COMPLEX ROTATION NUMBERS

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(Communicated by Nataliya Goncharuk)

ABSTRACT. We investigate the notion of complex rotation number which was introduced by V. I. Arnold in 1978. Let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving circle diffeomorphism and let $\omega \in \mathbb{C}/\mathbb{Z}$ be a parameter with positive imaginary part. Construct a complex torus by glueing the two boundary components of the annulus $\{z \in \mathbb{C}/\mathbb{Z} \mid 0 < \mathrm{Im}(z) < \mathrm{Im}(\omega)\}$ via the map $f+\omega$. This complex torus is isomorphic to $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some appropriate $\tau \in \mathbb{C}/\mathbb{Z}$.

According to Moldavskis [5], if the ordinary rotation number $\mathrm{rot}(f+\omega_0)$ is Diophantine and if ω tends to ω_0 non tangentially to the real axis, then τ tends to $\mathrm{rot}(f+\omega_0)$. We show that the Diophantine and non tangential assumptions are unnecessary: if $\mathrm{rot}(f+\omega_0)$ is irrational then τ tends to $\mathrm{rot}(f+\omega_0)$ as ω tends to ω_0.

This, together with results of N.Goncharuk [3], motivates us to introduce a new fractal set, given by the limit values of τ as ω tends to the real axis. For the rational values of $\mathrm{rot}(f+\omega_0)$, these limits do not necessarily coincide with $\mathrm{rot}(f+\omega_0)$ and form a countable number of analytic loops in the upper half-plane.

Notation:

- $\mathbb{H} = \mathbb{H}^+$ is the set of complex numbers with positive imaginary part.
- $\mathbb{H}^-$ is the set of complex numbers with negative imaginary part.
- If p/q is a rational number, then p and q are assumed to be coprime.
- If x and y are distinct points in $\mathbb{R}/\mathbb{Z}$, then (x,y) denotes the set of points $z \in \mathbb{R}/\mathbb{Z} - \{x,y\}$ such that the three points x,z,y are in increasing order and $[x,y] := (x,y) \cup \{x,y\}$.
- If $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a circle diffeomorphism, $D_f := \displaystyle\int_{\mathbb{R}/\mathbb{Z}} \left|\frac{f''(x)}{f'(x)}\right| \,\mathrm{d}x$.

Received July 14, 2013.

2010 *Mathematics Subject Classification:* Primary: 37E10; Secondary: 37E45 and 30C62.

Key words and phrases: complex tori, rotation numbers, diffeomorphisms of the circle, quasiconformal maps.

XB: Supported by the IUF.

NG: Supported by the following grants: RFBR projects 10-01-00739-a and 12-01-33020 mol_a_ved, joint RFBR/CNRS project 10-01-93115-CNRS_a, Moebius Contest Foundation for Young Scientists, Simons Foundation.

INTRODUCTION

Given an orientation preserving analytic circle diffeomorphism $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a parameter $\omega \in \mathbb{H}/\mathbb{Z}$, set

$$f_\omega := f + \omega\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} + \omega.$$

The circles $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}+\omega$ bound an annulus $A_\omega \subset \mathbb{C}/\mathbb{Z}$. Glueing the two sides of A_ω via f_ω, we obtain a complex torus $E(f_\omega)$, which may be uniformized as $\mathscr{E}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some appropriate $\tau \in \mathbb{H}/\mathbb{Z}$, the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $E(f_\omega)$ corresponding to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $\mathscr{E}_\tau$. The complex rotation number of f_ω is $\tau_f(\omega) := \tau$. It is the complex analog of the ordinary rotation number of $f+t$ for $t \in \mathbb{R}/\mathbb{Z}$.

V. I. Arnold's problem [1], generalized by R. Fedorov and E. Risler independently, is to study the relation of the ordinary rotation number of the circle diffeomorphism $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and the limit behaviour of the complex rotation number $\tau_f(\omega)$ as ω tends to 0.

According to work of Risler [6, Chapter 2, Proposition 2], the function

$$\tau_f\colon \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$$

is holomorphic. We shall show that there is a continuous extension of τ_f to

$$\overline{\mathbb{H}/\mathbb{Z}} := \mathbb{H}/\mathbb{Z} \cup \mathbb{R}/\mathbb{Z}.$$

The ordinary rotation number of a circle homeomorphism $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is defined as follows. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Such a lift is unique up to addition of an integer. The sequence of functions $\frac{1}{n}\big(F^{\circ n} - \mathrm{id}\big)$ converges uniformly to a constant function Θ. If we replace F by $F+k$ with $k \in \mathbb{Z}$, the limit Θ is replaced by $\Theta + k$, so that the value $\mathrm{rot}(f) \in \mathbb{R}/\mathbb{Z}$ of Θ modulo 1 only depends on f. This is the rotation number of f. Note that the rotation number is rational if and only if the circle homeomorphism has a periodic cycle.

Our main result, proved in Section 2.6, concerns the behavior of $\tau_f(\omega)$ as ω tends to $\mathbb{R}/\mathbb{Z}$. Recall that a periodic cycle of a circle diffeomorphism is called *parabolic* if its multiplier is 1, and it is called *hyperbolic* otherwise. A circle diffeomorphism with periodic cycles is called *hyperbolic* if it has only hyperbolic periodic cycles.

MAIN THEOREM. *Let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving analytic circle diffeomorphism. Then, the function $\tau_f\colon \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ has a continuous extension $\bar\tau_f\colon \overline{\mathbb{H}/\mathbb{Z}} \to \overline{\mathbb{H}/\mathbb{Z}}$. Assume $\omega \in \mathbb{R}/\mathbb{Z}$.*

- *If $\mathrm{rot}(f_\omega)$ is irrational, then $\bar\tau_f(\omega) = \mathrm{rot}(f_\omega)$.*
- *If $\mathrm{rot}(f_\omega) = p/q$ is rational, then $\bar\tau_f(\omega)$ belongs to the closed disk of radius $D_f/(\pi q^2)$ tangent to $\mathbb{R}/\mathbb{Z}$ at p/q; moreover*
  - *if f_ω has a parabolic cycle, then $\bar\tau_f(\omega) = \mathrm{rot}(f_\omega)$.*
  - *if f_ω is hyperbolic, then $\bar\tau_f(\omega) \in \mathbb{H}/\mathbb{Z}$, in particular $\bar\tau_f(\omega) \neq \mathrm{rot}(f_\omega)$.*

Our main contribution to this result is the case of irrational (yet not Diophantine) rotation number, and the continuous extension of τ_f to the whole boundary $\mathbb{R}/\mathbb{Z}$. The case of Diophantine rotation numbers was investigated earlier by

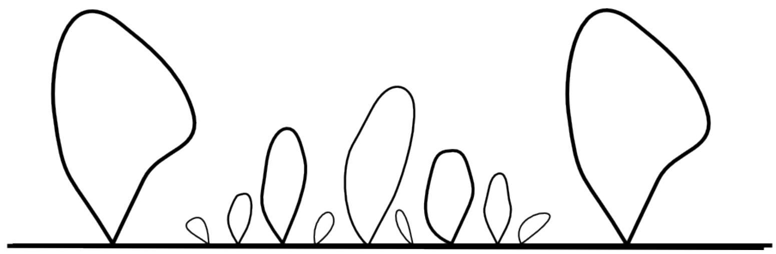

FIGURE 1. Bubbles. The sketch of the set $\bar{\tau}_f(\mathbb{R}/\mathbb{Z})$.

E.Risler [6, Chapter 2] and V.Moldavskis [5] independently. The case of parabolic cycles was studied by J.Lacroix (unpublished) and N.Goncharuk [3] independently. The case of hyperbolic diffeomorphisms was dealt first by Ilyashenko and Moldavskis [4], then this result was improved by N.Goncharuk [3]. For exact statements of these results, see Section 2.

In Appendix A, we shall also study the behavior of $\tau_f(\omega)$ as the imaginary part of ω tends to $+\infty$.

Bubbles: a new fractal set. The Main Theorem enables us to define a new interesting fractal set, related to the circle diffeomorphism, namely the set $\bar{\tau}_f(\mathbb{R}/\mathbb{Z})$. Due to the Main Theorem, this set contains $\mathbb{R}/\mathbb{Z}$ and a countable number of loops — "bubbles", the endpoints of bubbles are rational points of $\mathbb{R}/\mathbb{Z}$ (see the sketch at Fig. 1). Due to [3], these loops are analytic curves.

There arises a natural conjecture that $\bar{\tau}_f(\mathbb{R}/\mathbb{Z})$ is the boundary of $\tau_f(\mathbb{H}/\mathbb{Z})$, and τ_f is univalent. We disprove this conjecture, see Corollary 2.13 of Section 2.5.2.

There are still many open questions about the geometrical structure of the set $\bar{\tau}_f(\mathbb{R}/\mathbb{Z})$:

- What can be said about the shape and the size of a bubble? In particular, could a bubble be self-intersecting?
- Is it possible that different bubbles intersect each other?
- What can be said about the "bubble bundle", when several bubbles grow from the same point of the real axis?

