Robustness of equilibrium in the Kyle model of informed speculation

Alex Boulatov* and Dan Bernhardt

May 15, 2013

Abstract

We analyze a static Kyle (1983) model in which a risk-neutral informed trader can use arbitrary (linear or non-linear) deterministic strategies, and a finite number of market makers can use arbitrary pricing rules. We establish a strong sense in which the linear Kyle equilibrium is *robust*: the first variation in any agent's expected payoff with respect to a small variation in his conjecture about the strategies of others vanishes at equilibrium. Thus, small errors in a market maker's beliefs about the informed speculator's trading strategy do not reduce his expected payoffs. We also establish that if a non-linear equilibrium exists, then it is not robust.

JEL Classification Numbers: G12, G14, C62.
Keywords: Market microstructure, informed speculation, Bayesian Nash equilibrium, uniqueness, robustness, information.

*This study was carried out within "The National Research University Higher School of Economics" Academic Fund Program in 2012-2013, research grant No.11-01-0216.

1 Introduction

A central feature of information-based models of financial markets is that each strategic market participant considers not only knowledge about asset fundamentals, but also makes tremendously sophisticated and accurate assessments of the strategies that other agents employ. Typically, researchers assume away all errors in those assessments. However, in complicated financial market speculation settings, it seems likely that agents's beliefs about the strategies of others may be slightly mis-specified. Concretely, a market maker may get an informed speculator's strategy slightly wrong.

In this paper, we investigate the robustness of the static Kyle (1983) model of strategic trading to errors of this form. The Kyle (1983) model generalizes the Kyle (1985) model to a setting with a finite number J of market makers who obtain finite expected profits in equilibrium. In principle, Kyle (1983) allows a monopolistically-informed trader to choose a possibly non-linear trading strategy, and market makers to simultaneously choose possibly non-linear supply schedules. He establishes that there is a unique equilibrium in which the trading strategy and pricing rules are linear. With many market makers, $J \to \infty$, Kyle shows that market makers submit competitive supply schedules, and that the pricing rule becomes informationally efficient, i.e., the model reduces to the static Kyle (1985) model.

Our contribution is twofold. We first establish a remarkable robustness property of the linear Kyle equilibrium: the first variation of any agent's expected payoff with respect to a small variation of her *own* conjectures about the strategies of others vanishes in equilibrium. In particular, small errors in the beliefs of a market maker about the trading strategy of the informed speculator do not reduce the market maker's expected payoffs. In fact, we show that each market participant is also indifferent to small errors in the beliefs that others hold. As slight errors in beliefs do not affect agents' expected profits, this means that the linear Nash equilibrium remains an equilibrium; and it follows that considering the limit $J \to \infty$, the standard linear equilibrium of Kyle (1985) model is robust. We next establish that if a non-linear Nash equilibrium to the Kyle (1983) model exists, then it is not similarly robust to slight mis-specifications in beliefs.

Our notion of robustness is more demanding than that considered by other researchers, who typically only require that small perturbations of beliefs lead to ε-best responses and ε-equilibria for every nearby perturbation (see Stauber (2006, 2011) or Barelli (2009)). Of course, our analysis and characterization solely pertains to the Kyle model.

2 The Model

We begin with a review of the Kyle (1983) model. A single risk neutral informed trader, privately observes an asset's liquidation value v, which is drawn from a normal distribution with mean zero and variance σ_0^2. Liquidity traders trade a quantity u, which is drawn independently from a normal distribution with mean zero and variance σ_u^2. After observing the liquidation value v, but not the level of noise trading u, the informed trader chooses a quantity x to trade. This "market order" x does not depend on the equilibrium price, but does depend on the observed liquidation value v. The informed trader's strategy is thus a function $X(\cdot)$ that details for each value of v, the traded quantity $x = X(v)$.

The informed trader and liquidity traders trade in a market with $J \geq 3$ risk-neutral, profit-maximizing market makers. Market makers know the joint distribution of v and u, but do not see either realization. Each market maker $k = 1, \ldots J$ submits a limit order described by a non-discriminatory supply schedule $y_k(P)$ that details for each price the quantity it will supply. The equilibrium price clears the market, $Y(P) = \sum_{k=1}^{J} y_k(P) = x + u$: the *pricing rule* $P(\cdot)$ is defined as an inverse function of the aggregate demand, i.e., $P(Y(\xi)) = \xi$, for $\xi \in \mathbb{R}$.

