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LCK rank of locally conformally Kähler manifolds with potential

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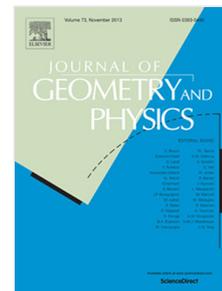# LCK rank of locally conformally Kähler manifolds with potential

Liviu Ornea[1], and Misha Verbitsky[2]

Abstract

An LCK manifold with potential is a compact quotient of a Kähler manifold X equipped with a positive Kähler potential f, such that the monodromy group acts on X by holomorphic homotheties and multiplies f by a character. The LCK rank is the rank of the image of this character, considered as a function from the monodromy group to real numbers. We prove that an LCK manifold with potential can have any rank between 1 and $b_1(M)$. Moreover, LCK manifolds with proper potential (ones with rank 1) are dense. Two *errata* to our previous work are given in the last Section.

Contents

1 Introduction

1.1 LCK manifolds

A complex manifold (M, I) is called **locally conformally Kähler** (LCK) if it admits a Hermitian metric g and a closed 1-form θ, called **the Lee form**, such that the fundamental 2-form $\omega(\cdot, \cdot) := g(\cdot, I\cdot)$ satisfies the integrability condition

$$d\omega = \theta \wedge \omega, \quad d\theta = 0. \tag{1.1}$$

The above definition is equivalent (see [DO]) to the existence of a covering $\tilde{M}$ endowed with a Kähler metric Ω which is acted on by the deck group $\mathrm{Aut}_M(\tilde{M})$ by holomorphic homotheties. Hence, if $\tau \in \mathrm{Aut}_M(\tilde{M})$, then $\tau^*\Omega = c_\tau \cdot \Omega$, where $c_\tau \in \mathbb{R}^{>0}$ is the scale factor. This defines a character

$$\chi : \mathrm{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}, \quad \chi(\tau) = c_\tau. \tag{1.2}$$

Two subclasses of LCK manifolds will be of interest to us.

The **Vaisman** class is formed by LCK manifold (M, ω, θ) with parallel Lee form with respect to the Levi-Civita connection of g. While the LCK condition is conformally invariant (if g is LCK, then any $e^f \cdot g$ is still LCK, with Lee form $\theta + df$), the Vaisman condition is not. The main example of Vaisman manifold is the diagonal Hopf manifold ([OV6]). Also, all compact complex submanifolds of Vaisman manifolds are Vaisman, too, [Ve1]. The Vaisman compact complex surfaces are classified in [Be].

We observed in [Ve1], [OV3] that the Kähler form of the universal cover of any Vaisman manifold has global potential represented by the square of the length of the Lee form. Moreover, the deck group acts on the potential by multiplying it with the character χ. This led us to introducing the larger class of LCK manifolds **with potential**. The precise definition requires the existence of a Kähler covering on which the Kähler metric has global, positive and proper potential function which is acted on by homotheties by the deck group. Besides Vaisman manifolds, there exist non-Vaisman examples, such as the non-diagonal Hopf manifolds, [OV3].

1.2 LCK manifolds with potential

"LCK manifolds with (proper) potential" can be defined as LCK manifolds (M, ω, θ) equipped with a smooth function $\psi \in \mathscr{C}^\infty(M)$ such that

$$\omega = d_\theta d_\theta^c \psi, \tag{1.3}$$

where $d_\theta(x) = dx - \theta \wedge x$, $d_\theta^c = I d_\theta I^{-1}$, and the following properties are satisfied:

(i) $\psi > 0$;

(ii) the class $[\theta] \in H^1(M, \mathbb{R})$ is proportional to a rational one.

$$(1.4)$$

For more details and historical context of this definition, please see Subsection 2.1. The differential d_θ is identified with the de Rham differential with coefficients in a flat line bundle L called **the weight bundle**. In this context, ψ should be considered as a section of L. After passing to the smallest covering $\tilde{M} \xrightarrow{\pi} M$ where θ becomes exact, the pull-back bundle $\pi^* L$ can be trivialized by a parallel section. Then the equation (1.3) becomes $\tilde{\omega} = d d^c \tilde{\psi}$, where $\tilde{\omega}$ is a Kähler form on $\tilde{M}$, and $\tilde{\psi}$ the Kähler potential.