1. DENJOY'S LEMMA

Before embarking into the proof of our results, we shall recall a classical result of Denjoy on the dynamics of circle diffeomorphisms. The *distortion* of a diffeomorphism $f\colon I \to J$ is

$$\operatorname{dis}_I(f) = \max_{x,y\in I} \log \frac{f'(x)}{f'(y)}.$$

If $f: I \to J$ and $g: J \to K$ are diffeomorphisms, then

$$\mathrm{dis}_J(f^{-1}) = \mathrm{dis}_I(f) \quad \text{and} \quad \mathrm{dis}_I(g \circ f) \le \mathrm{dis}_I(f) + \mathrm{dis}_J(g).$$

LEMMA 1.1 (Denjoy)**.** *Let $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving diffeomorphism and $I \subset \mathbb{R}/\mathbb{Z}$ be an interval such that I, $f(I)$, $f^{\circ 2}(I), \ldots, f^{\circ n}(I)$ are disjoint. Then,*

$$\mathrm{dis}_I(f^{\circ n}) \le D_f.$$

Proof. Let x and y be points in I. Set $x_k := f^{\circ k}(x)$ and $y_k := f^{\circ k}(y)$. Then,

$$\begin{aligned} \left|\log(f^{\circ n})'(x) - \log(f^{\circ n})'(y)\right| &= \left|\sum_{k=0}^{n-1} \log f'(x_k) - \log f'(y_k)\right| \\ &\le \sum_{k=0}^{n-1} \left|\int_{x_k}^{y_k} \frac{f''(x)}{f'(x)}\,\mathrm{d}x\right| \le \int_{\mathbb{R}/\mathbb{Z}} \left|\frac{f''(x)}{f'(x)}\right| \mathrm{d}x = D_f. \quad \square \end{aligned}$$

As a corollary, we have the following control on the multipliers of the periodic cycles of f. This result is surely known by specialists, but we include its proof due to the lack of a suitable reference.

LEMMA 1.2. *Let $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving diffeomorphism and ρ be the multiplier of a cycle of f. Then, $|\log \rho| \le D_f$.*

Proof. The average of the derivative $(f^{\circ q})'$ along the circle $\mathbb{R}/\mathbb{Z}$ is equal to 1. As a consequence, there exists a point $x_0 \in \mathbb{R}/\mathbb{Z}$ such that $(f^{\circ q})'(x_0) = 1$. Any periodic cycle $\{x, f(x), \ldots, f^{\circ q}(x) = x\}$ divides the circle into disjoint intervals $I_1, \ldots, I_q$ which are permuted by f. Without loss of generality, we may assume that I_1 contains x and x_0. Then, according to the previous Lemma,

$$|\log \rho| = \left|\log(f^{\circ q})'(x)\right| = \left|\log \frac{(f^{\circ q})'(x)}{(f^{\circ q})'(x_0)}\right| \le \mathrm{dis}_{I_1}(f^{\circ q}) \le D_f. \quad \square$$

2. BEHAVIOR OF τ_f NEAR $\mathbb{R}/\mathbb{Z}$

The proof of the Main Theorem goes as follows.

Step 1. Recall that a number $\theta \in \mathbb{R}/\mathbb{Z}$ is *Diophantine* if there are constants $c > 0$ and $\beta > 0$ such that for all rational numbers $p/q \in \mathbb{Q}/\mathbb{Z}$, we have

$$\left|x - \frac{p}{q}\right| > \frac{c}{q^{2+\beta}}.$$

THEOREM 2.1 (V. Moldavskis [5])**.** *If $\omega \in \mathbb{R}/\mathbb{Z}$ and if* $\mathrm{rot}(f_\omega)$ *is Diophantine, then*

$$\lim_{\substack{y \to 0 \\ y > 0}} \tau_f(\omega + \mathrm{i}y) = \mathrm{rot}(f_\omega).$$

Step 2. If $\omega \in \mathbb{R}/\mathbb{Z}$ and $\mathrm{rot}(f_\omega)$ is rational, then the conclusion of Theorem 2.1 is not true. This fact was first proved by Yu. Ilyashenko and V. Moldavkis [4]. We do not formulate their result since we will use its later generalized version.

THEOREM 2.2 (N. Goncharuk [3])**.** *If $\omega \in \mathbb{R}/\mathbb{Z}$, if* $\mathrm{rot}(f_\omega)$ *is rational and if f_ω is hyperbolic, then τ_f extends analytically to a neighborhood of ω.*

In the following, we shall denote by $\bar{\tau}_f(\omega)$ this extension of τ_f at ω.

Step 3. Recall that $\theta \in \mathbb{R}/\mathbb{Z}$ is *Liouville* if it is irrational but not Diophantine. We use the following result of Tsujii.

THEOREM 2.3 (M. Tsujii [7]). *The set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\operatorname{rot}(f_\omega)$ is Liouville has zero Lebesgue measure.*

It implies that almost every $\omega \in \mathbb{R}/\mathbb{Z}$ satisfies assumptions of either Theorem 2.1, or Theorem 2.2 (note that the set of ω such that f_ω has a parabolic cycle is countable).

Step 4. If f_ω has rational rotation number p/q, we denote by $\operatorname{Per}(f_\omega)$ the set of periodic points of $f_\omega\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. For $x \in \operatorname{Per}(f_\omega)$, we denote by ρ_x the multiplier of f as a fixed point of $f^{\circ q}$. Our contribution starts with the following result. It is an analog of the Yoccoz Inequality which bounds the multiplier of a fixed point of a polynomial in terms of its combinatorial rotation number [2].

LEMMA 2.4. *Assume that f_ω is a hyperbolic map with rational rotation number p/q. Then, $\bar{\tau}_f(\omega)$ belongs to the disk tangent to $\mathbb{R}/\mathbb{Z}$ at p/q with radius*

$$R_\omega := \frac{1}{\pi q \cdot \displaystyle\sum_{x \in \operatorname{Per}(f_\omega)} \frac{1}{|\log \rho_x|}}.$$

In addition, $R_\omega \le D_f/(\pi q^2)$.

The cardinal of $\operatorname{Per}(f_\omega)$ is at least q and according to Lemma 1.2, for each $x \in \operatorname{Per}(f_\omega)$ we have $|\log \rho_x| \le D_f$. This yields the upper bound $R_\omega \le D_f/(\pi q^2)$.

Step 5. Let $\bar{\tau}_f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ be defined by

- $\bar{\tau}_f(\omega) := \operatorname{rot}(f_\omega)$ if the rotation number of f_ω is irrational or if f_ω has a parabolic cycle and
- $\bar{\tau}_f(\omega) := \lim\limits_{\substack{y \to 0 \\ y > 0}} \tau_f(\omega + \mathrm{i}y)$ if f_ω is hyperbolic.

This definition agrees with the definition of $\bar{\tau}_f(\omega)$ for hyperbolic f_ω (see Step 2). We are going to prove that $\bar{\tau}_f$ is the continuous extension of τ_f to the real axis; so the coincidence of the notation with that of Main Theorem is not accidental and will not lead to confusion.

LEMMA 2.5. *The function $\bar{\tau}_f$ is continuous on $\mathbb{R}/\mathbb{Z}$.*

It is particularly difficult to prove the continuity of $\bar{\tau}_f$ at points $\omega \in \mathbb{R}/\mathbb{Z}$ for which f_ω has hyperbolic and parabolic cycles which bifurcate into complex conjugate cycles. The other cases follow easily from Theorem 2.2 and Lemma 2.4.

Step 6. The holomorphic map $\tau_f\colon \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ has radial limits on $\mathbb{R}/\mathbb{Z}$ almost everywhere, and those limits coincide with the continuous map $\bar{\tau}_f$. It follows easily that τ_f extends continuously by $\bar{\tau}_f$ to $\mathbb{R}/\mathbb{Z}$.

2.1. **The Diophantine case.** We include a proof of Theorem 2.1. The proof relies on the following lemma on quasiconformal maps which is classical.

LEMMA 2.6. *Suppose that there exists a K-quasiconformal map between two complex tori E_1 and E_2. Then*

$$\mathrm{dist}_{\mathbb{H}}(\tau(E_1),\tau(E_2)) \le \log K$$

where $\mathrm{dist}_{\mathbb{H}}$ is the hyperbolic distance in $\mathbb{H}$, and where $\tau(E_1)\in\mathbb{H}$ and $\tau(E_2)\in\mathbb{H}$ are moduli with respect to corresponding generators in $H_1(E_1)$ and $H_1(E_2)$.

Without loss of generality, we may assume that $f\colon \mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ has Diophantine rotation number $\theta\in\mathbb{R}/\mathbb{Z}$. A theorem of Yoccoz (see [8]) asserts that there is an analytic circle diffeomorphism $\phi\colon \mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ conjugating the rotation of angle θ to f: for all $x\in\mathbb{R}/\mathbb{Z}$, we have

$$\phi(x+\theta) = f\circ\phi(x).$$

Let $\hat\phi\colon \mathbb{C}/\mathbb{Z}\to\mathbb{C}/\mathbb{Z}$ be the homeomorphism defined by

$$\hat\phi(z) = \phi\big(\mathrm{Re}(z)\big) + \mathrm{i}\,\mathrm{Im}(z).$$

Then, $\hat\phi\colon \mathbb{C}/\mathbb{Z}\to\mathbb{C}/\mathbb{Z}$ is a K-quasiconformal homeomorphism with

$$K := \max\big(\|\phi'\|_\infty, \|1/\phi'\|_\infty\big).$$

Now, for any $y>0$,

$$\hat\phi(x+\theta+\mathrm{i}y) = f\big(\hat\phi(x)\big)+iy,$$

and so, $\hat\phi$ induces a K-quasiconformal homeomorphism between the complex tori $\mathbb{C}/\big(\mathbb{Z}+(\theta+\mathrm{i}y)\mathbb{Z}\big)$ and $E(f_{\mathrm{i}y})$. It follows that for $y>0$, the hyperbolic distance in $\mathbb{H}/\mathbb{Z}$ between $\theta+\mathrm{i}y$ and $\tau_f(\mathrm{i}y)$ is uniformly bounded and thus,

$$\lim_{\substack{y\to 0\\ y>0}} \tau_f(\mathrm{i}y) = \theta.$$

2.2. **The hyperbolic case.** We recall the arguments of the proof of Theorem 2.2 given in [3]. It is based on an auxiliary construction of a complex torus $\mathfrak{E}(f)$ when $f\colon \mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ has rational rotation number and is hyperbolic. This construction will be used again in the proofs of Lemmas 2.4 and 2.5.

