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Regular Carathéodory-type selectors under no convexity assumptions

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Abstract

We prove the existence of Carathéodory-type selectors (that is, measurable in the first variable and having certain regularity properties like Lipschitz continuity, absolute continuity or bounded variation in the second variable) for multifunctions mapping the product of a measurable space and an interval into compact subsets of a metric space or metric semigroup.

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1. Introduction

The purpose of this paper is to obtain the existence of Carathéodory-type selectors (i.e., measurable in the first variable and having certain regularity in the second variable) for multifunctions of two variables with compact values.

By the axiom of choice, any multifunction with nonempty images admits a selector. Usually one is interested in selectors satisfying some regularity properties such as, e.g.,

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continuity, measurability, etc. The fundamental result on the existence of continuous selectors for convex-valued multifunctions was given by Michael [24]. The existence of measurable selectors for multifunctions on a measurable space with closed values from a complete separable metric space was proved by Kuratowski and Ryll-Nardzewski [22]. The study of Carathéodory selectors for multifunctions of two variables was initiated by Castaing [3,4] and Cellina [6], which was motivated by applications to differential inclusions. This adopts methods and results developed in the theories of continuous and measurable selections (see Kucia [20] and references therein). In those works the attention was paid to convex-valued multifunctions (recall that a continuous multifunction with nonconvex values might admit no continuous selector [16,24]).

The existence of regular selectors (i.e., preserving the properties of multifunctions) for nonconvex-valued multifunctions of one real variable which are Lipschitzian, absolutely continuous or of bounded variation were treated by Belov and Chistyakov [1], Chistyakov [7–12], Hermes [17], Mordukhovich [25], Qiji [26] and Ślęzak [28]. The last author [28, Theorem 5] proved the existence of selectors for nonconvex valued multifunctions of two variables which are continuous in the first variable and Lipschitz continuous in the second variable.

Let F be a multifunction defined on the product of a measurable space and an interval. In this paper, we are interested in finding those selectors of F which preserve measurability in the first variable t and regularity properties like Lipschitz continuity, absolute continuity or bounded variation in the second variable x. In order to do this, we introduce a new multifunction with nonempty values (according to the results described in the last paragraph) which maps the measurable variable t into the set of regular selectors of the one-variable multifunction $x \mapsto F(t,x)$, and then apply the graph-conditioned measurable selection theorems due to Brown and Purves [2] and Leese [23] (see also the survey by Wagner [29]). This approach to Carathéodory-type selectors was proposed by Castaing in [4].

The paper is organized as follows. Section 2 contains preliminaries and the formulation of the problem. In Section 3, we prove the existence of selectors for Carathéodory-type multifunctions which are Lipschitz continuous or continuous and of bounded variation in the second variable (Theorem 1). In Section 4, we endow the spaces of mappings of bounded variation and absolutely continuous mappings with the structure of a metric semigroup (Theorem 2), which permits us to obtain, in Section 5, the existence of selectors for Carathéodory-type multifunctions which are absolutely continuous or of bounded variation in the second variable (Theorems 3 and 4). Some open problems are enlisted at the end of the paper.

2. Preliminaries and statement of the problem

We begin with reviewing certain definitions and known facts needed for our results.

Let X be an interval (open, closed, half-closed, bounded or not) on the real line $\mathbb{R}$ and (Y,d) be a metric space with metric d.

Given a mapping $f : X \to Y$, the quantity (finite or not)

$$V(f) \equiv V_d(f, X) = \sup\left\{\sum_{i=1}^{m} d\left(f(x_i), f(x_{i-1})\right) : P = \{x_i\}_{i=0}^{m}\right\}$$

with the supremum taken over all partitions $P = \{x_i\}_{i=0}^{m}$ of X (i.e. $m \in \mathbb{N}$, $\{x_0, x_1, \ldots, x_m\} \subset X$ and $x_{i-1} < x_i$, $i = 1, \ldots, m$) is called the *total* (*Jordan*) *variation of f on X*. If $V(f) < \infty$, we say that the mapping f is *of bounded variation* and write $f \in \mathrm{BV}(X, Y)$.

Recall that a mapping $f : X \to Y$ is said to be *Lipschitzian* if the following quantity, called the (*minimal*) *Lipschitz constant of f*, is finite:

$$L(f) \equiv L_d(f, X) = \sup\left\{d(f(x), f(x'))\big/\,|x - x'| : x, x' \in X, x \neq x'\right\};$$

in this case we write $f \in \mathrm{Lip}(X, Y)$.

A mapping $f : X \to Y$ is said to be *absolutely continuous* (in symbols, $f \in \mathrm{AC}(X, Y)$) if $f \in \mathrm{BV}(X, Y)$ (this condition is redundant if X is compact) and for each $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ (depending on f, in general) such that if $n \in \mathbb{N}$, $\{a_i, b_i\}_{i=1}^{n} \subset X$, $a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n$ and $\sum_{i=1}^{m} (b_i - a_i) \leqslant \delta(\varepsilon)$, then $\sum_{i=1}^{m} d(f(b_i), f(a_i)) \leqslant \varepsilon$. We will also call the mapping f (more precisely) $\delta(\cdot)$-*absolutely continuous*. Note that $\mathrm{Lip}(X, Y) \subset \mathrm{BV}(X, Y)$ if X is bounded, and $\mathrm{AC}(X, Y) \subset \mathrm{BV}(X, Y)$.

Below $\mathrm{C}(X, Y)$ stands for the family of all continuous mappings $f : X \to Y$, endowed with the compact-open topology (i.e. the uniform convergence on compact subsets of X). It is known that if Y is a Polish space (i.e. complete and separable), then $\mathrm{C}(X, Y)$ is metrizable by a complete metric and separable.

If $A, B \subset Y$ are nonempty subsets, the *Hausdorff distance* $D = D_d$ *between A and B* is defined by (e.g. [5, Chapter II])

$$D(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup\{\mathrm{dist}\,(y, B) : y \in A\}$ and $\mathrm{dist}\,(y, B) = \inf\{d(y, y') : y' \in B\}$. It is well known that the mapping $D(\cdot, \cdot)$ is a *metric*, called the *Hausdorff metric induced* (or *generated*) *by d*, on the set of all nonempty closed bounded subsets of Y and, in particular, on the set $\mathcal{K}(Y)$ of all nonempty compact subsets of Y. Note also that if Y is Polish, then $\mathcal{K}(Y)$ is also Polish (see [5, Theorem II-9]).

Let E and Z be two nonempty sets. A *multifunction* Φ *from E into Z* (in symbols, $\Phi : E \rightrightarrows Z$) is a mapping associating to each point $t \in E$ a nonempty subset $\Phi(t) \subset Z$, the *image of t under* Φ. By the *graph of* Φ we mean the set $\mathrm{Gr}\,\Phi = \{(t, z) \in E \times Z : z \in \Phi(t)\}$. A mapping $\varphi : E \to Z$ is called a *selector of* Φ if $\mathrm{Gr}\,\varphi \subset \mathrm{Gr}\,\Phi$ or, equivalently, if $\varphi(t) \in \Phi(t)$ for all $t \in E$.

