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Expanding Operators for the Independent Set Problem

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Abstract—The notion is introduced of an expanding operator for the independent set problem. This notion is a useful tool for the constructive formation of new cases with the efficient solvability of this problem in the family of hereditary classes of graphs and is applied to hereditary parts of the set $\mathcal{F}ree(\{P_5, C_5\})$. It is proved that if for a connected graph G the problem is polynomial-time solvable in the class $\mathcal{F}ree(\{P_5, C_5, G\})$ then it remains so in the class $\mathcal{F}ree(\{P_5, C_5, G \circ \overline{K}_2, G \oplus K_{1,p}\})$ for every p. We also find two new hereditary subsets of $\mathcal{F}ree(\{P_5, C_5\})$ with polynomially solvable independent set problem that are not a corollary of applying the revealed operators.

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INTRODUCTION

The present article is a continuation of [5], where the constructive approach was considered for exhibiting new cases of the polynomial solvability of the independent set problem in a representative family of hereditary graph classes. Recall that an *independent set* of a simple graph is an arbitrary set of its nonadjacent vertices. It is called *maximum* if it contains the greatest possible number of vertices. The number of vertices in a maximum independent set of a graph G is called the *independence number* of G and denoted by $\alpha(G)$. The independent set problem (Problem IS) for a given graph consists in finding its maximal independent set. It is known that, for any class of graphs, Problem IS is polynomially equivalent to the problem of computing the independence number.

A graph class $\mathcal{X}$ is called *hereditary* if $\mathcal{X}$ is closed under isomorphism and vertex removals. Every hereditary (and not only hereditary) graph class $\mathcal{X}$ can be defined by the set of its forbidden induced subgraphs $\mathcal{S}$. In this case, the notation $\mathcal{X} = \mathcal{F}ree(\mathcal{S})$ is used. The family of hereditary graph classes is a sufficiently representative continuous family and includes such familiar subfamilies as those closed under vertex and edge removals (also called *strongly hereditary*, or *monotone*) and *minor-closed* graph classes closed under the vertex and edge removals and the edge contractions.

In many articles, new cases are revealed of the efficient (polynomial) solvability of Problem IS, some of them also belong to the family of hereditary graph classes. Following the terminology in [7], we call a hereditary graph class with polynomially solvable problem IS as IS-*simple*. All results of works on finding new IS-simple classes known to the author are generalizations of previously known cases. Nevertheless (as was observed in [5]), it would be desirable to have some “universal” generalizations of this kind. Namely, in [5], we proposed to consider the transformations $f : \mathcal{S} \to \mathcal{S}'$ (the functions of one or several (unknown) arguments that are graphs in a part of $\mathcal{S}$) such that $\mathcal{F}ree(\mathcal{S}) \subset \mathcal{F}ree(\mathcal{S}')$ and the IS-simplicity of the class $\mathcal{F}ree(\mathcal{S})$ implies the IS-simplicity of $\mathcal{F}ree(\mathcal{S}')$. We call such transformations the *expanding operators* for Problem IS.

The usefulness of an expanding operator consists in the fact that the certain operators of this kind make it possible to constructively generate new IS-simple cases (in a “regular”manner). For example,

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if $\mathcal{F}ree(\mathcal{S})$ is a specific IS-simple class and f is an expanding operator for which $\mathcal{S}$ is contained in its domain then $\mathcal{F}ree(f(\mathcal{S}))$, $\mathcal{F}ree(f(f(\mathcal{S})))$, $\mathcal{F}ree(f(f(f(\mathcal{S}))))$ is a monotone increasing infinite chain (with respect to inclusion) of IS-simple classes. Note that such transformations can be especially useful when several expanding operators are known simultaneously (since the actions of their superpositions can be considered). Observe also that the notion of an expanding operator can be applied to any problem on graphs and not only to Problem IS.

In [5], we proved that the mapping $\{G\} \to \{G \oplus K_1\}$ is an expanding operator (by the *sum* $G_1 \oplus G_2$ we mean the union of graphs G_1 and G_2 with disjoint sets of vertices) and showed that the operator $\{P_5, C_5, G\} \to \{P_5, C_5, G \circ K_1\}$ is expanding (by the *product* $G_1 \circ G_2$ of graphs G_1 and G_2 we mean the graph $V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup V(G_1) \times V(G_2)$.) The interest to hereditary subclasses in $\mathcal{F}ree(\{P_5, C_5\})$ evinced in [5] is explained by a number of reasons. Thus, among all connected graphs G with five vertices, the case of a simple path is the only case for which the computational status of Problem IS in the class $Free(\{G\})$ remains an open question [21]. At the same time, there are dozens of articles in which, to the graph P_5, one or several other forbidden subgraphs are added and it is proved that this class is IS-simple (for example, see [8–10, 17, 18, 20–22]). It is also proved that, for every graph G with at most five vertices that differ from P_5 and C_5, Problem IS is polynomially solvable in the class $\mathcal{F}ree(\{P_5, G\})$ [21]. The question for $\mathcal{F}ree(\{P_5, C_5\})$ is still open. These circumstances motivate the author to consider the expanding operators for arguments $\mathcal{S}$ containing P_5 and C_5.

