 \wedge \dots \wedge h(v_p) \wedge \dots \wedge v_m \\ &= \sum_{g,h\in G, 1\le p\le m} a_g b_h v_1 \wedge \dots \wedge g(h(v_p)) \wedge \dots \wedge v_m \\ &+ \sum_{g,h\in G, 1\le p,q\le m, p\ne q} a_g b_h v_1 \wedge \dots \wedge h(v_p) \wedge \dots \wedge g(v_q) \wedge \dots \wedge v_m, \end{aligned}$$

whence $B_m([x, y]) = [B_m(x), B_m(y)]$. $\square$

Corollary. *The set of Lie elements $\mathcal{L}(V) \subset k[G]$ is a Lie subalgebra.*

Proposition 2. *For any $x, y \in k[G]$ and any m one has $yA_m(x)y^{-1} = {}_m(yxy^{-1})$ and $yB_m(x)y^{-1} = B_m(yxy^{-1})$.*

The proof is evident.

Corollary. *The set $\mathcal{L}(V) \subset k[G]$ is a representation of G where elements of the group act by conjugation.*

This note takes its origin from the paper [1]. The paper contains a formula for the so called Hurwitz generating function which lists factorizations of a cyclic permutation $(12\dots n)$ to a product of transpositions. The key ingredient of the proof of the formula is the fact that $1-(ij) \in \mathcal{L}(\mathbb{C}^n)$ where $\mathbb{C}^n$ is the permutation representation of the symmetric group (see Proposition 3 below). Any other element $x \in \mathcal{L}(\mathbb{C}^n)$ corresponds to a generalization of this result producing a formula listing factorizations of the cycle to a product of various permutations with various weights; the weights depend on x. Equivalently, the same formula lists graphs embedded into oriented surfaces so that their complement is homeomorphic to a disk; any $x \in \mathcal{L}(\mathbb{C}^n)$ generates a formula listing similar embeddings of multi-graphs (again, with the weights depending on x).

This note is a description of research in progress; see the list of questions and conjectures at the end.

2. The symmetric group case

Here we take $G = S_n$, $n = 2, 3, \dots$. Let $k = \mathbb{C}$ and V be an n-dimensional permutation representation of S_n (the group acts on elements of the basis $x_1, \dots, x_n \in \mathbb{C}^n$ permuting their indices). We'll be writing $\mathcal{L}_n$ for short, instead of $\mathcal{L}(\mathbb{C}^n)$.

Proposition 3 (cf. [1])**.** *$1-(ij) \in \mathcal{L}_n$ for all $1 \le i < j \le n$.*

Proof. Take any $v_1, \dots, v_m \in V$; then $A_m(1)v_1 \wedge \dots \wedge v_m = v_1 \wedge \dots \wedge v_m$ and $B_m(1)v_1 \wedge \dots \wedge v_m = mv_1 \wedge \dots \wedge v_m$.

It follows from Proposition 2 that without loss of generality one may assume $i = 1, j = 2$. Apparently, this is enough to take for v_s basic vectors: $v_s = x_{i_s}$ for all $s = 1, \dots, m$, where $1 \le i_1 < \dots < i_m \le n$ are any indices. Consider now three cases:

1. $i_1, \dots, i_m \ne 1, 2$. Then

$$A_m((12))(x_{i_1} \wedge \dots \wedge x_{i_m}) = x_{i_1} \wedge \dots \wedge x_{i_m},$$
$$B_m((12))(x_{i_1} \wedge \dots \wedge x_{i_m}) = mx_{i_1} \wedge \dots \wedge x_{i_m},$$

so that

$$A_m(1-(12))(x_{i_1} \wedge \dots \wedge x_{i_m}) = 0 = B_m(1-(12))(x_{i_1} \wedge \dots \wedge x_{i_m}).$$

2. $i_1 = 1$, $i_2, \dots, i_m \ne 1, 2$. Then

$$A_m((12))(x_1 \wedge x_{i_2} \wedge \dots \wedge x_{i_m}) = x_2 \wedge x_{i_2} \wedge \dots \wedge x_{i_m},$$
$$B_m((12))(x_2 + (m-1)x_1) \wedge x_{i_2} \wedge \dots \wedge x_{i_m}) = x_2 \wedge x_{i_2} \wedge \dots \wedge x_{i_m},$$

so that

$$\begin{aligned} A_m(1-(12))(x_1 \wedge x_{i_2} \wedge \dots \wedge x_{i_m}) &= (x_1 - x_2) \wedge x_{i_2} \wedge \dots \wedge x_{i_m} \\ &= B_m(1-(12))(x_1 \wedge x_{i_2} \wedge \dots \wedge x_{i_m}) \end{aligned}$$