A symmetric Nash equilibrium consists of a trading strategy $X^*(\cdot)$ and a supply schedule $Y^*(\cdot)$ that are mutual best responses, i.e., are profit maximizing for the insider and market makers. Kyle solves for the following linear equilibrium trading strategy and pricing rule:

$$X^*(v) = \left(\frac{J-2}{J-1}\right)\frac{\sigma_u}{\sigma_v}v, \quad \text{and} \quad Y^*(P) = 2\frac{\sigma_u}{\sigma_v}\left(\frac{J-2}{J-1}\right)P, \tag{1}$$

which we refer to as the *standard linear solution*. From (1), it follows that the pricing rule $P^*(\cdot)$ is linear and given by

$$P^*(Y) = \left(\frac{J-1}{J-2}\right)\frac{\sigma_v}{2\sigma_u}Y. \tag{2}$$

Note that (1) and (2) reduce to the linear solutions of Kyle (1985) in the limit as $J \to \infty$. In what follows, we normalize both σ_u and σ_v to one, so that the standard linear solution (1) takes the form

$$X^*(v) = \left(\frac{J-2}{J-1}\right)v, \quad \text{and} \quad P^*(Y) = \frac{1}{2}\left(\frac{J-1}{J-2}\right)Y. \tag{3}$$

We focus on *symmetric* Nash equilibria. To examine non-linear trading strategies, it is useful to develop notation that describes the reaction function of agents to possibly non-linear trading strategies of the others. The notation $y_k(P; X_{M,k}(\cdot))$ indicates that the supply schedule of market maker k depends on both a scalar argument given by the execution price P and the function argument given by market maker k's conjecture about the insider's demand function $X_{M,k}(\cdot)$, and given our focus on symmetric equilibria, beliefs are the same, so $X_{M,k}(\cdot) \equiv X_M(\cdot)$, $k = 1, ..J$.

In what follows, we make extensive use of functionals, i.e. functions mapping both scalars and other functions into scalars. To keep notation clear, we follow the above example by placing scalar arguments in front of functional arguments, separating the two types of arguments by a semi-colon, and using a dot to indicate function arguments.

2.1 Insider optimization

We consider the Nash equilibrium of Kyle (1983). When the insider observes the realization v, conjectures that market makers adopt strategy $y_I(\cdot)$ with a pricing rule $P_I(\cdot)$ and trades the quantity x, then the insider's expected payoff, $\Pi_I(v, x; P_I(\cdot))$, is

$$\begin{aligned}\Pi_I(v, x; P_I(\cdot)) &= E_u[(v - P_I(x + u))x] \\ &= (v - \overline{P}_I(x))x,\end{aligned} \tag{4}$$

where the expected price $\overline{P}_I(x)$ is defined as

$$\overline{P}_I(x) = E_u[P_I(x + u)]. \tag{5}$$

Define the insider's reaction functional to her conjecture about the market maker's strategy $y_I(\cdot)$ by

$$R_I(\cdot\ ; y_I(\cdot)) = \arg\max_x\ \Pi_I(v, x, P_I(\cdot))\,. \tag{6}$$

Making use of the above definitions, we obtain the following first-order condition for the informed trader's profit-maximization problem:

Proposition 1 The first-order condition describing the insider's strategy is

$$v = \overline{P}_I(X(v)) + X(v)\,\overline{P}_I'(X(v))\,, \tag{7}$$

which must hold pointwise for each v.

Proof: Evaluating the first variation of the payoff (4) with respect to x, yields (7).

The insider's first-order condition (7) has a simple economic interpretation: The marginal value from one more share on the left-hand side is equated to the marginal cost of increasing that position on the right-hand side.

Since the reaction functional $R_I(\cdot\,;y(\cdot))$ is monotonic in its first argument, it is invertible. Therefore, we can introduce an inverse reaction functional $V_I(x;y(\cdot))$ such that $V_I(R_I(v;y(\cdot));y(\cdot)) \equiv v$ and rewrite the first-order condition for the insider, equation (7), as

$$\begin{aligned} V_I(x;y_I(\cdot)) &= \overline{P}_I(x) + x\overline{P}_I'(x) \\ &= \frac{\partial}{\partial x}\left(x\overline{P}_I(x)\right), \end{aligned} \tag{8}$$

which explicitly relates the insider's reaction to the conjectured functional form of the expected pricing rule $\overline{P}_I(\cdot)$.

2.2 Market maker optimization

Suppose that when market maker k conjectures that all other market makers $j \neq k$ submit the schedule $y_{M_k}(P)$ that he supplies y_k. For each realization of v and u, the total liquidity supply from market makers of $Y = y_k + (J-1)y_{M_k}(P)$ must match the total or net demand of $X(v) + u$. This, in turn, determines the market-clearing price P as a function of y_k:

$$y_k + (J-1)y_{M_k}(P) = X(v) + u = Y. \tag{9}$$

We denote market maker k's expected payoff given his conjectures $y_{M_k}(\cdot), X_{M,k}$ by:

$$\begin{aligned} \Pi_{M,k}(y_k, P; y_{M_k}(\cdot), X_{M,k}(\cdot)) &= E_v[y_k(P-v); y_{M_k}(\cdot), X_{M,k}(\cdot)] \\ &= y_k(P - P_e(Y; X_{M,k}(\cdot))), \end{aligned} \tag{10}$$

where $P_e(Y; X_{M,k}(\cdot))$ is an informationally-efficient price from the perspective of market maker k:

$$P_e(Y; X_{M,k}(\cdot)) = E[v|Y; X_{M,k}(\cdot)]. \tag{11}$$

From (10), we obtain the first-order condition describing the strategy of market maker $k = 1, ..J$ in a symmetric Nash equilibrium:

Proposition 2 The first variation for market maker k's problem is

$$\begin{aligned} 0 &= \delta_{y_k}\Pi_{M,k}(y_k, P; y_{M_k}(\cdot), X_{M,k}(\cdot)) \\ &= \{P - P_e(Y; X_{M,k}(\cdot))\}\,\delta y_k + y_k \delta P, \end{aligned} \tag{12}$$

and the first-order condition describing her strategy is

$$y_k(P) = (J-1)\, y_{M_k}'(P)\,(P - P_e(Y; X_{M,k}(\cdot))). \tag{13}$$

Proof: Suppose that market maker k supplies y_k. Evaluating the first variation of the payoff functional (10) with respect to y_k, yields (12). Making use of the market-clearing condition (9) and noting that total demand does not change as a result of the deviation, we obtain

$$\delta y_k + (J-1) y'_{M_k}(P)\,\delta P = \delta Y = 0, \tag{14}$$

and therefore

$$\delta P = -\frac{1}{(J-1)} \frac{\delta y_k}{y'_{M_k}(P)}, \tag{15}$$

which means that the variation of the market-clearing price only depends on the variation of the strategy of market maker k and the functional form of the conjectured strategy $y_{M_k}(\cdot)$. Substituting (15) into (12) yields (13).

The FOC (13) implicitly defines the reaction function of market maker k,

$$R_{M_k}(P; y_{M_k}(\cdot), X_{M,k}(\cdot)) = (J-1)\, y'_{M_k}(P)\,(P - P_e(Y; X_{M,k}(\cdot))), \tag{16}$$

to his conjectures $y_{M_k}(\cdot)$ and $X_{M,k}(\cdot)$ about the strategies of other agents. Introducing the pricing rule as an inverse function of the equilibrium supply schedule, we can express the pricing rule in terms of the conjectured insider's strategy. That is, inverting (13) yields

$$P - \frac{1}{J-1} \frac{y_k}{y'_{M_k}(P)} = P_e(Y; X_{M,k}(\cdot)). \tag{17}$$

Introducing the pricing rule reaction $R_{MP}(Y; X_M(\cdot))$ as an inverse of the reaction function and using the fact that in a symmetric equilibrium, $y_{M_k}(P) \equiv y(P) = \frac{1}{J}Y$, equation (17) yields

$$R_{MP}(Y; X_M(\cdot)) - \frac{1}{J-1} Y R'_{MP}(Y; X_M(\cdot)) = P_e(Y; X_M(\cdot)). \tag{18}$$

In the competitive limit $J \to \infty$, the second term on the left-hand side vanishes and the price becomes informationally efficient, which means that the model reduces to Kyle (1985). Since the equilibrium pricing rule is monotonically increasing in its first argument, the pricing rule is steeper than the informationally efficient one for any finite number of market makers J. Solving the ODE (18), yields

$$R_{MP}(Y; X_M(\cdot)) = (J-1)\, Y^{J-1} \oint_Y^{+\infty} d\xi \xi^{-J} P_e(\xi; X_M(\cdot)), \tag{19}$$

which is an explicit characterization of the pricing rule in terms of the market makers' conjecture about the insider's strategy $X_M(\cdot)$ for any finite number of market makers $J \geq 3$. Note that with linear strategies, $P_e(Y; X_M(\cdot)) = \frac{1}{2}Y$, i.e., we reproduce the standard result for the equilibrium pricing rule (3).

3 Nash Equilibrium

The Nash equilibrium strategies of the insider and market makers, denoted $X^*(\cdot)$ and $y^*(\cdot)$ are defined by fixed-point conditions that can be expressed in terms of the reaction functionals $R_I(\cdot\,; y_I(\cdot))$ and $R_M(\cdot\,; y_M(\cdot), X_M(\cdot))$ as

$$\begin{aligned} X^*(\cdot) &= R_I(\cdot\,; y^*(\cdot)), \\ y^*(\cdot) &= R_M(\cdot\,; y^*(\cdot), X^*(\cdot)). \end{aligned} \tag{20}$$

The first condition states that in a Nash equilibrium, when the informed trader takes the equilibrium strategy of market makers as given, then the informed trader chooses the trading rule that market makers believe he is going to follow[1]. Similarly, the optimal trading strategy of each market maker is the one that he conjectures about the other market makers given their equilibrium beliefs about the insider's strategy. In other words, all agents' conjectures turn out to be correct at equilibrium and represent the optimal reactions of each agent to the strategies of the other agents.

Using (8) and (19), the conditions in (20) can be explicitly expressed in terms of the inverse insider's strategy $V^*(x)$ and the expected pricing rule $\overline{P}^*(x; X^*(\cdot))$ as

$$\begin{aligned} V^*(x) &= \frac{\partial}{\partial x}\left(x\overline{P}^*(x; X^*(\cdot))\right), \\ \overline{P}^*(x; X^*(\cdot)) &= (J-1)\, E_u\left[Y^{J-1}\oint_Y^{+\infty} d\xi\xi^{-J} P_e(\xi; X^*(\cdot))\right]. \end{aligned} \tag{21}$$

In the limit $J \to \infty$, the second condition yields $\overline{P}^*(x; X^*(\cdot)) = E_u[P_e(Y; X^*(\cdot))]$ and (21) reduces to the definition of Nash equilibrium in Kyle (1985)[2].

