Jump-diffusion Markets; Equivalent Measures and Optimality.[1]

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Abstract. One of the key questions in financial mathematics is the choice of an appropriate model for the financial market. There are a number of models available, such as Geometrical Brownian motion, different types of Levy processes and more general semimartingale models. As the model passes from the simple to the more general, it gains in its ability to model real data but loses computational tractability. The goal of this paper is to present a semimartingale model for which we can carry out computation of the optimal equivalent martingale measure, and which still retains the capacity to model processes with history-dependent increments. Since the inclusion of jump processes in the model leads to incomplete markets, we obtain the "best" equivalent martingale measure according to a number of popular criteria and show how these criteria are associated with particular choices of utility or distance functions..

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1 Introduction and History

Market models have evolved considerably since the beginning of the last century. The first and simplest model where a stock was modelled as a Brownian motion was introduced by Bachelier in 1900. Subsequently Samuelson (1965) expanded stock price modelling by arguing that geometrical Brownian motion is a better fit to real financial data. Based on this geometrical Brownian motion model, Black and Scholes (1973) obtained an explicit formula for pricing a European call

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option. There are numerous results extending the Black and Scholes formula to more general models. The main disadvantage of a geometrical Brownian motion pricing model is that jumps apparent in a real market cannot be described by continuous processes.

This disadvantage is a significant reason why a number of researchers started to work on models involving point processes to represent jumps. The first pure jump models were considered by Mandelbrot (1963). Press (1967) was one of the first to put Brownian motion and a Poisson process together as a model of the logarithm of stock prices. Sums of a Brownian motion and an independent Poisson (or compound Poisson process) were studied by many authors (Merton (1976), Aase (1988) and others) as a relatively good model because it reflected the diffusion and the jump part in a real market. But again the fixed jump sizes of a Poisson process are intuitively not the best fit to the variably sized jumps in a real market.

A further generalization leads us to Levy processes, and different forms for the jump part of Levy processes were of special interest. Variance-gamma processes (pure jump) introduced by Madan and Seneta (1990) have many advantages (such as existence of the moment generating function) and are currently used by many major investors in stock pricing models. Hyperbolic Levy processes were used by Eberlein and Keller (1995) based on the work of Barndorff-Nielsen(1977). Mittnik and Rachev (1999) studied the special case of stable asset returns. Some of these works employed subordinated processes where the time parameter is a stochastic process (Brownian motion, etc.) itself. In a real market the intensity of jumps could depend on the history of the price process. That is why the processes with independent increments are not necessarily a broad enough class to describe such market behavior. This observation now led to an even more general price model, the semimartingale structure.

Semimartingales are one of the most general choices available for modeling a stock price. There are also some popular non-semimartingale models, such as fractional Brownian motion, but we will not pursue those here. Among the first and most well-known steps in developing the semimartingale approach in option pricing were taken by Harrison and Pliska (1981). The authors established the price of the option in a market without arbitrage opportunities (the frictionless market) as the expectation with respect to an equivalent martingale measure (the measure under which the price process, a semimartingale, becomes a local martingale). This measure is also called the risk-free measure due to its relation to the economic concept of a risk-free pricing. The conditions for the existence and uniqueness of the equivalent martingale measures (the so-called fundamental theorem of option pricing) in the semimartingale market were developed in Delbaen and Schachermayer (1994). They showed that, under certain restrictions, in a market without arbitrage at least one equivalent martingale measure always exists. In a complete market (where every contingent claim is attainable) there is only one equivalent martingale measure (under some restrictions), but the construction of the desired equivalent martingale measure was a serious practical problem. Jacod and Shiryaev (1987) introduced semimartingale characteristics and described the density of the probability measure transformation in terms of these characteristics. Shiryaev (1999), Kallsen (2000) and others obtained equations for the density process parameters such that the transformed measure is the desired equivalent martingale measure. In a general semimartingale market, however, the solution of these equations can be quite difficult.

Another major problem is that in general the semimartingale market is incomplete, and there is more than one equivalent martingale measure. There are a number of ways to pick up the "optimal" measure among the candidates. Follmer and Schweizer (1991) chose the minimal martingale measure criterion. Schweizer (1995, 1996) introduced the variance optimal measure that minimizes the variance of the density process. Both of these measures are generally signed and this is one of the disadvantages of these methods. Gerber and Shiu (1994) introduced the so-called Esscher transform but only for a market with Levy processes. Kallsen and Shiryaev (2002) generalized this transform to the case of semimartingales. Another choice for optimality is the minimal relative entropy criterion introduced by a number of authors and completely studied by Fritelli (2000). A more general approach, utility maximization, was developed in Kallsen (1999, 2002). The optimal density process in this case is determined by a utility function. Choosing different utilities leads to different optimal measures. A similar concept is that of minimal distance measures as discussed in Goll and Ruchendorf (2001). There are also other ways to price options in incomplete markets such as super-hedging, and quantile hedging which we do not discuss here.

In passing from Levy to semimartingale models, we gain in realism of the model at the cost of tractable computations. This paper aims to develop an intermediate class of models – jump-diffusion processes – which retain the ability of semimartingales to represent past-dependent market behaviour (i.e. non-independent increments) but which also permit explicit computation of the density of the transformed measure. The markets are again generally incomplete, and we establish formulae for the density (and consequently the optimal strategy) under a number of optimality criteria. Indeed for a jump-diffusion market with constant jump intensities, we obtain formulae for the option price of a European call that clearly generalize that of Black and Scholes. We also identify utility and distance functions for each case and show that a number of these optimality criteria coincide for our class of market models.

The next section lays out the basic definitions used in this paper. Section 3 defines our class of jump-diffusion models and develops the class of equivalent martingale measures to a given model. Section 5 examines a series of optimality criteria and determines the optimal transform. That section ends with utility and distance optimization, and classifies the previous optimality criteria by the choice of optimality or distance function. Section 6 provides examples of the application of the methods of this paper to continuous markets, jump-diffusion markets with constant jump intensities and to a non-Levy market.

2 Basic definitions.

Consider a financial market $S = (S_0, ..., S_m)$, where S_0 is a bond (riskless asset) and there are m stocks $S_1, ..., S_m$. We assume here that S_i for $i = 0, ..., m$ are positive cadlag semimartingales on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$. We will consider a finite time horizon T, thus $t \in [0, T]$. We follow the definitions of Shiryaev (1998).

- The stochastic process $\pi = (\pi_0, ..., \pi_m)$ is said to be a **trading strategy** if $\pi_i(t)$ are $\mathbf{F}$-predictable and the and the stochastic integral $\int_0^T \pi_i(s) dS_i(s)$ exists for $i = 0, ..., m$.

- The accumulated gain or loss up to the instant t is called the **gains process generated by** π and is given by

$$G_\pi(t) := \sum_{i=0}^{m} \int_0^t \pi_i(s) dS_i(s).$$

- We call our holdings in the assets with values $S_0, ..., S_m$ our **portfolio**. We represent the portfolio by $\pi = (\pi_0, ..., \pi_m)$ and the associated **value (wealth) process** is defined to be

$$V_\pi(t) := \sum_{i=0}^{m} \pi_i(t) S_i(t).$$

- Let $L(S)$ denote the space of predictable processes which are integrable with respect to S. On the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ we call $(S, L(S), P)$ a **pricing model**.

- A trading strategy π is said to be **self-financing** ($\pi \in SF(S)$) if there is no investment or consumption at any time $t > 0$. That is

$$V_\pi(t) = V_\pi(0) + G_\pi(t) \;\; a.s. \; \forall t \in [0, T].$$

- If we assume that S_0 is strictly positive we can define a **discounted** process $\tilde{S} = (1, \frac{S_1}{S_0}, ..., \frac{S_m}{S_0})$. Let π be a strategy for S. If the process $\frac{1}{S_0}$ is predictable and of finite variation then $\tilde{S}$ is a semimartingale and $\pi \in SF(S)$ if and only if $\pi \in SF(\tilde{S})$ (Numeraire invariance in Protter (2001)). Hence from now on we can assume without loss of generality that $S_0 \equiv 1$ and $S \equiv \tilde{S}$.

- $\Pi_a(S) = \{\pi \in SF(S) : V_\pi(t) \geq -a, t \in [0, T]\}$. This class restricts the maximum loss with strategy π over the entire period $[0, T]$ to a, the so-called class of a**-admissible strategies**.

- $\Pi_+ = \cup_{a \geq 0} \Pi_a(S)$, the so-called class of **admissible strategies**.

- $S = (S_0, ..., S_m)$ has the **no-arbitrage (NA) property** if for any strategy $\pi \in SF(S)$ with $V_\pi(0) = 0$ we have:

$$P(V_\pi(T) \geq 0) = 1 \Longrightarrow P(V_\pi(T) = 0) = 1.$$

- We introduce $L_\infty^+(\Omega, \mathcal{F}_T, P)$ as the subspace of non-negative random variables of a Banach space $L_\infty(\Omega, \mathcal{F}_T, P)$ equipped with norm

$$\| x \|_\infty = \inf\{0 \leq c < \infty : P(|x| > c) = 0\}.$$

  $\bar{\Psi}_+(S)$ is the closure with respect to norm $\| \, . \, \|_\infty$ of the set

$$\Psi_+ = \{x \in L_\infty(\Omega, \mathcal{F}_T, P) : x \leq G_\pi(T), \; where \; \pi \in \Pi_+(S)\}.$$

  The definition of no-arbitrage can be also rewritten in the following form:

$$\Psi_+(S) \cap L_\infty^+(\Omega, \mathcal{F}_T, P) = \{0\}.$$

- $S = (S_0, ..., S_m)$ has the $\widetilde{NA}_+$ **property** (also called **no free lunch with vanishing risk**) if

$$\bar{\Psi}_+(S) \cap L_\infty^+(\Omega, \mathcal{F}_T, P) = \{0\}.$$

- A **contingent claim** is an $\mathcal{F}_T$-measurable nonnegative bounded random variable f_T.

- A contingent claim is said to be **attainable** if there exists *a hedging strategy* $\pi \in SF(S)$ such that $V_\pi(T) = f_T$ (P-a.s.).

- A market S is said to be **complete** if every contingent claim is attainable.

- If there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\tilde{P}$ is equivalent to $(\sim)$ P, and S is a martingale (local martingale) with respect to this measure $\tilde{P}$, we say that an **Equivalent Martingale Measure (Equivalent Local Martingale Measure)** exists and the **EMM (ELMM)** property holds.

The following theorems give us the relationship between economic ideas and the EMM property. We draw it statement from Shiryaev (1998.VII.2)

Theorem. 1 (Delbaen F., W Schachermayer) *For a nonnegative cadlag semimartingale $S = (1, S_1, ..., S_m)$ with bounded (locally bounded) components:*

$$E(L)MM \Longleftrightarrow \widetilde{NA}_+.$$

On $(\Omega, \mathcal{F}_T)$ let us denote the class of EMMs by $\mathcal{M} = \mathcal{M}(P)$. If there exists a unique EMM we write $|\mathcal{M}(P)| = 1$.

Theorem. 2 (Delbaen F., W Schachermayer) *Let the class of EMM consist of only one measure $\tilde{P}$. Then $\exists\pi \in SF(S)$ such that $V_\pi(T) = f_T$ (P-a.s.) for a contingent claim f_T with $E_{\tilde{P}}|f_T| < \infty$. Moreover*

$$|\mathcal{M}(P)| = 1 \Longleftrightarrow completeness$$

From now on when we talk about a contingent claim f_T in a complete NA market we assume that $E_{\tilde{P}}|f_T| < \infty$. Let $\tilde{P}$ be a unique EMM. Then the **(hedging) price** is

$$\mathcal{C}(f_T) = E_{\tilde{P}} f_T.$$

For this definition of a hedging price, our problem is to construct an equivalent martingale measure and to calculate the expectation of a contingent claim with respect to this measure. Explicit formulas of EMM parameters exist for Levy processes and general results for semimartingale models. The next section defines a class of processes between these two classes and for which calculations are still tractable.