The present article consists of the three sections: In Section 1, we establish some auxiliary results. In Section 2, we prove that some two hereditary subsets $\mathcal{F}ree(\{P_5, C_5\})$ (not contained in the results about IS-simple classes known to the author) are IS-simple. In Section 3, we prove that

$$\{P_5, C_5, G\} \to \{P_5, C_5, G \oplus K_{1,p}, G \circ \overline{K}_2\}$$

for each natural p is an expansion operator under the assumption that G is connected.

We adopt the following notation:

$N_k(x)$ is the set of vertices exactly at distance k from x, $N_1(x) = N(x)$;

ring$_1$, ring$_2$, ring$_3$ are the graphs obtained by adding a vertex x to a simple cycle $(y_1, y_2, y_3, \dots, y_7)$ and edges (x, y_1), (x, y_3) (for ring$_1$), (x, y_1), (x, y_2), and (x, y_3) (for ring$_2$), and (x, y_1), (x, y_2), (x, y_3), and (x, y_4) (for ring$_3$).

All main notions and denotations of graph theory that we do not explain can be found, for example, in [2, 3, 6, 11, 13].

1. AUXILIARY RESULTS

In [5], we prove that every connected graph of class $\mathcal{F}ree(\{P_5\})$ has radius at most 2 (Lemma 2). The main idea of the proof is that a vertex x with the greatest value of $|\{x\} \cup N(x) \cup N_2(x)|$ is considered and it is showed that the distance from x to every other vertex is at most 2. The proof is based on the following observation: If, for some vertex y_1, the set $N_3(y_1)$ is nonempty then there exists a vertex $y_2 \in N_2(y_1)$ such that

$$\{y_1\} \cup N(y_1) \cup N_2(y_1) \subset \{y_2\} \cup N(y_2) \cup N_2(y_2).$$

This fact enables us to prove rather simply the following assertion that strengthens Lemma 2 in [5]:

Lemma 1. *For every connected graph $G \in \mathcal{F}ree(\{P_5\}) \setminus \mathcal{F}ree(\{K_{1,p}\})$, there exists a vertex x such that, for every $y \in V(G)$, the distance from x to y is at most 2 and $N(y)$ contains at least p independent vertices.*

One of the auxiliary procedures in solving Problem IS consists in the simplification, "contraction" of the current graph. Here, instead of the initial graph, its induced subgraph is considered such that their independence numbers coincide. Sometimes ideas of such kind lead to the creation of an algorithm for solving Problem IS. For example, for nonempty chordal graphs (i.e., graphs without generated cycles of length 4 and more), it is always possible to remove a so-called adjacently absorbing vertex [1].

A vertex a is said to *adjacently absorb* b if a and b are adjacent and $N(b) \setminus \{a\} \subseteq N(a) \setminus \{b\}$. The meaning of this notion is that, after the removal of every adjacently absorbing vertex from the graph, the independence number remains unchanged [1]. The possibility of applying the idea of vertex removal with the preservation of the independence number is proved below.

Lemma 2. *Suppose that, in a connected graph $G = (V, E)$ in $\mathcal{F}ree(\{P_5\})$, the vertex x is central and there exist $y_1, y_2 \in N(x)$ such that*

(i) $N(z) \setminus (N(x) \cup \{x\}) \subseteq N(y_1) \setminus (N(x) \cup \{x\})$ *for every vertex* $z \in N(x) \setminus (N(y_1) \cup \{y_1\})$*; moreover, for* $z = y_2$*, the inclusion is strict*;

(ii) *no vertex in* $N_2(y_1) \setminus (N(y_1) \cup N(x))$ *is adjacent to any vertex in* $N(y_1) \setminus (N(x) \cup \{x\})$;

(iii) *each vertex in* $N(y_2) \setminus (N(x) \cup \{x\})$ *is adjacent to all vertices in* $N(y_1) \setminus (N(x) \cup \{x\})$.

Then $\alpha(G) = \alpha(G[V \setminus \{x\}])$.