Let us assume $f\colon \mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ has rational rotation number p/q and has only hyperbolic periodic cycles. The number $m\ge 1$ of attracting cycles is equal to the number of repelling cycles. Denote by α_j, $j\in\mathbb{Z}/(2mq)\mathbb{Z}$, the periodic points of f, ordered cyclically; even indices correspond to attracting periodic points and odd indices to repelling periodic points. Note that $f(\alpha_j)=\alpha_{j+2mp}$.

Let ρ_j be the multiplier of α_j as a fixed point of $f^{\circ q}$ and $\phi_j\colon (\mathbb{C},0)\to(\mathbb{C}/\mathbb{Z},\alpha_j)$ be the linearizing map which conjugates multiplication by ρ_j to $f^{\circ q}$:

$$f^{\circ q}\circ\phi_j(z) = \phi_j(\rho_j z)$$

and is normalized by $\phi_j'(0)=1$. Then,

$$f\circ\phi_j(z) = \phi_{j+2mp}(\lambda_j\cdot z) \quad\text{with}\quad \lambda_j := f'(\alpha_j).$$

In addition, if $\varepsilon > 0$ is small enough, the linearizing map ϕ_j extends univalently to the strip $\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \varepsilon\}$ and

$$\phi_j(\mathbb{R}) = (\alpha_{j-1}, \alpha_{j+1}).$$

For each $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, let x_j be a point in (α_j, α_{j+1}), so that

- $f(x_j) \in (\alpha_{j+2pm}, x_{j+2pm})$ if the orbit of α_j attracts (i.e. j is even) and
- $f(x_j) \in (x_{j+2pm}, \alpha_{j+2pm+1})$ if the orbit of α_j repels (i.e. j is odd).

This is possible since $f^{\circ q}(x_j) \in (\alpha_j, x_j)$ when j is even and $f^{\circ q}(x_j) \in (x_j, a_{j+1})$ when j is odd. Similarly, let ε_j be a point on the negative imaginary axis if j is even and on the positive imaginary axis if j is odd, so that for all $j \in \mathbb{Z}/(2mp\mathbb{Z})$,

- $|\varepsilon_j| < \varepsilon$, $|\lambda_j \varepsilon_j| < \varepsilon$ and
- $\lambda_j \varepsilon_j$ is above ε_{j+2mp}.

Let C_j be the arc of circle with endpoints $\phi_j^{-1}(x_{j-1})$ and $\phi_j^{-1}(x_j)$ passing through ε_j and set

$$\gamma := \bigcup_{j \in \mathbb{Z}/(2mq\mathbb{Z})} \phi_j(C_j).$$

Then, γ is a simple closed curve in $\mathbb{C}/\mathbb{Z}$ and f is univalent in a neighborhood of γ.

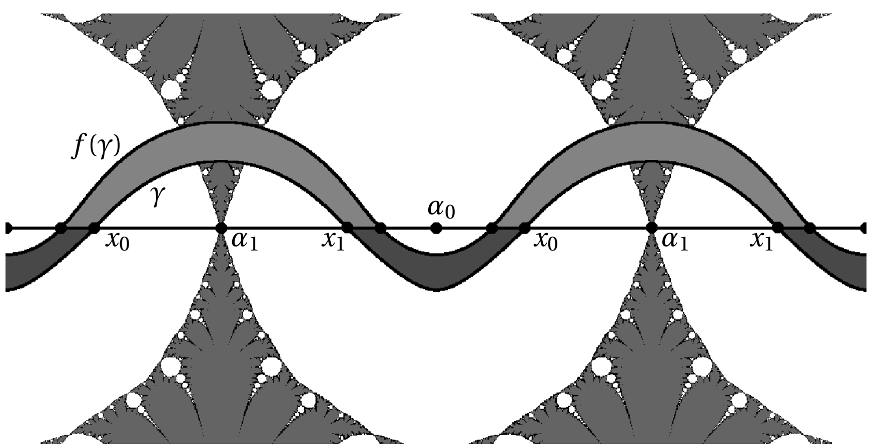

FIGURE 2. A possible choice of curve γ for the map $f\colon \mathbb{C}/\mathbb{Z} \ni z \mapsto z + \frac{1}{4\pi}\sin(2\pi x) \in \mathbb{C}/\mathbb{Z}$ which restricts as a hyperbolic circle diffeomorphism $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. The curve $f(\gamma)$ lies above γ in $\mathbb{C}/\mathbb{Z}$. The essential annulus between γ and $f(\gamma)$ is colored (light grey in the upper half-plane and dark grey in the lower half-plane). The map f has an attracting fixed point at $\alpha_0 := 0 \in \mathbb{R}/\mathbb{Z}$ and a repelling fixed point at $\alpha_1 := 1/2 \in \mathbb{R}/\mathbb{Z}$. The basin of attraction of α_0 in $\mathbb{C}/\mathbb{Z}$ is white; its complement is the Julia set of f.

The attracting cycles of f are above γ in $\mathbb{C}/\mathbb{Z}$ and the repelling cycles are below γ in $\mathbb{C}/\mathbb{Z}$. In addition,

$$f(\gamma) = \bigcup_{j\in\mathbb{Z}/(2mq\mathbb{Z})} \phi_{j+2mp}(\lambda_j C_j)$$

and so, $f(\gamma)$ lies above γ in $\mathbb{C}/\mathbb{Z}$.

For ω sufficiently close to 0, the curve $f_\omega(\gamma) = f(\gamma)+\omega$ remains above γ in $\mathbb{C}/\mathbb{Z}$. The curves γ and $f_\omega(\gamma)$ bound an essential annulus in $\mathbb{C}/\mathbb{Z}$. Glueing the two sides via f_ω, we obtain a complex torus $\mathfrak{E}(f_\omega)$, which may be uniformized as $\mathscr{E}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some appropriate $\tau \in \mathbb{H}/\mathbb{Z}$, the homotopy class of γ in $\mathfrak{E}(f_\omega)$ corresponding to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $\mathscr{E}_\tau$. We set $\bar\tau_f(\omega) := \tau \in \mathbb{H}/\mathbb{Z}$.

According to Risler [6, Chapter 2, Proposition 2], the map $\omega \mapsto \bar\tau_f(\omega)$ is holomorphic. When $\omega \in \mathbb{H}/\mathbb{Z}$, the complex torus $\mathfrak{E}(f_\omega)$ is isomorphic to $E(f_\omega)$ and the homotopy class of γ in $\mathfrak{E}(f_\omega)$ corresponds to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $E(f_\omega)$ (see [3] for details). As a consequence, $\bar\tau_f(\omega) = \tau_f(\omega)$ when $\omega \in \mathbb{H}/\mathbb{Z}$ is sufficiently close to 0. This completes the proof of Theorem 2.2 for $\omega = 0$.

2.3. **The Liouville case: Tsujii's theorem.** For completeness, we now present a proof of Tsujii's Theorem 2.3 which we believe is a simplification of the original one, although the ideas are essentially the same. The main argument in Tsujii's proof is the following.

PROPOSITION 2.7. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a $\mathscr{C}^2$-smooth orientation preserving circle diffeomorphism with irrational rotation number $\theta \in \mathbb{R}/\mathbb{Z}$. If p/q is an approximant to θ given by the continued fraction algorithm, then there is an $\omega \in \mathbb{R}/\mathbb{Z}$ satisfying*

$$|\omega| < e^{D_f}\cdot|\theta - p/q| \quad \textit{and} \quad \mathrm{rot}(f_\omega) = p/q.$$

Proof. According to a Theorem of Denjoy, there is a homeomorphism $\phi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $\phi(x+\theta) = f\circ\phi(x)$ for all $x \in \mathbb{R}/\mathbb{Z}$.

Without loss of generality, let us assume that $\theta < p/q$ and set $\delta := p - q\theta$. Let $T \subset \mathbb{R}/\mathbb{Z}$ be the union of intervals

$$T := \bigcup_{1\le j\le q} T_j \quad \text{with} \quad T_j := (j\theta, j\theta+\delta).$$

Since p/q is an approximant of θ, this is a disjoint union of q intervals of length δ. According to Lemma 2.8 below, we may choose $t \in \mathbb{R}/\mathbb{Z}$ such that the Lebesgue measure of $\phi(T+t)$ is at most $q\delta$.

Now, set $x := \phi(t)$ and for $j \in \mathbb{Z}$, set

$$x_j := f^{\circ j}(x) = \phi(t+j\theta) \quad \text{and} \quad I_j := (x_j, x_{j-q}) = \phi(T_j).$$

The intervals I_1, $I_2 = f(I_1)$, $\ldots$, $I_q = f^{\circ q}(I_1)$ are disjoint and the sum of their lengths satisfies

$$\sum_{j=1}^{q} |I_j| \le q\delta = q^2\cdot|\theta - p/q|.$$

As $\omega \in \mathbb{R}/\mathbb{Z}$ increases from 0, the rotation number $\mathrm{rot}(f_\omega) \in \mathbb{R}/\mathbb{Z}$ increases from θ, and there is a first ω_0 such that $\mathrm{rot}(f_{\omega_0}) = p/q$. For $j \in [0,q]$, set

$$y_j := (f_{\omega_0})^{\circ j}(x) \quad \text{and} \quad z_j := f^{\circ(q-j)}(y_j).$$

Finally, for $j \in [1,q]$, set

$$J_j := \big(f(y_{j-1}), y_j\big) = \big(f(y_{j-1}), f(y_{j-1}) + \omega_0\big) \quad \text{and} \quad K_j := (z_{j-1}, z_j).$$

Then, $(z_0, z_1, \ldots, z_q)$ is a subdivision of (z_0, z_q) (see Figure 3).