If $X \subset \mathbb{R}$ is an interval and (Y, d) is a metric space, a multifunction $F : X \rightrightarrows Y$ with compact images is said to be *of bounded variation on X* (respectively, *Lipschitzian on X* or $\delta(\cdot)$-*absolutely continuous on X*) if it is of bounded variation (respectively, Lipschitzian or $\delta(\cdot)$-absolutely continuous) as a mapping $F : X \to (\mathcal{K}(Y), D)$ in the above sense, where D is the Hausdorff metric induced by d. For such a multifunction we will use the notation $F \in \mathrm{BV}(X, \mathcal{K}(Y))$ (respectively, $F \in \mathrm{Lip}(X, \mathcal{K}(Y))$ or $F \in \mathrm{AC}(X, \mathcal{K}(Y))$).

The following theorem gives sufficient conditions for the existence of regular selectors for a compact-valued multifunction defined on an interval.

Theorem A. *Suppose that $X \subset \mathbb{R}$ is an interval, (Y, d) is a metric space, D is the Hausdorff metric on $\mathcal{K}(Y)$ induced by d, $x_0 \in X$ and $y_0 \in Y$ are fixed, and $F : X \to \mathcal{K}(Y)$ is a given multifunction. We have*:

(i) *if F is of bounded variation, then it admits a selector $f : X \to Y$ of bounded variation such that*

$$d(y_0, f(x_0)) = \mathrm{dist}\,(y_0, F(x_0)) \quad \textit{and} \quad V_d(f, X) \leqslant V_D(F, X); \tag{1}$$

(ii) *if F is of bounded variation and continuous, then it admits a continuous selector f of bounded variation satisfying* (1);

(iii) *if F is Lipschitzian, then it admits a Lipschitzian selector f satisfying* (1) *and such that $L_d(f, X) \leqslant L_D(F, X)$*;

(iv) *if F is $\delta(\cdot)$-absolutely continuous, then it has a $\delta(\cdot)$-absolutely continuous selector satisfying* (1).

Theorem A(i) is due to Chistyakov ([7, Theorem 9.1]; [8, Theorem 6.1(b)]; [9, Theorem 4] if Y is a Banach space, $\dim Y \leqslant \infty$ and $\mathrm{Gr}\, F$ is compact) and Belov and Chistyakov ([1, Theorem 2] in the general case). Continuous selectors in Theorem A(ii) and Lipschitzian selectors in Theorem A(iii) were established by Hermes ([17, Theorem 2] if $\dim Y < \infty$), Kikuchi and Tomita ([19, Theorem 1] if $\dim Y < \infty$ and F is convex-valued), Mordukhovich ([25, Theorem D1.8] if Y is a Banach space, $\dim Y \leqslant \infty$ and $\mathrm{Gr}\, F$ is compact) and Ślęzak ([28, Theorem 1] in the general case). In addition, the existence of selectors of bounded variation in Theorem A(ii), (iii) and property (1) was proved by Chistyakov [7, Theorem 9.2]; [8, Theorem 6.1(a), (c)]; [10, Theorem 13] and Belov and Chistyakov [1, Theorem 3(a), (b)]. Theorem A(iv) is due to Kikuchi and Tomita ([19, Theorem 2] with continuous selectors, $\dim Y < \infty$ and F convex-valued), Qiji ([26] if $\dim Y < \infty$), Chistyakov ([8, Theorem 6.1(d)]; [11, Theorem 5.1] if $\dim Y \leqslant \infty$ and $\mathrm{Gr}\, F$ is compact) and Belov and Chistyakov ([1, Theorem 3(c)] in the general case).

It is well known that a continuous compact-valued multifunction defined on an interval need not have a continuous selector (see the $\sin 1/x$ example of Michael [24] or Hermes [16, Example 1]). Even a Hölder continuous of any exponent $0 < \alpha < 1$ compact-valued multifunction on an interval may have no continuous selector (see [13, Proposition 8.2]).

In order to formulate our main results, we introduce further terminology.

Let $(T, \mathcal{M})$ be a measurable space with the σ-field $\mathcal{M}$ and Z be a topological space. We denote by $\mathcal{B}(Z)$ the Borel σ-field on Z.

Recall that a mapping $\varphi : T \to Z$ is said to be *measurable* if for each open $U \subset Z$ the inverse image $\varphi^{-1}(U) = \{t \in T : \varphi(t) \in U\}$ belongs to $\mathcal{M}$. We say that a multifunction $\Phi : T \rightrightarrows Z$ is *measurable* if for each open $U \subset Z$ the preimage $\Phi^{-}(U) = \{t \in T : \Phi(t) \cap U \neq \emptyset\}$ belongs to $\mathcal{M}$. If Z is metrizable and separable, and Φ is measurable and closed-valued, then $\mathrm{Gr}\,\Phi \in \mathcal{M} \otimes \mathcal{B}(Z)$ (see e.g. Himmelberg [18, Theorem 3.5]), where $\mathcal{M} \otimes \mathcal{B}(Z)$ is the product σ-field of σ-fields $\mathcal{M}$ and $\mathcal{B}(Z)$.

Denote by $\mathfrak{N}$ and $\mathfrak{N}^\star$, respectively, the sets of infinite and finite sequences of positive integers. Let $\mathcal{F}$ be a family of sets. We say $\mathcal{F}$ is a *Suslin family* if for each function $A : \mathfrak{N}^\star \to \mathcal{F}$ the set defined as

$$\bigcup_{\sigma \in \mathfrak{N}} \bigcap_{n=1}^{\infty} A(\sigma_1, \ldots, \sigma_n)$$

is also in $\mathcal{F}$ (Christensen [14] says that $\mathcal{F}$ is closed with respect to the Suslin operation).

Recall that a σ-field $\mathcal{M}$ is a Suslin family provided one of the following three conditions is satisfied [21, p. 95]; [14, pp. 23–24]: (i) $\mathcal{M}$ is complete with respect to a σ-finite measure; (ii) T is a topological space and $\mathcal{M}$ is the family of all sets with the Baire property; (iii) T is a locally compact topological space and $\mathcal{M}$ is the family of all subsets of T measurable with respect to a Radon measure on T.

In order to prove the existence of regular selectors for Carathéodory-type multifunctions corresponding to Theorem A we shall use the following graph-conditioned measurable selection theorem:

Theorem B. (i) *Let $(T, \mathcal{M})$ be a measurable space, Z be a Polish space, and $\Phi : T \rightrightarrows Z$ be a multifunction. If $\mathcal{M}$ is a Suslin family and* $\mathrm{Gr}\,\Phi \in \mathcal{M} \otimes \mathcal{B}(Z)$, *then Φ has a measurable selector.*

(ii) *Suppose that T and Z are Polish spaces, $\mathcal{M} = \mathcal{B}(T)$, and $\Phi : T \rightrightarrows Z$ is a σ-compact-valued multifunction with Borel graph. Then Φ admits a Borel-measurable selector.*

Theorem B(i) was given by Leese [23, Corollary to Theorem 7]; it is a generalization of results of Yankov and von Neumann (see [29]). Theorem B(ii) is due to Brown and Purves [2].