Proof. Consider an arbitrary maximum independent set in G which we denote by IS. If none of the vertices in the set $N(y_2) \setminus (N(x) \cup \{x\})$ (which generates a complete subgraph in G) belongs IS then conditions (i), (ii) implies that $IS \setminus \{x\} \cup \{y_2\}$ is a maximum independent set in G. If at least one vertex $z \in IS$ belongs to $N(y_2) \setminus (N(x) \cup \{x\})$ then

$$(IS \cap N(y_1)) \setminus (N(x) \cup \{x\}) = \{z\}$$

by condition (iii). By condition (i), there is a vertex $z' \in N(y_1) \setminus (N(y_2) \cup N(x))$ such that $(z', z) \in E$. Conditions (i) and (ii) imply that $IS \setminus \{y_2, z'\} \cup \{x, z\}$ is a maximum independent set in G. In both possible cases, $\alpha(G) = \alpha(G[V \setminus \{x\}])$. Lemma 2 is proved. □

Lemmas 1 and 2 are auxiliary results necessary for proving one of the main assertions of the present article, which is in turn a useful tool for constructing new cases of the polynomial solvability of Problem IS. Other tools of this kind can be given by some joins of already known IS-simple classes. One of such transformations is considered below. Namely, we introduce the notion of the composition of hereditary graph classes and prove that the efficient solvability of Problem IS is preserved under composition.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two hereditary classes. We refer as the *composition* of $\mathcal{X}_1$ and $\mathcal{X}_2$ (which we denote by $\mathcal{X}_1 \star \mathcal{X}_2$) to the set of those graphs G for which there are graphs $G_1 \in \mathcal{X}_1$ and $G_2 \in \mathcal{X}_2$ together with subsets $V_1 \subseteq V(G_1)$ and $V_2 \subseteq V(G_2)$ such that

$$V(G) = V(G_1) \cup V(G_2), \qquad E(G) = E(G_1) \cup E(G_2) \cup V_1 \times V_2.$$

Lemma 3. *If $\mathcal{X}_1$ and $\mathcal{X}_2$ are* IS-*simple classes then their composition $\mathcal{X}_1 \star \mathcal{X}_2$ is an* IS-*simple class too.*

Proof. Clearly, the composition $\mathcal{X}_1$ and $\mathcal{X}_2$ is a hereditary class. Prove that this class is computed in polynomial time in $|V(G)|$. Suppose that a graph $G = (V, E)$ belongs to the class $\mathcal{X}_1 \star \mathcal{X}_2$ and $G_1 = (V_1', E_1')$, $G_2 = (V_2', E_2')$, V_1, and V_2 are the graphs and sets that are involved in the definition of composition for G. Obviously, if $V_1 \times V_2 = \varnothing$ (i.e., if $V_1 = \varnothing$ or $V_2 = \varnothing$) then $\alpha(G) = \alpha(G_1) + \alpha(G_2)$, and hence the independence number of G is calculated for a time polynomial in $|V(G)|$. If $V_1 \times V_2 \neq \varnothing$ then every independent set in G contains vertices of at most one of the sets V_1 and V_2. Thus,

$$\alpha(G) = \alpha(G[V_1' \setminus N(x)]) + \alpha(G[V \setminus V_2]) \qquad \big(\alpha(G) = \alpha(G[V_2' \setminus N(y)]) + \alpha(G[V \setminus V_1])\big)$$

if some maximum independent set in G contains at least one vertex of $x \in V_1$ ($y \in V_2$). If none of the vertices in $V_1 \cup V_2$ belongs to one maximum set in G then

$$\alpha(G) = \alpha(G[V \setminus V_1]) + \alpha(G[V \setminus V_2]).$$

Hence, for nonempty $V_1 \times V_2$, we have

$$\alpha(G) = \max(\max_{x \in V_1} \alpha(G[V_1' \setminus N(x)]) + \alpha(G[V \setminus V_2]), \\ \max_{y \in V_2} \alpha(G[V_2' \setminus N(y)]) + \alpha(G[V \setminus V_1]), \alpha(G[V \setminus V_1]) + \alpha(G[V \setminus V_2])).$$

Thus, the independence number of G is computed for a polynomial time regardless of the emptiness of $V_1 \times V_2$.