3. $i_1 = 1, i_2 = 2$. Then

$$A_m((12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}) = -x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m},$$
$$B_m((12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}) = (m-2)x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m},$$

so that

$$\begin{aligned} A_m(1-(12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}) &= 2x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m} \\ &= B_m(1-(12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}). \end{aligned}$$

□

Denote by $\iota_n : S_n \to S_{n+1}$ a standard embedding: for any permutation $\sigma \in S_n$ take $\iota_n(\sigma)(k) = \sigma(k)$ for any $1 \le k \le n$ and $\iota_n(\sigma)(n+1) = n+1$. The embedding can be extended by linearity to an algebra homomorphism $\iota_n : \mathbb{C}[S_n] \to \mathbb{C}[S_{n+1}]$.

Proposition 4. $\iota_n(\mathcal{L}_n) \subset \mathcal{L}_{n+1}$.

Proof. Let $u = \sum_{\sigma \in S_n} a_\sigma \sigma \in \mathcal{L}_n$. Like in Proposition 3 above, it is enough to consider the action of $\iota_n(u)$ on $x \overset{\text{def}}{=} x_{i_1} \wedge \cdots \wedge x_{i_m}$ where $1 \le i_1 < \cdots < i_m \le n+1$. Consider two cases.

1. $i_m \le n$. Then $A_m(\iota_n(u))(x) = A_m(u)(x) = B_m(u)(x) = B_m(\iota_n(u))(x)$, so that $\iota_n(u) \in \mathcal{L}_{n+1}$.

2. $i_m = n+1$. Then $A_m(\iota_n(u))(x) = A_{m-1}(u)(x_{i_1} \wedge \cdots \wedge x_{i_{m-1}}) \wedge x_{n+1}$. On the other hand,

$$B_m(\iota_n(u))(x) = \left(\sum_{\sigma \in S_n} a_\sigma \sum_{p=1}^{n} x_{i_1} \wedge \cdots \wedge x_{\sigma(i_p)} \wedge \cdots \wedge x_{i_{m-1}}\right) \wedge x_{n+1} + \sum_{\sigma \in S_n} a_\sigma \cdot x.$$

One has $A_0(u) = \sum_{\sigma \in S_n} a_\sigma$ and $B_0(u) = 0$. Once $u \in \mathcal{L}_n$, the last term in the equation above is zero, so

$$B_m(\iota_n(u))(x) = B_{m-1}(u)(x_{i_1} \wedge \cdots \wedge x_{i_{m-1}}) \wedge x_{n+1},$$

whence $A_m(\iota_n(u))(x) = B_m(\iota_n(u))(x)$, and again $\iota_n(u) \in \mathcal{L}_{n+1}$. □

3. $\mathcal{L}_n$ FOR SMALL n

One has $\dim \mathcal{L}_2 = 1$. The space is spanned by $1-(12) \in \mathbb{C}[S_2]$, is a trivial Lie algebra and a trivial representation of $S_2 = \mathbb{Z}/2\mathbb{Z}$.

The space $\mathcal{L}_3$ contains elements $1-(12)$, $1-(23)$ and $1-(13)$ by Proposition 3. By the corollary of Proposition 1 it also contains $[1-(12), 1-(23)] = (123)-(132)$ (by $(i_1 \dots i_k) \in S_n$ we mean a cyclic permutation sending every i_s to $i_{s+1 \bmod k}$). Easy calculations show that these elements form a basis in $\mathcal{L}_3$, so that $\dim \mathcal{L}_3 = 4$. The space $\mathcal{L}_3$ splits, as a representation of S_3, to the trivial representation V_0 (spanned by $1-(12)/3-(13)/3-(23)/3$), sign representation V_1 (spanned by $(123)-(132)$) and a two-dimensional representation V_2 (spanned by $(12)-(13)$, $(13)-(23)$ and $(23)-(12)$; the elements sum up to zero, and any two of them form a basis). As a Lie algebra $\mathcal{L}_3$ is a direct sum of the center V_0 and a three-dimensional subalgebra spanned by $V_1 \cup V_2$. (This statement is partly true for any n: $\mathcal{L}_n$ contains a trivial representation, which lies in its center as a Lie algebra.)