4 Robustness and Linearity

We now analyze the robustness of the Kyle (1983) model of strategic trading. We focus on a particularly demanding notion of robustness of equilibrium payoffs to small errors that

[1] Although our reaction-function notation emphasizes the choice of the function $X(\cdot)$, the condition (20) leads to a definition of Nash equilibrium logically equivalent to that in Kyle (1983) and Kyle (1985). The two definitions are equivalent since the informed trader's optimization problem decomposes into separate state-by-state optimization problems for each realization of v.

[2] This follows from the following representation of the Dirac's delta function, $\delta(\cdot)$:

$$\lim_{J\to\infty} (J-1)\, Y^{J-1} (Y+z)^{-J} = \delta(z).$$

agents make in their conjectures about the strategies that other agents adopt:

Definition 1 An equilibrium is **robust** if and only if the first variation of an agent's expected payoffs with respect to a small variation in his conjecture about the strategies of others vanishes at equilibrium.

Our notion of robustness demands that all market participants be indifferent to small errors in their beliefs about what others will do.

To describe the expected payoffs sensitivity to the variation of the conjectured insider's strategy, we use the notion of *functional differentiation* commonly used in functional analysis.

Definition 2 *The functional differential of the price with respect to the strategy* $X(\cdot)$ *is*

$$\delta_X \overline{P}\left(x; X\left(\cdot\right), \delta X\left(\cdot\right)\right) = \lim_{\varepsilon\to 0}\left\{\frac{\overline{P}\left(x; X\left(\cdot\right)+\varepsilon\delta X\left(\cdot\right)\right)-\overline{P}\left(x; X\left(\cdot\right)\right)}{\varepsilon}\right\}, \tag{22}$$

provided that the limit (22) exists for every $\delta X(\cdot)$ *(from the same functional space), and defines a functional, linear and bounded in* $\delta X(\cdot)$.

The above definition corresponds to the *strong*, or *Frechet differential* (see, e.g., Kolmogorov and Fomin (1999)). Note that (22) can be viewed as an extension of the *directional derivative* of functions depending on several variables, familiar from standard calculus, to the case when some of the arguments can be functions.

4.1 Robustness

The Nash equilibrium in Kyle (1983) model is constructed as follows. Before observing the realization of the fundamental v, the insider defines his optimal reaction $R_I(\cdot\,; X_M(\cdot))$ to the market makers' conjecture $X_M(\cdot)$ of her true trading strategy in order to maximize expected payoffs. Simultaneously, each market maker $k = 1, ..J$ makes a conjecture $X_{M_k}(\cdot) \equiv X_M(\cdot)$, of the true strategy $X(\cdot)$ and submits her schedule $y_k\left(P; y_{M_k}(\cdot), X_{M_k}(\cdot)\right) \equiv y\left(P; X_M(\cdot)\right)$ which is a function of the market-clearing price and a functional of the conjecture. The pricing rule is defined by inversion of $y\left(P\,; X_M(\cdot)\right)$ and can be viewed as a reaction function to total order flow. In the competitive limit considered in Kyle (1985), market makers submit informationally-efficient pricing rules and earn zero profits at equilibrium.

By definition, equilibrium requires that $X_M(\cdot) = X^*(\cdot)$, i.e., each market maker's conjecture about the insider's strategy is correct. Now, suppose that we found the Nash equilibrium described above and consider a small variation in the conjecture of market maker k:

$$X_{M_k}(\cdot) = X^*(\cdot) + \delta X_{M_k}(\cdot). \tag{23}$$

Economically, this means that the conjecture of one market maker is "slightly off", i.e., there is a small deviation from the constructed Nash equilibrium. As a result of the variation of beliefs, the trading strategy $y_k(P; y_{M_k}(\cdot), X_{M_k}(\cdot))$ shifts by $\delta y_k(P; y_{M_k}(\cdot), X_{M_k}(\cdot))$, which leads to a shift of the market-clearing price δP according to (15). In addition, the market maker's estimate of the informationally efficient price also shifts, which also affects her expected payoffs.

Since the market maker's k conjecture turns out to be wrong and the insider actually does not deviate, the total order flow Y remains the same as the one in the original Nash equilibrium, and the variation in the market maker's k supply δy_k is completely absorbed by the adjustment of the aggregate supply due to the shift of the market clearing price P. Because the FOC in the original Nash equilibrium is satisfied, the effects of demand and price shifts exactly offset each other and do not lead to any change of the expected market maker's k payoff. Therefore, the expected market maker's payoff may only change due to the shift of the estimated informationally efficient price in her information set. We show that this contribution vanishes if and only if the original Nash equilibrium is linear.

Our main results are summarized by the following:

Theorem 1 *1.* The standard linear equilibrium of Kyle (1983) is robust with respect to small conjecture errors of the market makers or the informed trader.

2. The only equilibrium of Kyle (1983) that is robust in the sense of Definition 1 is the standard linear equilibrium.

Proof: See Appendix.