Lemma 3 is proved. □

2. POLYNOMIAL SOLVABILITY OF PROBLEM IS IN GRAPH CLASSES $\mathcal{F}ree(\{P_5, C_5, \overline{\mathrm{ring}}_1, \overline{\mathrm{ring}}_3, \overline{P_2 \oplus P_4}\})$ AND $\mathcal{F}ree(\{P_5, C_5, \overline{\mathrm{ring}}_2, \overline{\mathrm{ring}}_3, \overline{2P_3}\})$

In this section, we prove that the independent set problem is efficiently solvable for graphs without collections of generated subgraphs $P_5, C_5, \overline{\mathrm{ring}}_1, \overline{\mathrm{ring}}_3, \overline{P_2 \oplus P_4}$ and $P_5, C_5, \overline{\mathrm{ring}}_2, \overline{\mathrm{ring}}_3, \overline{2P_3}$. This new result is one of the main assertions in this article and has no overlap with the other IS-simple subsets in the class $\mathcal{F}ree(\{P_5\})$ known to the author (for example, see [8–10, 14, 17–22]).

Theorem 1. *The graph classes*

$$\mathcal{F}ree(\{P_5, C_5, \overline{\mathrm{ring}}_1, \overline{\mathrm{ring}}_3, \overline{P_2 \oplus P_4}\}), \qquad \mathcal{F}ree(\{P_5, C_5, \overline{\mathrm{ring}}_2, \overline{\mathrm{ring}}_3, \overline{2P_3}\})$$

are IS-*simple.*

Proof. Recall that a graph is called *perfect* if it hereditarily possesses the property of the equality of the chromatic and clique numbers (which is equivalent to the equality of the independence number and the clique cover number [16]). It is known [12] that the set of perfect graphs coincides with the set of graphs in $\mathcal{F}ree(\{C_5, \overline{C_5}, C_7, \overline{C_7}, \dots\})$ and that some classical extremal problems on graphs (including the independent set problem [15]) are efficiently solvable for the perfect graphs. At the same time, we observe that all graphs in the classes

$$\mathcal{F}ree\big(\{P_5, C_5, \overline{\mathrm{ring}}_1, \overline{\mathrm{ring}}_3, \overline{P_2 \oplus P_4}\}\big) \cap \mathcal{F}ree\big(\{\overline{C_7}\}\big),$$
$$\mathcal{F}ree\big(\{P_5, C_5, \mathrm{ring}_2, \overline{\mathrm{ring}}_3, \overline{2P_3}\}\big) \cap \mathcal{F}ree(\{\overline{C_7}\})$$

are perfect. Therefore, for proving the theorem, it suffices to consider only those graphs in these classes containing the generated subgraph $\overline{C_7}$.

Suppose that G_1 is an arbitrary graph in the class

$$\mathcal{F}ree\big(\{P_5, C_5, \overline{\mathrm{ring}}_1, \overline{\mathrm{ring}}_3, \overline{P_2 \oplus P_4}\}\big),$$

G_2 is an arbitrary graph in

$$\mathcal{F}ree(\{P_5, C_5, \overline{\mathrm{ring}}_2, \overline{\mathrm{ring}}_3, \overline{2P_3}\})$$

each of which includes a generated subgraph $\overline{C_7}$. Clearly, the graph

$$H_1 = \overline{G_1} \in \mathcal{F}ree(\{\overline{P_5}, \overline{C_5} = C_5, \mathrm{ring}_1, \mathrm{ring}_3, P_2 \oplus P_4\})$$

includes C_7 as a generated subgraph. The same is true for the graph

$$H_2 = \overline{G_2} \in \mathcal{F}ree(\{\overline{P_5}, C_5, \mathrm{ring}_2, \mathrm{ring}_3, 2P_3\}).$$

Show that every vertex in $V(H_1) \setminus V(C_7)$ and every vertex in $V(H_2) \setminus V(C_7)$ is either adjacent to each of the vertices of the corresponding cycle C_7 or is adjacent to neither of its vertices. Let us prove this assertion only for H_1 since for H_2 it is proved similarly.

Assume that there exists some $x \in V(H_1)$ without this property. Consider the set $N(x) \cap V(C_7)$ cyclically ordered by the order of its elements in the cycle. The distance (in the sense of the usual distance in graphs) between two consecutive vertices (in the sense of the just-introduced ordering) in $N(x) \cap V(C_7)$ cannot equal 3 (otherwise H_1 would include a generated subgraph C_5). Since H_1 does not contain a generated subgraph $\overline{P_5}$, the graph $N(x) \cap V(C_7)$ does not contain consecutive values y_1, y_2, and y_3 such that the distance between y_1 and y_2 (between y_2 and y_3) is equal to 2 and the distance between y_2 and y_3 (between y_1 and y_2) is equal to 1. Since $H_1 \in \mathcal{F}ree(\{\mathrm{ring}_3\})$, the vertex x is adjacent to at most three vertices of the cycle C_7. Moreover, if $|N(x) \cap V(C_7)| = 3$ then this set is composed of three consecutive vertices y_1, y_2, and y_3 of C_7; and if $|N(x) \cap V(C_7)| = 2$ then it is composed of two vertices the distance between which in the cycle equals 2. Both cases are impossible because, in the first case, H_1 contains the subgraph $P_2 \oplus P_4$ generated by the set of vertices $\{x, y_2\} \cup V(C_7) \setminus \{y_1, y_2, y_3\}$ and, in the second case, H_1 contains the subgraph ring_1 generated by the set of vertices $\{x\} \cup V(C_7)$. We are left with the only case in which x is adjacent to exactly one vertex in C_7. In this case, H_1 contains a generated subgraph $P_2 \oplus P_4$; a contradiction to the assumption.