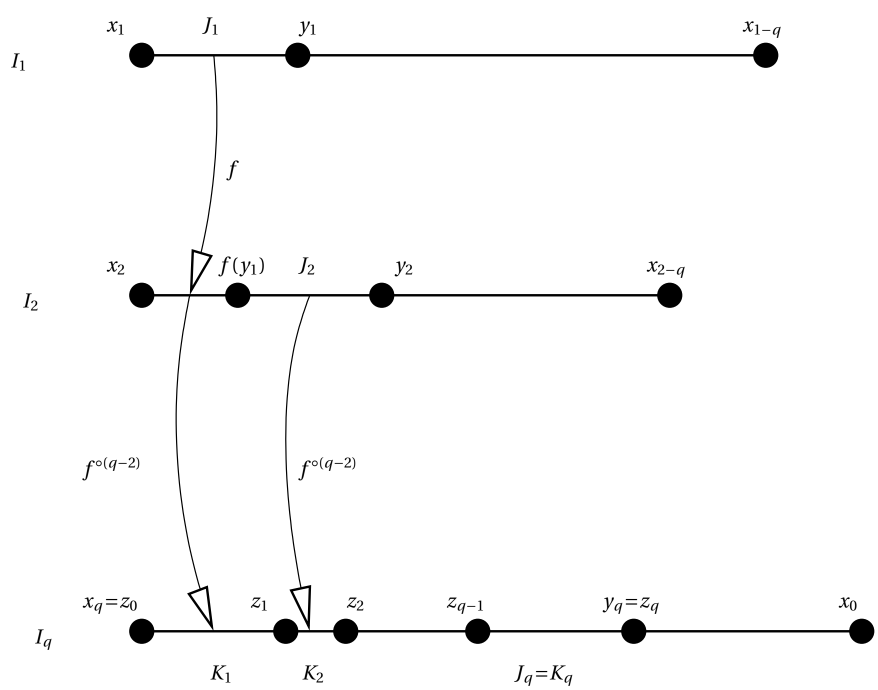

FIGURE 3. The intervals I_j, J_j and K_j.

As ω increases from 0 to ω_0, the point $(f_\omega)^{\circ q}(x)$ increases from x_q to y_q but remains in I_q since $\mathrm{rot}(f_\omega)$ remains less than p/q. Thus, $(z_0, z_q) = (x_q, y_q) \subseteq I_q$ and so,

$$|I_q| \ge |z_q - z_0| = \sum_{j=1}^{q} |K_j|.$$

In addition, $J_j \subset I_j$ and $K_j = f^{\circ(q-j)}(J_j)$. It follows from Denjoy's Lemma 1.1 that

$$\frac{|K_j|}{|I_q|} \ge e^{-D_f} \frac{|J_j|}{|I_j|} = e^{-D_f} \frac{\omega_0}{|I_j|}.$$

Now, according to the Cauchy-Schwarz Inequality, we have

$$q^2 = \left(\sum_{j=1}^{q} \sqrt{|I_j|} \cdot \frac{1}{\sqrt{|I_j|}} \right)^2 \le \left(\sum_{j=1}^{q} |I_j| \right) \cdot \left(\sum_{j=1}^{q} \frac{1}{|I_j|} \right) \le q^2 \cdot |\theta - p/q| \cdot \sum_{j=1}^{q} \frac{1}{|I_j|}.$$

Thus,

$$|I_q| \geq \sum_{j=1}^{q} |K_j| \geq e^{-D_f}\omega_0 |I_q| \cdot \sum_{j=1}^{q} \frac{1}{|I_j|} \geq \frac{e^{-D_f}\omega_0 |I_q|}{|\theta - p/q|}$$

and so,

$$\omega_0 \leq e^{D_f} \cdot |\theta - p/q|. \qquad \square$$

LEMMA 2.8. *Let $\phi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism. Then, for any measurable set $T \subseteq \mathbb{R}/\mathbb{Z}$, there is a $t \in \mathbb{R}/\mathbb{Z}$ such that*

$$\mathrm{Leb}\big(\phi(T+t)\big) \leq \mathrm{Leb}(T).$$

Proof. Let μ be the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. According to Tonelli's theorem,

$$\begin{aligned}
\int_{t\in\mathbb{R}/\mathbb{Z}} \mu\big(\phi(T+t)\big)\, \mathrm{d}t &= \int_{t\in\mathbb{R}/\mathbb{Z}} \left(\int_{u\in T+t} \mathrm{d}(\phi^*\mu)\right) \mathrm{d}\mu \\
&= \int_{u\in\mathbb{R}/\mathbb{Z}} \left(\int_{t\in -T+u} \mathrm{d}\mu\right) \mathrm{d}(\phi^*\mu) \\
&= \int_{u\in\mathbb{R}/\mathbb{Z}} \mu(T)\ \mathrm{d}(\phi^*\mu) \\
&= \mu(T)\cdot\mu\big(\phi(\mathbb{R}/\mathbb{Z})\big) = \mu(T).
\end{aligned}$$

So, the average of $\mu\big(\phi(T+t)\big)$ with respect to t is equal to $\mu(T)$ and the result follows. $\square$

Theorem 2.3 follows easily from Proposition 2.7: for $\beta > 0$, let S_β be the set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is irrational and such that there are infinitely many $p, q \in \mathbb{Z}$ satisfying $\big|\mathrm{rot}(f_\omega) - p/q\big| < 1/q^{2+\beta}$. The set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is Liouville is the intersection of the sets S_β. So, it is sufficient to show that the $\mathrm{Leb}(S_\beta) = 0$ for all $\beta > 0$. Note that

$$S_\beta = \limsup_{q\to+\infty} S_{\beta,q}$$

where $S_{\beta,q}$ is the set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is irrational and such that $\big|\mathrm{rot}(f_\omega) - p/q\big| < 1/q^{2+\beta}$ for some approximant p/q of $\mathrm{rot}(f_\omega)$.

Proposition 2.7 implies that $S_{\beta,q}$ is located in the $C/q^{2+\beta}$-neighborhood of the union of q intervals where the rotation number is rational with denominator q, where $C := e^{D_f}$. So,

$$\mathrm{Leb}(S_{\beta,q}) \leq 2q \cdot \frac{C}{q^{2+\beta}} = \frac{2C}{q^{1+\beta}}.$$

In particular, for all $\beta > 0$,

$$\mathrm{Leb}(S_\beta) = \mathrm{Leb}\left(\limsup_{q\to+\infty} S_{\beta,q}\right) \leq \limsup_{q\to+\infty} \sum_{r\geq q} \frac{2C}{r^{1+\beta}} = 0.$$

2.4. **Back to the hyperbolic case.** We now come to our main contribution, starting with the proof of Lemma 2.4. Assume $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has rational rotation number p/q and has only hyperbolic periodic cycles. As in Section 2.2, consider a simple closed curve γ oscillating between the attracting cycles of f (which are above γ in $\mathbb{C}/\mathbb{Z}$) and the repelling cycles of f (which are below γ in $\mathbb{C}/\mathbb{Z}$), so that $f(\gamma)$ lies above γ in $\mathbb{C}/\mathbb{Z}$.

The curves γ and $f(\gamma)$ bound an essential annulus in $\mathbb{C}/\mathbb{Z}$. Glueing the curves via f, we obtain a complex torus $\mathfrak{E}(f)$ isomorphic to $\mathscr{E}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\tau := \bar{\tau}_0(f) \in \mathbb{H}/\mathbb{Z}$, the class of γ in $\mathfrak{E}(f)$ corresponding to the class of $\mathbb{R}/\mathbb{Z}$ in $\mathscr{E}_\tau$.

The projection of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f)$ consists of $2m$ topological circles cutting $\mathfrak{E}(f)$ into $2m$ annuli associated to the cycles of f. More precisely, each attracting (respectively repelling) cycle c has a basin of attraction B_c for f (respectively for f^{-1}) and the projection of $\mathbb{H}^- \cap B_c$ (respectively $\mathbb{H}^+ \cap B_c$) in $\mathfrak{E}(f)$ is an annulus A_c of modulus

$$\operatorname{mod} A_c = \frac{\pi}{|\log \rho_c|},$$

where ρ_c is the multiplier of c as a cycle of f.