The proof proceeds in two steps. First, we prove that the equilibrium expected payoffs of a market maker do not change as a result of a small variation of her own conjecture only if the equilibrium in question is linear. That is, we prove that linearity is a necessary condition for robustness, i.e. no non-linear equilibrium can be robust. We then show that the standard linear equilibrium is indeed robust with respect to small deviations in any agent's conjectures.

Because the Kyle (1985) model can be viewed as a limiting case of Kyle (1983) with $J \to \infty$, the above results apply to Kyle (1985). One may argue that the variation (26) vanishes in the limit $J \to \infty$ and therefore the proposed robustness condition may not be relevant as $J \to \infty$. However, this concern is misplaced. First, (26) is finite for any finite J. One can view Kyle (1985) as the continuous limiting case of Kyle (1983) as J grows infinitely large. Second, we may consider the situation in which a finite fraction $q < 1$ of market makers deviates from the equilibrium conjecture with the same $\delta X_{M,k}(\cdot)$. In this case, all results hold with the change that the factor $\frac{1}{J}$ in the right-hand side of the last expression in (26) is replaced with q, which remains finite in the limit $J \to \infty$.

In fact, we establish an even stronger robustness notion than the one we define, showing robustness not only to an agent's own errors, but also to small errors that others may make:

Proposition 3 In the standard linear Nash equilibrium of Kyle (1983), the first variations of all agent's expected payoffs with respect to variations in the conjectures of any agent vanishes.

Proof: See Appendix.

The robustness result in Proposition 3 is stronger than the notion defined in Definition 1, because it says that in a standard linear equilibrium, each market participant is indifferent to small errors in his or her own beliefs, *and* to errors in the beliefs that others hold.

5 Conclusion

In this note, we establish a very strong sense in which the standard linear Nash equilibrium of the Kyle (1983, 1985) model is robust. We say that a Nash equilibrium *robust* if the first variations of each agent's expected payoff with respect to small variations in their conjectures about the strategies of others vanishes at equilibrium. We prove that each market participant is indifferent to small errors in his or her own beliefs and to small errors in the beliefs that others hold. Further, the only robust Nash equilibrium of Kyle (1983) model is the standard linear one.

The notion of robustness that we establish is a particularly appealing one, as action spaces are continuous, and the strategic interactions in this financial market speculation game are especially complex, rendering it implausible that market makers fully understand the nature of the trading strategy that the speculator adopts. Fortunately, we establish that the equilibrium is unaffected when the conjectures that market makers make are slightly wrong.

6 Appendix: Proof of Theorem 1

1. Impact of MM k conjecture error on her own expected payoffs.

The expected profits of market maker k are given by the functional

$$\begin{aligned} \overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right) &= E_Y\left[\Pi_{M,k}\left(y_k,P;y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right)\right] \\ &= E_Y\left[y_k(P-P_e\left(Y;X_{M,k}\left(\cdot\right)\right))\right]. \end{aligned} \tag{24}$$

Evaluating the first variation of (24) with respect to $\delta X_{M_k}\left(\cdot\right)$, yields

$$\begin{aligned} &\delta_{X_{M,k}}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right) \\ =\;& E_Y\left[\delta y_k(P-P_e\left(Y;X_{M,k}\left(\cdot\right)\right))+y_k\delta P-y_k\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right]. \end{aligned} \tag{25}$$

Taking into account that the actual total demand does not change as a result of a wrong conjecture and making use of the FOC (12) at equilibrium, observe that the first two terms on the right-hand side of (25) cancel and the following "envelope theorem" result holds:

$$\begin{aligned} \delta_{X_{M,k}}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right) &= -E_Y\left[y\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right] \\ &= -\frac{1}{J}E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right]. \end{aligned} \tag{26}$$

Therefore, the robustness condition reduces to

$$E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right]=0. \tag{27}$$

Introduce the marginal p.d.f. $Z_P\left(Y;X\left(\cdot\right)\right)$ as

$$Z_P\left(Y;X\left(\cdot\right)\right)\equiv\int dv' e^{-\frac{(v')^2}{2}}e^{-\frac{(Y-X(v'))^2}{2}}. \tag{28}$$

At a symmetric equilibrium, $P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\equiv P_e\left(Y;X_M^*\left(\cdot\right)\right)$ and $Z_P\left(Y;X\left(\cdot\right)\right)\equiv Z_P\left(Y;X^*\left(\cdot\right)\right)$. Since the optimal insider's strategy is fixed, we use the short-hand notation $P_e\left(Y;X_M^*\left(\cdot\right)\right)=P_e\left(Y\right)$ and $Z_P\left(Y;X^*\left(\cdot\right)\right)=Z_P\left(Y\right)$. We have

$$\begin{aligned} &E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right)\right] \\ =\;& E_{u,v}\left[Y\frac{\int dv' e^{-\frac{(v')^2}{2}}e^{-\frac{(Y-X_M(v'))^2}{2}}\left(v'-P_e\left(Y\right)\right)\left(Y-X_M\left(v'\right)\right)\delta X_{M,k}\left(v'\right)}{Z_P\left(Y\right)}\right] \\ =\;& \int dv\int dY Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right). \end{aligned} \tag{29}$$