Clearly, C_7 is not a generated subgraph in H_1 and H_2. Therefore, if H_1 contains two cycles of length 7 with disjoint sets of vertices as subgraphs then there are two adjacent vertices x and y belonging to different cycles of such kind. Consequently, x is adjacent to all vertices of one cycle, and y is adjacent to all vertices of the other. This and the assertion proved in the previous paragraph imply that, in H_1, each vertex in every cycle C_7 is adjacent to any other cycle C_7. The same holds for H_2.

Let $\mathcal{X}_1$ be the set of graphs whose each connected component is a subgraph in C_7, and let $\mathcal{X}_2$ be the set of all perfect graphs. Both classes are IS-simple. By the above-proved,

$$\mathcal{F}ree(\{P_5, C_5, \overline{\text{ring}}_1, \overline{\text{ring}}_3, \overline{P_2 \oplus P_4}\}) \cup \mathcal{F}ree(\{P_5, C_5, \overline{\text{ring}}_2, \overline{\text{ring}}_3, \overline{2P_3}\}) \subseteq \mathcal{X}_1 \star \mathcal{X}_2.$$

Hence, by Lemma 3, the classes

$$\mathcal{F}ree(\{P_5, C_5, \overline{\text{ring}}_1, \overline{\text{ring}}_3, \overline{P_2 \oplus P_4}\}), \qquad \mathcal{F}ree(\{P_5, C_5, \overline{\text{ring}}_2, \overline{\text{ring}}_3, \overline{2P_3}\})$$

are IS-simple, which completes the proof of Theorem 1. □

3. A NEW EXPANDING OPERATOR FOR THE INDEPENDENT SET PROBLEM

One of the possible approaches to solving Problem IS consists in the use of branching at a vertex chosen in some way. The essence of this method is that, in a graph $G = (V, E)$, some vertex x is chosen, and Problem IS for G is reduced to the same problem for the graphs $G_1 = G[V \setminus \{x\}]$ and $G_2 = G[V \setminus N(x)]$. This recursive rule is based on the validity of the obvious inequality

$$\alpha(G) = \max(\alpha(G_1), \alpha(G_2)).$$

Unfortunately, in general, this process is not polynomial in time but sometimes applying the idea of branching (at an appropriately chosen vertex) really leads to efficient algorithms. In [4], it is shown that, for every graph $G \in \mathcal{F}ree(\{P_5, C_5\})$, there exists a vertex x such that the clique number of G_2 is less than that of G. This immediately leads to the possibility of solving Problem IS in polynomial time for graphs in $\mathcal{F}ree(\{P_5, C_5, K_p\})$ for every fixed p. Another (more general) example of a successful application of branching consists in choosing a vertex x such that at least one of the graphs G_1 and G_2 belongs to an appropriately chosen case of polynomial solvability of Problem IS. By analogy with the proof of Theorem 3 in [4], it is easy to prove that, in this case, the time complexity is also polynomial. A successful choice of a vertex for branching enables us to prove one of the main assertions of the present article. It states that, for each p, the transformation

$$\{P_5, C_5, H\} \to \{P_5, C_5, H \circ \overline{K}_2, H \oplus K_{1,p}\}$$

is an expanding operator under the extra condition of the connectedness of H.

Let G be a connected graph of the class $\mathcal{F}ree(\{P_5, C_5, H \circ \overline{K}_2, H \oplus K_{1,p}\})$. We may assume that G does not contain adjacently absorbing vertices. Since, for each s, the class $\mathcal{F}ree(\{P_5, K_{1,s}\})$ is IS-simple [18], we may also assume that G contains $K_{1,p+1}$ as a generated subgraph. By Lemma 1, there exists a central vertex x in G (of radius at most 2) for which $N(x)$ contains $p+1$ independent vertices $y_1, y_2, \ldots, y_{p+1}$. For each vertex $y \in N(x)$, denote by $N'(y)$ the set $N(y) \setminus (N(x) \cup \{x\})$. Since G does not contain adjacently absorbing vertices, for each such y, the set $N'(y)$ is nonempty. On the set of vertices $N(x)$, introduce a quasiorder relation R as follows:

$$uRv \;\Leftrightarrow\; N'(u) \subseteq N'(v).$$

In [4, Lemma 2], we proved that if a graph of class $\mathcal{F}ree(\{P_5, C_5\})$ contains a subgraph P_3 generated by vertices a, b, and c (in the indicated order) then, in this graph,

$$\text{either} \quad N(a) \setminus N(b) \subseteq N(c) \qquad \text{or} \quad N(c) \setminus N(b) \subseteq N(a).$$