Those annuli wind around the class of γ in $\mathfrak{E}(f)$ with combinatorial rotation number $-p/q$. It follows from a classical length-area argument (see [2, Proposition 3.3] for example) that there is a representative $\tilde{\tau} \in \mathbb{H}$ of $\tau \in \mathbb{H}/\mathbb{Z}$ such that

$$\sum_{c \text{ cycle of } f} \operatorname{mod} A_c \le \frac{\operatorname{Im}(\tilde{\tau})}{|-p+q\tilde{\tau}|^2}.$$

As a consequence,

$$\frac{|\tilde{\tau}-p/q|^2}{\operatorname{Im}\tilde{\tau}} \le R_\omega := \frac{1}{\pi q^2 \cdot \displaystyle\sum_{c \text{ cycle of } f} \operatorname{mod} A_c},$$

which yields Lemma 2.4 since

$$\sum_{c \text{ cycle of } f} \operatorname{mod} A_c = \sum_{c \text{ cycle of } f} \frac{\pi}{|\log \rho_c|} = \frac{1}{q} \sum_{x \in \operatorname{Per}(f)} \frac{\pi}{|\log \rho_x|}.$$

Before going further, we shall establish a result that will be used in the proof of Lemma 2.5. Recall that the curve γ intersects the interval (α_j, α_{j+1}) at the point x_j, belongs to the lower half-plane below the segment (x_{j-1}, x_j) if j is even and to the upper half-plane above the segment (x_{j-1}, x_j) if j is odd.

Recall that m is the number of attracting cycles of f. The projection of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f^{\circ q})$ cuts the torus in $2mq$ annuli A_j, $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, which wind around the class of γ with combinatorial rotation number 0 and have moduli

$$\operatorname{mod} A_j = m_j := \frac{\pi}{|\log \rho_j|}.$$

Let $S_j \subset \mathbb{C}$ and $B_j \subset \mathbb{C}/\mathbb{Z}$ be defined by

$$S_j := \{z \in \mathbb{C} \mid 0 < \operatorname{Im}(z) < m_j\} \quad \text{and} \quad B_j := S_j/\mathbb{Z}.$$

Set

$$\tilde{r}_j := \frac{\log\phi_j^{-1}(x_j)}{\log\rho_j} \quad \text{and} \quad \tilde{s}_j := \frac{\log|\phi_j^{-1}(x_{j-1})|}{\log\rho_j} + \frac{\mathrm{i}\pi}{|\log\rho_j|}.$$

The class r_j of $\tilde{r}_j$ in $\mathbb{C}/\mathbb{Z}$ belongs to the lower boundary component $C_j^- := \mathbb{R}/\mathbb{Z}$ of B_j and the class s_j of $\tilde{s}_j$ in $\mathbb{C}/\mathbb{Z}$ belongs to the upper boundary component $C_j^+ := (\mathbb{R}+\mathrm{i}m_j)/\mathbb{Z}$ of B_j. The map $z \mapsto \phi_j \circ \exp(z\cdot\log\rho_j)$ induces an isomorphism $\chi_j : B_j \to A_j$ which extends analytically to the boundary, sends r_j to the class of x_j in $\mathfrak{E}(f^{\circ q})$ and s_j to the class of x_{j-1} in $\mathfrak{E}(f^{\circ q})$ (see Figure 4).

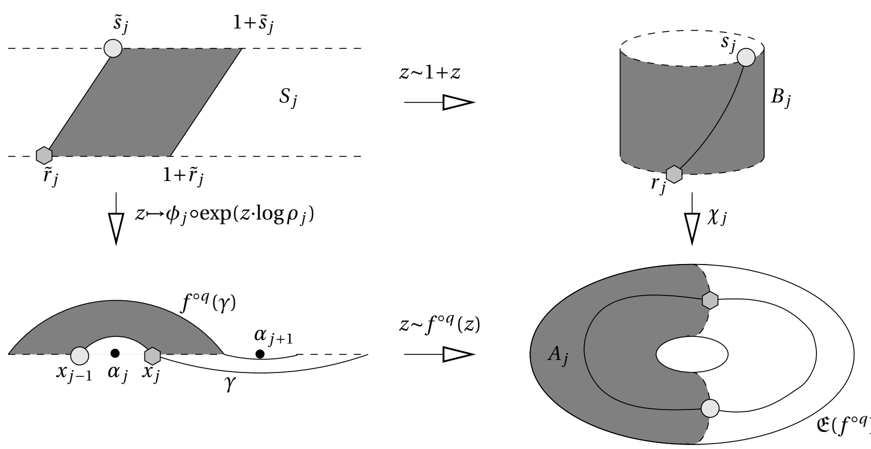

FIGURE 4. The projection of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f^{\circ q})$ cuts the torus in $2mq$ annuli A_j, $j \in \mathbb{Z}/(2mq)\mathbb{Z}$.

LEMMA 2.9. *We have that*

$$\mathrm{dist}_{\mathbb{H}/\mathbb{Z}}\left(q\tau, -\frac{1}{\sigma}\right) \le 5D_f \quad \textit{with} \quad \sigma := \sum_{j\in\mathbb{Z}/2mqZ} \tilde{s}_j - \tilde{r}_j.$$

Proof. It will be more convenient to work with $f^{\circ q}$ whose rotation number is $0/1$. The diffeomorphism f induces an automorphism of $\mathfrak{E}(f^{\circ q})$ of order q. The quotient of $\mathfrak{E}(f^{\circ q})$ by this automorphism is isomorphic to $\mathfrak{E}(f)$. The class of γ in $\mathfrak{E}(f)$ has q disjoint preimages in $\mathfrak{E}(f^{\circ q})$ which map with degree 1 to γ. It follows that $\mathfrak{E}(f^{\circ q})$ is isomorphic to $\mathscr{E}_{q\tau} := \mathbb{C}/(\mathbb{Z}+q\tau\mathbb{Z})$, the class of γ in $\mathfrak{E}(f^{\circ q})$ corresponding to the class of $\mathbb{R}/\mathbb{Z}$ in $\mathscr{E}_{q\tau}$.

Set $\mathscr{E}_\sigma := \mathbb{C}/(\mathbb{Z}+\sigma\mathbb{Z})$. We will now construct a K-quasiconformal map

$$\psi : \mathfrak{E}(f^{\circ q}) \to \mathscr{E}_\sigma$$

which sends the class of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f^{\circ q})$ to the class of $\sigma\mathbb{R}/\sigma\mathbb{Z}$ in $\mathscr{E}_\sigma$. We will also show that $\log K \le 5D_f$. The result then follows from Lemma 2.6.

On the one hand, glueing the lower boundary component C_j^- of B_j with the upper boundary component C_{j+1}^+ of B_{j+1} via the analytic diffeomorphism

$$\xi_j := \chi_{j+1}^{-1} \circ \chi_j : C_j^- \to C_{j+1}^+,$$

we obtain a complex torus E which is isomorphic to $\mathfrak{E}(f^{\circ q})$. Let δ_j be the projection of the segment $[\tilde r_j, \tilde s_j]$ to E. The homotopy class of the simple closed curve

$$\delta := \bigcup_{j\in\mathbb{Z}/(2mq)\mathbb{Z}} \delta_j$$

in E corresponds to the homotopy class of γ in $\mathfrak{E}(f^{\circ q})$.

On the other hand, glueing the lower boundary component C_j^- of B_j with the upper boundary component C_{j+1}^+ of B_{j+1} via the translation by $z\mapsto z-r_j+s_{j+1}$, we obtain a complex torus E' which is isomorphic to $\mathscr{E}_\sigma$. Let δ'_j be the projection of the segment $[\tilde r_j, \tilde s_j]$ to E'. The homotopy class of the simple closed curve

$$\delta' := \bigcup_{j\in\mathbb{Z}/(2mq)\mathbb{Z}} \delta'_j$$

in E' corresponds to the homotopy class of $\sigma\mathbb{R}/\sigma\mathbb{Z}$ in $\mathscr{E}_\sigma$.

The homeomorphism

$$\psi_j := \xi_j - s_{j+1} + r_j \colon C_j^- \to C_j^-$$

fixes $r_j \in C_j^-$. Let $\tilde\psi_j\colon \mathbb{R}\to\mathbb{R}$ be the lift of $\psi_j\colon C_j^-\to C_j^-$ which fixes $\tilde r_j$ and let $\Psi_j\colon S_j\to S_j$ be the extension to S_j defined by

$$\Psi_j(x+\mathrm{i}y) := \frac{y}{m_j}(x+\mathrm{i}m_j) + \left(1-\frac{y}{m_j}\right)\tilde\psi_j(x).$$

The homeomorphism $\Psi_j\colon \overline{S}_j\to\overline{S}_j$ restricts to the identity on $\mathbb{R}+\mathrm{i}m_j$ and descends to a homeomorphism $\psi_j\colon \overline{B}_j\to\overline{B}_j$. By construction, the following diagram commutes:

$$\begin{tikzcd} C_j^- \arrow[r, "\psi_j"] \arrow[d, "\xi_j"'] & C_j^- \arrow[d, "z\mapsto z-r_j+s_{j+1}"] \\ C_{j+1}^+ \arrow[r, "\psi_{j+1}"'] & C_{j+1}^+. \end{tikzcd}$$

So, the collection of homeomorphisms $\psi_j\colon \overline{B}_j\to\overline{B}_j$ induces a global homeomorphism $\psi\colon E\to E'$. Since Ψ_j fixes $\tilde r_j$ and $\tilde s_j$, the homeomorphism ψ sends the homotopy class of δ in E to the homotopy class of δ' in E'. The proof is completed by Lemma 2.10 below. □

LEMMA 2.10. *The homeomorphism $\psi\colon E\to E'$ is e^{5D_f}-quasiconformal.*

Proof. The image of the curves $C_j^\pm$ in E are analytic (because the glueing map ξ_j is analytic), therefore quasiconformally removable. So, it is enough to prove that each $\psi_j\colon B_j\to B_j$ is e^{5D_f}-quasiconformal. Equivalently, we must prove that

$$\left\|\frac{\partial\Psi_j/\partial\bar z}{\partial\Psi_j/\partial z}\right\|_\infty \le k<1 \quad\text{with}\quad \mathrm{dist}_{\mathbb{D}}(0,k) < 5D_f,$$

where $\mathrm{dist}_{\mathbb{D}}$ is the hyperbolic distance within the unit disk.