Changing the order of integration and making use of (28) yields

$$\begin{aligned}
& E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right)\right] \qquad (30)\\
= & \int dY Z_P\left(Y\right)Y\frac{\int dv' e^{-\frac{(v')^2}{2}}e^{-\frac{(Y-X_M(v'))^2}{2}}\left(v'-P_e\left(Y\right)\right)\left(Y-X_M\left(v'\right)\right)\delta X_{M,k}\left(v'\right)}{Z_P\left(Y\right)}\\
= & \int dv e^{-\frac{v^2}{2}}\delta X_{M,k}\left(v\right)\int dy e^{-\frac{(Y-X_M(v))^2}{2}}Y\left(v-P_e\left(Y\right)\right)\left(Y-X_M\left(v'\right)\right)\\
= & E_v\left[\delta X_{M,k}\left(v\right)E_u\left[\left(u+X_M\left(v\right)\right)\left(v-P_e\left(Y\right)\right)\left(Y-X_M\left(v\right)\right)\right]\right]\\
= & E_v\left[\delta X_{M,k}\left(v\right)E_u\left[u\left(u+X\left(v\right)\right)\left(v-P_e\left(Y\right)\right)\right]\right].
\end{aligned}$$

Now, we have

$$\begin{aligned}
E_u\left[u\left(u+X\left(v\right)\right)\left(v-P_e\left(Y\right)\right)\right] &= E_u\left[\frac{\partial}{\partial Y}\left(Y\left(v-P_e\left(Y\right)\right)\right)\right] \qquad (31)\\
&= v-\overline{P}_e\left(X\left(v\right)\right)-X\left(v\right)\overline{P}_e'\left(X\left(v\right)\right)-\overline{P}_e''\left(X\left(v\right)\right),
\end{aligned}$$

where the last equality is obtained by integrating the noise trade out and using Stein's lemma. Combining the last equality of (31) with the insider's FOC in Proposition 1 and substituting into (30) and (26), reveals that at equilibrium

$$\delta_{X_{M,k}}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right)=\frac{1}{J}E_v\left[\delta X_{M,k}\left(v\right)\overline{P}_e''\left(X\left(v\right)\right)\right], \qquad (32)$$

which can be viewed as a "double envelope theorem" result since it makes use of envelope properties with respect to optimization by both the insider and market makers.

Taking into account that $\delta X_{M,k}\left(\cdot\right)$ is an arbitrary variation and making use of the basic lemma of Variation Calculus (see, e.g., Kolmogorov and Fomin (1999)), we conclude that the functional variation in (32) vanishes if and only if $\overline{P}_e''\left(X\left(v\right)\right)=0$, $\forall X\left(v\right)\in R$, which means that at equilibrium, the expected payoff of each market maker is insensitive with respect to a small deviation in her own conjecture.

From the above analysis, it follows that the robustness with respect to the conjectures by market makers is equivalent to the linearity of the pricing rule: the above condition says that at equilibrium the market makers' expected profit cannot vary with a small variation of their own conjecture (23) only if the equilibrium is linear. Therefore, linearity is a necessary condition for robustness, i.e. no equilibrium save for a linear one can be robust. We next show that the standard linear equilibrium is indeed robust.

2. Impact of MM k conjecture error on the payoffs of other agents.

Now, suppose that the initial equilibrium is linear, market maker k's conjecture is slightly "off", and let us analyze the equilibrium expected payoffs of market makers $j \neq k$ and the insider. The expected payoffs of market maker $j \neq k$ are given by

$$\overline{\Pi}_{M,j}\left(y\left(\cdot\right), X_{M,k}\left(\cdot\right)\right) = E_Y\left[y_j(P - P_e\left(Y; X_{M,j}\left(\cdot\right)\right))\right]. \tag{33}$$

Evaluating the first variation of (33) with respect to $\delta X_{M_k}\left(\cdot\right)$ and using the fact that (2) implies that the linear equilibrium is characterized by a linear relation between the pricing rule and the market efficient price, $P = \frac{J-1}{J-2} P_e\left(Y; X_M\left(\cdot\right)\right)$ we obtain

$$\begin{aligned}
\delta_{X_{M,k}} \overline{\Pi}_{M,j}\left(y\left(\cdot\right), X_{M,j}\left(\cdot\right)\right) &= E_Y\left[y_j \delta P\right] \\
&= \frac{1}{J} E_Y\left[Y \delta_{X_{M,k}} P_e\left(Y; X_M\left(\cdot\right)\right)\right] \\
&= \frac{1}{J} E_v\left[\delta X_{M,k}\left(v\right) \overline{P}_e''\left(X\left(v\right)\right)\right] = 0,
\end{aligned} \tag{34}$$

which establishes that the payoffs of other market makers are insensitive with respect to a small error in the conjecture by market maker k.