This means that the quasiorder R on $A = \{y_1, y_2, \ldots, y_{p+1}\}$ is linear. Therefore, there exists a vertex $y^* \in A$ such that $N'(y') \subseteq N'(y^*)$ for every $y' \in A$. Consider the quasiorder R on

$$\{y \in N(x) \mid N'(y^*) \subseteq N'(y)\}.$$

On this set, R is not necessarily linear but must have a maximal element y_1. Clearly, for each vertex $y \in N(x)$ nonadjacent to y_1, we have the inclusion $N'(y) \subseteq N'(y_1)$, which is strict if

$$G[N'(y_1)] \in \mathcal{F}ree(\{H\})$$

(the set $B = N(x) \setminus (N(y_1) \cup \{y_1\})$ is nonempty since otherwise y_1 would be an adjacently absorbing vertex).

Let C be the set of the vertices in G lying at distance 2 from y_1 and not belonging to $N(x) \cup N(y_1)$.

Lemma 4. *The graph $G[C]$ belongs to the class $\mathcal{F}ree(\{H\})$.*

Proof. If $C = \varnothing$ then the assertion is obvious. If $C \neq \varnothing$ then no vertex in C is adjacent to a vertex in $A \cup B$ (since $N'(y) \subseteq N'(y_1)$ for each vertex $y \in A \cup B$). Therefore, $G[C] \in \mathcal{F}ree(\{H\})$ since otherwise some vertices in C, every p vertices in A, and x would generate a subgraph $H \oplus K_{1,p}$ in G, which completes the proof. □

Lemma 5. *If there exist adjacent vertices $z_1 \in C$ and $z_2 \in N'(y_1)$ then $G[B] \in \mathcal{F}ree(\{H\})$.*

Proof. Each vertex in B is adjacent to z_2 since the existence of a vertex $y \in B$ nonadjacent to z_2 means that the vertices y, x, y_1, z_2, and z_1 generate P_5 (recall that $(y, z_1) \notin E(G)$). Hence, $G[B] \in \mathcal{F}ree(\{H\})$ since otherwise some vertices in B and x and z_2 would generate a subgraph $H \circ \overline{K}_2$ in G. Lemma 5 is proved. □

Consider the vertex y_2—a maximal element of B with respect to R. Clearly, $G[N'(y_2)] \in \mathcal{F}ree(\{H\})$ and if $G[N'(y_1)] \notin \mathcal{F}ree(\{H\})$ then $N'(y_2) \subset N'(y_1)$. It is also clear that, for every $y \in B \setminus (A \cup N(y_2))$, we have $N'(y) \subseteq N'(y_2)$.

Let D denote the set $N'(y_1) \setminus N'(y_2)$. In Lemma 6, we assume that D is nonempty.

Lemma 6. *If*

$$G[B \setminus A] \notin \mathcal{F}ree(\{H\}), E(G[N'(y_2)]) \times E(G[D]) \nsubseteq E(G)$$

or $E(G[N'(y_2)]) \times E(G[D]) \subseteq E(G)$ and $G[N'(y_2)]$ is not complete then

$$G[N'(y_1)] \in \mathcal{F}ree(\{H\}) \star \mathcal{F}ree(\{H\}).$$

Proof. We first prove that each vertex in $N'(y_2)$ is either adjacent to each vertex in D or is adjacent to no such vertex. Suppose that there is $z_1 \in N'(y_2)$ adjacent to some $z_2 \in D$ and not adjacent to $z_3 \in D$. Note that each vertex $y \in B \setminus A$ nonadjacent to y_2 is adjacent to z_1 since otherwise G would contain the subgraph P_5 generated by y, x, y_2, z_1, and z_2. At the same time, each vertex y' in $N(y_2) \cap (B \setminus A)$ must be adjacent to z_1 since otherwise G would contain the subgraph P_5 generated by y', y_2, z_1, y_1, and z_3 (if $(y', z_3) \notin E(G)$) or a subgraph C_5 (if $(y', z_3) \in E(G)$). Hence, each of the vertices of $B \setminus A$ is simultaneously adjacent both to x and z_1. But since G contains no generated subgraph $H \circ \overline{K}_2$, the graph $G[B \setminus A]$ contains no generated subgraph H; a contradiction to the hypothesis. Therefore, the assumption about the existence of vertices z_1, z_2, and z_3 fails.