3 The Model.

3.1 A Class of jump-diffusion semimartingales.

Let $W = (W_1, \ldots, W_d)$ be a Brownian motion and let $N = (N_1(t,z), \ldots, N_{k-d}(t,z))$, $j = 1, ..., k-d$ be an independent vector of independent marked point processes with characteristics $\lambda_j(t)$ and $\phi_j(t,z)$ (see Bremaud(1981) for background on marked point processes). Here the λ_j are the stochastic intensities of the jumps (intensities of the corresponding counting processes) and the $\phi_j(t,z)$ are the probability density functions of the jump size (absolutely continuous distributions) under the condition that this jump occurs at time t. Each $N_j(t,z)$ is a sequence of pairs $(t_n^{(j)}, z_n^{(j)}; n \geq 1)$, where $t_n^{(j)}$ is the time of n-th jump, and $z_n^{(j)}$, the size of this jump, is a random variable with the density function $\phi_j(t,z)$. We assume that each $\lambda_j(t)$ is strictly positive and bounded.

At this point we want to characterize a general class of semimartingales that can be represented as stochastic integrals with respect to the processes W and N. In the next subsection we will define a market model in terms of this class of semimartingales.

Proposition 1 *We fix truncation functions $h_i(z)$ (bounded with compact support and $h_i(z) = z$ in a neighborhood of zero). Let $b_i(t), \sigma_{ij}^W(t)$ be predictable processes. Let $\sigma_{ij}^N(t,z)$ be a predictable function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}$, and so that $\sigma_{ij}^N(t,z)$ as a function of z for any fixed t and ω is strictly monotone and differentiable in z. Hence it has the inverse function $(\sigma_{ij}^N)^{-1}(t,\hat{z})$ (i.e. $\hat{z} = \sigma_{ij}^N(t,z)$) with derivative $(\sigma_{ij}^N)_{\hat{z}}^{-1}(t,\hat{z})$. Let $\mathcal{A}_{loc}^+$ be the class of adapted processes with locally integrable variation. We assume that for every i:*

$$
\begin{aligned}
&\int_0^t |b_i(s)|ds \in \mathcal{A}^+_{loc},\\
&\sum_{j=1}^{d}\int_0^t (\sigma^W_{ij}(s))^2 ds \in \mathcal{A}^+_{loc},\\
&\sum_{j=1}^{k-d}\int_0^t \lambda_j(s)\int_{\mathbb{R}} \min(|\sigma^N_{ij}(s,z)|^2,1)\phi_j(s,z)dzds \in \mathcal{A}^+_{loc}. \qquad (1)
\end{aligned}
$$

Let

$$
\begin{aligned}
R_i &= \int_0^t b_i(s)ds + \sum_{j=1}^{d}\int_0^t \sigma^W_{ij}(s)dW_j(s) + \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} h_{ij}(\sigma^N_{ij}(s,z))dN_j(s,z)\\
&+\sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}}(\sigma^N_{ij}(s,z) - h_{ij}(\sigma^N_{ij}(s,z)))dN_j(s,z), i=1,...,m. \qquad (2)
\end{aligned}
$$

Then R is a d-dimensional semimartingale with characteristics:

$$
\begin{aligned}
B^R_i &= \int_0^t \left(b_i(s) + \sum_{j=1}^{k-d}\lambda_j(s)\int_{\mathbb{R}} h_{ij}(\sigma^N_{ij}(s,z))\phi_j(s,z)dz\right)ds\\
C^R_{i_1 i_2} &= \sum_{j=1}^{d}\int_0^t \sigma^W_{i_1 j}(s)\sigma^W_{i_2 j}(s)ds\\
\nu^R_i &= \sum_{j=1}^{k-d}\frac{\lambda_j(s)}{|(\sigma^N_{ij})_z(s,z)|}\phi_j(s,\sigma^N_{ij}(s,z)). \qquad (3)
\end{aligned}
$$

Moreover if for every i:

$$
\sum_{j=1}^{k-d}\int_0^t \lambda_j(s)\int_{\mathbb{R}} \min(|\sigma^N_{ij}(s,z)|^2, |\sigma^N_{ij}(s,z)|)\phi_j(s,z)dzds \in \mathcal{A}^+_{loc} \qquad (4)
$$

then R is a special semimartingale and we can rewrite (2) *as*

$$
R_i = \int_0^t b_i(s)ds + \sum_{j=1}^{d}\int_0^t \sigma^W_{ij}(s)dW_j + \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} \sigma^N_{ij}(s,z)dN_j(s,z). \qquad (5)
$$

Additionally if for every i:

$$
\sum_{j=1}^{k-d}\int_0^t \lambda_j(s)\int_{\mathbb{R}} |\hat{\sigma}^N_{ij}(s,z)|^2\phi_j(s,z)dzds \in \mathcal{A}^+_{loc} \qquad (6)
$$

then R is a square-integrable semimartingale.

Proof. We are looking for conditions on the predictable parameters of the processes R_i in (2) such that the R_i are semimartingales. Following Jacod and Shiryaev (1987, II.2) every semimartingale has a canonical representation:

$$R_i = R_i(0) + B_i^R + R_i^c + h_i(z) \star (\mu_i^R - \nu_i^R) + (z - h_i(z)) \star \mu_i^R, \tag{7}$$

where R_i^c is the continuous part of the martingale part R_i^M, μ_i^R is a jump random measure with compensator ν_i^R and $h_i(z)$ is a truncation function. Parameters of the representation (7) are subject to several conditions, which we will verify shortly after we develop formula for these characteristics.

We will rewrite (2) in the form (7).

Let $N_j^c(t)$ be the counting process with intensity $\lambda_j(t)$ corresponding to the marked point process $N_j(t,z)$. Let also $\hat{z} = \hat{z}_t(z) = \sigma_{ij}^N(t,z)$ be a random function of z for every fixed time t and

$$\hat{z}_n^{(j)} = \sigma_{ij}^N(t_n^{(j)}, z_n^{(j)}).$$

Now we introduce a marked point process

$$\hat{N}_{ij}(t,\hat{z}) = \sum_{n=1}^{N_j^c(t)} \sigma_{ij}^N(t_n^{(j)}, z_n^{(j)}) = \sum_{n=1}^{N_j^c(t)} \hat{z}_n^{(j)} \tag{8}$$

The characteristics (the intensity and the jump size distribution density) of $\hat{N}_{ij}(t,\hat{z})$ are

$$\lambda_j(s) \quad \text{and} \quad |(\sigma_{ij}^N)_{\hat{z}}^{-1}(t,\hat{z})|\phi_j(t,\hat{z}).$$

In (2) we can rewrite the jump part of the process R_i as a sum of integrals with respect to the marked point processes $\hat{N}_{ij}(t,\hat{z})$;

$$\begin{aligned} & \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\sigma_{ij}^N(s,z)) dN_j(s,z) \\ + & \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\sigma_{ij}^N(s,z) - h_{ij}(\sigma_{ij}^N(s,z))) dN_j(s,z) \\ & = \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\hat{z})) d\hat{N}_j(s,\hat{z}) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\hat{z} - h_{ij}(\hat{z}) d\hat{N}_j(s,\hat{z}). \end{aligned} \tag{9}$$

From (2), using (9) we obtain

$$
\begin{aligned}
R_i &= \sum_{j=1}^{d} \int_0^t \sigma_{ij}^W(s) dW_j(s) \\
&+ \int_0^t \left(b_i(s) + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} h_{ij}(\hat{z}) |(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})| \phi_j(s,\hat{z})\, d\hat{z} \right) ds \\
&+ \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\hat{z}) \left(d\hat{N}_{ij}(s,\hat{z}) - \lambda_j(s) |(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})| \phi_j(s,\hat{z})\, d\hat{z} ds \right) \\
&+ \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\hat{z} - h_{ij}(\hat{z})) d\hat{N}_{ij}(s,\hat{z}). \qquad (10)
\end{aligned}
$$

Hence the Jacod-Shiryaev characteristics (B, C, ν) of the process R are

$$
\begin{aligned}
B_i^R(h) &= \int_0^t \left(b_i(s) + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} h_{ij}(\hat{z}) |(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})| \phi_j(s,\hat{z}) d\hat{z} \right) ds \\
C_{i_1 i_2}^R &= \sum_{j=1}^{d} \int_0^t \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\
\nu_i^R &= \sum_{j=1}^{k-d} \lambda_j(s) |(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})| \phi_j(s,\hat{z}), \qquad (11)
\end{aligned}
$$

which is equivalent to (3).

We recall the conditions on "good" characteristics B^R, C^R, ν^R so that R is a semimartingale.
1. B^R is a $\mathbf{F}$-predictable process of finite variation over finite intervals and $B_i(0) = 0$.
2. C^R is a $\mathbf{F}$-predictable, continuous, $C^R(0) = 0$, and $C^R(t) - C^R(s)$ is a nonegative symmetric matrix for $s \leq t$.
3. ν^R is a $\mathbf{F}$-predictable random measure with $\nu^R(0, z) = \nu^R(t, 0) = 0$ and such that

$$\int_0^t \int_{\mathbb{R}} \min(|x|^2, 1) d\nu^R \in \mathcal{A}_{loc}^+$$

From (11) we can see that in our case these three conditions are equivalent to (1). Conditions (4) and (6) are straightforward from the definitions and (1). □

Remark 1 *Technically, the dependence on $k - d$ point processes could be expressed as a single more complex expression in terms of a single underlying point process. The parametric form of the above model is simpler both to work with and to interpret.*

- *In this form R has no predictable jumps (it is quasi-left continuous) since the drift function is continuous. This means that this model cannot be used to describe* *markets where there are jumps at predictable times even if the jump sizes at those times are unpredictable – e.g. weekly federal decisions. It may be possible to add a predictable jump component to the continuous drift term in the model to compensate.*

- *By Jacod and Shiryaev (1987, II.2) one can write the characteristics of the semimartingale R in the following form*

$$
\begin{aligned}
B^R(t) &= \int_0^t b^R(s)dA^R(s) + \int_0^t \int_{\mathbb{R}} (h(z) - z)d\nu^R; \\
C^R(t) &= \int_0^t c^R(s)dA^R(s); \\
d\nu^R(t,z) &= F^R(t,dz)dA^R(t),
\end{aligned}
\tag{12}
$$

where $A^R \in \mathcal{A}^+_{loc}$, b^R is a predictable processes, c^R is a predictable non-negative symmetric matrix and F^R is a transition kernel. The above proposition also shows that in our case, (3), we have the typical choice of $A^R(t) = t$ (e.g. for Levy processes, diffusions, Ito processes and etc.). For R_i in the form (2) one can also interpret $b_i(t)$ as a drift rate, $\sigma^W_{ij}(t)$ for every j as diffusion coefficients and $\sigma^N_{ij}(t,z)\lambda_j(t)\phi(t,z)$ for every j as local jump coefficients.

Definition 1 *Under conditions* (1) *we call R in* (2) ***a jump-diffusion semimartingale****. We also call the conditions* (1) ***jump-diffusion semimartingale conditions****.*

We can compare our definition of jump-diffusion semimartingales to the similar class of processes given by Jacod and Shiryaev (1987, III.2c). Our definition is more general, since we do not require the corresponding counting process to be a one-dimensional Poisson process.

We can show that almost every semimartingale can be represented in a jump-diffusion form. By the uniqueness of the Brownian motion W and of the multivariate marked point process N with given compensator ν, we can apply the fundamental representation theorem (Jacod and Shiryaev (1987 III.4d)) and obtain that every local martingale M adapted to the joint filtration of W, N has the $(W, N - \nu)$-representation property:

$$
M = M_0 + \int f^W dW + \int\int f^N dN - \int\int f^N d\nu,
$$

where f^W and f^N are predictable such that

$$
\begin{aligned}
\int (f^W)^2 &\in \mathcal{A}^+_{loc}, \\
\int\int |f^N| d\nu &\in \mathcal{A}^+_{loc}.
\end{aligned}
\tag{13}
$$

This means that under (13) every semimartingale R with decomposition $R = R_0 + A + M$ where A is a process of finite variation and M is a local martingale can be represented in the following form:

$$R = R_0 + \hat{A} + \int f^W dW + \int\int f^N dN,$$

where $\hat{A} = A - \int\int f^N d\nu$ is obviously a process of finite variation. Hence almost every semimartingale adapted to the joint filtration of W, N can be represented in the form given by (2). Jump-diffusion semimartingale conditions are more restrictive than (13) (mainly because we require f^N to be monotone and differentiable) but it is clear that the class of jump-diffusion semimartingales is a very wide subclass of semimartingales.

3.2 A Market with jump-diffusion semimartingales.

Consider the discounted market consisting of the bond $B \equiv 1$ and m stocks S_i and assume $m \leq k$. We consider a jump-diffusion semimartingale R given by (2) and an accepted model for stock prices:

$$S_i = \exp(R_i), \ \ i = 1, ..., m.$$

The stock price is always positive, but if R_i is a local martingale S_i may not share this property.