We shall also use the following projection theorem (see, e.g., [14, Theorem 1.3]):

Theorem C. *Let $(T, \mathcal{M})$ be a measurable space and Z be a Polish space. If $\mathcal{M}$ is a Suslin family, then for each $A \in \mathcal{M} \otimes \mathcal{B}(Z)$ its projection* $\mathrm{Proj}_T A$ *belongs to $\mathcal{M}$.*

Let $X \subset \mathbb{R}$ be an interval, (Y, d) be a metric space and f be a mapping from the product $T \times X$ into Y. If $x \in X$ is fixed, we define the mapping $f(\cdot, x) : T \to Y$ by $f(\cdot, x)(t) = f(t, x)$ for all $t \in T$, and similarly, if $t \in T$ is fixed, we define the mapping $f(t, \cdot) : X \to Y$ by $f(t, \cdot)(x) = f(t, x)$ for all $x \in X$. We say that $f : T \times X \to Y$ is *Carathéodory* (*Carathéodory–Lipschitz*) if $f(\cdot, x)$ is measurable for all $x \in X$ and

$f(t,\cdot) \in \mathrm{C}(X,Y)$ (respectively, $f(t,\cdot) \in \mathrm{Lip}(X,Y)$) for all $t \in T$; in other words, this is expressed as: f is measurable in the first variable $t \in T$ and is continuous (respectively, Lipschitzian) in the second variable $x \in X$. A similar definition applies to a multifunction $F : T \times X \rightrightarrows Y$ with compact images.

We shall deal with the following problem (cf. Theorem A). Given a multifunction $F : T \times X \to \mathcal{K}(Y)$, $x_0 \in X$ and $\xi : T \to Y$ a measurable mapping, does there exist a selector f of F satisfying the condition

$$d\left(\xi(t), f(t,x_0)\right) = \operatorname{dist}\left(\xi(t), F(t,x_0)\right) \quad \text{for all} \quad t \in T \tag{2}$$

and the same regularity properties as the multifunction F (e.g., measurability in the first variable t and Lipschitz continuity in the second variable x)? Note that if ξ is a (measurable) selector of $F(\cdot,x_0)$, then condition (2) can be rewritten in the form $f(t,x_0) = \xi(t)$. The answers to the above question are contained in the three main theorems of this paper, namely, Theorem 1 in Section 3 (corresponding to Theorem A(iii), (ii)) and Theorems 3 and 4 in Section 5 (which are parametrized versions of Theorem A(iv), (i)), and the selectors obtained for F are called *regular*. Section 4 is preparatory for Theorems 3 and 4.

A few final remarks are in order. A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is measurable in the first and is of bounded variation in the second variable need not be Lebesgue measurable: for instance, the characteristic function of the Sierpinski set (see Gelbaum and Olmsted [15, Example 10.21]) is upper semicontinuous and of bounded variation in each variable separately, but it is not Lebesgue measurable. On the other hand, a mapping measurable in the first variable and continuous in the second variable is product-measurable. More precisely, let $(T,\mathcal{M})$ be a measurable space, X and Y be two metric spaces with X separable, and $f : T \times X \to Y$ be measurable in the first and continuous in the second variable. Then f is $\mathcal{M} \otimes \mathcal{B}(X)$-measurable. Let us note that assumptions on X and Y can be weakened (see [20, Corollary 1, p. 315]).

3. Carathéodory-type selectors: part 1

The first main result of this paper is a theorem on the existence of Carathéodory–Lipschitz selectors and Carathéodory selectors of bounded variation in the second variable corresponding to Theorem A(iii) and (ii):

Theorem 1. *Let $(T,\mathcal{M})$ be a measurable space, $X \subset \mathbb{R}$ an interval, (Y,d) a Polish space, D the Hausdorff metric on $\mathcal{K}(Y)$ induced by d, $x_0 \in X$ fixed, $\xi : T \to Y$ a measurable mapping, and $F : T \times X \to \mathcal{K}(Y)$ be a Carathéodory multifunction. We have*:

(a) *if either $\mathcal{M}$ is a Suslin family or T is Polish and $\mathcal{M} = \mathcal{B}(T)$, and F is Carathéodory –Lipschitz, then F admits a Carathéodory–Lipschitz selector $f : T \times X \to Y$*

satisfying the inequality

$$V_d\,(f(t,\cdot),X) \leqslant V_D\,(F(t,\cdot),X) \quad \textit{for all} \quad t\in T, \tag{3}$$

condition (2) *and such that* $L_d\,(f(t,\cdot),X) \leqslant L_D\,(F(t,\cdot),X)$ *for all* $t\in T$;

(b) *if* $\mathcal{M}$ *is a Suslin family and* $F(t,\cdot)\in \mathrm{BV}(X,\mathcal{K}(Y))$ *for all* $t\in T$, *then* F *admits a Carathéodory selector* f *of bounded variation in the second variable satisfying conditions* (2) *and* (3).

Proof. (a) Define a new multifunction Φ from T into $\mathrm{C}(X,Y)$ by

$$\begin{aligned}\Phi(t)=\{u\in \mathrm{C}(X,Y)\,:\, u \text{ is a Lipschitzian selector of } F(t,\cdot) \text{ such that}\\ d(\xi(t),u(x_0))=\operatorname{dist}(\xi(t),F(t,x_0)),\\ L_d(u,X)\leqslant L_D(F(t,\cdot),X) \text{ and}\\ V_d(u,X)\leqslant V_D(F(t,\cdot),X)\},\quad t\in T.\end{aligned}$$

Because of Theorem A(iii), $\Phi(t)\neq\emptyset$ for all $t\in T$ (if, for some $t\in T$, $V_D(F(t,\cdot),X)$ is not finite, the last inequality in the definition of $\Phi(t)$ is redundant). Note that if $\varphi: T\to \mathrm{C}(X,Y)$ is a measurable selector of Φ, then the function $f: T\times X\to Y$ given by $f(t,x)=\varphi(t)(x)$ is a required Carathéodory–Lipschitz selector of F. In fact, it suffices to verify that the mapping $f(\cdot,x)$ is measurable for each $x\in X$. Given $x\in X$ and $U_Y\subset Y$ open, we set $U=\{u\in \mathrm{C}(X,Y): u(x)\in U_Y\}$, so that

$$(f(\cdot,x))^{-1}\,(U_Y)=\{t\in T:\varphi(t)(x)\in U_Y\}=\{t\in T:\varphi(t)\in U\}=\varphi^{-1}(U).$$

Since U is open in $\mathrm{C}(X,Y)$ and φ is measurable, $\varphi^{-1}(U)\in\mathcal{M}$.

Hence, we shall look for measurable selectors of Φ. In order to apply the graph-conditioned measurable selection Theorems B(i) and (ii), we will show that $\operatorname{Gr}\Phi\in \mathcal{M}\otimes\mathcal{B}(\mathrm{C}(X,Y))$ and Φ is compact-valued.