The space $\mathcal{L}_4$ contains, by Proposition 3, the 6 elements $1-(ij)$, $1 \le i < j \le 4$. By Propositions 1 and 2 it also contains all the elements $(ijk)-(ikj) = [1-(ij), (1-$

$(jk)]$, $1 \le i < j < k \le 4$ (totally 4), and the elements $\gamma_1 = [1-(14), (123)-(132)] = (1234) + (1432) - (1243) - (1342)$ and $\gamma_2 = [1-(24), (123)-(132)] = (1243) + (1342) - (1324) - (1423)$. Easy computer-assisted computations show that these 12 elements form a basis in $\mathcal{L}_4$.

As a representation of S_4, $\mathcal{L}_4$ contains a 6-dimensional representation spanned by $1-(ij)$, $1 \le i < j \le 4$; it splits into a trivial representation spanned by $1 - \frac{1}{6}\sum_{1\le i<j\le 4}(ij)$, a 3-dimensional representation of the type $(3,1)$ and a 2-dimensional representation of the type $(2,2)$. Another 4-dimensional subrepresentation of $\mathcal{L}_4$ is spanned by $(ijk)-(ikj)$, $1 \le i < j < k \le 4$; it splits into a sign representation (spanned by $\sum_{1\le i<j<k\le 4}(ijk)-(ikj))$ and a 3-dimensional representation of the type $(2,1,1)$. The elements γ_1 and γ_2 span a 2-dimensional subrepresentation. Totally, $\mathcal{L}_4$ contains a trivial representation, a sign representation, two copies of a 2-dimensional representation and two nonisomorphic 3-dimensional representations.

4. Questions and conjectures

4.1. **Dimension and representations.** For an arbitrary n, what is the dimension of $\mathcal{L}_n$? A refinement of the question: find the Frobenius character $R_n = \sum_{|\lambda|=n} a_\lambda \chi_\lambda$ of the representation $\mathcal{L}_n$; here the sum runs over all partitions of n, a_λ is the multiplicity in $\mathcal{L}_n$ of the irreducible representation of S_n of the type λ, and χ_λ is the Schur polynomial corresponding to λ.

4.2. **Generators.**

Conjecture. *The Lie algebra $\mathcal{L}_n$ is generated by the elements $\nu_{ij} = 1-(ij)$, $1 \le i < j \le n$.*

Computations confirm the conjecture for $n \le 5$.

4.3. **Action on the original representation.** The elements of $\mathcal{L}(V) \subset k[G]$ act in the original representation V of the group G. This action may have a kernel. These kernels and quotients of $\mathcal{L}(V)$ by them sometimes exhibit interesting properties:

Conjecture. *Let K_n be a kernel of the action of $\mathcal{L}_n$ in the permutation representation $\mathbb{C}^n$. Then $\dim \mathcal{L}_n/K_n = (n-1)!$. The repeated commutators*

$$[\dots[\nu_{1i_1}, \nu_{2i_2}], \nu_{3i_3}], \dots], \nu_{n-1,i_{n-1}}]$$

for all $i_1, \dots, i_{n-1}$ such that $s+1 \le i_s \le n$ for all $s = 1, \dots, n-1$ form a basis in $\dim \mathcal{L}_n/K_n$.

Grants and acknowledgements

The final stage of the work was supported by the RFBR grant NSh-5138.2014.1 “Singularities theory and its applications”, by the Higher School of Economics (HSE) Scientific foundation grant 12-01-0015 “Differential geometry on graphs and discrete path integration”, by the by Dobrushin professorship grant 2013 (the Independent University of Moscow) and by the Simons foundation grant (autumn 2013).

References

[1] Yu.Burman, D.Zvonkine, Cycle factorizations and 1-faced graph embeddings, *European Journal of Combinatorics*, 31, no. 1 (2010), pp. 129–144.

Independent University of Moscow (119002, 11, B.Vlassievsky per., Moscow, Russia) and Higher School of Economics (National Research University; 101000, 20, Myasnitskaya str., Moscow, Russia)

E-mail address: `burman@mccme.ru`