To be specific, the i-th stock S_i $i = 1, ..., m$

$$\begin{aligned} S_i = \quad & \exp\left(\int_0^t b_i(s)ds + \sum_{j=1}^{d}\int_0^t \sigma_{ij}^W(s)dW_j(s)\right) \\ \times \quad & \exp\left(\sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} h_{ij}(\sigma_{ij}^N(s,z))dN_j(s,z)\right) \\ \times \quad & \exp\left(\sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} (\sigma_{ij}^N(s,z) - h_{ij}(\sigma_{ij}^N(s,z)))dN_j(s,z)\right). \end{aligned} \tag{14}$$

where $b_i(t), \sigma_{ij}^W(t), \sigma_{ij}^N(t,z)$ satisfy the jump-diffusion conditions (1) for fixed truncation functions h_{ij}.

We want to find a special semimartingale (i.e. with bounded jumps) $\hat{R}$ so that $S_i = \mathcal{E}(\hat{R}_i)$ (the Dolean-Dade exponential, see Shiryaev (1998, VII.3d) for definitions). In this case whenever $\hat{R}$ is a local martingale, so is S_i. Our next proposition shows this is possible under certain conditions.

Proposition 2 *Assume that for every i,j and t:*

$$\int_0^t\int_{\mathbb{R}}\left(|\sigma_{ij}^N(s,z)|I_{|\sigma_{ij}^N(s,z)|\leq 1} + e^{\sigma_{ij}^N(s,z)}I_{|\sigma_{ij}^N(s,z)|>1})\phi_j(s,z)dz\right)ds \tag{15}$$

belongs to $\mathcal{A}_{loc}^+$. Then $S_i = \mathcal{E}(\hat{R}_i)$ where

$$\hat{R}_i = \int_0^t \hat{b}_i(s)ds + \sum_{j=1}^{d}\int_0^t \sigma_{ij}^W(s)dW_j + \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)dq_j(s,z), \tag{16}$$

$$\hat{b}_i(t) = b_i(t) + \frac{1}{2}\sum_{j=1}^{d}(\sigma_{ij}^W(t))^2 + \sum_{j=1}^{k-d}\lambda_j(s)\int_{\mathbb{R}}(\hat{\sigma}_{ij}^N(t,z) - h_{ij}(\sigma_{ij}^N(s,z)))\phi_j(t,z)dz$$

$$dq_j(t,z) = dN_j(t,z) - \lambda_j(t)\phi_j(t,z)dzdt,$$

and

$$\hat{\sigma}_{ij}^N(t,z) = e^{\sigma_{ij}^N(t,z)} - 1.$$

Under condition (15), $\hat{R}$ is a special semimartingale with characteristics

$$\begin{aligned}
B_i^{\hat{R}} &= \int_0^t \left(b_i(s) + \frac{1}{2}\sum_{j=1}^{d}(\sigma_{ij}^W(s))^2\right)ds \\
&\quad + \int_0^t \sum_{j=1}^{k-d}\lambda_j(s)\int_{\mathbb{R}}(\hat{\sigma}_{ij}^N(s,z) - h_{ij}(\sigma_{ij}^N(s,z)))\phi_j(s,z)dzds, \\
C_{i_1 i_2}^{\hat{R}} &= \sum_{j=1}^{d}\int_0^t \sigma_{i_1 j}^W(s)\sigma_{i_2 j}^W(s)ds \\
\nu_i^{\hat{R}} &= \sum_{j=1}^{k-d}\lambda_j(s)\frac{e^{-\sigma_{ij}^N(s,z)}}{|(\sigma_{ij}^N)_z(s,z)|}\phi_j(s, e^{\sigma_{ij}^N(t,z)} - 1).
\end{aligned} \tag{17}$$

Proof. Recalling the characteristics of the semimartingale R_i given in (11) and the definition of the Dolean-Dade exponent we obtain (Shiryaev (1998, VII.3d)):

$$\begin{aligned}
\hat{R}_i &= B_i^R + \sum_{j=1}^{d}\int_0^t \sigma_{ij}^W(s)dW_j + \frac{1}{2}C_{ii}^R \\
&\quad + \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} h_{ij}(\hat{z})\left(d\hat{N}_{ij}(s,\hat{z}) - \lambda_j(s)|(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})|\phi_j(s,\hat{z})d\hat{z}ds\right) \\
&\quad + \sum_{j=1}^{k-d}\left(\int_0^t\int_{\mathbb{R}}(\hat{z} - h_{ij}(\hat{z}))d\hat{N}_{ij}(s,\hat{z}) + \int_0^t\int_{\mathbb{R}}(e^{\hat{z}} - 1 - \hat{z})d\hat{N}_{ij}(s,\hat{z})\right).
\end{aligned}$$

To simplify the above expression we can use condition (15) or equivalently :

$$\int_0^t\int_{\mathbb{R}}\left(|\hat{z}|I_{|\hat{z}|\le 1} + e^{\hat{z}}I|\hat{z}| > 1)|(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})|\phi_j(s,\hat{z})d\hat{z}\right)ds \in \mathcal{A}_{loc}^+$$

for every i,j and t. Under this condition we can rewrite $\hat{R}_i$ as:

$$\begin{aligned}
\hat{R}_i &= B_i^{\hat{R}} + \sum_{j=1}^{d}\int_0^t \sigma_{ij}^W(s)dW_j(s) \\
&\quad + \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}}(e^{\hat{z}} - 1)\left(d\hat{N}_{ij}(s,\hat{z}) - \lambda_j(s)|(\sigma_{ij}^N)_{\hat{z}}^{-1}(s,\hat{z})|\phi_j(s,\hat{z})d\hat{z}ds\right),
\end{aligned} \tag{18}$$

where

$$
\begin{aligned}
B_i^{\hat{R}} &= B_i^R + \frac{1}{2}C_{ii}^R + \int_0^t (b_i(s) + \frac{1}{2}\sum_{j=1}^d (\sigma_{ij}^W(s))^2 \\
&= \int_0^t \left(b_i(s) + \frac{1}{2}\sum_{j=1}^d (\sigma_{ij}^W(s))^2 \right) ds \\
&\quad + \int_0^t \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (e^{\sigma_{ij}^N(s,z)} - 1 - h_{ij}(\sigma_{ij}^N(s,z)))\phi_j(s,z)dzds. \quad (19)
\end{aligned}
$$

We can see that (18) is equivalent to

$$
\begin{aligned}
\hat{R}_i &= B_i^{\hat{R}} + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s)dW_j(s) \\
&\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)(dN_{ij}(s,z) - \lambda_j(s)\phi_j(s,z)dzds), \quad (20)
\end{aligned}
$$

Thus under condition (15) $\hat{R}_i$ become a special semimartingale.

To obtain the characteristics of the process $\hat{R}$ we rewrite (18) in canonical form. It is easy to see that if $\hat{\hat{z}} = \hat{\sigma}_{ij}^N(t,z)$ then for $\hat{\hat{z}} > -1$

$$
z = (\sigma_{ij}^N)^{-1}(t, \log(\hat{\hat{z}} + 1)).
$$

Now if we substitute σ_{ij}^N with $\hat{\sigma}_{ij}^N$ we can apply Proposition 1 for a special semimartingale $\hat{R}_i$ (condition (15) is an analog of (4)). We obtain the following characteristics of $\hat{R}$:

$$
\begin{aligned}
B_i^{\hat{R}} &= \int_0^t \left(b_i(s) + \frac{1}{2}\sum_{j=1}^d (\sigma_{ij}^W(s))^2 \right) ds \\
&\quad + \int_0^t \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (e^{\sigma_{ij}^N(s,z)} - 1 - h_{ij}(\sigma_{ij}^N(s,z)))\phi_j(s,z)dzds, \\
C_{i_1 i_2}^{\hat{R}} &= \sum_{j=1}^d \int_0^t \sigma_{i_1 j}^W(s)\sigma_{i_2 j}^W(s)ds \\
\nu_i^{\hat{R}} &= \sum_{j=1}^{k-d} \lambda_j(s) \frac{|(\sigma_{ij}^N)_{\hat{z}}^{-1}(s, \log(\hat{\hat{z}}+1))|}{\hat{\hat{z}}+1} \phi_j\left(s, \hat{\hat{z}}\right),
\end{aligned}
$$

which is equivalent to (17). Hence using (20) we can rewrite (14) as (16). □

Remark 2 *We can note here that under condition* (15) *R remains a semimartingale, although not necessarily a special semimartingale, since* σ^N *can be large and negative.*

- *Since $\hat{R}_i$ is a special semimartingale then by the property of the stochastic exponential, $S = \mathcal{E}(\hat{R})$ is also a special semimartingale. We can also call R **exponentially special** (Kallsen and Shiryaev (2002)).*

Now define

$$Q_{ij}(t) = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z) dq_j(s,z).$$

We can rewrite (16) as the stochastic differential equation:

$$\frac{dS_i}{S_i(t-)} = \hat{b}_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}^W(t)dW_j + \sum_{j=1}^{k-d} dQ_{ij}(t).$$

Since most methods of optimizing the equivalent martingale measure deal directly with the process S itself and not with its (stochastic) logarithm we want to obtain the characteristics of S using the characteristics of its (stochastic) logarithm. First we recall the stochastic differential equation for S in its integral form:

$$\begin{aligned} S_i(t) &= S_i(0) + \int_0^t \hat{b}_i(s)S_i(s-)ds + \sum_{j=1}^{d} \int_0^t \sigma_{ij}^W(s)S_i(s-)dW_j \\ &\quad + \sum_{j=1}^{k-d} \int_0^t S_i(s-) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)dq_j(s,z). \end{aligned} \tag{21}$$

Knowing that $S_i(s-)$ is a predictable process we can obtain

Corollary 1 *Under the conditions of Proposition 2 the characteristics of S are:*

$$\begin{aligned} B_i^{\hat{R}} &= \int_0^t \hat{b}_i(s)S_i(s-)ds, \\ C_{i_1 i_2}^{\hat{R}} &= \sum_{j=1}^{d} \int_0^t S_{i_1}(s-)S_{i_2}(s-)\sigma_{i_1 j}^W(s)\sigma_{i_2 j}^W(s)ds \\ \nu_i^{\hat{R}} &= \sum_{j=1}^{k-d} \lambda_j(s)S_i(s-)\frac{e^{-\sigma_{ij}^N(s,z)}}{|(\sigma_{ij}^N)_z(s,z)|}\phi_j(s, e^{\sigma_{ij}^N(t,z)} - 1). \end{aligned} \tag{22}$$

We want to mention here a very important property of our model which we will use later on. We can rewrite (21) as

$$S_i = S_i^0 + S_i^M + S_i^A,$$

where

$$\begin{aligned} S_i^A &= \int_0^t \hat{b}_i(s) S_i(s-) ds; \\ S_i^M &= \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) S_i(s-) dW_j(s) \\ &\quad + \sum_{j=1}^{k-d} \int_0^t S_i(s-) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z) dq_j(s,z). \end{aligned} \tag{23}$$

Under the jump-diffusion conditions and (15) S^M is a square-integrable local martingale. We introduce the predictable processes

$$\begin{aligned} \varsigma_{i_1 i_2}(t) &= \frac{d\langle S_{i_1}^M, S_{i_2}^M \rangle}{S_{i_1}(t-) S_{i_2}(t-) dt} \\ &= \sum_{j=1}^d \sigma_{i_1 j}^W(t) \sigma_{i_2 j}^W(t) + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{i_1 j}^N(t,z) \hat{\sigma}_{i_2 j}^N(t,z) \phi_j(t,z) dz; \\ \hat{\varsigma}_{i_1 i_2}(t) &= \varsigma_{i_1 i_2}(t) S_{i_1}(t-) S_{i_2}(t-). \end{aligned}$$

.