Note that $\lambda: T\to\mathbb{R}$ defined by $\lambda(t)=L_D(F(t,\cdot),X)$, $t\in T$, is a measurable function. Indeed, $\lambda(t)=\sup\{\mu(t,x,x'):x,x'\in X, x\neq x'\}$, where μ is a real-valued function defined on the product of T and $X\times X$ without the diagonal by the formula $\mu(t,x,x')=D(F(t,x),F(t,x'))/|x-x'|$, and so, μ is measurable in t and continuous in (x,x'). Consequently,

$$\lambda(t)=\sup\left\{\mu(t,x,x'):x,x'\in X \text{ are rational and } x\neq x'\right\},$$

which proves that λ is measurable.

Let α, δ, γ, $v: T\times \mathrm{C}(X,Y)\to\mathbb{R}\cup\{\infty\}$ and $\beta: \mathrm{C}(X,Y)\to\mathbb{R}\cup\{\infty\}$ be auxiliary functions defined for $t\in T$ and $u\in \mathrm{C}(X,Y)$ by

$$\alpha(t,u)=\sup\{\operatorname{dist}(u(x),F(t,x)):x\in X\},$$

$$\beta(u) = \sup\{d(u(x), u(x'))/|x - x'| : x,\, x' \in X,\, x \neq x'\},$$

$$\delta(t, u) = |d(\xi(t), u(x_0)) - \mathrm{dist}\,(\xi(t), F(t, x_0))|,$$

$$v(t, u) = V_d(u, X) - V_D(F(t, \cdot), X) \text{ if } V_D(F(t, \cdot), X) \text{ is finite, and}$$

$$v(t, u) = 0 \text{ otherwise},$$

$$\gamma(t, u) = \sup\{\alpha(t, u), \delta(t, u), \beta(u) - \lambda(t), v(t, u)\}.$$

We have $\mathrm{Gr}\,\Phi = \{(t, u) \in T \times \mathrm{C}(X, Y) : \gamma(t, u) \leqslant 0\}$. Now we show that γ is $\mathcal{M} \otimes \mathcal{B}(\mathrm{C}(X, Y))$-measurable and lower semicontinuous in u.

The function $(t, u, x) \mapsto \mathrm{dist}\,(u(x), F(t, x))$ is measurable in t and continuous in u and x. Thus, for each fixed $x \in X$ it is $\mathcal{M} \otimes \mathcal{B}(\mathrm{C}(X, Y))$-measurable. We have $\alpha(t, u) = \sup\{d(u(x), F(t, x)) : x \in X \text{ is rational}\}$ by virtue of the continuity in x. Consequently, α is product-measurable. Note that for each fixed $t \in T$, $\alpha(t, \cdot)$ is lower semicontinuous, being the supremum of continuous functions $u \mapsto \mathrm{dist}\,(u(x), F(t, x))$, $x \in X$.

Similarly, for each x, $x' \in X$, $x \neq x'$, the function $u \mapsto d(u(x), u(x'))/|x - x'|$ is continuous. Thus, β is lower semicontinuous.

The real-valued function δ is measurable in t and continuous in u, and hence, product-measurable.

The function $u \mapsto V_d(u, X)$ is lower semicontinuous (see [8, Proposition 2.1(V7)]). Since F is Carathéodory, the function $t \mapsto V_D(F(t, \cdot), X)$ is measurable. In fact, for each partition $P = \{x_i\}_{i=0}^m$ of X the sum $\sum_{i=1}^m D(F(t, x_i), F(t, x_{i-1}))$ depends measurably on t. Since the total variation $V_D(F(t, \cdot), X)$ is the supremum of these sums over all rational partitions P of X, it is measurable. We conclude that the function v is $\mathcal{M} \otimes \mathcal{B}(\mathrm{C}(X, Y))$-measurable and lower semicontinuous in u.

In this way we have proved that γ is product-measurable and lower semicontinuous in u. Hence, $\mathrm{Gr}\,\Phi \in \mathcal{M} \otimes \mathcal{B}(\mathrm{C}(X, Y))$.

If $\mathcal{M}$ is a Suslin family, then Theorem B(i) yields the existence of a measurable selector φ of Φ.

Now assume that T is a Polish space and $\mathcal{M} = \mathcal{B}(T)$. We have already established that $\mathrm{Gr}\,\Phi \in \mathcal{B}(T) \otimes \mathcal{B}(C(X, Y))$. Note that this product σ-field coincides with $\mathcal{B}(T \times C(X, Y))$, and so, the graph of Φ is Borel. Since γ is lower semicontinuous in u, $\Phi(t)$ is closed in $\mathrm{C}(X, Y)$ for each $t \in T$. Given $u \in \Phi(t)$ and $x \in X$, we have: $L_d(u, X) \leqslant \lambda(t)$, and $u(x)$ belongs to the compact set $F(t, x)$. By the Arzelà–Ascoli theorem, $\Phi(t)$ is a compact subset of $\mathrm{C}(X, Y)$. An application of Theorem B(ii) completes the proof of item (a).

(b) Let a multifunction $\Phi : T \rightrightarrows \mathrm{C}(X, Y)$ be defined by

$$\begin{aligned}\Phi(t) = \{u \in \mathrm{C}(X, Y) :\ & u \text{ is a selector of } F(t, \cdot) \text{ such that} \\ & d(\xi(t), u(x_0)) = \mathrm{dist}\,(\xi(t), F(t, x_0)) \text{ and} \\ & V_d(u, X) \leqslant V_D(F(t, \cdot), X)\}, \quad t \in T.\end{aligned}$$

By Theorem A(ii), the image $\Phi(t)$ is nonempty for every $t \in T$. Since, as is already shown, the functions α, δ and v from step (a) are $\mathcal{M} \otimes \mathcal{B}(\mathrm{C}(X, Y))$-measurable, the function $\gamma : T \times \mathrm{C}(X, Y) \to [0, \infty]$ defined by

$$\gamma(t, u) = \sup\{\alpha(t, u), \delta(t, u), v(t, u)\}, \ t \in T, \ u \in \mathrm{C}(X, Y),$$

is also product-measurable. Consequently, the graph $\mathrm{Gr}\,\Phi = \{(t, u) \in T \times \mathrm{C}(X, Y) : \gamma(t, u) \leqslant 0\}$ of Φ is measurable. By Theorem B(i), Φ admits a measurable selector $\varphi : T \times X \to Y$. Now, setting $f(t, x) = \varphi(t)(x)$, $(t, x) \in T \times X$, we obtain a Carathéodory selector f of F with desired properties. $\square$

We note that for multifunctions $F : T \times X \to \mathcal{K}(Y)$ with T a topological space, continuous selectors, which are Lipschitzian in the second variable, were established by Ślęzak [28, Theorem 5].

Before we turn to parametrized counterparts of Theorem A(i), (iv), we are going to study metric semigroups of mappings of bounded variation.