Since $\tilde{M} \xrightarrow{\pi} M$ is the smallest covering where θ becomes exact, its monodromy is equal to $\mathbb{Z}^k$, where k is the rank of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contain $[\theta]$. In particular, the condition (1.4) (ii) means precisely that $\tilde{M} \xrightarrow{\pi} M$ is a $\mathbb{Z}$-covering. This implies that the definition (1.3)-(1.4) is equivalent to the historical one (Definition 2.1).

However, the condition (1.4) (ii) is more complicated: there are examples of LCK manifolds satisfying (1.3) and not (1.4) (ii) (Subsection 2.3). Still, any complex manifold (M, ω, θ) admitting an LCK metric with potential ψ satisfying (1.3), admits an LCK metric satisfying (1.3)-(1.4) in any $\mathscr{C}^\infty$-neighbourhood of (ω, θ). Therefore the condition (1.4) (ii) is not restrictive, and for most applications, unnecessary.

It makes sense to modify the notion of LCK manifold with potential to include the following notion (Subsection 2.3):

Definition 1.1: Let (M, ω, θ) be an LCK manifold, and $\psi \in \mathscr{C}^\infty(M)$ a strictly positive function satisfying $d_\theta d_\theta^c \psi = \omega$. Denote by k the rank of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contain $[\theta]$. Then ψ is called **proper potential** if $k = 1$ and **improper potential** if $k > 1$.

1.3 Some errors found

This paper is much influenced by Paul Gauduchon, who discovered an error in our result mentioned as obvious in [OV4]. In [OV4], we claimed erroneously that an LCK metric is pluricanonical (*i.e.* $(\nabla \theta)^{1,1} = 0$, see [Kok]), if and only if it admits an LCK potential. This was obvious because (as we claimed) the equations for LCK with potential and for pluricanonical metric are the same. Unfortunately, a scalar multiplier was missing in our equation for the pluricanonical metric.

From an attempt to understand what is brought by the missing multiplier, this paper grew, and we found an even stronger result: any compact pluricanonical manifold is Vaisman. Very recently, Andrei and Sergiu Moroianu gave a simple, direct proof of this result, using elegant tensor computations, [MM].

However, during our work trying to plug a seemingly harmless mistake, we discovered a much more offensive error, which has proliferated in a number of our papers.

In [OV1], we claimed that any Vaisman manifold admits a $\mathbb{Z}$-covering which is Kähler. This is true for locally conformally hyperkähler manifolds, as shown in [Ve1]. However, this result is false for more general Vaisman manifolds, such as a Kodaira surface (Theorem 3.4).

It is easiest to state this problem and its solution using the notion of "LCK rank" (Definition 2.5), defined in [GOPP] and studied in [PV]. Briefly, the LCK rank is the smallest r such that there exists a $\mathbb{Z}^r$-covering $\tilde{M}$ of M such that the pullback of the LCK metric is conformally equivalent to a Kähler metric on $\tilde{M}$.

It turns out that the LCK rank of a Vaisman manifold can be any number between 1 and $b_1(M)$ (Theorem 3.4). Moreover, for each r, the set of all Vaisman metrics of LCK rank r is dense in the space of all Vaisman metrics (say, with $\mathscr{C}^\infty$-topology).

It is disappointing to us (and even somewhat alarming) that nobody has discovered this important error earlier.

However, not much is lost, because the metrics which satisfy the Structure Theorem of [OV1] are dense in the space of all LCK metrics, hence all results of complex analytic nature remain true. To make the remaining ones correct, we need to add "Vaisman manifold of LCK rank 1" or "Vaisman manifold with proper potential" (Subsection 2.3) to the set of assumptions whenever [OV1] is used.

Still, we want to offer our apologies to the mathematical community for managing to mislead our colleagues for such a long time.