Proposition 3 *Let S be given by* (21)*. Then*

$$S_i = S_i^0 + S_i^M + \int_0^t \frac{\hat{b}_i(s)}{S_i(s-)\varsigma_{ii}(s)} d\langle S^M, S^M \rangle. \tag{24}$$

Proof. Since S^M is a square-integrable local martingale we obtain by (23):

$$\begin{aligned} \langle S_i^M, S_i^M \rangle &= \sum_{j=1}^d \int_0^t (\sigma_{ij}^W(s) S_i(s-))^2 ds \\ &\quad + \sum_{j=1}^{k-d} \int_0^t (S_i(s-))^2 \lambda_j(s) \int_{\mathbb{R}} (\hat{\sigma}_{ij}^N(s,z))^2 \phi_j(s,z) dz ds \end{aligned}$$

which equals

$$\int_0^t (S_i(s-))^2 \left(\sum_{j=1}^d (\sigma_{ij}^W(s))^2 + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (\hat{\sigma}_{ij}^N(s,z))^2 \phi_j(s,z) dz \right) ds.$$

$$\begin{aligned} \langle S_{i_1}^M, S_{i_2}^M \rangle &= \int_0^t S_{i_1}(s-) S_{i_2}(s-) \sum_{j=1}^d \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \qquad (25) \\ &\quad + \int_0^t \sum_{j=1}^{k-d} S_{i_1}(s-) S_{i_2}(s-) \lambda_j(s) \int_{\mathbb{R}} \hat{\sigma}_{i_1 j}^N(s,z) \hat{\sigma}_{i_2 j}^N(s,z) \phi_j(s,z) dz ds \end{aligned}$$

Hence the have the following expression for S^A,

$$S_i^A = \int_0^t \frac{\hat{b}_i(s)}{S_i(s-)\varsigma_{ii}(s)} d\langle S_i^M, S_i^M\rangle = \int_0^t \frac{\hat{b}_i(s)S_i(s-)}{\hat{\varsigma}_{ii}(s)} d\langle S_i^M, S_i^M\rangle, \tag{26}$$

and hence we obtain (24). □

We recall that by the property of Dolean-Dade exponential, if $\hat{R}_i$ is a local martingale under some measure $\tilde{P}$, then S_i also is a local martingale. Hence if under $\tilde{P}$, $B_i^{\hat{R}} = 0$, then S_i becomes a local martingale under this measure (since $\hat{R}_i - \hat{B}_i^R$ is a local martingale). In the next section we look for this equivalent martingale measure $\tilde{P}$. We will see that in general we may have more than one equivalent martingale measure. We will describe the family of equivalent martingale measures $\tilde{P}$ for this market and in the subsequent section find the "optimal" measure under additional criteria.

4 The general class of equivalent martingale measures.

There are number of ways to obtain the class of equivalent martingale measures for a special semimartingale. The most general way is described in Jacod and Shiryaev (1987). In our simpler case we will instead follow Bardhan and Chao (1996), where a similar model is considered. The best way to find the equivalent martingale measure $\tilde{P}$ is through its density process, $Z(t)$, such that $\hat{R}_i Z$ (and therefore $S_i Z$) are local martingales for each i. As a density process $Z(t)$ is a local martingale itself and hence in our case there exists a local martingale $M(t)$ such that $Z = \mathcal{E}(M)$ (see for example Shiryaev, 1998 VII.3). Consequently any equivalent measure $\tilde{P}$ can be uniquely specified by M. From now on we are looking only for equivalent measures such that M is a square-integrable local martingale.

Let $g_j = g_j(t,z)$ be a predictable function. For each i and j we introduce the predictable processes

$$\alpha_{ij}^g(t) = \lambda_j(t) \int_{\mathbb{R}} g_j(t,z)\hat{\sigma}_{ij}^N(t,z)\phi_j(t,z)dz.$$

If we assume that:

$$\int_0^t \int_{\mathbb{R}} |g_j(s,z)|^2 \phi_j(s,z)dzds \in \mathcal{A}_{loc}^+, \tag{27}$$

then from condition (15) and the Cauchy inequality, $\alpha_{ij}^g(t)$ is well defined and

$$\int_0^t \alpha_{ij}^g(s)ds \in \mathcal{A}_{loc}^+.$$

Now let Σ be the m by k matrix where the first d rows are defined by the elements (σ_{ij}^W) and the remaining $k-d$ rows by the elements (α_{ij}^g).

The following proposition will provide us with the representation for M corresponding to an EMM. It is an analog of a result of Schweizer(1995), but in our case we have the exponential form of stock prices and we are looking for M in a slightly different form.

Proposition 4 *We assume that the jump-diffusion semimartingale conditions (1) along with* (15) *are fulfilled. We also require that* $\det(\Sigma\Sigma^{tr}) \neq 0$. *Let* $(\theta_j^W(t))_{j=1}^d$ *and* $(\theta_j^N(t))_{j=1}^{k-d}$ *be predictable processes such that:*

$$\hat{b}_i(t) = \sum_{j=1}^{d} \sigma_{ij}^W(t)\theta_j^W(t) + \sum_{j=1}^{k-d} \alpha_{ij}^g(t)\theta_j^N(t). \tag{28}$$

If $g_j(t,z)$ *is predictable such that* (27) *holds then* $\tilde{P}$ *is an equivalent martingale measure if* $M(t)$ *equals*

$$-\sum_{j=1}^{d}\int_0^t \theta_j^W(s)dW_j - \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} g_j(s,z)\theta_j^N(s)dq_j(s,z) + H(t), \tag{29}$$

where H *is a local martingale orthogonal for every* i *to the martingale part* $\hat{R}_i^M$ *of the process* $\hat{R}_i$.

Proof. We know that $d\langle W_j, W_j\rangle = dt$ and $\langle q_j(t,z), q_j(t,z)\rangle = \lambda_j(t)\phi_j(t,z)$. Shiryaev(1998, VII.3) (corollary of the Girsanov theorem) showed that R_i is a local martingale if

$$\int_0^t \hat{b}_i(s)ds + \left\langle M, \sum_{j=1}^{d}\int_0^t \sigma_{ij}^W(s)dW_j + \sum_{j=1}^{k-d} Q_{ij}(t)\right\rangle = 0. \tag{30}$$

This is equivalent to requiring that

$$\left\langle M + \sum_{j=1}^{d}\int_0^t \theta_j^W(s)dW_j + \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} g_j(s,z)\theta_j^N(s)dq_j, \right.$$
$$\left. \sum_{j=1}^{d}\int_0^t \sigma_{ij}^W(s)dW_j + \sum_{j=1}^{k-d} Q_{ij} \right\rangle$$

be equal to zero. Define the local martingale H by

$$H(t) = M + \sum_{j=1}^{d}\int_0^t \theta_j^W(s)dW_j + \sum_{j=1}^{k-d}\int_0^t \theta_j^N(s)\int_{\mathbb{R}} g_j(s,z)dq_j(s,z) \tag{31}$$

This proves the proposition.□

Again consider $Z = \mathcal{E}(M)$. Since M is a square integrable martingale it has a representation (Jacod and Shiryaev (1987, III.4)):

$$M(t) = -\sum_{j=1}^{d}\int_0^t \upsilon_j(s)dW_j - \sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} \eta_j(s,z)dq_j(s,z) \tag{32}$$

for some predictable $\upsilon_j(t)$ and $\eta_j(t,z)$ such that:

$$\int_0^t (\upsilon_j(s))^2 ds \in \mathcal{A}^+_{loc},$$

$$\int_0^t \lambda_j(s) \int_{\mathbb{R}} |\eta_j(s,z)|^2 \phi_j(s,z) dz ds \in \mathcal{A}^+_{loc} \tag{33}$$

Since $\mathcal{E}(M)$ is well defined we know that $\Delta M > -1$ and hence $-\eta_j(s,t) > -1$. From the above conditions we also have that:

$$\int_0^t \lambda_j(s) \int_{\mathbb{R}} |\eta_j(s,z)\hat{\sigma}^N_{ij}(s,z)| \phi_j(s,z) dz ds \in \mathcal{A}^+_{loc}.$$

The next proposition uses Proposition 4 to establish when parameters υ_j and η_j leads to a martingale density.

Proposition 5 *We assume that all the conditions of* Proposition 4 *and the decomposition leading to* (33) *hold. The process*

$$Z = \mathcal{E}\left(-\sum_{j=1}^{d} \int_0^t \upsilon_j(s) dW_j - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \eta_j(s,z) dq_j(s,z) \right) \tag{34}$$

is the density process corresponding to an equivalent martingale measure $\tilde{P}$ if the following conditions holds for every i,j and t:

$$\begin{aligned} &\hat{b}_i(t) - \sum_{j=1}^{d} \sigma^W_{ij}(t)\upsilon_j(t) - \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}^N_{ij}(t,z)\eta_j(t,z)\phi_j(t,z)dz = 0; \\ &\eta_j(t,z) < 1 \end{aligned} \tag{35}$$

Proof.

From (31) and (32) we have:

$$\begin{aligned} H(t) &= \sum_{j=1}^{d} \int_0^t (\theta^W_j(s) - \upsilon_j(s)) dW_j \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \left(g_j(s,z)\theta^N_j(s) - \eta_j(s,z) \right) dq_j(s,z). \end{aligned} \tag{36}$$

Now from Proposition 4 $\tilde{P}$ is a martingale measure if

$$\left\langle H(t), \sum_{j=1}^{d} \int_0^t \sigma^W_{ij}(s) dW_j + \sum_{j=1}^{k-d} Q_{ij}(t) \right\rangle = 0 \tag{37}$$

and the lefthandside is

$$\begin{aligned} &\int_0^t \hat{b}_i(s) ds - \sum_{j=1}^{d} \int_0^t \sigma^W_{ij}(s)\upsilon_j(s) ds \\ &- \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \hat{\sigma}^N_{ij}(s,z)\eta_j(s,z)\phi_j(t,z) dz ds. \end{aligned} \tag{38}$$

Since the above equation holds for every $t \in [0,T]$ we can now remove the integration and obtain the condition (35). □

We introduce two vectors of positive predictable processes $(r_j(t))_{j=1}^{k-d}$ and $(\psi_j(t,z))_{j=1}^{k-d}$ such that $\eta_j(s,t) = 1 - r_j(t)\psi_j(t,z)$. We rewrite (35) as:

$$\hat{b}_i(t) = \sum_{j=1}^{d} \sigma_{ij}^W(t)\upsilon_j(t) + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t,z)(1 - r_j(t)\psi_j(t,z))\phi_j(t,z)dz.$$
$$r_j(t)\psi_j(t,z) > 0, \tag{39}$$

for every i, j and t.

The next result shows that after the measure transformation (34) the processes W_j and Q_{ij} become local martingales. We also obtain the formula for S_i with respect to the transformed processes.

Proposition 6 *Under conditions of* Proposition 4*,* (33) *and* (39)*:*

$$\tilde{W}_j(t) = W_j(t) + \int_0^t \upsilon_j(s)ds \tag{40}$$

are $\tilde{P}$-local martingales for each $j = 1, ..., d$. Moreover $\tilde{W}_j(t)$ is a Brownian motion under $\tilde{P}$. The processes

$$\tilde{Q}_{ij}(t) = Q_{ij}(t) + \int_0^t \lambda_j(s) \int_{\mathbb{R}} (1 - r_j(s)\psi_j(s,z))\hat{\sigma}_{ij}^N(s,z)\phi_j(s,z)dzdt \tag{41}$$

are $\tilde{P}$-local martingales for $i = 1, ..., m$ and $j = 1, ..., k-d$. Moreover $\tilde{\lambda}_j(t) = r_j(t)\lambda_j(t)$, $\tilde{\phi}_j(t,z) = \psi_j(t,z)\phi_j(t,z)dz$ are the new characteristics (the intensity and the density function of the jump distribution) for point processes N_j ($j = 1, ..., k-d$) under the new measure $\tilde{P}$. The stock prices follow the equations:

$$S_i(t) = \mathcal{E}\left(\sum_{j=1}^{d} \int_0^t \sigma_{ij}^W(s)d\tilde{W}_j + \sum_{j=1}^{k-d} \tilde{Q}_{ij}(t) \right) \tag{42}$$

and hence they are also $\tilde{P}$-local martingales.

Proof. To prove that $\tilde{W}_j$ is a $\tilde{P}$-local martingale it is enough to show that $\mathcal{E}(\tilde{W}_j)Z$ is a martingale. Indeed since Q_{ij} and W_j are orthogonal, and using $Z = \mathcal{E}(M)$,

$$\mathcal{E}(\tilde{W}_j)Z = \mathcal{E}\left(W_j(t) - \sum_{j=1}^{d} \int_0^t \upsilon_j(s)dW_j \right), \tag{43}$$

which is a local martingale. Since $[\tilde{W}_j, \tilde{W}_j] = [W_j, W_j] = t$ we have that $\tilde{W}_j$ is Brownian motion. Similarly we can prove that $\tilde{Q}_{ij}(t)$ is a $\tilde{P}$-local martingale.