Assume that there exists a vertex $a \in N'(y_2)$ adjacent to no vertex in D. The above arguments imply that a is adjacent to all vertices in $N(y_2) \cap (B \setminus A)$ simultaneously. Since the graph $G[N(y_2) \cap (B \setminus A)]$ does not belong to the class $\mathcal{F}ree(\{H\})$ and $G[B \setminus A]$ does belong to it, the connectedness of H implies the existence of two adjacent vertices $b \in N(y_2) \cap (B \setminus A)$ and $c \in B \setminus A$ such that $(a, b) \in E(G)$ and $(a, c) \notin E(G)$. Then the vertex b must be adjacent to all vertices in D simultaneously; otherwise G would contain the subgraph P_5 generated by the vertices c, b, a, y_1, and some vertex in D. Therefore, $G[D] \in \mathcal{F}ree(\{H\})$ because all vertices in D are simultaneously adjacent to the nonadjacent vertices b and y_1. Hence,

$$G[N'(y_1)] \in \mathcal{F}ree(\{H\}) \star \mathcal{F}ree(\{H\}).$$

Suppose that $E(G[N'(y_2)]) \times E(G[D]) \subseteq E(G)$ and the graph $G[N'(y_2)]$ is incomplete. Then there exist two nonadjacent vertices u and v that are adjacent to all vertices in D simultaneously. Thus,

$G[D]$ does not contain H as a generated subgraph. This and what has been said above imply that $G[N'(y_1)]$ belongs to the composition of the classes $\mathcal{F}ree(\{H\})$ and $\mathcal{F}ree(\{H\})$.

Lemma 6 is proved. □

One of the main results of the article is the following

Theorem 2. *If the class $\mathcal{F}ree(\{P_5, C_5, H\})$ for some connected graph H is* IS-*simple then the class*

$$\mathcal{F}ree(\{P_5, C_5, H \circ \overline{K}_2, H \oplus K_{1,p}\})$$

is IS-*simple for every p.*

Proof. We use the idea of branching at a vertex v of the current graph so that either its subgraph without v or its subgraph without $N(v)$ belongs to some IS-simple extension of $\mathcal{F}ree(\{P_5, C_5, H\})$. This guarantees the possibility of solving Problem IS in the class $\mathcal{F}ree(\{P_5, C_5, H \circ \overline{K}_2, H \oplus K_{1,p}\})$ for polynomial time. We may assume that the current graph $G = (V, E)$ belongs to this class and satisfies all conditions at the beginning of Section 3: connectedness, the absence of adjacently absorbing vertices, and the absence of membership in $\mathcal{F}ree(\{K_{1,p+1}\})$. Describe the rule for choosing the vertex v.

1. If $G[B \setminus A] \in \mathcal{F}ree(\{P_5, C_5, H\})$, which is checked in polynomial time in $|V(G)|$, then put $v = y_1$. In this case,

$$G[V \setminus N(y_1)] = G[C] \oplus G[(B \setminus A) \cup (A \setminus N(y_1))]$$

and $G[C] \in \mathcal{F}ree(\{P_5, C_5, H\})$ by Lemma 4.

At the same time, in the graph $G[(B \setminus A) \cup (A \setminus N(y_1))]$, consider all independent subsets of the set $A \setminus N(y_1)$ (obviously, there are at most 2^{p+1} of them). For each of these subsets, consider the subgraph in $G[B \setminus A]$ generated by all vertices not adjacent to the vertices of the chosen subset. This subgraph belongs to the class $\mathcal{F}ree(\{P_5, C_5, H\})$. Hence, Problem IS for $G[V \setminus N(y_1)]$ is solved in polynomial time in the number of its vertices.

2. If $G[B \setminus A] \notin \mathcal{F}ree(\{P_5, C_5, H\}), E(G[N'(y_2)]) \times E(G[D]) \nsubseteq E(G)$ or if

$$G[B \setminus A] \notin \mathcal{F}ree(\{P_5, C_5, H\}), E(G[N'(y_2)]) \times E(G[D]) \subseteq E(G)$$

and the graph $G[N'(y_2)]$ is incomplete then put $v = x$. Lemma 5 implies

$$G[V \setminus (N(x) \cup \{x\})] = G[C] \oplus G[N'(y_1)],$$

and $G[C] \in \mathcal{F}ree(\{P_5, C_5, H\})$ by Lemma 4.