For more details about our error and an explanation where the arguments of [OV1] failed, please see Subsection 3.2.

2   LCK manifolds: properness of the potential

2.1   LCK manifolds with potential: historical definition

When the notion of LCK manifold with potential was introduced in [OV3], we assumed properness of the potential. Later, it was "proven" that the potential is always proper ([OV5]). Unfortunately, the proof was false (see the *Errata* to this paper, Section 3). In view of this error and other results in Section 3, it makes sense to generalize the notion of LCK manifold with potential to include the manifolds

with LCK rank > 1. For the old notion of LCK with potential we should attach "proper" to signify that the potential is a proper function on the minimal Kähler covering.

Definition 2.1: ([OV3]) An **LCK manifold with proper potential** is a manifold which admits a Kähler covering $(\tilde{M}, \tilde{\omega})$ and a smooth function $\varphi : \tilde{M} \to \mathbb{R}^{>0}$ (the **LCK potential**) satisfying the following conditions:

(i) φ is proper, *i.e.* its level sets are compact;

(ii) The deck transform group acts on φ by multiplication with the character χ (see (1.2)): $\tau^*(\varphi) = \chi(\tau)\varphi$, where $\tau \in \mathrm{Aut}_M(\tilde{M})$ is any deck transform map[1].

(iii) φ is a Kähler potential, *i.e.* $dd^c\varphi = \tilde{\omega}$.

Remark 2.2: In this situation, we can fix a choice of LCK metric on M by writing $\pi^*\omega = \varphi^{-1}dd^c\varphi$. Further on, we shall tacitly assume that this choice is used whenever we work with an LCK manifold with potential. In this case, the Lee form is written as $\pi^*\theta = d\log\varphi$, and $d^c(\pi^*\theta) = \pi^*\theta \wedge I(\pi^*\theta) - \pi^*\omega$ ([OV2]).

Remark 2.3: Positivity of the potential cannot be relaxed, as the following simple example (for which we thank V. Vuletescu) shows. On $\mathbb{C}^2 \setminus 0$, with $\mathbb{Z}$ acting as $(z_1, z_2) \mapsto (2z_1, 2z_2)$ (the quotient being the usual Hopf surface, which is Vaisman) take $\varphi(z_1, z_2) = |z_1|^2 + |z_2|^2 - \frac{1}{3}(z_1 + \overline{z}_1)^2$. Then: $\partial\overline{\partial}\varphi = \frac{1}{3}dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2$ and $\varphi(2z_1, 2z_2) = 4\varphi(z_1, z_2)$, and hence the potential is automorphic. But $\varphi(1, 0) = -\frac{1}{3}$, $\varphi(0, 1) = 1$, and $\varphi^{-1}(0)$ is non-empty.

More general examples are obtained by starting from any automorphic potential and adding the real part of a convenient holomorphic function, automorphic with the same automorphy factor as the potential.

2.2 Properness of the LCK potential

In [OV3], it was also shown that the properness condition is equivalent to the following condition on the deck transform group of $\tilde{M}$. Recall that a group is **virtually cyclic** if it contains $\mathbb{Z}$ as a finite index subgroup. We obtain the following claim (which proof we include for convenience):

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[1]In general, differential forms $\eta \in \Lambda^\bullet \tilde{M}$ which satisfy $\tau^*\eta = \chi(\tau)\eta$ are called **automorphic**. In particular, so is the Kähler form on $\tilde{M}$.

Claim 2.4: Let M be a compact manifold, $\tilde{M}$ a covering, and $\varphi : \tilde{M} \to \mathbb{R}^{>0}$ an automorphic function, that is, a function which satisfies $\gamma^* \varphi = c_\gamma \varphi$ for any deck transform map γ, where $c_\gamma = \chi(\gamma) \neq 1$ is constant (χ being the character (1.2)). Then φ is proper if and only if the deck transform group $\Gamma := \operatorname{Aut}_M(\tilde{M})$ of $\tilde{M}$ is virtually cyclic.