Now recalling the definition of $Q_{ij}(t)$ and $q(t,dz)$ we have that

$$\tilde{Q}_{ij}(t) = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)dN_j - \int_0^t \lambda_j(s)r_j(s)\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)\psi_j(s,z)\phi_j(s,z)dzds.$$

Since $\tilde{Q}_{ij}(t)$ is a $\tilde{P}$-local martingale, the $\tilde{P}$-compensator of $\int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)dN_j$ is

$$\int_0^t \lambda_j(s)r_j(s)\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)\psi_j(s,z)\phi_j(s,z)dzds. \tag{44}$$

Now using uniqueness of the compensator we can find versions of $r_j(t)$ and $\psi_j(t,z)$ such that $\tilde{\lambda}_j(t) = r_j(t)\lambda_j(t)$ and $\tilde{\phi}_j = \psi_j(t,z)\phi_j(t,z)dz$ are the new characteristics of the point processes N_j under the new measure $\tilde{P}$.

Using (40), (41) and (39) we obtain (42).□

Since $\tilde{\phi}_j$ is a probability density function of jumps we obtain the following property for $j = 1,...,k-d$:

$$\int_{\mathbb{R}} \psi_j(t,z)\phi_j(t,z)dz = 1. \tag{45}$$

We call (33), (39) and (45) the ***general EMM conditions***. We can now conclude that the density process Z and hence the corresponding EMM can be determined by parameters $(\upsilon_j(t))_{j=1}^d$, $(r_j(t))_{j=1}^{k-d}$ and $(\psi_j(t,z))_{j=1}^{k-d}$. Under the general EMM conditions the density process, $Z(t)$, is given by

$$\begin{aligned} &\mathcal{E}\left(-\sum_{j=1}^d \int_0^t \upsilon_j(s)dW_j - \sum_{j=1}^{k-d}\int_0^t \int_{\mathbb{R}}(1-r_j(t)\psi_j(t,z))dq_j(s,z)\right) \quad (46)\\ = \;& \exp\left(-\sum_{j=1}^d \left(\int_0^t \upsilon_j(s)dW_j - \frac{1}{2}\int_0^t \upsilon_j^2(s)ds\right)\right)\\ &\times \exp\left(\sum_{j=1}^{k-d}\int_0^t \int_{\mathbb{R}} \log((r_j(s)\psi_j(s,z))dN_j(s,z)\right)\\ &\times \exp\left(\int_0^t \int_{\mathbb{R}}(r_j(s)\psi_j(s,z)-1)\lambda_j(s)\phi_j(s,z)dzds\right) \quad (47)\end{aligned}$$

whenever the following is true:

$$\int_0^t \int_{\mathbb{R}} \log((r_j(s)\psi_j(s,z))dN_j(s,z) \in \mathcal{A}_{loc}^+.$$

Since the equations on the parameters υ_j, r_j, ψ_j, (39), may have more than one solution, the general EMM conditions do not always provide us with a unique EMM. Hence for option pricing we have to find an EMM that is optimal in some sense so we can then find the value of the contingent claim with respect to this measure.

5 Optimality.

We want to optimize the choice of the characteristics $(\upsilon_j(t))_{j=1}^d$, $(r_j(t))_{j=1}^{k-d}$ and $(\psi_j(t,z))_{j=1}^{k-d}$ and so specify a "best" EMM. There are a number of ways to do this. We assume in this section that the stocks are discounted semimartingales given by (16) and that the jump-diffusion special semimartingale conditions and the general EMM conditions hold.

5.1 General Esscher transformation.

Let us first consider our market in the form (14) i.e. $S = \exp(R)$ where R is a semimartingale given by (2). Kallsen and Shiryaev (2002) obtain an Esscher transform for such processes. If we directly apply their results we will obtain a transform and a correspondent measure which may not be unique in the multidimensional case. We consider instead a different transform based on our market in the stochastic exponential form (16) i.e. $S = \mathcal{E}(\hat{R})$. We apply however Theorem 2.18 and Theorem 4.4. of Kallsen and Shiryaev (2002).

Definition 2 (Kallsen, Shiryaev (2002)) *Let $\vartheta \in L(R)$ be such that*

$$S^{\vartheta} = \exp\left(\int_0^t \vartheta_i(s) dR_i\right)$$

is a special semimartingale (i.e. $\int_0^t \vartheta_i(s)dR_i$ is exponentially special). The ***Laplace cumulant process*** *is $\tilde{K}^R_{\vartheta}$ is defined as the compensator of the special semimartingale $\mathcal{L}og(S^{\vartheta})$. The* ***modified Laplace cumulant process*** *$K^R_{\vartheta} := log(\mathcal{E}(\tilde{K}^R_{\vartheta}))$*

Proposition 7 (Kallsen, Shiryaev (2002))
Let $\hat{\vartheta}_i \in L(S_i)$. $\int_0^t \hat{\vartheta}_i(s)d\hat{R}_i$ is exponentially special if and only if

$$\sum_{j=1}^{k-d} \int_0^t \lambda_j \int_{\mathbb{R}} \left(\exp\left(\sum_{i=1}^m \hat{\vartheta}_i(s)\hat{\sigma}^N_{ij}\right) - 1 - \sum_{i=1}^m \hat{\vartheta}_i(s)\hat{\sigma}^N_{ij}\right) \phi_j(s,z)dzds \qquad (48)$$

belongs to $\mathcal{A}^+_{loc}$.

Proposition 8 *Let $\int_0^t \vartheta_i(s)d\hat{R}_i$ be exponentially special then in our market model, the Laplace cumulant process for $\hat{R}$:*

$$\begin{aligned} \tilde{K}^{\hat{R}}_{\hat{\vartheta}} &= K^{\hat{R}}_{\hat{\vartheta}} \qquad (49) \\ &= \int_0^t \sum_{i=1}^m \hat{\vartheta}_i(s)\hat{b}_i(s)ds + \frac{1}{2}\int_0^t \sum_{j=1}^d \sum_{i_1,i_2} \hat{\vartheta}_{i_1}(s)\hat{\vartheta}_{i_2}(s)\sigma^W_{i_1 j}(s)\sigma^W_{i_2 j}(s)ds \\ &\quad + \sum_{j=1}^{k-d}\int_0^t \lambda_j \int_{\mathbb{R}} \left(e^{\sum_{i=1}^m \hat{\vartheta}_i(s)\hat{\sigma}^N_{ij}} - 1 - \sum_{i=1}^m \hat{\vartheta}_i(s)\hat{\sigma}^N_{ij}\right)\phi_j(s,z)dzds \end{aligned}$$

and this is a continuous predictable process. Moreover the density process

$$\begin{aligned} Z^{\hat{\vartheta}}(t) &= \exp\left(\sum_{i=1}^{m}\int_0^t \hat{\vartheta}_i(s)d\hat{R}_i - K_{\hat{\vartheta}}^{\hat{R}}\right) \\ &= \mathcal{E}(\sum_{j=1}^{d}\int_0^t \left(\sum_{i=1}^{m}\hat{\vartheta}_i(s)\sigma_{ij}^W(s)\right) dW_j(s) \\ &\quad + \sum_{j=1}^{k-d}\int_0^t \lambda_j(s)\int_{\mathbb{R}}\left(\exp\left(\sum_{i_1=1}^{m}\hat{\vartheta}_i(s)\hat{\sigma}_{ij}^N(s,z)\right)-1\right)dq_j(s,z)) \end{aligned} \tag{50}$$

provides us with the EMM $\tilde{P}$ if and only if

$$\sum_{j=1}^{k-d}\int_0^t \lambda_j \int_{\mathbb{R}} \hat{\sigma}_{ij}^N \exp\left(\sum_{i=1}^{m}\hat{\vartheta}_i(s)\hat{\sigma}_{ij}^N\right)\phi_j(s,z)dzds \in \mathcal{A}_{loc}^+. \tag{51}$$

and

$$\begin{aligned} &\hat{b}_i(t) + \sum_{j=1}^{d}\sigma_{ij}^W(t)\sum_{i_0=1}^{m}\hat{\vartheta}_{i_0}(t)\sigma_{i_0 j}^W(t) \\ &+ \sum_{j=1}^{k-d}\lambda_j(t)\int_{\mathbb{R}}\left(\exp\left(\sum_{i_0=1}^{m}\hat{\vartheta}_{i_0}(t)\hat{\sigma}_{i_o j}^N(t,z)\right)-1\right)\hat{\sigma}_{ij}^N(t,z))\phi_j(t,z)dz \\ = &\ 0. \end{aligned} \tag{52}$$

The EMM provided by the density process *(50)* under conditions *(51)* and *(52)* we call a **general Esscher transformation for jump-diffusion processes**.

If the solution of (52) exists for every i then the parameters of the general Esscher transformation are

$$\begin{aligned} \upsilon_j(t) &= -\sum_{i=1}^{m}\hat{\vartheta}_i(t)\sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}}\exp\left(\sum_{i=1}^{m}\hat{\vartheta}_i(t)\hat{\sigma}_{ij}^N(t,z)\right)\phi_j(t,z)dz, \\ \psi_j(t,z) &= \frac{\exp\left(\sum_{i=1}^{m}\hat{\vartheta}_i(t)\hat{\sigma}_{ij}^N(t,z)\right)}{\int_{\mathbb{R}}\exp\left(\sum_{i=1}^{m}\hat{\vartheta}_i(t)\hat{\sigma}_{ij}^N(t,y)\right)\phi_j(t,dy)}, \end{aligned} \tag{53}$$

In our case the general Esscher transform is unique. For more general cases uniqueness may fail.

5.2 Variance-optimal EMM.

Another way to find an optimal EMM is the variance optimal martingale measure of Schweizer (1996) . Let f_T be a contingent claim. We assume that $E(H)^2 < \infty$

(i.e. $f_T \in L^2(P)$). In this section we will consider strategies π such that the gains process $G_\pi(T) \in L^2(P)$. We will call this strategy ***square-integrable***.

Schweizer (1996)) defined a signed measure Q to be a **signed martingale measure** if $Q(\Omega) = 1$, $Q \ll P$ with $\frac{dQ}{dP} \in L^2(P)$ and

$$E_Q(G_\pi(T)) = 0$$

for all admissible strategies π. We denote by Z_Q its density. A signed martingale measure Q_V is called a **signed variance optimal** if Q_V minimizes:

$$Var\left(\frac{dQ}{dP}\right) = E\left(\left(\frac{dQ}{dP}\right)^2\right) - 1 = E_Q(Z_Q) - 1.$$

The signed variance optimal measure is not always a probability measure. To avoid this problem we can give another definition of variance optimality for our market model.

Let $\tilde{P}$ be an EMM given by the density process given by *(46)*. Then we call $\tilde{P}$ a **variance optimal measure** if $\tilde{P}$ minimizes:

$$Var(Z) = E_{\tilde{P}}(Z) - 1$$

over all υ, r, ψ satisfying *(39)* and *(45)*. To compare this definition to that of Schweizer we recall that under (39) the price S is given by (42) and the gains process is

$$\begin{aligned} G_\pi(t) &= \sum_{i=1}^{m} \int_0^t \pi_i(s) d\tilde{S}_i(s) & (54) \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{d} \int_0^t \sigma_{ij}^W(s)\pi_i(s)S_i(s-)d\tilde{W}_j + \sum_{j=1}^{k-d} \int_0^t \pi_i(s)S_i(s-)d\tilde{Q}_{ij}(s) \right), \end{aligned}$$

where $\tilde{W}$ is a $\tilde{P}$-Brownian motion and $\tilde{Q}$ is a $\tilde{P}$-compensated point process (martingale). Hence $E_{\tilde{P}}(G_\pi(t)) = 0$ and therefore this EMM is the candidate for Schweizer's signed variance optimal measure.

The variance optimal measure is interesting from the economic point of view because it provides us with the probability EMM whose difference (the density process) from the original measure P has the smallest variation.

The next proposition gives us explicitly the parameters of the variance optimal measure we seek. We use the optimization procedure introduced by Chan (1999).