4. Macses of mappings of bounded variation

Recall (cf. [12]) that a triple $(Y, d, +)$ is called a *metric additive commutative semigroup* (*macs*, for short) if (Y, d) is a metric space, $(Y, +)$ is an additive commutative semigroup and d is translation invariant, i.e. $d(y + y'', y' + y'') = d(y, y')$ for all $y, y', y'' \in Y$. A macs $(Y, d, +)$ is called *complete* (respectively, *Polish*) if (Y, d) is a complete (respectively, Polish) metric space.

Simple examples of macses are any normed linear space $(Y, \|\cdot\|)$ with metric $d(y, y') = \|y - y'\|$, $y, \ y' \in Y$, and a nonempty convex cone $K \subset Y$ (i.e., $K + K \subset K$ and $\lambda K \subset K$, $\lambda \geqslant 0$). The following example is more interesting. Let $\mathcal{K}_{\mathrm{c}}(Y)$ be the family of all nonempty compact convex subsets of a real normed linear space $(Y, \|\cdot\|)$ equipped with the Hausdorff metric D generated by the norm $\|\cdot\|$. Since D is translation invariant on $\mathcal{K}_{\mathrm{c}}(Y)$ (cf. [27, Lemma 3]), the triple $(\mathcal{K}_{\mathrm{c}}(Y), D, +)$ is a macs, where the addition operation (the Minkowski sum) is defined by $A + B = \{a + b : a \in A, \ b \in B\}$ for $A, \ B \in \mathcal{K}_{\mathrm{c}}(Y)$. Recall also that if Y is a Banach (separable Banach) space, then this macs is a complete (Polish) macs (see [5, Theorems II-9 and II-14]).

If $(Y, d, +)$ is a macs, then, due to the translation invariance of d and the triangle inequality for d, for all $u, v, p, q \in Y$ we have

$$d(u, v) \leqslant d(p, q) + d(u + p, v + q), \tag{4}$$

$$d(u + p, v + q) \leqslant d(u, v) + d(p, q). \tag{5}$$

Inequality (5) implies that the addition operation $(u, v) \mapsto u + v$ is a continuous mapping from $Y \times Y$ into Y; more generally, if $u_n \to u$, $v_n \to v$, $p_n \to p$ and $q_n \to q$ in Y as $n \to \infty$, then $\lim_{n\to\infty} d(u_n + v_n, p_n + q_n) = d(u + v, p + q)$.

Let $X \subset \mathbb{R}$ be an interval, $x^0 \in X$ be fixed and $(Y, d, +)$ be a macs.

We endow the sets $\mathrm{BV}(X, Y)$ and $\mathrm{AC}(X, Y)$ with the pointwise addition operation $+$ and a metric d_V as follows:

$$d_V(f, g) \equiv d_V(f, g, X) = d(f(x^0), g(x^0)) + \Delta(f, g), \tag{6}$$

where

$$\Delta(f, g) \equiv \Delta(f, g, X) = \sup \sum_{i=1}^{m} d\left(f(x_i) + g(x_{i-1}), g(x_i) + f(x_{i-1})\right) \tag{7}$$

and the supremum is taken over all partitions $\{x_i\}_{i=0}^m \subset X$ ($m \in \mathbb{N}$) of X. We note that a metric of form (6)–(7) on the space $\mathrm{BV}(X, Y)$ (with $(S, \|\cdot\|)$ a Banach space, $X = [a, b]$, $Y = \mathcal{K}_\mathrm{c}(S)$, $d = D_{\|\cdot\|}$ and $+$ the Minkowski sum) was used by Zawadzka [30] for the characterization of set-valued Lipschitzian (Nemytskii) operators of substitution mapping $\mathrm{BV}(X, S)$ into $\mathrm{BV}(X, \mathcal{K}_\mathrm{c}(S))$.

The mapping $\Delta(\cdot, \cdot)$ is a translation invariant semimetric on $\mathrm{BV}(X, Y)$ (see Theorem 2(a); the triangle inequality for $\Delta(\cdot, \cdot)$ is a consequence of (4) and the translation invariance of d). Its main properties are gathered in the following:

Theorem 2. *For any two mappings $f, g \in \mathrm{BV}(X, Y)$ we have*:

(a) $|d(f(x), g(x)) - d(f(x'), g(x'))| \leqslant d(f(x) + g(x'), g(x) + f(x')) \leqslant \Delta(f, g)$ *for all* $x, x' \in X$;
(b) $\sup_{x \in X} d(f(x), g(x)) \leqslant d_V(f, g)$;
(c) *if* $\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty \subset \mathrm{BV}(X, Y)$ *and* $f_n \to f$, $g_n \to g$ *in metric d pointwise on X as* $n \to \infty$, *then* $\Delta(f, g) \leqslant \liminf_{n\to\infty} \Delta(f_n, g_n)$;
(d) $|V(f) - V(g)| \leqslant \Delta(f, g) \leqslant V(f) + V(g)$;
(e) $\Delta(f, g, X) = \Delta(f, g, X \cap (-\infty, x]) + \Delta(f, g, X \cap [x, \infty))$.

Consequently, if $(Y, d, +)$ is a (complete) macs, then $(\mathrm{BV}(X, Y), d_V, +)$ is also a (complete) macs. A similar assertion holds for the space $\mathrm{AC}(X, Y)$.

Proof. 1. (a), (b) The first inequality follows from (4), and the second one is a consequence of the definition of $\Delta(\cdot, \cdot)$. Item (b) readily follows from the definition of d_V and item (a).

(c) Suppose that $\lambda = \liminf_{n\to\infty} \Delta(f_n, g_n)$ is finite. Then there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that $\lambda = \lim_{k\to\infty} \Delta(f_{n_k}, g_{n_k})$. The pointwise convergence of f_{n_k} to f and g_{n_k} to g as $k \to \infty$ and the continuity of the addition operation in Y yield, for all $x, x' \in X$,

$$\lim_{k\to\infty} d\left(f_{n_k}(x) + g_{n_k}(x'), g_{n_k}(x) + f_{n_k}(x')\right) = d\left(f(x) + g(x'), g(x) + f(x')\right).$$

From the definition of $\Delta(f_{n_k}, g_{n_k})$ for any partition $\{x_i\}_{i=0}^m$ of X we have

$$\sum_{i=1}^{m} d\left(f_{n_k}(x_i) + g_{n_k}(x_{i-1}), g_{n_k}(x_i) + f_{n_k}(x_{i-1})\right) \leqslant \Delta(f_{n_k}, g_{n_k}), \quad k \in \mathbb{N}.$$

Passing to the limit as $k \to \infty$ in this inequality and then taking the supremum over all partitions of X we arrive at the desired inequality.