Proof: By passing to a smaller cover, we may suppose $\Gamma = \mathbb{Z}$ and let γ be a generator of $\mathbb{Z}$, such that $\gamma^* \varphi = \lambda \varphi$, and $\pi : \tilde{M} \longrightarrow \tilde{M}/\Gamma = M$ the quotient map. Then $\varphi^{-1}([1, \lambda[)$ is a fundamental domain of the Γ-action. Therefore $\pi : \varphi^{-1}([1, \sqrt{\lambda}]) \longrightarrow M$ is bijective onto its image, which is compact, and hence $\varphi^{-1}([1, \sqrt{\lambda}])$ is also compact. This implies that the preimage of any closed interval is compact.

Conversely, suppose φ is proper and, by absurd, assume $\Gamma \neq \mathbb{Z}$. Then Γ is a dense subgroup of $\mathbb{R}^{>0}$. Fix $x \in \tilde{M}$ and a nonempty interval $]a, b[\subset \mathbb{R}^{>0}$, and let

$$\mathfrak{H} := \left\{ \gamma \in \Gamma; \, \varphi(\gamma(x)) \in]a, b[\right\}.$$

Since Γ is dense, $\mathfrak{H}$ is infinite. However, $\varphi(\mathfrak{H} \cdot x) \subset [a, b]$, hence the infinite discrete set $\mathfrak{H} \cdot x$ is contained in a compact $\varphi^{-1}([a, b])$. This contradiction ends the proof. ∎

Definition 2.5: Let (M, ω, θ) be an LCK manifold. Define the **LCK rank** as the dimension of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that the Lee class $[\theta]$ lies in $V \otimes_{\mathbb{Q}} \mathbb{R}$.

Remark 2.6: The character $\chi : \operatorname{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}$ is defined on any LCK manifold, because the Kähler form $\tilde{\omega}$ is automorphic by definition: $\tau^*(\tilde{\omega}) = \chi(\tau)\tilde{\omega}$. Then one can see that the LCK rank as defined above coincides with the rank of the image of $\chi : \operatorname{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}$ which is also called **the weight monodromy group** of the LCK manifold. See also [GOPP] for another interpretation of the LCK rank and see [PV] for examples on non-Vaisman compact LCK manifolds with Kähler rank greater than 1. Clearly, LCK rank 0 corresponds to globally conformally Kähler structures.

From Claim 2.4 above it follows that condition (i) in Definition 2.1 is equivalent to M being of LCK rank 1.

In [OV2], we managed to get rid of the need to take the covering in Definition 2.1, by using the Morse-Novikov (twisted) differential $d_\theta := d - \theta \wedge \cdot$, where $\theta \wedge \cdot(x) = \theta \wedge x$, and θ is the Lee form. In [OV2] the definition of LCK manifold with potential was restated equivalently as follows.

Definition 2.7: Let (M, ω, θ) be an LCK manifold of LCK rank 1. Then M is called **LCK manifold with (proper) potential** if there exists a positive function $\varphi_0 \in \mathscr{C}^\infty(M)$ satisfying $d_\theta d_\theta^c(\varphi_0) = \omega$, where $d_\theta^c = I d_\theta I^{-1}$.

Claim 2.8: Definition 2.1 is equivalent to Definition 2.7.

Proof: To see that Definition 2.1 and Definition 2.7 are equivalent, consider the smallest covering $\pi : \tilde{M} \longrightarrow M$ such that $\pi^*\theta$ is exact, and take a function ψ satisfying $d\psi = \pi^*\theta$. Since $\pi^*\theta$ is invariant under the deck transform group Γ, for each $\gamma \in \Gamma$ one has $\gamma^*\psi = \psi + c_\gamma$, where c_γ is a constant. Consider the multiplicative character $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ given by $\chi(\gamma) = e^{-c_\gamma}$. Let $\Lambda_\chi^*(M)$ denote the space of automorphic forms on $\tilde{M}$ which satisfy $\gamma^*\eta = \chi(\gamma)\eta$. The map $\Lambda^*(M) \xrightarrow{\Psi} \Lambda_\chi^*(M)$ mapping η to $e^{-\psi}\pi^*\eta$ makes the following diagram commutative:

$$
\begin{array}{ccc}
\Lambda^*(M) & \xrightarrow{\ \Psi\ } & \Lambda_\chi^*(M) \\
{\scriptstyle d_\theta}\big\downarrow & & d\,\big\downarrow \\
\Lambda^*(M) & \xrightarrow{\ \Psi\ } & \Lambda_\chi^*(M)
\end{array}
$$