Proposition 9 *Let $l(t) = (l_1(t), ..., l_m(t))$, if it exists, be nonnegative and de-*

termined by the linear system of equations and inequalities: for each $i = 1, ..., m$

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^{d} \sigma_{ij}^W(t) \left(\sum_{n=1}^{m} l_n(t) \sigma_{nj}^W(t) \right) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t,z) \left(\sum_{n=1}^{m} l_n(t) \hat{\sigma}_{nj}^N(t,z) \right) \phi_j(t,z) dz, \\ \text{and} \quad & \sum_{i=1}^{m} l_i(t) \hat{\sigma}_{ij}^N(t,z) < 1. \end{aligned} \tag{55}$$

Then a version of the variance optimal measure is characterized by:

$$\begin{aligned} &\upsilon_j = \sum_{i=1}^{m} l_i(s) \sigma_{ij}^W(s), \\ &r_j(s) = 1 - \sum_{i=1}^{m} l_i(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z) \phi_j(s,z) dz, \\ &\psi_j(s,z) = \frac{1 - \sum_{i=1}^{m} l_i(s) \hat{\sigma}_{ij}^N(s,z)}{1 - \sum_{i=1}^{m} l_i(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,y) \phi_j(s,dy)}. \end{aligned} \tag{56}$$

If such l *does not exist then the minimum of density variance cannot be achieved and variance optimal measure does not exist.*

Proof. From (46), Proposition 6 and the definition of the Dolean-Dade exponent we can see that:

$$\begin{aligned} E_{\tilde{P}}(Z) &= 1 + E_{\tilde{P}} \sum_{j=1}^{d} \int_0^t (\upsilon_j(s))^2 Z(s-) ds \\ &+ E_{\tilde{P}} \int_0^t \lambda_j(s) Z(s-) \int_{\mathbb{R}} (r_j(s)\psi_j(s,z) - 1)^2 \phi_j(s,z) dz ds. \end{aligned} \tag{57}$$

Our goal is to minimize (57) with respect to υ_j, r_j and ψ_j subject to constraints (39). Let $l(s) = (l_1(s), ..., l_m(s))$ be Lagrange multipliers for (39). Following Chan(1999) we fix ω and without loss of generality remove $E_{\tilde{P}}$. Since $Z(s-) > 0$ we can remove $Z(s-)$ and the integration over s and fix s without loss of generality. Hence we want to minimize the following:

$$\sum_{j=1}^{d} (\upsilon_j(s))^2 ds + \lambda_j(s) \int_{\mathbb{R}} (r_j(s)\psi_j(s,z) - 1)^2 \phi_j(s,z) dz ds. \tag{58}$$

First we fix η_j for $j = 1, ..., k-d$ and υ_j for $j = 1, ..., j_0 - 1, j_0 + 1, ..., d$ and minimize (58) with respect to υ_{j_0} subject to the constraints (35). Hence we want to minimize the Lagrange function:

$$L_W(l, \upsilon_{j_0}) = \upsilon_{j_0}^2(s) - 2 \sum_{i=1}^{m} l_i(s) \sigma_{ij_0}^W(s) \upsilon_{j_0}(s) \tag{59}$$

and the minimizer is:

$$\upsilon_{j_0} = \sum_{i=1}^{m} l_i(s)\sigma^W_{ij_0}(s), \quad j_0 = 1, ..., d. \tag{60}$$

Since j_0 is arbitrary between 1 and d, this is true for each j.

Now we fix υ_j for $j = 1, ..., d$ and η_j for $j = 1, ..., j_1 - 1, j_1 + 1, ..., k - d$ and minimize (58) with respect to η_{j_1} with constraint (35). Again we can remove the integration over s and fix s without loss of generality. Similarly we want to minimize the corresponding Lagrange function:

$$\begin{aligned} L_N(l, \eta_{j_1}) &= \int_{\mathbb{R}} (\eta_{j_1}(s,z))^2 \lambda_{j_1}(s)\phi_{j_1}(s,dz) \\ & \quad -2\sum_{i=1}^{m} l_i(s)\lambda_{j_1}(s)\int_{\mathbb{R}} (e^{\sigma^N_{ij_1}(s,z)} - 1)\eta_{j_1}(s,z)\phi_{j_1}(s,dz). \end{aligned} \tag{61}$$

Again following Chan(1999) we require

$$\frac{d}{dt} L_N(l, \eta_{j_1} + tF) \mid_{t=0} = 0$$

for all F, which gives

$$\eta_{j_1}(s,z) = \sum_{i=1}^{m} l_i(s)(e^{\sigma^N_{ij_1}(s,z)} - 1). \tag{62}$$

Again j_1 is not fixed and hence this is true for any j between 1 and $k - d$. Observing $L_N(\eta_{j_1})$ is a convex function of η, we have a minimum.

Now we plug (60) and (62) into condition (35) to obtain (65). By (60), (62) and (45) we obtain (56). $\Box$

Generally this measure will be a signed measure. If (39) holds for our market model the variance optimal measure is a probability measure.

5.3 Minimal relative entropy.

We consider our model $S = \exp(R)$. Let $\tilde{P}$ be a EMM with the density process Z and define the EMM that minimizes the **relative entropy** ($E_{\tilde{P}}(\log Z)$) to be the **minimal relative entropy measure**. Fritelli (2000) investigated the properties of the minimal relative entropy measure (MRE) and obtained conditions for existence and uniqueness. His main result states that if there exists an EMM $\tilde{P}$ such that $(E_{\tilde{P}}(\log Z)) < \infty$ and $\hat{R}$ is bounded, then the minimal relative entropy measure exists and it is unique. In our case the MRE measure is characterized by its parameters υ, r, ψ. We will find these parameters of the MRE measure using the optimization procedure of Chan (1999) as in the proof of Proposition 9.

Proposition 10 *Let $l(t) = (l_1(t), ..., l_m(t))$, whenever it exists, be determined by the following system:*

$$\begin{aligned}\hat{b}_i(t) &= \sum_{j=1}^{d} \sigma_{ij}^W(t) \left(\sum_{n=1}^{m} l_n(t)\sigma_{nj}^W(t) \right) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t,z) \left(1 - \exp\left(\sum_{n=1}^{m} l_n(t)\hat{\sigma}_{nj}^N(t,z) \right) \right) \phi_j(t,z)dz,\end{aligned} \tag{63}$$

for each $i = 1, ..., m$.

Then a version of the minimal relative entropy measure is characterized by:

$$\begin{aligned}\upsilon_j &= \sum_{i=1}^{m} l_i(s)\sigma_{ij}^W(s), \\ r_j(s) &= \int_{\mathbb{R}} \exp\left(\sum_{i=1}^{m} l_i(s)\hat{\sigma}_{ij}^N(s,z) \right) \phi_j(s,z)dz, \\ \psi_j(s,z) &= \frac{\exp\left(\sum_{i=1}^{m} l_i(s)\hat{\sigma}_{ij}^N(s,z)\right)}{\int_{\mathbb{R}} \exp\left(\sum_{i=1}^{m} l_i(s)\hat{\sigma}_{ij}^N(s,y)\right) \phi_j(s,dy)}.\end{aligned} \tag{64}$$

If such l does not exist then the minimum of relative entropy cannot be achieved and the minimal relative entropy measure does not exist.

Hence we can conclude that the minimal relative entropy measure is equal to the measure obtained by the general Esscher transform.

5.4 Exponential MRE.

We also consider a different approach to minimization of relative entropy. The EMM that minimizes the expectation of the stochastic logarithm of the density process, $E_{\tilde{P}}(\mathcal{L}ogZ)$, we call an **exponential minimal relative entropy measure (exponential MRE)**.

Proposition 11 *Let $l(t) = (l_1(t), ..., l_m(t))$, if it exists, be nonnegative and determined by the linear system:*

$$\begin{aligned}\hat{b}_i(t) &= \sum_{j=1}^{d} \sigma_{ij}^W(t) \left(\sum_{n=1}^{m} l_n(t)\sigma_{nj}^W(t) \right) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t,z) \left(\sum_{n=1}^{m} l_n(t)\hat{\sigma}_{nj}^N(t,z) \right) \phi_j(t,z)dz, \\ \text{and} \quad & \sum_{n=1}^{m} l_n(t)\hat{\sigma}_{nj}^N(t,z) < 1\end{aligned} \tag{65}$$

for each $i = 1, ..., m$. Then a version of the exponential MRE measure is characterized by:

$$\upsilon_j = \sum_{i=1}^{m} l_i(s)\sigma_{ij}^W(s),$$

$$r_j(s) = 1 - \sum_{i=1}^{m} l_i(s)\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,z)\phi_j(s,z)dz,$$

$$\psi_j(s,z) = \frac{1 - \sum_{i=1}^{m} l_i(s)\hat{\sigma}_{ij}^N(s,z)}{1 - \sum_{i=1}^{m} l_i(s)\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s,y)\phi_j(s,dy)}. \tag{66}$$

If such l does not exist then the minimum of the stochastic logarithm of the density cannot be achieved and the exponential MRE measure does not exist.

Proof. In this case the condition that $r_j\psi_j > 0$ is essential. From (34) we can see that:

$$\begin{aligned} & E_{\tilde{P}}(\mathcal{L}ogZ) & (67)\\ = \quad & -E_{\tilde{P}}\sum_{j=1}^{d}\int_0^t \upsilon_j(s)dW_j - E_{\tilde{P}}\sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}} \eta_j(s,z)dq_j(s,z) \end{aligned}$$

Now we can compare this to (57) and the proof of Proposition 9 gives us the same result.□

Here we observe that the exponential MRE measure is the variance optimal measure. Again we can notice that in general the exponential MRE can be a signed measure, but if (65) holds, this is in fact a probability measure.

5.5 Reverse relative entropy.

As in Goll and Kallsen (2000), an EMM is the **minimal reverse relative entropy measure** if it minimizes $E(-\log Z)$. The Goll and Kallsen result applied to our model can be stated as follows

Proposition 12 *Let $\varpi = (\varpi_1, ..., \varpi_m)$ be a predictable process such that*

$$1 + \sum_{i_1=1}^{m} \varpi_{i_1}(t)\hat{\sigma}_{i_1 j}^N(t,z) > 0,$$

$$\int_0^t \lambda_j(s)\int_{\mathbb{R}} \frac{\hat{\sigma}_{ij}^N(s,z)}{1 + \sum_{i_1=1}^{m} \varpi_{i_1}(s)\hat{\sigma}_{i_1 j}^N(s,z)}\phi_j(s,z)dz \in \mathcal{A}_{loc}^+,$$

and

$$\begin{aligned} \hat{b}_i(t) \quad & -\sum_{j=1}^{d}\sigma_{ij}^W(t)\left(\sum_{i_1=1}^{m}\varpi_{i_1}(t)\sigma_{i_1 j}^W(t)\right) \\ & +\sum_{j=1}^{k-d}\lambda_j(t)\int_{\mathbb{R}}\hat{\sigma}_{ij}^N(t,z)\left(\frac{1}{1+\sum_{i_1=1}^{m}\varpi_{i_1}(t)\hat{\sigma}_{i_1 j}^N(t,z)} - 1\right)\phi_j(t,z)dz \\ & = 0 \end{aligned} \tag{68}$$

almost everywhere for $i = 1, ..., m$. Moreover assume that the ϖ exists and define the equivalent measure $\tilde{P}$ by the density process $Z^{\varpi}(t) = \mathcal{E}(M^{\varpi})$ where:

$$\begin{aligned} M^{\varpi}(t) &= -\sum_{j=1}^{d} \int_0^t \left(\sum_{i=1}^{m} \varpi_i(s) \sigma_{ij}^W(s) \right) dW_j(s) \qquad (69) \\ &+ \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t,z) \left(\frac{1}{1 + \sum_{i_1=1}^{m} \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t,z)} - 1 \right) dq_j(s,z). \end{aligned}$$

Then $\tilde{P}$ is an EMM minimizing the reverse relative entropy and its parameters are

$$\begin{aligned} \upsilon_j(t) &= \sum_{i=1}^{m} \varpi_i(t) \sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}} \frac{1}{1 + \sum_{i_1=1}^{m} \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t,z)} \phi_j(t,z) dz, \\ \psi_j(t,z) &= 1 \Bigg/ \int_{\mathbb{R}} \frac{1 + \sum_{i_1=1}^{m} \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t,z)}{1 + \sum_{i_1=1}^{m} \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t,y)} \phi_j(t,dy) \; . \qquad (70) \end{aligned}$$

5.6 Minimal martingale measure.

Follmer and Schweizer (1991) and Schweizer (1995) provide three characterizations of the minimal martingale measure. The main disadvantage of the minimal martingale measure is that it is, in general, a signed martingale measure. Here we will try to construct only probability minimal martingale measures.