(d) Applying (4) for all $x, x' \in X$ we have

$$d\left(f(x), f(x')\right) \leqslant d\left(g(x), g(x')\right) + d\left(f(x) + g(x'), g(x) + f(x')\right)$$

and so, $V(f) \leqslant V(g) + \Delta(f, g)$, which implies the first inequality. To obtain the second inequality, it suffices to note from (5) that, for all $x, x' \in X$,

$$d\left(f(x) + g(x'), g(x) + f(x')\right) \leqslant d\left(f(x), f(x')\right) + d\left(g(x), g(x')\right).$$

(e) Let us fix $x \in X$ and denote by $\Delta[f, g, P]$ the finite sum under the supremum sign in (7) corresponding to the partition $P = \{x_i\}_{i=0}^m$ of X. If P_1 and P_2 are arbitrary partitions of $X \cap (-\infty, x]$ and $X \cap [x, \infty)$, respectively, we set $\tilde{P}_j = P_j \cup \{x\}$, $j = 1, 2$, so that $\tilde{P}_1 \cup \tilde{P}_2$ is a partition of X. Consequently,

$$\begin{aligned}\Delta[f, g, P_1] + \Delta[f, g, P_2] &\leqslant \Delta[f, g, \tilde{P}_1] + \Delta[f, g, \tilde{P}_2] \\ &= \Delta[f, g, \tilde{P}_1 \cup \tilde{P}_2] \leqslant \Delta(f, g, X),\end{aligned}$$

which implies that the right-hand side in (e) does not exceed the left-hand side. In order to prove the reverse inequality, let $P = \{x_i\}_{i=0}^m$ be a partition of X. Clearly, $\Delta[f, g, P]$ is less then or equal to the right-hand side in (e) if $x \in P$ or $x < x_0$ or $x_m < x$. So, suppose that $x_{k-1} < x < x_k$ for some $k \in \{1, \ldots, m\}$. By virtue of (4) and the translation invariance of d, we get

$$\begin{aligned}d\,(f(x_k) + g(x_{k-1}), g(x_k) + f(x_{k-1})) &\leqslant d\,(f(x) + g(x_{k-1}), g(x) + f(x_{k-1})) \\ &+ d\,(f(x_k) + g(x_{k-1}) + g(x) + f(x_{k-1}), g(x_k) + f(x_{k-1}) + f(x) + g(x_{k-1})) \\ &= d\,(f(x) + g(x_{k-1}), g(x) + f(x_{k-1})) + d\,(f(x_k) + g(x), g(x_k) + f(x)).\end{aligned}$$

It follows that

$$\begin{aligned}\Delta[f, g, P] &\leqslant \Delta[f, g, \{x_i\}_{i=0}^{k-1} \cup \{x\}] + \Delta[f, g, \{x\} \cup \{x_i\}_{i=k}^m] \\ &\leqslant \Delta(f, g, X \cap (-\infty, x]) + \Delta(f, g, X \cap [x, \infty)).\end{aligned}$$

2. That $\mathrm{BV}(X, Y)$ is a macs follows from the above definition, item (b), and triangle inequalities for and the translation invariance of $d(\cdot, \cdot)$ and $\Delta(\cdot, \cdot)$.

Suppose that Y is complete. Let us prove that $\mathrm{BV}(X, Y)$ is also complete. Let $\{f_n\}_{n=1}^\infty \subset \mathrm{BV}(X, Y)$ be a Cauchy sequence:

$$d_V(f_n, f_m) = d(f_n(x^0), f_m(x^0)) + \Delta(f_n, f_m) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{8}$$

By item (b), the sequence $\{f_n(x)\}_{n=1}^\infty$ is Cauchy in Y for all $x \in X$, and so, since Y is complete, there is a mapping $f : X \to Y$ such that $d(f_n(x), f(x)) \to 0$ as $n \to \infty$ for all $x \in X$. By item (d) and (8), the sequence $\{V(f_n)\}_{n=1}^\infty$ is Cauchy in $\mathbb{R}$, and so, it is bounded and convergent. Taking into account the lower semicontinuity of $V(\cdot)$ (e.g. [8, Proposition 2.1(V7)]) we find that $V(f) \leqslant \liminf_{n\to\infty} V(f_n) < \infty$, that is, f is of bounded variation on X. Item (c) implies

$$\Delta(f_n, f) \leqslant \liminf_{m\to\infty} \Delta(f_n, f_m) \leqslant \lim_{m\to\infty} d_V(f_n, f_m) \in [0, \infty), \quad n \in \mathbb{N}.$$

Again, because $\{f_n\}_{n=1}^\infty$ is Cauchy in $\mathrm{BV}(X, Y)$, we have

$$\limsup_{n\to\infty} \Delta(f_n, f) \leqslant \lim_{n\to\infty} \lim_{m\to\infty} d_V(f_n, f_m) = 0,$$

whence we conclude that $d_V(f_n, f) \to 0$ as $n \to \infty$.

3. The same arguments as in step 2 apply for the space $\mathrm{AC}(X, Y)$. The only thing we need to verify is that the limit f is in $\mathrm{AC}(X, Y)$ provided $\{f_n\}_{n=1}^\infty \subset \mathrm{AC}(X, Y)$ is Cauchy. Given $x \in X$ and $n \in \mathbb{N}$, we set $X_x = X \cap (-\infty, x]$, $v_n(x) = V(f_n, X_x)$ and $v(x) = V(f, X_x)$. By item (d),

$$|v_n(x^0) - v(x^0)| \quad \text{and} \quad |V(v_n - v, X)| \leqslant \Delta(f_n, f, X) \to 0 \quad \text{as} \quad n \to \infty \tag{9}$$

(the second inequality in (9) will be proved later in this proof), and so, the sequence of real absolutely continuous functions v_n (cf. [8, Lemma 3.3]) tends to v in the norm of $\mathrm{AC}(X, \mathbb{R})$ which is given by $\|v\|_{\mathrm{AC}} = |v(x^0)| + V(v, X)$. Since $\mathrm{AC}(X, \mathbb{R})$ is a Banach space, $v \in \mathrm{AC}(X, \mathbb{R})$, which, in turn (cf. [8, Lemma 3.2]), gives $f \in \mathrm{AC}(X, Y)$.

To prove the second inequality in (9), let $\{x_i\}_{i=0}^m \subset X$ be a partition of X of the form $x_0 < x_1 < \cdots < x_{m-1} < x_m$. Applying items (e) and (d), we have

$$\begin{aligned}\sum_{i=1}^m |(v_n - v)(x_i) - (v_n - v)(x_{i-1})| &= \sum_{i=1}^m |V(f_n, [x_{i-1}, x_i]) - V(f, [x_{i-1}, x_i])| \\ &\leqslant \sum_{i=1}^m \Delta(f_n, f, [x_{i-1}, x_i]) = \Delta(f_n, f, [x_0, x_m]) \\ &\leqslant \Delta(f_n, f, X).\end{aligned}$$

It remains to take into account the arbitrariness of the partition $\{x_i\}_{i=0}^m$. $\square$

5. Carathéodory-type selectors: part 2

Our next result deals with Carathéodory selectors which are absolutely continuous in the second variable.

Theorem 3. *Let* $(T, \mathcal{M})$ *be a measurable space with* $\mathcal{M}$ *a Suslin family,* $X \subset \mathbb{R}$ *an interval,* $(Y, d, +)$ *a Polish macs, D the Hausdorff metric on* $\mathcal{K}(Y)$ *generated by d,* $x_0 \in X$ *fixed,* $\xi : T \to Y$ *a measurable mapping, and* $F : T \times X \to \mathcal{K}(Y)$ *be a multifunction measurable in t and absolutely continuous in x. Suppose that* $(\mathrm{AC}(X, Y), d_V)$ *is separable. Then F has a selector* $f : T \times X \to Y$ *measurable in t, absolutely continuous in x and satisfying conditions* (2) *and* (3).