The equilibrium expected payoffs of the insider are

$$\overline{\Pi}_I\left(X\left(\cdot\right), P\left(\cdot\right)\right) = E_v[X\left(v\right)\left(v - \overline{P}\left(X\left(v\right)\right)\right)], \tag{35}$$

where $\overline{P}\left(\cdot\right)$ is the equilibrium pricing rule. If market maker k makes an incorrect conjecture, this should affect the pricing rule and hence the insider's expected payoffs. We have

$$\begin{aligned}
\delta_{X_{M,k}} \overline{\Pi}_I\left(X\left(\cdot\right), P\left(\cdot\right)\right) &= -E\left[X\left(v\right) \delta_{X_{M,k}} \overline{P}\left(X\left(v\right)\right)\right] \\
&= -\frac{J-1}{J-2} E_{v,u}\left[X\left(v\right) \delta_{X_{M,k}} P_e\left(Y; X_M\left(\cdot\right)\right)\right].
\end{aligned} \tag{36}$$

Now, we have

$$\begin{aligned}
\delta_{X_M} P\left(y; X_M\left(\cdot\right)\right) &= E_{v'|y}\left[\left(v' - P\left(y; X_M\left(\cdot\right)\right)\right)\left(y - X_M\left(v'\right)\right) \delta X_M\left(v'\right)\right] \\
&= \int dv' f\left(v'; X_M\left(\cdot\right)|y\right)\left(v' - P\left(y; X_M\left(\cdot\right)\right)\right)\left(y - X_M\left(v'\right)\right) \delta X_M\left(v'\right),
\end{aligned} \tag{37}$$

where the conditional p.d.f. $f\left(v; X_M\left(\cdot\right)|y\right)$ is defined by

$$f(v; X\left(\cdot\right)|X\left(v\right) + u = y) = \frac{1}{f\left(y; X\left(\cdot\right)\right)} \exp\left[-\frac{\left(y - X\left(v\right)\right)^2}{2\Sigma_u}\right] \exp\left[-\frac{v^2}{2\Sigma_0}\right], \tag{38}$$

the marginal distribution density function

$$f(y; X(\cdot)) = \int_{-\infty}^{+\infty} dv' \exp\left[-\frac{(y - X(v'))^2}{2\Sigma_u}\right] \exp\left[-\frac{(v')^2}{2\Sigma_0}\right], \quad (39)$$

and the parameters normalized to one, $\Sigma_0 = \Sigma_u = 1$. Therefore,

$$\delta_{X_M} E_v \left[X\overline{P}\right] = E_{v,u} X(v)\, \delta_{X_M} P, \quad (40)$$

and

$$\delta_{X_M} P = \frac{\int dv' e^{-\left(\frac{v'^2}{2} + \frac{(y - X_M(v'))^2}{2}\right)} (v' - P(y))(y - X_M(v'))\, \delta X_M(v')}{\int dv'' e^{-\left(\frac{v''^2}{2} + \frac{(y - X_M(v''))^2}{2}\right)}}, \quad (41)$$

where we dropped the functional arguments to simplify notation. For example, $P(y)$ is short hand for $P(y; X_M(\cdot))$. Making use of (37) and changing the order of integration, yields

$$E_{v,u}\left[X(v)\, \delta_{X_M} P(y; X_M(\cdot))\right] = E_v\left[\delta X_M(v)\, E_u\left[Q(y; X_M(\cdot))(v - P(y; X_M(\cdot)))(y - X_M(v))\right]\right],$$

where

$$Q(y; X_M(\cdot)) = E_{v|y}[X(v)] = \frac{\int dv e^{-\left(\frac{v^2}{2} + \frac{(y - X_M(v))^2}{2}\right)} X(v)}{\int dv' e^{-\left(\frac{v'^2}{2} + \frac{(y - X_M(v'))^2}{2}\right)}}. \quad (42)$$

In short-hand notation, we have

$$\begin{aligned} \delta_{X_M} E_v\left[X\overline{P}\right] &= E_v\left[\delta X_M(v)\, E_u\left[Q(y)(v - P(y))(y - X_M(v))\right]\right] \\ &= E_v\left[\delta X_M(v)\, E_u\left[\frac{\partial}{\partial y}(Q(y)(v - P(y)))\right]\right]. \end{aligned} \quad (43)$$

In a linear equilibrium, $Q(y) = \beta P(y) = \lambda\beta y$. Therefore, (43) yields

$$\delta_{X_M} E_v\left[X\overline{P}\right] = \lambda\beta E_v\left[\delta X_M(v)\, E_u\left[\frac{\partial}{\partial y}(y(v - P(y)))\right]\right]. \quad (44)$$

From Stein's lemma, we obtain

$$E_u\left[\frac{\partial}{\partial y}(y(v - P(y)))\right] = v - \overline{P}(X(v)) - X(v)\, \overline{P}'(X(v)) - \overline{P}''(X(v)) = 0,$$

where the last equality follows from the insider's FOC and linearity. Substituting (43) back into (44) and then into (36), we finally obtain $\delta_{X_{M,k}} \overline{\Pi}_I(X(\cdot), P(\cdot)) = 0$, which proves that the insider's expected equilibrium payoffs are insensitive to a small variation in a conjecture by one of the market makers.