We assume that there exists predictable processes $\ell = (\ell_1, ..., \ell_m)$, $\ell \in L^2(S^M)$ such that for every $i = 1, ..., m$, we have that $\hat{b}_i(t) S_i(t-)$ equals

$$\begin{aligned} & \sum_{i_1=1}^{m} \hat{\varsigma}_{i_1 i}(t) \ell_{i_1}(t) \\ = \; & S_i(t-) \sum_{j=1}^{d} \left(\sum_{i_1=1}^{m} S_{i_1}(s-) \sigma_{i_1 j}^W(t) \ell_{i_1}(t) \right) \sigma_{ij}^W(t) \\ & + S_i(t-) \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(\sum_{i_1=1}^{m} S_{i_1}(t-) \hat{\sigma}_{i_1 j}^N(t,z) \ell_{i_1}(t) \right) \hat{\sigma}_{ij}^N(t,z) \phi_j(t,z) dz. \end{aligned}$$

Setting $\hat{\ell}_i(t) = S_i(t-) \ell_i(t)$, we obtain

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^{d} \left(\sum_{i_1=1}^{m} \sigma_{i_1 j}^W(t) \hat{\ell}_{i_1}(t) \right) \sigma_{ij}^W(t) \qquad (71) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(\sum_{i_1=1}^{m} \hat{\sigma}_{i_1 j}^N(t,z) \hat{\ell}_{i_1}(t) \right) \hat{\sigma}_{ij}^N(t,z) \phi_j(t,z) dz. \end{aligned}$$

If S satisfies (24) and (71) we say that S satisfies the ***structure conditions***.

Proposition 13 (Schweizer (1995))
Suppose S has a square-integrable martingale part S^M. Let Z, a density of a EMM $\tilde{P}$, be a square-integrable martingale. Then for $\hat{\ell}_i(t)$ defined by (71) *we have*

$$Z(t) = \mathcal{E}\left(-\sum_{i=1}^{m}\int_0^t \frac{\hat{\ell}_i(s)}{S_i(s-)}dS_i^M + \hat{H}\right), \tag{72}$$

where $\hat{H}$ is a square-integrable local martingale orthogonal to S_i^M for each i. Moreover if

$$\sum_{i=1}^{m}\frac{\hat{\ell}_i(s)}{S_i(s-)}\Delta S^M < 1 \quad P-a.s., \tag{73}$$

then

$$Z^{MM}(t) = \mathcal{E}\left(-\sum_{i=1}^{m}\int_0^t \frac{\hat{\ell}_i(s)}{S_i(s-)}dS_i^M\right)$$

is the density of an EMM.

The EMM with the density Z^{MM} is the **minimal martingale measure**. Generally the minimal martingale measure (MMM) is a signed measure, but under condition (73) it is a probability measure. We can rewrite (73) as

$$\sum_{i=1}^{m}\hat{\ell}_i(s)\hat{\sigma}_{ij}^N(s,z) < 1 \quad P-a.s., \tag{74}$$

We can also see that

$$\begin{aligned} Z^{MM}(t) \quad = \quad & \mathcal{E}\left(-\sum_{j=1}^{d}\int_0^t\left(\sum_{i=1}^{m}\hat{\ell}_i(s)\sigma_{ij}^W(s)\right)dW_j(s)\right. \\ & \left.-\sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}}\left(\sum_{i=1}^{m}\hat{\ell}_i(s)\hat{\sigma}_{ij}^N(s,z)\right)dq_j(s,z)\right) \end{aligned} \tag{75}$$

Now we can determine parameters υ, r, ψ that correspond to the MMM.

Proposition 14 *Let $\hat{\ell}_i(s)$ satisfy the equations* (71) *and* (74)*. Then a version of the minimal martingale measure is characterized by:*

$$\begin{aligned} \upsilon_j &= \sum_{i=1}^{m}\hat{\ell}_i(s)\sigma_{ij}^W(s), \\ r_j(s) &= 1 - \int_{\mathbb{R}}\sum_{i=1}^{m}\hat{\ell}_i(s)\hat{\sigma}_{ij}^N(s,z)\phi_j(s,z)dz, \\ \psi_j(s,z) &= \frac{1-\sum_{i=1}^{m}\hat{\ell}_i(s)\hat{\sigma}_{ij}^N(s,z)}{1-\int_{\mathbb{R}}\sum_{i=1}^{m}\hat{\ell}_i(s)\hat{\sigma}_{ij}^N(s,z)\phi_j(s,z)dz}. \end{aligned} \tag{76}$$

If such $\hat{\ell}$ does not exist then the minimum martingale measure does not exist.

5.7 EMM based on utility maximization.

Recently, Kallsen (1999, 2002) introduced and developed portfolio optimization by maximizing expected local utility. This is now one of the most common approaches for optimization over a class of equivalent martingale measures.

We will consider the market of discounted stocks S with characteristics (22). Let $\pi = (\pi_1, ..., \pi_m)$ be a trading strategy and G_π be the gain process for the strategy π. To apply Kallsen's results (which employ characteristics of S instead of $\hat{R}$) we consider strategies of the form:

$$\pi_i = \frac{\hat{\pi}_i}{S_i(s-)}.$$

We call $\hat{\pi}$ an ***exponential strategy*** since it specifies holdings as a proportion of past stock values. We assume that

$$\sum_{i=1}^{m} \int_0^t |\hat{\pi}_i(s)\hat{b}_i(s)|ds + \sum_{j=1}^{d} \sum_{i_1,i_2}^{m} \int_0^t \hat{\pi}_{i_1}(s)\hat{\pi}_{i_2}(s)\sigma^W_{i_1 j}(s)\sigma^W_{i_2 j}(s)ds \tag{77}$$
$$+\sum_{i=1}^{m} \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \min((\hat{\pi}_i(s)\hat{\sigma}^N_{ij}(s,z))^2, |\hat{\pi}_i(s)\hat{\sigma}^N_{ij}(s,z)|)\phi_j(s,z)dzds \in$$

belongs to $\mathcal{A}^+_{loc}$. Following Kallsen (2002), we call a function $u : \mathbb{R} \to \mathbb{R}$ a **utility function** if

(i) u is twice continuously differentiable.

(ii) the derivatives u', u'' are bounded and $\lim_{x\to\infty} u'(x) = 0$;

(iii) $u(0) = 0$,$u'(0) = 1$;

(iv) $u'(x) > 0$ for any $x \in \mathbb{R}$;

(v) $u''(x) < 0$ for any $x \in \mathbb{R}$.

For any exponential strategy $\hat{\pi}$, the random variable

$$\begin{aligned} \xi_{\hat{\pi}}(t) &= \sum_{i=1}^{m} \hat{\pi}_i(t)\hat{b}_i(t) + \frac{u''(0)}{2} \sum_{j=1}^{d} \sum_{i_1,i_2} \hat{\pi}_{i_1}(t)\hat{\pi}_{i_2}(t)\sigma^W_{i_1 j}(t)\sigma^W_{i_2 j}(t) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(u'\left(\sum_{i=1}^{m} \hat{\pi}_i(t)\hat{\sigma}^N_{ij}(t,z)\right) - \sum_{i=1}^{m} \hat{\pi}_i(t)\hat{\sigma}^N_{ij}(t,z)\right) \phi_j(t,z)dz \end{aligned} \tag{78}$$

is termed the **local utility** of the exponential strategy $\hat{\pi}$ at t, and we call an exponential strategy $\hat{\pi}$ **u-optimal** if

$$E\left(\int_0^T \xi_{\hat{\pi}}(s)ds\right) \geq E\left(\int_0^T \xi_{\tilde{\pi}}(s)ds\right) \tag{79}$$

for any other exponential strategy $\tilde{\pi}$.

Kallsen (2002) considered a market with characteristics of the form (12) and defined u-optimal strategies for this market. There is a correspondence between a u-optimal exponential strategy and a u-optimal strategy. For the second one the expression in *(78)* will involve an $S(t-)$ term. The next proposition follows from the results of Kallsen(2002) applied to our market model.

Proposition 15 *An exponential trading strategy $\hat{\pi}$ is u-optimal if and only if*

$$\hat{b}_i(t) + u''(0)\sum_{j=1}^{d}\sigma_{ij}^W(t)\left(\sum_{i_1=1}^{m}\hat{\pi}_{i_1}(t)\sigma_{i_1 j}^W(t)\right) \tag{80}$$

$$+\sum_{j=1}^{k-d}\lambda_j(t)\int_{\mathbb{R}}\hat{\sigma}_{ij}^N(t,z)\left(u'\left(\sum_{i_1=1}^{m}\hat{\pi}_{i_1}(t)\hat{\sigma}_{i_1 j}^N(t,z)\right)-1\right)\phi_j(t,z)dz = 0$$

almost everywhere for $i = 1, ..., m$.
Moreover assume that the u-optimal exponential trading strategy $\hat{\pi}$ exists and define the equivalent measure $\tilde{P}$ by the density process $Z^{\hat{\pi}}(t) = \mathcal{E}(M^{\hat{\pi}})$ where:

$$\begin{aligned} M^{\hat{\pi}}(t) &= u''(0)\sum_{j=1}^{d}\int_0^t\left(\sum_{i=1}^{m}\hat{\pi}_i(s)\sigma_{ij}^W(s)\right)dW_j(s) \\ &\quad +\sum_{j=1}^{k-d}\int_0^t\int_{\mathbb{R}}\left(u'\left(\sum_{i=1}^{m}\hat{\pi}_i(s)\hat{\sigma}_{ij}^N(s,z)\right)-1\right)dq_j(s,z). \end{aligned} \tag{81}$$

Then $\tilde{P}$ is an EMM with parameters

$$\begin{aligned} \upsilon_j(t) &= -u''(0)\sum_{i=1}^{m}\hat{\pi}_i(t)\sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}}u'\left(\sum_{i=1}^{m}\hat{\pi}_i(t)\hat{\sigma}_{ij}^N(t,z)\right)\phi_j(t,z)dz, \\ \psi_j(t,z) &= \frac{u'\left(\sum_{i=1}^{m}\hat{\pi}_i(t)\hat{\sigma}_{ij}^N(t,z)\right)}{\int_{\mathbb{R}}u'\left(\sum_{i=1}^{m}\hat{\pi}_i(t)\hat{\sigma}_{ij}^N(t,y)\right)\phi_j(t,dy)}. \end{aligned} \tag{82}$$

We call $\tilde{P}$ with parameters (82) *the* ***maximum utility measure****.*

Proof. The equation (80) follows from Kallsen (2002) specialized to our model as does (81). Since the density given by (46) describes the general class of EMMs for our market model we can compare it to the density given by (81). Now we can conclude that the EMM that maximizes the local utility is given by the parameters (82), assuming that $\hat{\pi}$ satisfying (80) exists. Note that for the parameters given by (82), the conditions (39) and (80) are equivalent. Thus $\tilde{P}$ is indeed a EMM. □

Remark 3 *Kallsen made an assumption that $Z^{\hat{\pi}}$ is a martingale. In our case W is a martingale and if we assume that q is also a martingale (previously we defined it as a local martingale), then $M^{\hat{\pi}}$ is also a martingale and*

$$Z^{\hat{\pi}}(t) = 1 + \int_0^t Z^{\hat{\pi}}(s-)dM^{\hat{\pi}}(s)$$

is a martingale.

A common way to classify optimal EMMs is through the utility function. As can be seen from Sections 5.1-5.6 the density and the parameters of all optimal EMMs satisfy (81) and (82) for different utility functions. Some of the optimal EMMs considered above must have an additional restriction on the domain of the utility function to be classified as probability measures. Goll and Kallsen (2000) worked with a slightly different definition of utility where the domain can be bounded from below. We will call $\hat{u}$ a ***restricted utility function*** if it is defined on $(\bar{x}, \infty)$ with

$$\hat{u}'(\bar{x}) = \lim_{x \downarrow \bar{x}} \hat{u}'(x) = \infty$$

and the rest of the properties of usual utility function are fulfilled on the domain $(\bar{x}, \infty)$. In terms of our definition of utility Goll and Kallsen (2000) showed that the corresponding utility function for the reverse relative entropy is $u(x) = \log(1 + x)$, $x > -1$ and obtained the optimal strategy. We change the domain of the utility functions for other optimal EMMs where it is necessary. We summarize these results in Table 1.

EMM	Utility function
Minimal martingale	$x - x^2/2$, $x < 1$
Variance optimal	$x - x^2/2$, $x < 1$
Exponential MRE	$x - x^2/2$, $x < 1$
general Esscher transform	$1 - e^{-x}$
MRE	$1 - e^{-x}$
Reverse Relative Entropy	$\log(1 + x)$, $x > -1$

Table 1: Utility functions corresponding to standard EMMs.