Proof. We argue in the same way as in the proof of Theorem 1. Let a new multifunction $\Phi : T \rightrightarrows \mathrm{AC}(X, Y)$ be defined by

$$\begin{aligned}\Phi(t) = \{u \in \mathrm{AC}(X, Y) \ : u \text{ is a selector of } F(t, \cdot) \text{ such that}\\ d(\xi(t), u(x_0)) = \mathrm{dist}\,(\xi(t), F(t, x_0)) \text{ and}\\ V_d(u, X) \leqslant V_D(F(t, \cdot), X)\}, \quad t \in T.\end{aligned}$$

We know from Theorem A(iv) that Φ has nonempty values. In order to apply Theorem B(i), we have to prove that $\mathrm{Gr}\,\Phi \in \mathcal{M} \otimes \mathcal{B}(\mathrm{AC}(X, Y))$.

Let α, δ, v and γ be the restrictions to $T \times \mathrm{AC}(X, Y)$ of corresponding auxiliary functions from the proof of Theorem 1(b). The graph of Φ is given by $\mathrm{Gr}\,\Phi = \{(t, u) \in T \times \mathrm{AC}(X, Y) : \gamma(t, u) \leqslant 0\}$. Since the topology of $\mathrm{AC}(X, Y)$ generated by the metric d_V is stronger than the topology induced from $\mathrm{C}(X, Y)$, the trace σ-field $\mathrm{AC}(X, Y) \cap \mathcal{B}(\mathrm{C}(X, Y))$ is contained in $\mathcal{B}(\mathrm{AC}(X, Y))$. Consequently, the functions α, δ, v and γ are $\mathcal{M} \otimes \mathcal{B}(\mathrm{AC}(X, Y))$-measurable. Hence, the graph of Φ is product-measurable. □

Remark 1. If X is a compact interval then the space $\mathrm{AC}(X, \mathbb{R}^n)$ is separable. In this case the d_V metric is generated by the norm

$$\|f\|_{\mathrm{AC}} = \|f(x^0)\| + \int_X \|f'(x)\| dx,$$

where $\|\cdot\|$ is the norm on $\mathbb{R}^n$.

Finally, we give a parametrized version of Theorem A(i). Let $(Y, d, +)$ be a macs and $F : T \times X \to \mathcal{K}(Y)$ a multifunction. Suppose that $v : T \to [0, \infty]$ is a measurable majorant of the total variation of $F(t, \cdot)$, i.e., $V_D(F(t, \cdot), X) \leqslant v(t)$ for all $t \in T$. Similarly as in the previous proofs we define a new multifunction $\Phi : T \rightrightarrows \mathrm{BV}(X, Y)$ by

$$\begin{aligned}\Phi(t) = \{u \in \mathrm{BV}(X, Y) \ : u \text{ is a selector of } F(t, \cdot) \text{ such that}\\ d(\xi(t), u(x_0)) = \mathrm{dist}\,(\xi(t), F(t, x_0)) \text{ and}\\ V_d(u, X) \leqslant v(t)\}, \quad t \in T,\end{aligned}$$

and show that Φ has a measurable selector. The space $\mathrm{BV}(X, Y)$ is considered with the metric d_V.

Theorem 4. *Let $(T, \mathcal{M})$ be a measurable space with $\mathcal{M}$ a Suslin family, $X \subset \mathbb{R}$ an interval, $(Y, d, +)$ a Polish macs, D the Hausdorff metric on $\mathcal{K}(Y)$ generated by d, $x_0 \in X$ fixed, $\xi : T \to Y$ a measurable mapping, and $F : T \times X \to \mathcal{K}(Y)$ be a $\mathcal{M} \otimes \mathcal{B}(X)$-measurable multifunction such that $F(t, \cdot) \in \mathrm{BV}(X, \mathcal{K}(Y))$ for each $t \in T$. Suppose that $\Phi(T) = \bigcup\{\Phi(t) : t \in T\}$ is a separable subspace of $\mathrm{BV}(X, Y)$. Then F admits a $\mathcal{M} \otimes \mathcal{B}(X)$-measurable selector f, which is of bounded variation in the second variable x, and satisfies conditions* (2) *and $V_d(f(t, \cdot), X) \leqslant v(t)$ for every $t \in T$.*

Proof. Let Z be the closure of $\Phi(T)$ in $\mathrm{BV}(X, Y)$. Such Z is a Polish space. We are going to prove that $\mathrm{Gr}\,\Phi \in \mathcal{M} \otimes \mathcal{B}(Z)$. Let α, $\delta : T \times Z \to [0, \infty]$ be defined as in the proof of Theorem 1(a). Define $\gamma : T \times Z \to [0, \infty]$ by

$$\gamma(t, u) = \sup\{\alpha(t, u), \delta(t, u), V_d(u, X) - v(t)\}.$$

Again $\mathrm{Gr}\,\Phi = \{(t, u) \in T \times Z : \gamma(t, u) \leqslant 0\}$.

Let $\beta : T \times X \times Z \to [0, \infty]$ be given by $\beta(t, x, u) = \mathrm{dist}\,(u(x), F(t, x))$. Fix $u \in Z$. Being a function of bounded variation, u has at most a countable set of discontinuity points (cf. [8, Theorem 4.3]). Hence, it is Borel-measurable. Consequently, $\beta(\cdot, u)$ is measurable in (t, x). For each real number r we have

$$\begin{aligned}\{t \in T : \alpha(t, u) > r\} &= \{t \in T : \beta(t, x, u) > r \text{ for some } x \in X\} \\ &= \mathrm{Proj}_T\{(t, x) \in T \times X : \beta(t, x, u) > r\}.\end{aligned}$$

By the Projection Theorem C, the last set belongs to $\mathcal{M}$, and hence, α is measurable in t. Let us show that it is continuous in u. Given $t \in T$ and $u_1, u_2 \in Z$, we have

$$\begin{aligned}|\alpha(t, u_1) - \alpha(t, u_2)| &\leqslant \sup\{|\beta(t, x, u_1) - \beta(t, x, u_2)| : x \in X\} \\ &\leqslant \sup\{d(u_1(x), u_2(x)) : x \in X\} \leqslant d_V(u_1, u_2).\end{aligned}$$

Thus, α is $\mathcal{M} \otimes \mathcal{B}(Z)$-measurable. It is immediate that δ is also product-measurable. Finally, the function $(t, u) \mapsto V_d(u, X) - v(t)$ is $\mathcal{M} \otimes \mathcal{B}(Z)$-measurable. This is a consequence of the measurability of v and the lower semicontinuity of $u \mapsto V(u, X)$. It follows that γ is product-measurable, and $\mathrm{Gr}\,\Phi \in \mathcal{M} \otimes \mathcal{B}(Z)$.