For readibility, we drop the index j in the following computation:

$$\begin{aligned}\frac{\partial\Psi/\partial\bar z}{\partial\Psi/\partial z}(x+\mathrm{i}y) &= \frac{\partial\Psi/\partial x+\mathrm{i}\partial\Psi/\partial y}{\partial\Psi/\partial x-\mathrm{i}\partial\Psi/\partial y}(x+\mathrm{i}y)\\ &= \frac{\left(1-\frac{y}{m}\right)\cdot\left(\tilde\psi'(x)-1\right)-\frac{\mathrm{i}}{m}\left(\tilde\psi(x)-x\right)}{2+\left(1-\frac{y}{m}\right)\cdot\left(\tilde\psi'(x)-1\right)+\frac{\mathrm{i}}{m}\left(\tilde\psi(x)-x\right)}.\end{aligned}$$

This last quantity is of the form $(a-1)/(\bar a+1)$ with

$$\mathrm{Re}(a)=1+\left(1-\frac{y}{m}\right)\cdot\left(\tilde\psi'(x)-1\right)\quad\text{and}\quad \mathrm{Im}(a)=\frac{\tilde\psi(x)-x}{m}.$$

Note that $\left|\frac{a-1}{\bar a+1}\right|=\left|\frac{a-1}{a+1}\right|$ and the Möbius transformation $a\mapsto\frac{a-1}{a+1}$ sends the right half-plane into the unit disk. So, it is enough to show that a belongs to the right half-plane $\{z\in\mathbb{C}\mid \mathrm{Re}(z)>0\}$ and that the hyperbolic distance within this half-plane between 1 and a is at most $5D_f$.

This hyperbolic distance is bounded from above by $\left|\mathrm{Im}(a)\right|+\left|\log\mathrm{Re}(a)\right|$. Since $\tilde\psi\colon\mathbb{R}\to\mathbb{R}$ is an increasing diffeomorphism which fixes $p+\mathbb{Z}\in\mathbb{R}$, we have that $\tilde\psi'(x)>0$ and $\left|\tilde\psi(x)-x\right|<1$. In addition, $0<1-y/m<1$, and so,

$$0<\min_{\mathbb{R}}\tilde\psi'\le\mathrm{Re}(a)\le\max_{\mathbb{R}}\tilde\psi'\quad\text{and}\quad\left|\mathrm{Im}(a)\right|\le\frac{1}{m}=\frac{|\log\rho|}{\pi}\le|\log\rho|\le D_f.$$

The last inequality is given by Lemma 1.2. The average of $\tilde\psi'$ on $[0,1]$ is equal to $\tilde\psi(1)-\tilde\psi(0)=1$. So, $\tilde\psi'$ takes the value 1 and

$$-\mathrm{dis}_{\mathbb{R}}(\xi)=-\mathrm{dis}_{\mathbb{R}}(\tilde\psi)<\log\min_{\mathbb{R}}(\tilde\psi')\le0\le\log\max_{\mathbb{R}}(\tilde\psi')<\mathrm{dis}_{\mathbb{R}}(\tilde\psi)=\mathrm{dis}_{\mathbb{R}}(\xi).$$

The proof is completed by Lemma 2.11 below. □

LEMMA 2.11. *For any $j\in\mathbb{Z}/(2mq)\mathbb{Z}$, the distortion of ξ_j is bounded by $4D_f$.*

Proof. The map $\xi_j\colon C_j^-\to C_{j+1}^+$ is induced by the following composition

$$\mathbb{R}\xrightarrow{e_j}(0,+\infty)\xrightarrow{\phi_j}(\alpha_j,\alpha_{j+1})\xrightarrow{\phi_{j+1}^{-1}}(-\infty,0)\xrightarrow{e_{j+1}^{-1}}\mathbb{R}+\mathrm{i}m_{j+1}.$$

with

$$e_j(z):=\exp(z\cdot\log\rho_j)\quad\text{and}\quad e_{j+1}(z)=\exp(z\cdot\log\rho_{j+1}).$$

The distortion of e_j on any interval of length 1 is $|\log\rho_j|$ which is at most D_f according to Lemma 1.2. Similarly, the distortion of e_{j+1} on any interval of length 1 is $|\log\rho_{j+1}|\le D_f$.

Let x be any point in (α_j,α_{j+1}) and let $I\subset\mathbb{R}/\mathbb{Z}$ be the interval whose extremities are x and $f(x)$. To complete the proof, it is enough to show that

$$\mathrm{dis}_I(\phi_j^{-1})\le D_f\quad\text{and}\quad\mathrm{dis}_I(\phi_{j+1}^{-1})\le D_f.$$

We will only prove this result for ϕ_j in the case where α_j is attracting. The other cases are dealt similarly and left to the reader.

On I, the linearizing map ϕ_j is the limit of the maps $\varphi_n:=(f^{\circ qn}-\alpha_j)/\rho_j^n$. Since I is disjoint from all its iterates, Denjoy's Lemma 1.1 yields

$$\mathrm{dis}_I\,\varphi_n=\mathrm{dis}_I\,f^{\circ qn}\le D_f.$$

Passing to the limit as n tends to ∞ shows that $\operatorname{dis}_I \phi_j \le D_f$ as required. □

2.5. **Continuity of $\bar\tau_f$.** We now prove Lemma 2.5. It is enough to prove that $\bar\tau_f$ is continuous at $\omega = 0$. We shall see that when $\operatorname{rot}(f)$ is irrational, the continuity follows from Lemma 2.4, but when $\operatorname{rot}(f)$ is rational, the situation is more subtle.

2.5.1. *Irrational rotation number.* If $\operatorname{rot}(f)$ is irrational, then $\bar\tau_f(0) = \operatorname{rot}(f)$ due to the definition of $\bar\tau_f$.

Let $I \subset \mathbb{R}/\mathbb{Z}$ be a small neighborhood of 0 such that for $\omega \in I$, the periods of the periodic cycles of f_ω are at least N. For $\omega \in I$, either $\bar\tau_f(\omega) = \operatorname{rot}(f_\omega)$, or according to Lemma 2.4,

$$\left|\bar\tau_f(\omega) - \operatorname{rot}(f_\omega)\right| \le \frac{D_f}{N^2}.$$

Thus, $\bar\tau_f(I)$ is located within D_f/N^2-neighborhood of $\left\{\operatorname{rot}(f_\omega) \,\middle|\, \omega \in I\right\}$. The result follows since $\omega \mapsto \operatorname{rot}(f_\omega)$ is continuous.

2.5.2. *Rational rotation number.* If f is hyperbolic, then the continuity of $\bar\tau_f$ at 0 follows directly from Theorem 2.2.

Let us assume f has at least one parabolic cycle. We will only prove that

$$\lim_{\omega>0,\omega\to 0} \bar\tau_f(\omega) = \frac{p}{q} = \bar\tau_f(0).$$

Applying this result to the diffeomorphism $x \mapsto -f(-x)$ yields

$$\lim_{\omega<0,\omega\to 0} \bar\tau_f(\omega) = \frac{p}{q} = \bar\tau_f(0).$$

There are three different cases.

1. All q-periodic orbits of f disappear as ω increases, so that, $\operatorname{rot}(f_\omega) > p/q$ for $\omega > 0$. In this case, the proof is literally the same as in the case of irrational rotation number.
2. At least one parabolic cycle of f bifurcates into real hyperbolic cycles. In this case, the multipliers of these real hyperbolic cycles tend to 1 as ω tends to 0. The result follows from Lemma 2.4.
3. All parabolic cycles of f bifurcate into complex conjugate cycles as $\omega > 0$ increases but the rotation number stays unchanged because f has hyperbolic cycles.

The rest of the Section is devoted to the treatment of the third case.

LEMMA 2.12. *Under the assumptions of case (3) above, the curve $\bar\tau_f(\omega)$ is tangent to the segment $\left[\frac{p}{q}, \frac{p}{q}+\varepsilon\right) \subset \mathbb{R}/\mathbb{Z}$; moreover, it is located between two horocycles tangent to $\mathbb{R}/\mathbb{Z}$ at $\frac{p}{q}$.*

Proof. According to Lemma 2.4, we know that for $\omega > 0$ close to ω, $\bar\tau_f(\omega)$ remains in a subdisk of $\mathbb{H}/\mathbb{Z}$ tangent to the real axis at p/q. So, it is enough to prove that $q\bar\tau_f(\omega)$ tends to 0 tangentially to the segment $[0,\varepsilon) \in \mathbb{R}/\mathbb{Z}$ and is located in between two horocycles tangent to $\mathbb{R}/\mathbb{Z}$ at the point 0.

According to Lemma 2.9, the hyperbolic distance in $\mathbb{H}/\mathbb{Z}$ between $q\bar{\tau}_f(\omega)$ and $-1/\sigma$ (where $\sigma = \sigma_\omega$ depends on ω) is uniformly bounded as $\omega > 0$ tends to 0. So, it is enough to show that the imaginary part of σ_ω is bounded and that the real part of σ_ω tends to $-\infty$.

Now we recall the definition of σ, and at the same time we introduce some notation. This new notation is similar to that of Section 2.2. The main difference is, that f is *not* hyperbolic.