Then Ψ maps a "potential" φ_0 in the sense of Definition 2.7 to a potential ψ in the sense of Definition 2.1 and vice versa. Properness of $\Psi(\varphi_0)$ is equivalent to Γ being virtually cyclic, as Claim 2.4 implies. The existence of a Kähler covering with virtually cyclic deck transform group is clearly equivalent to M having LCK rank 1. ∎

Example 2.9: On a diagonal Hopf manifold $(\mathbb{C}^n \setminus 0)/\mathbb{Z}$, with LCK form and Lee form written on $\mathbb{C}^n \setminus 0$ respectively $\omega = |z|^{-2} \sum dz^i \wedge d\overline{z}^i$ and $\theta = -d\log|z|^2$, the function $\varphi_0 \in \mathscr{C}(M)$ is the constant function 1, while the potential function on $\mathbb{C}^n \setminus 0$ is $|z|^2$.

We now show that automorphic potentials can be approximated by proper ones. The following argument is taken from [OV2].

Claim 2.10: Let (M, ω, θ) be an LCK manifold, and $\varphi \in \mathscr{C}^\infty(M)$ a positive function satisfying $d_\theta d_\theta^c(\varphi) = \omega$. Then M admits an LCK structure (ω', θ') of LCK rank 1, approximating (ω, θ) in $\mathscr{C}^\infty$-topology.

Proof: Replace θ by a form θ' with rational cohomology class $[\theta']$ in a sufficiently small $\mathscr{C}^\infty$-neighbourhood of θ, and let $\omega' := d_{\theta'} d_{\theta'}^c(\varphi)$. Then ω' approxi-

mates ω in $\mathscr{C}^\infty$-topology, hence for θ' sufficiently close to θ, the form ω' is positive. It is $d_{\theta'}$-closed, because $d^2_{\theta'} = 0$, hence $0 = d_{\theta'}\omega' = d\omega' - \theta' \wedge \omega'$. This implies that (ω',θ') is an LCK structure. The Kähler rank of an LCK manifold is the dimension of the smallest rational subspace $W \subset H^1(M,\mathbb{Q})$ such that $W \otimes_\mathbb{Q} \mathbb{R}$ contains the cohomology class of the Lee form. Since $[\theta']$ is rational, (M,ω',θ') has LCK rank 1. $\blacksquare$

2.3 LCK manifolds with proper and improper potential

It seems now that the equation $d_\theta d^c_\theta \psi = \omega$ (on the LCK manifold M itself) is more fundamental than the notion of LCK manifold with (proper) potential. For most applications, this (more general) condition is already sufficient.

The relation between manifolds with $d_\theta d^c_\theta \psi = \omega$ and LCK with potential is similar to the relation between general Vaisman manifolds and quasiregular ones[2]. One could always deform an irregular Vaisman manifold to a quasiregular one, and quasiregular Vaisman manifolds are dense in the space of all Vaisman manifolds.

The notion of "LCK manifold with improper potential" is similar, in this regard, to the notion of irregular Vaisman or irregular Sasakian manifold[3], [BG].

Definition 2.11: Let (M,θ,ω) be an LCK manifold, and ψ a strictly positive function which satisfies $d_\theta d^c_\theta \psi = \omega$. Then (M,θ,ω) is called **a manifold with improper LCK potential** if its LCK rank is $\geqslant 2$, and **a manifold with proper LCK potential** if it has LCK rank 1.