By Theorem B(i), Φ has a measurable selector $\varphi : T \to Z$. In order to complete the proof, we show that $f : T \times X \to Y$, given by $f(t, x) = \varphi(t)(x)$, is $\mathcal{M} \otimes \mathcal{B}(X)$-measurable. Let $W : Z \times X \to Y$ be the valuation function, i.e., $W(u, x) = u(x)$. Note that $f(t, x) = W(\varphi(t), x)$. The valuation W is continuous in u and Borel-measurable in x. Consequently, it is $\mathcal{B}(Z) \otimes \mathcal{B}(X)$-measurable. Thus f is product-measurable, as the superposition of W and measurable function $(t, x) \mapsto (\varphi(t), x)$. $\square$

Remark 2. The assumption of the separability of $\Phi(T)$ is rather restrictive. Therefore Theorem 4 may be considered as the first step in the study of selectors of bounded variation depending measurably on a parameter.

Some open problems

(1) It would be interesting to know whether Theorems 1, 3 and 4 hold for an arbitrary measurable space $(T, \mathcal{M})$.

(2) In Theorem 3 we have assumed the separability of $\mathrm{AC}(X, Y)$. Is this property implied automatically by the separability of Y?

(3) Is it possible to obtain a parametrized version of Theorem A(i) under less restrictive assumptions than in Theorem 4?

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Note added in proof

(1) More general assertions than Theorem A above are presented in a recent paper by V.V. Chistyakov, Selections of bounded variation, J. Appl. Anal. 10 (1) (2004) 1–82.
(2) A deeper insight into functions of two variables which are measurable in the first variable and of bounded variation in the second variable (cf. the last paragraph in Section 2) can be gained from a recent paper by M. Balcerzak, A. Kucia, A. Nowak, Regular dependence of total variation on parameters, Real Anal. Exchange 29 (2) (2003/2004) 921–930.
(3) Concerning the remark following the proof of Theorem 1, we should note also that for multifunctions $F : T \times X \to \mathcal{K}(Y)$ regular (and continuous) selections, which are continuous in the first variable and are of bounded generalized Riesz variation in the second varialbe, were obtained by V.V. Chistyakov, On multi-valued mappings of finite generalized, variation, Mat. Zametki 71 (4) (2002) 611–632 (in Russian) (English transl.: Math. Notes 71 (3–4) (2002) 556–575), cf. Theorem 15.

References

[1] S.A. Belov, V.V. Chistyakov, A selection principle for mappings of bounded variation, J. Math. Anal. Appl. 249 (2) (2000) 351–366.
[2] L.D. Brown, R. Purves, Measurable selections of extrema, Ann. Statist. 1 (1973) 902–912.
[3] C. Castaing, Sur l'existence des sections séparément mesurables et séparément continues d'une multi-application, Séminaire d'Analyse Convexe, Montpellier 1975, Exposé No. 14.

[4] C. Castaing, À propos de l'existence des sections séparément mesurable et séparément continues d'une multiapplication séparément mesurable et séparément semi continue inferieurement, Séminaire d'Analyse Convexe, Montpellier 1976, Exposé No. 6.
[5] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, vol. 580, Springer, Berlin, 1977.
[6] A. Cellina, A selection theorem, Rend. Sem. Mat. Univ. Padova 55 (1976) 143–149.
[7] V.V. Chistyakov, On mappings of bounded variation, J. Dynam. Control Systems 3 (2) (1997) 261–289.
[8] V.V. Chistyakov, On the theory of multivalued maps of bounded variation of one real variable, Mat. Sb. 189 (5) (1998) 153–176 (in Russian) (English transl.: Sbornik: Math. 189 (5) (1998) 797–819).
[9] V.V. Chistyakov, On mappings of bounded variation with values in a metric space, Uspekhi Mat. Nauk 54 (3) (1999) 189–190 (in Russian) (English transl.: Russian Math. Surveys 54 (3) (1999) 630–631).
[10] V.V. Chistyakov, Mappings of bounded variation with values in a metric space: generalizations, Pontryagin Conference, vol. 2, Nonsmooth Analysis and Optimization (Moscow, 1998). J. Math. Sci. (New York) 100 (6) (2000) 2700–2715.
[11] V.V. Chistyakov, Generalized variation of mappings with applications to composition operators and multifunctions, Positivity 5 (4) (2001) 323–358.
[12] V.V. Chistyakov, On mappings of finite generalized variation and nonlinear operators, Real Anal. Exchange 24th Summer Symposium Conference Reports, May 2000, pp. 39–43.
[13] V.V. Chistyakov, O.E. Galkin, On maps of bounded p-variation with $p > 1$, Positivity 2 (1) (1998) 19–45.
[14] J.P.R. Christensen, Topology and Borel Structure, Mathematics Studies, vol. 10, North-Holland, Amsterdam, 1974.
[15] B.R. Gelbaum, J.M.H. Olmsted, Counterexamples in Analysis, Holden-Day, San Francisco, London, Amsterdam, 1964.
[16] H. Hermes, Existence and properties of solutions of $\dot{x} \in R(t, x)$, Stud. Appl. Math. 5 (1969) 188–193.
[17] H. Hermes, On continuous and measurable selections and the existence of solutions of generalized differential equations, Proc. Amer. Math. Soc. 29 (3) (1971) 535–542.
[18] C.J. Himmelberg, Measurable relations, Fund. Math. 87 (1975) 53–72.
[19] N. Kikuchi, Y. Tomita, On the absolute continuity of multi-functions and orientor fields, Funkcial. Ekvac. 14 (3) (1971) 161–170.
[20] A. Kucia, Some results on Carathéodory selections and extensions, J. Math. Anal. Appl. 223 (2) (1998) 302–318.
[21] K. Kuratowski, Topology I, Academic Press and Polish Scientific Publishers, New York, London, Warsaw, 1966.
[22] K. Kuratowski, C. Ryll-Nardzewski, A general theorem on selectors, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 13 (6) (1965) 397–403.
[23] S.J. Leese, Multifunctions of Souslin type, Bull. Austral. Math. Soc. 11 (1974) 395–411.
[24] E.A. Michael, Continuous selections. I, Ann. Math. 63 (2) (1956) 361–382.
[25] B.Sh. Mordukhovich, Approximation Methods in the Problems of Optimization and Control, Nauka, Moscow, 1988 (in Russian).
[26] Z. Qiji, Single-valued representation of absolutely continuous set-valued mappings, Kexue Tongbao 31 (7) (1986) 443–446 (in Chinese).
[27] H. Rådström, An embedding theorem for spaces of convex sets, Proc. Amer. Math. Soc. 3 (1) (1952) 165–169.
[28] W.A. Ślęzak, Concerning continuous selectors for multifunctions with nonconvex values, Zeszyty Nauk. Wyższej Szkoły Pedagog. Bydgoszcz. Problemy Mat. 9 (1987) 85–104.
[29] D.H. Wagner, Survey of measurable selection theorems, SIAM J. Control Optim. 15 (5) (1977) 859–903.
[30] G. Zawadzka, On Lipschitzian operators of substitution in the space of set-valued functions of bounded variation, Rad. Mat. 6 (1990) 279–293.