If D is empty then $G[N'(y_1)]$ belongs to $\mathcal{F}ree(\{P_5, C_5, H\})$, while if D is nonempty then $G[N'(y_1)]$ belongs to $\mathcal{F}ree(\{P_5, C_5, H\}) \star \mathcal{F}ree(\{P_5, C_5, H\})$ (by Lemma 6). Therefore, from Lemma 3 it follows that Problem IS for the graph $G[V \setminus N(x)]$ is solved in polynomial time in the number of its vertices.

3. If $G[B \setminus A] \notin \mathcal{F}ree(\{P_5, C_5, H\}), E(G[N'(y_2)]) \times E(G[D]) \subseteq E(G)$, and $G[N'(y_2)]$ is a complete graph then $\alpha(G) = \alpha(G \setminus \{x\})$ by Lemma 2.

Thus, the third case is reduced to the first two by removing the redundant vertex.

Theorem 2 is proved. □

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REFERENCES

1. V. E. Alekseev and V. A. Talanov, *Graphs and Algorithms. Data Structures. Models of Computations* (BINOM, Lab. Znanii, Moscow, 2006) [in Russian].
2. V. A. Emelichev, O. I. Melnikov, V. I. Sarvanov, and R. I. Tyshkevich, *Lecture Notes on Graph Theory* (Nauka, Moscow, 1990) [in Russian].
3. A. A. Zykov, *Basics of Graph Theory* (Nauka, Moscow, 1987) [in Russian].
4. D. S. Malyshev, "Polynomial Solvability of the Independent Set Problem in a Class of Graphs Without Induced Path and Cycle With Five Vertices and a Big Clique," Diskretn. Anal. Issled. Oper. **19** (3), 58–64 (2012).
5. D. S. Malyshev, "Polynomial Solvability of the Independent Set Problem for One Class of Graphs With Small Diameter," Diskretn. Anal. Issled. Oper. **19** (4), 66–72 (2012).
6. F. Harary, *Graph Theory* (Addison-Wesley, Reading, MA, 1969; Mir, Moscow, 1982).
7. V. E. Alekseev, "On Easy and Hard Hereditary Classes of Graphs With Respect to the Independent Set Problem," Discrete Appl. Math. **132**, 17–26 (2004).
8. C. Arbib and R. Mosca, "On $(P_5, diamond)$-Free Graphs," Discrete Math. **250**, 1–22 (2002).
9. A. Brandstadt and R. Mosca, "On the Structure and Stability Number of P_5- and Co-Chair-Free Graphs," Discrete Appl. Math. **132**, 47–65 (2003).
10. A. Brandstadt, H.-O. Le, and R. Mosca, "Chordal Co-Gem-Free and (P_5, gem)-Free Graphs Have Bounded Clique-Width," Discrete Appl. Math. **145** (2), 232 (2005).
11. A. Bondy and U. Murty, *Graph Theory* (Springer, Heidelberg, 2008).
12. M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, "The Strong Perfect Graph Theorem," Ann. Math. **164**, 51–229 (2002).
13. R. Diestel, *Graph Theory* (Heidelberg, Springer, 2010).
14. M. Gerber and V. Lozin, 'On the Stable Set Problem in Special P_5-Free Graphs," Discrete Appl. Math. **125**, 215–224 (2003).
15. M. Grotschel, L. Lovasz, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization (Algorithms and Combinatorics)* (Springer, Berlin, 1993).
16. L. Lovasz, "A Characterization of Perfect Graphs," J. Comb. Theory, Ser. B. **13**, 95–98 (1972).
17. V. Lozin and R. Mosca, "Maximum Independent Sets in Subclasses of P_5-Free Graphs," Inform. **109** (6), 319–324 (2009).
18. R. Mosca, "Polynomial Algorithms for the Maximum Stable Set Problem on Particular Classes of P_5-Free Graphs," Inform. Process. Lett. **61** (3), 137–143 (1997).
19. R. Mosca, "Stable Sets in Certain P_6-Free Graphs," Discrete Appl. Math. **92**, 177–191 (1999).
20. R. Mosca, "Some Results on Maximum Stable Sets in Certain P_5-Free Graphs," Discrete Appl. Math. **132**, 175–183 (2003).
21. R. Mosca, "Some Observations on Maximum Weight Stable Sets in Certain P_5-Free Graph," European J. Oper. Res. **184** (3), 849–859 (2008).
22. C. D. Simone and R. Mosca, "Stable Set and Clique Polytopes of (P_5, gem)-Free Graphs," Discrete Math. **307** (22), 2661–2670 (2007).