3. Impact of Insider's conjecture error on her own expected payoffs.

Suppose the insider's conjecture about some market maker's strategy is slightly "off". As a result, the insider's conjecture of the pricing rule is "off," and the insider incorrectly assumes that the pricing rule $P(\cdot)$ deviates from the Nash equilibrium one $P^*(\cdot)$ to $P(\cdot) = P^*(\cdot) + \delta P(\cdot)$. The insider reacts to the conjectured pricing rule by deviating from the original equilibrium strategy $X^*(\cdot)$. From the optimization by the insider, we conclude that the shift of insider's demand due to the shift in the pricing rule conjecture $\delta P(\cdot)$ is

$$\delta x = \widehat{O}\delta P(x) = \left(-\frac{1}{2}\right)\frac{1}{\overline{P}'(x)}\left(1 + x\frac{\partial}{\partial x}\right)\delta P(x), \tag{45}$$

which means that the shift of the insider's strategy is commensurate with the insider's conjecture error. The actual pricing rule is unaffected by the insider's conjecture, so the "ex-post" variation of the insider's expected payoffs is given by

$$\begin{aligned} & \delta\overline{\Pi}_I(X(\cdot), P(\cdot)) \\ = \; & E\left[\delta X(v)\left(v - \overline{P}(X(v)) - X(v)\right)\overline{P}'(X(v))\right] = 0, \end{aligned} \tag{46}$$

where the last equality holds due to the FOC that is satisfied at the symmetric Nash equilibrium.

4. Impact of Insider's conjecture error on expected payoffs of market makers.

We must evaluate the variation of the market maker's expected equilibrium payoffs. These expected profits shift because the insider's optimal strategy shifts due to her conjecture error, which shifts the demand for liquidity. As a result, both the aggregate liquidity supply and market-clearing price shift, which could shift market maker expected payoffs. Importantly, the market makers do not change the functional form of their strategies and therefore they all supply the same amount $y_k = \frac{1}{J}Y$ and the pricing rule $P(\cdot)$ is the one defined in the symmetric Nash equilibrium. We have

$$\begin{aligned} \overline{\Pi}_{M,k}(y(\cdot), X_M(\cdot)) &= E\left[\frac{1}{J}Y(P(Y; X_M(\cdot)) - v)\right] \\ &= \frac{1}{J}E_{v,u}\left[(X(v) + u)(P(X(v) + u; X_M(\cdot)) - v)\right]. \end{aligned} \tag{47}$$

With the use of the Stein's lemma, the last expression yields

$$\overline{\Pi}_{M,k}(y(\cdot), X_M(\cdot)) = \frac{1}{J}E_v\left[X(v)\left(\overline{P}(X(v); X_M(\cdot)) - v\right) + \overline{P}'(X(v); X_M(\cdot))\right], \tag{48}$$

which is simply a normalized-by-$\frac{1}{J}$ difference between the losses of liquidity traders and the profits of informed. Since, as discussed above, the market makers' conjecture about the informed's strategy is unaffected by the informed's error, the functional form of the pricing rule does not change. Therefore, we drop the functional arguments in (48). The first variation of expected market maker's profits takes the form

$$\delta\overline{\Pi}_{M,k}\left(y\left(\cdot\right)\right) = \frac{1}{J}E_v\left[\delta X\left(v\right)\left(\left(\overline{P}\left(X\left(v\right)\right) + X\left(v\right)\overline{P}'\left(X\left(v\right)\right) - v\right) + \overline{P}''\left(X\left(v\right)\right)\right)\right]. \quad (49)$$

Note that the first three terms on the right-hand side of (49) cancel due to the FOC of the insider's problem, and we finally obtain an envelope theorem result

$$\delta\overline{\Pi}_{M,k}\left(y\left(\cdot\right)\right) = \frac{1}{J}E_v\left[\delta X\left(v\right)\overline{P}''\left(X\left(v\right)\right)\right]. \quad (50)$$

Analogous to the first part of the proof, we argue that since $\delta X\left(\cdot\right)$ is an arbitrary variation (defined in an appropriate functional space), the basic lemma of Variation Calculus states that the variation (50) vanishes if and only if $\overline{P}''\left(X\left(v\right)\right) = 0$, $\forall X\left(v\right) \in R$. In particular, this means that in a linear equilibrium, the expected payoff of each market maker is insensitive to a small error in the insider's conjecture. *Q.E.D.*

7 Bibliography

Barelli, P., 2009, "Consistency of Beliefs and Epistemic Conditions for Nash and Correlated Equilibria," *Games and Economic Behavior*, 67(2), 363-375.

Kolmogorov, A.N., and S.V. Fomin, 1999, *Elements of the Theory of Functions and Functional Analysis*, New York: Dover.

Kyle, A.S., 1983, "Continuous auctions and insider trading," *Mimeo*, Princeton University.

Kyle, A.S., 1985, "Continuous auctions and insider trading," *Econometrica*, 53, 1315-1335.

Stauber, R., 2006, "Belief Types and Equilibria for Cautious Players," *Working paper*, The Australian National University.

Stauber, R., 2011, "Knightian games and robustness to ambiguity," *Journal of Economic Theory*, 146(1), 248-274.

Weisstein, Eric W. “Hermite Polynomial." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/HermitePolynomial.html