Let m be the number of attracting hyperbolic cycles of f and order cyclically the hyperbolic periodic points α_j, $j \in \mathbb{Z}/(2mq)\mathbb{Z}$. For each $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, let x_j be a point in (α_j, α_{j+1}), so that

- $f(x_j) \in (\alpha_{j+2pm}, x_{j+2pm})$ if the orbit of α_j attracts (i.e. j is even) and
- $f(x_j) \in (x_{j+2pm}, \alpha_{j+2pm+1})$ if the orbit of α_j repels (i.e. j is odd).

Note that since the parabolic cycles disappear as $\omega > 0$ increases, the graph of $f^{\circ q} - \mathrm{id}$ lies above the diagonal near those points. As a consequence, each parabolic periodic point lies in an interval of the form (α_j, α_{j+1}) with α_j repelling and α_{j+1} attracting.

For $\omega > 0$ close enough to 0, f_ω has a hyperbolic point $\alpha_j(\omega)$ close to α_j. We denote by $\rho_{\omega,j}$ the corresponding multiplier and by $\phi_{\omega,j}$ the corresponding linearizing map. Finally, using the points x_j chosen above which do not depend on ω, set

$$\tilde{r}_{\omega,j} := \frac{\log \phi_{\omega,j}^{-1}(x_j)}{\log \rho_{\omega,j}}, \quad \tilde{s}_{\omega,j} := \frac{\log |\phi_{\omega,j}^{-1}(x_{j-1})|}{\log \rho_{\omega,j}} + \frac{\mathrm{i}\pi}{|\log \rho_{\omega,j}|}$$

and

$$\sigma_\omega := \sum_{j \in \mathbb{Z}/(2mq)\mathbb{Z}} \tilde{s}_{\omega,j} - \tilde{r}_{\omega,j}.$$

This definition agrees with the notation of Lemma 2.9.

Now, we prove that the imaginary part of σ_ω is bounded and that the real part of σ_ω tends to $-\infty$.

Since

$$\mathrm{Im}(\tilde{r}_{\omega,j}) = 0 \quad \text{and} \quad \mathrm{Im}(\tilde{s}_{\omega,j}) \underset{\omega>0,\omega\to 0}{\longrightarrow} \mathrm{Im}(\tilde{s}_j),$$

we see that the imaginary part remains bounded as $\omega > 0$ tends to 0.

If f has no parabolic periodic point on the interval (α_j, α_{j+1}), then $\phi_{\omega,j}^{-1} \to \phi_j^{-1}$ on the interval (α_j, α_{j+1}). It follows that $\mathrm{Re}(\tilde{r}_{\omega,j})$ and $\mathrm{Re}(\tilde{s}_{\omega,j+1})$ remain bounded. If f has a parabolic periodic point on the interval (α_j, α_{j+1}), then α_j is repelling and α_{j+1} is attracting. Either the two quantities $\log \phi_{\omega,j}^{-1}(x_j)$ and $\log |\phi_{\omega,j+1}^{-1}(x_j)|$ tend to $+\infty$, or one remains bounded and the other tends to $+\infty$. Since $\log \rho_{\omega,j} \to \log \rho_j > 0$ and $\log \rho_{\omega,j+1} \to \log \rho_{j+1} < 0$, in both cases,

$$\mathrm{Re}(\tilde{s}_{\omega,j+1} - \tilde{r}_{\omega,j}) \underset{\omega>0,\omega\to 0}{\longrightarrow} -\infty. \qquad \square$$

As announced in the introduction, we derive the existence of orientation preserving analytic circle diffeomorphisms f for which τ_f fails to be univalent.

COROLLARY 2.13. *Assume that $x - f(x)$ has two local maxima at points x_1 and x_2 with $x_1 - f(x_1) \neq x_2 - f(x_2)$. Then, τ_f is not injective.*

Proof. Let y_1 and y_2 be the respective values of $x - f(x)$ at x_1 and x_2. Suppose that $y_1 < y_2$. Then the map f_ω for $y_1 < \omega < y_2$ has zero rotation number, and it has parabolic fixed points for $\omega = y_1$ and $\omega = y_2$. When ω increases from y_1 to $y_1 + \varepsilon$, the parabolic fixed point disappears, thus due to Lemma 2.12, the curve $\omega \mapsto \bar{\tau}_f(\omega)$ is tangent to $[y_1, y_1 + \varepsilon)$. When $\omega < y_2$ tends to y_2, the two hyperbolic fixed points merge into a parabolic fixed point. Thus, according to Lemma 2.4, the curve $\omega \mapsto \bar{\tau}_f(\omega)$ enters any horocycle as $\omega < y_2$ tends to y_2. But if τ_f were injective, the pair of germs of the curve $\bar{\tau}_f|_{\mathbb{R}/\mathbb{Z}}$ at y_1 and y_2 (both passing through 0) would be oriented clockwise. The contradiction shows that τ_f is not injective in the upper half-plane. □

2.6. **Proof of the Main Theorem.** The map

$$\mathbb{C}/\mathbb{Z} \ni z \mapsto \exp(2\pi \mathrm{i} z) \in \mathbb{C} - \{0\}$$

is an isomorphism of Riemann surfaces. It conjugates $\tau_f : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ to a holomorphic function $g : \mathbb{D} - \{0\} \to \mathbb{D} - \{0\}$ and $\bar{\tau}_f : \mathbb{R}/\mathbb{Z} \to \overline{\mathbb{H}/\mathbb{Z}}$ to a continuous function $h : \partial\mathbb{D} \to \overline{\mathbb{D}}$. Since g is bounded, it extends holomorphically at 0. According to the previous study,

$$\text{for almost every } t \in \mathbb{R}/\mathbb{Z}, \qquad \lim_{r \to 1, r < 1} g(re^{2\pi \mathrm{i} t}) = h(e^{2\pi \mathrm{i} t}).$$

The Main Theorem is therefore a consequence of the following classical result.

LEMMA 2.14. *Let $g : \mathbb{D} \to \mathbb{C}$ be a bounded holomorphic function and $h : \partial\mathbb{D} \to \mathbb{C}$ be a continuous function such that:*

$$\textit{for almost every } t \in \mathbb{R}/\mathbb{Z}, \qquad \lim_{r \to 1, r < 1} g(re^{2\pi \mathrm{i} t}) = h(e^{2\pi \mathrm{i} t}).$$

Then, h extends g continuously to $\overline{\mathbb{D}}$.

Proof. The real and imaginary parts of g are harmonic functions. Due to the Poisson formula (applied to both $\operatorname{Re} g$ and $\operatorname{Im} g$) for $|z| < r$ we have

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\alpha}) P(re^{i\alpha}, z) \, \mathrm{d}\alpha, \tag{2.1}$$

where P is the Poisson kernel,

$$P(re^{i\alpha}, Re^{i\beta}) = \frac{r^2 - R^2}{r^2 + R^2 - 2rR\cos(\alpha - \beta)}.$$

The integrand in (2.1) is bounded as r tends to 1, and it tends to $h(e^{i\alpha})P(e^{i\alpha}, z)$ almost everywhere. Due to the Lebesgue bounded convergence theorem,

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\alpha}) P(e^{i\alpha}, z) \, \mathrm{d}\alpha.$$

Due to the Poisson theorem, the right-hand side provides the solution of the Dirichlet boundary problem for Laplace equation. Thus $\operatorname{Re} g$ and $\operatorname{Im} g$ satisfy

$$\lim_{z\to e^{i\alpha}} \operatorname{Re} g(z) = \operatorname{Re} h(e^{i\alpha}), \quad \lim_{z\to e^{i\alpha}} \operatorname{Im} g(z) = \operatorname{Im} h(e^{i\alpha}). \qquad \square$$

APPENDIX A. BEHAVIOR OF τ_f NEAR $+\mathrm{i}\infty$

Here, we study the behavior of $\tau_f(\omega)$ as the imaginary part of ω tends to $+\infty$. The map $\mathbb{C}/\mathbb{Z} \ni z \mapsto \exp(2\pi \mathrm{i} z) \in \mathbb{C} - \{0\}$ is an isomorphism of Riemann surfaces. Thus, $\mathbb{C}/\mathbb{Z}$ may be compactified as a Riemann surface $\overline{\mathbb{C}/\mathbb{Z}}$ isomorphic to the Riemann sphere, by adding two points $+\mathrm{i}\infty$ anf $-\mathrm{i}\infty$ (the notation suggests that $\pm\mathrm{i}\infty$ is the limit of points $z \in \mathbb{C}/\mathbb{Z}$ whose imaginary part tends to $\pm\infty$). We shall denote by

$$\overline{\mathbb{H}^\pm/\mathbb{Z}} = \mathbb{H}^\pm/\mathbb{Z} \cup \mathbb{R}/\mathbb{Z} \cup \{\pm\mathrm{i}\infty\}$$

the closure of $\mathbb{H}^\pm/\mathbb{Z}$ in $\overline{\mathbb{C}/\mathbb{Z}}$.

The following construction is usually referred to as *conformal welding*. It is customarily studied in the case of non-smooth circle homeomorphisms and is trivial in the case of analytic circle diffeormorphisms.

The analytic circle diffeomorphism f may be viewed as an analytic diffeomorphism between the boundary of $\overline{\mathbb{H}^+/\mathbb{Z}}$ and the boundary of $\overline{\mathbb{H}^-/\mathbb{Z}}$. If we glue $\overline{\mathbb{H}^+/\mathbb{Z}}$ to $\overline{\mathbb{H}^-/\mathbb{Z}}$ via f, we obtain a Riemann surface which is isomorphic to $\overline{\mathbb{C}/\mathbb{Z}}$. We may choose the isomorphism ϕ such that $\phi(\pm\mathrm{i}\infty) = \pm\mathrm{i}\infty$. Such an isomorphism is not unique, but it is unique up to addition of a constant in $\mathbb{C}/\mathbb{Z}$. It restricts to univalent maps $\phi^\pm\colon \mathbb{H}^\pm/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ which extend univalently to neighborhoods of $\overline{\mathbb{H}^\pm/\mathbb{Z}}$ and satisfy $\phi^- \circ f = \phi^+$ near the boundary of $\overline{\mathbb{H}^+/\mathbb{Z}}$.