Remark 2.12: The expressions "proper potential" and "improper potential", when used for solutions of the equation $d_\theta d^c_\theta \psi = \omega$, as in the above Definition, do not refer to the properness of $\psi : M \longrightarrow \mathbb{R}$, which is not plurisubharmonic and is always proper if M is compact.

Note that "LCK with potential" was previously used instead of "manifold with proper LCK potential"; now (in light of the discovery of Vaisman manifolds having improper potential, see Section 3) it makes sense to change the terminology by including improper potentials in the definition of LCK with potential.

Claim 2.10 can be rephrased as follows.

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[2](Quasi)Regularity and irregularity of a Vaisman manifold refers to the (quasi)regularity and irregularity of the 2-dimensional canonical foliation generated by $\theta^\sharp$ and $I\theta^\sharp$.

[3]Here (ir)regularity refers to the 1-dimensional foliation generated by the Reeb field.

Proposition 2.13: Let $(M, \omega, \theta, \psi)$ be a compact LCK manifold with improper LCK potential. Then (ω, θ, ψ) can be approximated in the $\mathscr{C}^\infty$-topology by an LCK structure with proper LCK potential. ∎

Remark 2.14: We have just proven that existence of an LCK metric with improper LCK potential implies existence of a metric with proper LCK potential on the same manifold. The converse is clearly false: when $H^1(M, \mathbb{R})$ is 1-dimensional, any Lee class is proportional to an integral cohomology class, and any LCK structure has LCK rank 1, hence M admits no metrics with improper LCK potentials.

However, in all other situations improper potentials do exist.

Proposition 2.15: Let $(M, \omega, \theta, \psi)$ be an LCK manifold with potential, and suppose $b_1(M) > 1$. Then M admits an LCK metric $(M, \omega', \theta', \psi)$ with improper potential and arbitrary LCK rank between 2 and $b_1(M)$. Moreover, (ω', θ') can be chosen in arbitrary $\mathscr{C}^\infty$-neighbourhood of (ω, θ).

Proof: Choose a closed θ' in a sufficiently small neighbourhood of θ, and let V_θ be the smallest rational subspace of $H^1(M, \mathbb{R})$ such that $V_\theta \otimes_\mathbb{Q} \mathbb{R}$ contains θ. Since the choice of the cohomology class $[\theta']$ is arbitrary in a neighbourhood of $[\theta]$, the dimension of V_θ can be chosen in arbitrary way. Choosing θ' sufficiently close to θ, we can assume that the (1,1)-form $\omega' := d_{\theta'} d_{\theta'}^c(\psi)$ is positive definite. Then $(M, \omega', \theta', \psi)$ is an LCK manifold with improper potential and arbitrary LCK rank. ∎

3 *Errata*

3.1 **Pluricanonical condition revisited**

In Section 3 of [OV4] the following erroneous claim was made: "We now prove that the pluricanonical condition is equivalent with the existence of an automorphic potential on a Kähler covering."

Then we proceeded to make calculations purporting to show that pluricanonical condition (*i.e.* $(\nabla\theta)^{1,1} = 0$, see [Kok]) is equivalent to the LCK with potential condition $d(I\theta) = \omega - \theta \wedge I\theta$. Here, the scalar term is lost: the correct equation (in the notation of [OV4]) is $d(I\theta) = |\theta|^2 \omega - \theta \wedge I\theta$.

It is of course true that this equation, indeed, implies $d(I\theta) = \omega - \theta \wedge I\theta$.

However, the converse statement is false: not all LCK manifolds with potential admit a pluricanonical LCK structure, but only Vaisman ones, see [MM].

3.2   LCK rank of Vaisman manifolds

Recall that the **LCK rank** of an LCK manifold (M, ω, θ) is the rank of the smallest rational subspace V in $H^1(M, \mathbb{R})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contains the cohomology class $[\theta]$. When the LCK rank is 1, the manifold admits a $\mathbb{Z}$-covering which is Kähler (Section 2).

In several papers published previously ([OV1], [OV2], [OV5]) we claimed that a Vaisman manifold and an LCK manifold with potential always have LCK rank 1. This is in fact false. In this section we produce a counterexample to these claims, and explain the error.