Holomorphy of $\phi^\pm$ near $\pm\mathrm{i}\infty$ yields that

$$\phi^\pm(z) = z + C^\pm + o(1) \text{ as } z \to \pm\mathrm{i}\infty$$

for appropriate constants $C^\pm \in \mathbb{C}/\mathbb{Z}$. Since ϕ is unique up to addition of a constant, the difference

$$C_f := C^+ - C^-$$

only depends on f and will be referred as the *welding constant* of f.

PROPOSITION A.1. *Let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving analytic circle diffeomorphism and let C_f be its welding constant. As ω tends to $+\mathrm{i}\infty$ in $\mathbb{C}/\mathbb{Z}$,*

$$\tau_f(\omega) = \omega + C_f + o(1).$$

The proof goes as follows.

Step 1. The isomorphism between the complex torus $E(f_\omega)$ and $\mathcal{E}_{\tau_f(\omega)}$ induces a univalent map $\phi_\omega\colon A_\omega \to \mathbb{C}/\mathbb{Z}$ which extends univalently to a neighborhood of the closed annulus $\overline{A}_\omega$, with $\phi_\omega(f_\omega) = \phi_\omega + \tau_f(\omega)$ in a neighborhood of $\mathbb{R}/\mathbb{Z}$.

Step 2. As $\omega \to +\mathrm{i}\infty$, the sequence of univalent maps

$$\phi_\omega^+\colon z \mapsto \phi_\omega(z) - \phi_\omega(0)$$

converges locally uniformly in $\mathbb{H}^+/\mathbb{Z}$ to a limit $\phi^+ \colon \mathbb{H}^+/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$, and the sequence of univalent maps

$$\phi_\omega^- \colon z \mapsto \phi_\omega(z+\omega) - \phi_\omega\big(f(0)+\omega\big)$$

converges locally uniformly in $\mathbb{H}^-/\mathbb{Z}$ to a limit $\phi^- \colon \mathbb{H}^-/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$. In addition, the maps $\phi^\pm \colon \mathbb{H}^+/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ form a pair of univalent maps provided by the welding construction.

Step 3. Comparing constant Fourier coefficients of ϕ_ω, ϕ^+ and ϕ^-, we deduce that as $\omega \to +\mathrm{i}\infty$, we have

$$C^+ + \phi_\omega(0) = -\omega + C^- + \phi_\omega\big(f(0)+\omega\big) + o(1),$$

whence

$$\tau_f(\omega) = \phi_\omega\big(f(0)+\omega\big) - \phi_\omega(0) = \omega + C^+ - C^- + o(1) = \omega + C_f + o(1).$$

A.1. **The map ϕ_ω.** Let $\delta > 0$ be sufficiently tiny so that $f \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ extends univalently to the annulus $B_\delta := \{z \in \mathbb{C}/\mathbb{Z} \mid \delta > |\mathrm{Im}(z)|\}$. Set

$$A_\omega^+ := A_\omega \cup B_\delta \cup \big(\omega + f(B_\delta)\big).$$

The complex torus $E(f_\omega)$ is the quotient of A_ω^+ where $z \in B_\delta$ is identified to $f_\omega(z) \in f(B_\delta) + \omega$.

An isomorphism between $E(f_\omega)$ and $\mathscr{E}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ sending the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $E(f_\omega)$ to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $\mathscr{E}_{\tau_f(\omega)}$ will lift to a univalent map $\phi_\omega \colon A_\omega^+ \to \mathbb{C}/\mathbb{Z}$ sending $\mathbb{R}/\mathbb{Z}$ to a curve homotopic to $\mathbb{R}/\mathbb{Z}$, preserving orientation. The following relation then holds on B_δ:

$$\phi_\omega(f_\omega) = \phi_\omega + \tau_f(\omega).$$

A.2. **Convergence of $\phi_\omega^\pm$.** As $\omega \to +\mathrm{i}\infty$, the open sets A_ω^+ eat every compact subset of $\mathbb{H}^+/\mathbb{Z} \cup B_\delta$. The sequence of univalent maps $\phi_\omega^+ \colon A_\omega^+ \to \mathbb{C}/\mathbb{Z}$ defined by

$$\phi_\omega^+(z) := \phi_\omega(z) - \phi_\omega(0)$$

is normal and any limit value $\phi^+ \colon \mathbb{H}^+/\mathbb{Z} \cup B_\delta$ satisfies $\phi^+(0) = 0$. It cannot be constant since each ϕ_ω^+ sends $\mathbb{R}/\mathbb{Z}$ to a homotopically nontrivial curve in $\mathbb{C}/\mathbb{Z}$ passing through 0. So, any limit value $\phi^+ \colon \mathbb{H}^+/\mathbb{Z} \cup B_\delta \to \mathbb{C}/\mathbb{Z}$ is univalent.

Similarly, as $\omega \to +\mathrm{i}\infty$, the open sets

$$A_\omega^- := -\omega + A_\omega^+$$

eat every compact subset of $\mathbb{H}^-/\mathbb{Z} \cup f(B_\delta)$. In addition, the sequence of univalent maps $\phi_\omega^- \colon A_\omega^- \to \mathbb{C}/\mathbb{Z}$ defined by

$$\phi_\omega^-(z) := \phi_\omega(z+\omega) - \phi_\omega\big(f(0)+\omega\big)$$

is normal and any limit value $\phi^- \colon \mathbb{H}/\mathbb{Z} \cup f(B_\delta) \to \mathbb{C}/\mathbb{Z}$ is univalent and satisfies $\phi^-\big(f(0)\big) = 0$.

Passing to the limit on the following relation, valid on B_δ:

$$\begin{aligned}\phi_\omega^- \circ f(z) &= \phi_\omega\big(f(z)+\omega\big) - \phi_\omega\big(f(0)+\omega\big)\\ &= \phi_\omega(z) + \tau_f(\omega) - \phi_\omega\big(f(0)+\omega\big) = \phi_\omega(z) - \phi_\omega(0) = \phi_\omega^+(z),\end{aligned}$$

we get the following relation, valid on B_δ:

$$\phi^- \circ f = \phi^+.$$

It follows that the pair (ϕ^-, ϕ^+) induces an isomorphism from $\big(A_\omega^+ \sqcup A_\omega^-\big)/f$ (we identify $z \in B_\delta \subseteq A_\omega^+$ to $f(z) \in f(B_\delta) \subseteq A_\omega^-$) to $\mathbb{C}/\mathbb{Z}$. Therefore, ϕ^- and ϕ^+ coincide with the unique isomorphisms arising from the welding construction, normalized by the conditions $\phi^+(0) = \phi^-\big(f(0)\big) = 0$. This uniqueness shows that there is only one possible pair of limit values. Thus, the sequences $\phi_\omega^-\colon A_\omega^- \to \mathbb{C}/\mathbb{Z}$ and $\phi_\omega^+\colon A_\omega^+ \to \mathbb{C}/\mathbb{Z}$ are convergent.

A.3. **Comparing Fourier coefficients.** Note that $z \mapsto \phi_\omega^\pm(z) - z$ and $z \mapsto \phi^\pm(z)$ are well-defined on $\mathbb{R}/\mathbb{Z}$ with values in $\mathbb{C}$. The previous convergence implies:

$$C_\omega^+ := \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega^+(z) - z\big)\,\mathrm{d}z \underset{\omega\to+\mathrm{i}\infty}{\longrightarrow} C^+ := \int_{\mathbb{R}/\mathbb{Z}} \big(\phi^+(z) - z\big)\,\mathrm{d}z$$

and

$$C_\omega^- := \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega^-(z) - z\big)\,\mathrm{d}z \underset{\omega\to+\mathrm{i}\infty}{\longrightarrow} C^- := \int_{\mathbb{R}/\mathbb{Z}} \big(\phi^-(z) - z\big)\,\mathrm{d}z.$$

Since ϕ_ω is holomorphic on A_ω^+, we have

$$\int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega(z) - z\big)\,\mathrm{d}z = \int_{\omega+\mathbb{R}/\mathbb{Z}} \big(\phi_\omega(z) - z\big)\,\mathrm{d}z = \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega(t+\omega) - t\big)\,\mathrm{d}t - \omega.$$

Thus,

$$\begin{aligned}
C_\omega^+ &:= \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega^+(z) - z\big)\,\mathrm{d}z \\
&= \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega(z) - z\big)\,\mathrm{d}z - \phi_\omega(0) \\
&= \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega(t+\omega) - t\big)\,\mathrm{d}t - \omega - \phi_\omega(0) \\
&= \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega^-(t) - t\big)\,\mathrm{d}t - \omega + \phi_\omega\big(f(0)+\omega\big) - \phi_\omega(0) = C_\omega^- - \omega + \tau_f(\omega).
\end{aligned}$$

As $\omega \to +\mathrm{i}\infty$, we therefore have

$$C^+ + o(1) = C^- + o(1) - \omega + \tau_f(\omega)$$

which yields

$$\tau_f(\omega) = \omega + C^+ - C^- + o(1) = \omega + C_f + o(1).$$

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