Notice, however, that, as we prove below, any complex manifold which admits a structure of a Vaisman manifold (or LCK manifold with potential) *also* admits a structure of a Vaisman manifold (or LCK manifold with potential) with LCK rank one. This means that all problems arising because of this error are of differential-geometrical nature; results of complex geometry remain valid. This is probably the reason why the error was not noticed for so many years. Moreover, *the set of Vaisman (or LCK with potential) structures with LCK rank 1 on a given manifold is dense in the set of all Vaisman (or LCK with potential) structures.*

LCK manifolds with potential can have arbitrary LCK rank, as follows from Proposition 2.15. To construct a Vaisman manifold with an LCK rank bigger than 1, we proceed as follows.

Let us recall some facts from Vaisman geometry used in this construction. Any Vaisman manifold is equipped with a canonical holomorphic foliation Σ, generated by the Lee field $\theta^\sharp$ and $I(\theta^\sharp)$ ([Va1], [Ts1]). This foliation might have a global leaf space (in this case the Vaisman manifold is called **quasiregular**), or have non-closed leaves (**irregular Vaisman**). Locally, the leaf space always exists. **Transversal forms** are forms which are lifted (locally) from the leaf space of Σ. K. Tsukada in [Ts2] proved the following decomposition theorem.

Theorem 3.1: The space of harmonic forms on a compact Vaisman manifold M can be expressed as

$$\mathcal{H}^*(M) = \theta \wedge \mathcal{H}^*_{tr}(M) \oplus \mathcal{H}^*_{tr}(M)$$

where $\mathcal{H}^*_{tr}(M)$ is the space of transversal harmonic forms. ∎

The Vaisman manifold is transversally Kähler, that is, the leaf space of the canonical foliation Σ is locally equipped with a complex structure and a globally defined transversally Kähler form. This allows us to use the Hodge decomposition theorem for transversal harmonic forms ([EG]), entirely similar to the usual Hodge decomposition theorem in Kähler geometry. In particular, any transversal

harmonic 1-form on a Vaisman manifold is the sum of a transversal holomorphic form and a transversal antiholomorphic form.

This leads to the following useful corollary.

Corollary 3.2: Let M be a compact Vaisman manifold. Then the space of harmonic 1-forms can be decomposed as $\mathscr{H}^1(M,\mathbb{R}) = \langle\theta\rangle \oplus \mathrm{Re}(H^0(\Omega^1_{tr}(M)))$, where $\Omega^1_{tr}(M)$ denotes the sheaf of holomorphic transversal 1-forms. ∎

Proposition 3.3: Let (M,θ,ω) be a compact Vaisman manifold, α a harmonic 1-form, and $\theta' := \theta + \alpha$. Consider the (1,1)-form $\omega' := d_{\theta'} d^c_{\theta'}(1)$ obtained as a deformation of $\omega = d_\theta d^c_\theta(1)$. Assume that α is chosen sufficiently small in such a way that ω' is Hermitian (Proposition 2.15). Then ω' is conformally equivalent to a Vaisman form.

Proof: Consider the holomorphic flow F generated by the Lee field $\theta^\sharp$. It fixes ω and θ and its lift $\tilde{F}$ to the universal cover $\tilde{M}$ acts by non-trivial homotheties with respect to the Kähler metric $\tilde{\omega}$.

We shall show that: (1) F preserves ω' and (2) $\tilde{F}$ acts by non-trivial homotheties with respect to the Kähler metric $\tilde{\omega}'$ corresponding to ω'.

As α is the sum of $\lambda\theta$ and a transversal form (Corollary 3.2), it is preserved by F, too. Then θ' is preserved by F, and also $I(\theta')$ is preserved, because F is holomorphic. As d commutes with the action of the flow and 1 is a constant function, ω' is preserved by F. This proves (1).

As for (2), if $\tilde{\omega}$ is the Kähler form on $\tilde{M}$ corresponding to (M,ω), then note that $\tilde{M}/\langle e^{\tilde{F}}\rangle$ is Vaisman, too, and hence has odd b_1, [Va1]. If $\tilde{F}$ acts by isometries on $\tilde{\omega}'$, then this Kähler metric descends to a Kähler metric on $\tilde{M}/\langle e^{\tilde{F}}\rangle$, contradiction.

By [KO, Theorem A], (1) and (2) imply that ω' is conformally equivalent to a Vaisman metric. ∎

This gives the following unexpected result:.

Theorem 3.4: Let M be a compact Vaisman manifold or LCK manifold with potential and let $\mathscr{L} \subset H^1(M,\mathbb{R})$ be the set of cohomology classes of all Lee forms for the Vaisman (LCK with potential) structures on M. Then $\mathscr{L}$ is open in $H^1(M,\mathbb{R})$.

Proof: This is Proposition 3.3 for Vaisman manifolds and Proposition 2.15 for LCK manifolds with potential. ∎

We have shown that for a general Vaisman structure (M,ω,θ), its LCK rank is equal to $b_1(M)$, and any number between 1 and $b_1(M)$ can be obtained as an

LCK rank for an appropriate choice of θ.

Theorem 3.4 has the following consequences.

Corollary 3.5: Let M be a compact complex manifold which admits a structure of a Vaisman manifold (or LCK manifold with potential) (M, ω, θ). Then M admits a structure of a Vaisman manifold (or LCK manifold with potential) (M, ω', θ') with proper potential, that is, of LCK rank one. Moreover, such ω' and θ' can be chosen in any neighbourhood of (M, ω, θ). ∎

Now, let us explain where the proof of [OV1] (later refined in [OV5]) failed.

Let M be a compact Vaisman manifold, and $\theta^\sharp$ its Lee field. Then $\theta^\sharp$ acts on M by holomorphic isometries, and on its smallest Kähler covering $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties. Denote by G the closure of the group generated by $e^{t\theta^\sharp}$. This group is a compact Lie group, because isometries form a compact Lie group on a compact Riemannian manifold, and a closed subgroup of a Lie group is a Lie group by Cartan's theorem. Moreover, it is commutative, because $\langle e^{t\theta^\sharp} \rangle$ is commutative, and this gives $G = (S^1)^k$.

Let $\tilde{G}$ be the group of pairs $(\tilde{f} \in \text{Aut}(\tilde{M}),\ f \in G)$, making the following diagram commutative:

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\ \tilde{f}\ } & \tilde{M} \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
M & \xrightarrow{\ f\ } & M
\end{array}
$$

Then $\tilde{G}$ is a covering of G, and the kernel of this projection is $\tilde{G} \cap \text{Aut}_M(\tilde{M})$, where $\text{Aut}_M(\tilde{M})$ is the deck transform group of the covering $\tilde{M} \longrightarrow M$.

Consider the homomorphism $\chi : \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ mapping an element of $\pi_1(M)$ considered as an automorphism of $\tilde{M}$, to the Kähler homothety constant, $\gamma \mapsto \frac{\gamma^* \tilde{\omega}}{\tilde{\omega}}$. Since $\tilde{M}$ is the smallest Kähler covering, we identify $\text{Aut}_M(\tilde{M})$ with $\chi(\pi_1(M)) \subset \mathbb{R}^{>0}$.

Now, let $\tilde{G}_0 \subset \tilde{G}$ be the subgroup acting on $\tilde{M}$ by isometries. Since the group $\tilde{G} \cap \text{Aut}_M(\tilde{M})$ is a subgroup of $\text{Aut}_{\tilde{M}}(M)$, $\tilde{G}_0$ maps to its image in G bijectively.

We assumed that G_0 (being the subgroup of elements of $\tilde{G}$ acting by isometries on both $\tilde{M}$ and M) is closed in G. Then, if $\tilde{G}_0 \cong S^{k-1}$, this would imply that $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$, proving that M is a quotient of $\tilde{M}$ by a $\mathbb{Z}$-action.

However, this is false, because G_0 is closed in $\tilde{G}$, but not closed in G. This is where the argument fails.

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