In our market model the Minimal martingale, Variance optimal and exponential MRE measures are the same. The EMM obtained by the general Esscher transform and the MRE are also the same.

5.8 Minimal distance martingale measures.

In addition to classification by utility function Goll and Ruchendorf (2001) combined some of known optimal EMMs in the class of minimal distance martingale measures. Let $Q \ll P$ and let $f : (0, \infty) \to \mathbb{R}$ be a continuous, strictly convex and differentiable function. We assume also that $f(0) = \lim_{x \downarrow 0} f(x)$. Then

EMM	function f(y)
Variance optimal	y^2
Exponential MRE	$y\mathcal{L}og(y)$
MRE	$y \log y$
Reverse Relative Entropy	$-\log y$

Table 2: Distance functions corresponding to standard EMMs.

***f*-divergence** between Q and P is defined as

$$f(Q||P) = Ef\left(\frac{dQ}{dP}\right),$$

if $Ef\left(\frac{dQ}{dP}\right)$ exists or $f(Q||P) = \infty$ if does not. Let $\mathcal{K}$ denote a convex set of probability measures dominated by P. A measure $\tilde{P} \in \mathcal{K}$ is called the ***f*-projection** of P on $\mathcal{K}$ if $f(Q||P) = \inf_{Q\in\mathcal{K}} f(Q||P) = f(\mathcal{K}||P)$. Suppose now that $f'(0) = -\infty$. Assume the existence of a measure $Q \in \mathcal{K}$ such that $Q \sim P$ and $f(Q||P) < \infty$. If $\tilde{P}$ is the f-projection of P, then Goll and Ruchendorf showed that $\tilde{P} \sim P$. Let $\mathcal{M}^s$ be the set of signed EMMs. As above we have that $G_\pi(t)$ is a $\tilde{P}$-martingale and so $E_{\tilde{P}}(G_\pi(t)) = 0$. Goll and Ruchendorf provide the characterization of minimal distance martingale measures. The f-projection of P on $\mathcal{M}^s$ is a ***minimal distance martingale measure***. Generally the minimal distance martingale measure is a signed EMM and Goll and Ruchendorf obtain the conditions when $\tilde{P}$ is the f-projection of P on $\mathcal{M}^s$. Using our market model we can represent the optimal EMMs as minimal distance martingale measures for different functions f in Table 2. It is not clear how to classify the minimal martingale measure and the general Esscher transformation. Goll and Ruchendorf (2001) obtained a form for f for the Esscher transformation only in the case of Levy processes.

6 Examples.

6.1 Continuous market model.

We apply our results here to a continuous market, i.e. the market where the price of the stock is determined as a continuous semimartingale. We know that every (multidimensional) continuous martingale can be represented as an integral with respect to a (multidimensional) Brownian motion (e.g. Shiryaev (1987)). As discussed in Section 3.1,

$$\hat{R}_i = \int_0^t \hat{b}_i(s)ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s)dW_j(s)$$

with square-integrable σ^W is a general type of continuous semimartingale. Hence the price given by $S_i(t) = \mathcal{E}(\hat{R}_i)$ is a general form of a positive continuous semimartingale. Equation (35) can be written as

$$\hat{b}_i(t) = \sum_{j=1}^{d} \sigma_{ij}^W(t)\upsilon_j(t) \tag{83}$$

for $i = 1, .., m,, m < d$. Hence the solution of the above linear system of m equations, if it exists, will provide us with the EMM with density of the following form

$$Z(t) \quad = \quad \mathcal{E}\left(-\sum_{j=1}^{d}\int_0^t \upsilon_j(s)dW_j(s)\right),$$

and so an EMM is characterized by d parameters $\upsilon_j(t)$. If (83) has a unique solution then we have only one EMM in our market and our market is complete. If we have more than one solution we need an optimization procedure to obtain the optimal EMM. All optimal measures considered here are the same when the jump part disappears, and the EMM parameters are given by

$$\upsilon_j(t) = -u''(0)\sum_{i=1}^{m}\hat{\pi}_i(t)\sigma_{ij}^W(t),$$

where $\hat{\pi}_i \in L(S_i)$ are predictable processes and $\hat{\pi}_i(t)/S_i(t)$ are optimal strategies for the utility function $u(x)$. From (83) we obtain $\hat{b} = (\Sigma\Sigma^{tr})\hat{\pi}$, and since we require $\det(\Sigma\Sigma^{tr}) \neq 0$ (see Proposition 4) we always have a unique solution for $\hat{\pi}$. Hence we have a unique optimal EMM and strategy, and option pricing in this case becomes fairly easy since we know the law of W_j under the new measure:

$$\tilde{W}_j = W_j + \int_0^t \upsilon_j(s)ds$$

is a Brownian motion under $\tilde{P}$.

In the case of a single Brownian motion, we can recover the Black-Scholes formula. In the multidimentsional case, similar formulae can be obtained, but the probabilities of non-spherical regions under a multivariate normal must be computed numerically.

6.2 Jump-diffusion with constant intensities.

Now we apply our results in the case when the stock prices are geometric Levy processes. Chan(1999), Shiryaev (1999) and Fujiwara and Miyahara (2003) described optimal EMMs for the case of Levy processes. In order to make R in our model into a Levy process we require conditions such as

$$\begin{aligned} \sigma_{ij}^N(t,z) &= \sigma_{ij}^N(z); \\ \sigma_{ij}^W(t) = \sigma_{ij}^W &= const; \\ b_i(t) &= b_i; \\ \lambda_j(t) = \lambda_j &= const; \\ \phi_j(t,z)dz &= \phi_j(dz). \end{aligned}$$

We then obtain the Levy process

$$\hat{R}_i = \hat{b}_i t + \sum_{j=1}^{d} \sigma_{ij}^W W_j(t) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (e^{\sigma_{ij}^N(z)} - 1) dq_j(s, z)$$

with

$$\hat{b}_i = b_i + \frac{1}{2} \sum_{j=1}^{d} (\sigma_{ij}^W)^2 + \sum_{j=1}^{k-d} \lambda_j \int_{\mathbb{R}} (e^{\sigma_{ij}^N(z)} - 1 - h_{ij}(\sigma_{ij}^N(z)))\phi_j(dz).$$

In other words the price process is determined by a Brownian motion and a compound Poisson process.

For a utility function u the maximum utility EMM $\tilde{P}$ will be determined by the density $Z = \mathcal{E}(M^{\hat{\pi}})$ where

$$\begin{aligned} M^{\hat{\pi}}(t) &= u''(0) \sum_{j=1}^{d} \sum_{i=1}^{m} \sigma_{ij}^W \int_0^t \hat{\pi}_i(s) dW_j(s) \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \left(u' \left(\sum_{i=1}^{m} \hat{\pi}_i(s) \hat{\sigma}_{ij}^N(z) \right) - 1 \right) dq_j(s, z) \end{aligned}$$

and

$$\begin{aligned} &\hat{b}_i + u''(0) \sum_{j=1}^{d} \sum_{i_1=1}^{m} \sigma_{ij}^W \sigma_{i_1 j}^W \hat{\pi}_{i_1}(t) \\ &+ \sum_{j=1}^{k-d} \lambda_j \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(z) \left(u' \left(\sum_{i_1=1}^{m} \hat{\pi}_{i_1}(t) \hat{\sigma}_{i_1 j}^N(z) \right) - 1 \right) \phi_j(dz) = 0 \end{aligned} \tag{84}$$

From the above condition on $\hat{\pi}$ we can conclude that $\hat{\pi}$ does not depend on t in this case. Thus $\tilde{P}$ has parameters that do not depend on time

$$\begin{aligned} \upsilon_j &= -u''(0) \sum_{i=1}^{m} \hat{\pi}_i \sigma_{ij}^W, \\ r_j &= \int_{\mathbb{R}} u' \left(\sum_{i=1}^{m} \hat{\pi}_i \hat{\sigma}_{ij}^N(z) \right) \phi_j(dz), \\ \psi_j(z) &= \frac{u' \left(\sum_{i=1}^{m} \hat{\pi}_i \hat{\sigma}_{ij}^N(z) \right)}{\int_{\mathbb{R}} u' \left(\sum_{i=1}^{m} \hat{\pi}_i \hat{\sigma}_{ij}^N(y) \right) \phi_j(dy)}. \end{aligned} \tag{85}$$

In the case of a minimal martingale, variance optimal and exponential MRE EMM, the utility function is $u(x) = x - \frac{x^2}{2}, x < 1$, and the equation for the optimal exponential strategy will be a solution of

$$\hat{b}_i \quad - \sum_{j=1}^{d} \sum_{i_1=1}^{m} \sigma_{ij}^W \sigma_{i_1 j}^W \hat{\pi}_{i_1} - \sum_{j=1}^{k-d} \lambda_j \sum_{i_1=1}^{m} \hat{\pi}_{i_1} \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(z) \hat{\sigma}_{i_1 j}^N(z) \phi_j(dz) = 0 \tag{86}$$

under the condition

$$\sum_{i_1=1}^{m} \hat{\pi}_{i_1} \hat{\sigma}_{i_1 j}^{N}(z) < 1. \tag{87}$$

This is a linear system of m equations with m unknowns $\pi_1, ..., \pi_m$ with a unique solution if $\det(\Sigma\Sigma^{tr}) \neq 0$, where Σ is defined similarly as in the section 3.3 for the function $g \equiv 1$. If this condition satisfies (87) then we have an optimal EMM, otherwise we have a signed optimal EMM.

The above example also shows that sometimes the stock price after transformation remains within the same "class". Indeed, since the EMM parameters are constants, the stock price after transformation will again be determined by the sum of a Brownian motions and compound Poisson processes with coefficients. This may be a useful feature for estimation problems.

Now let us further simplify the above situation so as to achieve an explicit formula directly comparable that of Balck and Scholes. We assume that N_j, $j = 1, ..., k-d$ are Poisson processes without any marks and with constant intensities λ_j. We also assume that $\sigma_{ij}^N(z) = \sigma_{ij}^N$. In other words the price process S_i for this model will be given by

$$S_i(t) = \exp\left(b_i t + \sum_{j=1}^{d} \sigma_{ij}^W W_j(t) + \sum_{j=1}^{k-d} \sigma_{ij}^N N_j(t)\right). \tag{88}$$

We apply Proposition 2 and obtain

$$S_i(t) \quad = \quad \mathcal{E}\left(\hat{b}_i t + \sum_{j=1}^{d} \sigma_{ij}^W W_j(t) + \sum_{j=1}^{k-d} \left(e^{\sigma_{ij}^N} - 1\right) q_j(t)\right), \tag{89}$$

where

$$\hat{b}_i = b_i + \frac{1}{2}\sum_{j=1}^{d} (\sigma_{ij}^W)^2 + \sum_{j=1}^{k-d} \lambda_j (e^{\sigma_{ij}^N} - 1 - \sigma_{ij}^N)$$

and

$$dq_j(t) = dN_j(t) - \lambda_j dt.$$

The parameters of the minimal martingale (the variance optimal, exponential MRE) measure $\tilde{P}$ here will be:

$$\upsilon_j = \sum_{i=1}^{m} \hat{\pi}_i \sigma_{ij}^W,$$

$$r_j = 1 - \sum_{i=1}^{m} \hat{\pi}_i \left(e^{\sigma_{ij}^N} - 1\right), \tag{90}$$

where $\hat{\pi}_i$ satisfies

$$\begin{aligned}
&\hat{b}_i - \sum_{j=1}^{d}\sum_{i_1=1}^{m}\sigma_{ij}^W\sigma_{i_1j}^W\hat{\pi}_{i_1} - \sum_{j=1}^{k-d}\lambda_j\sum_{i_1=1}^{m}\hat{\pi}_{i_1}\left(e^{\sigma_{i_1j}^N}-1\right)\left(e^{\sigma_{ij}^N}-1\right) = 0 \\
&\sum_{i_1=1}^{m}\hat{\pi}_{i_1}\left(e^{\sigma_{ij}^N}-1\right) < 1. \qquad (91)
\end{aligned}$$

Here we can notice that since there is no jump-size distribution involved the parameter $\psi_j \equiv 1$. Let Σ be a $m \times k$ matrix with rows

$$\left(\sigma_{i1}^W, ..., \sigma_{id}^W, \sqrt{\lambda_1}\left(e^{\sigma_{i1}^N}-1\right), ..., \sqrt{\lambda_{k-d}}\left(e^{\sigma_{i(k-d)}^N}-1\right)\right),$$

$i = 1, ..., m$. If $det(\Sigma\Sigma^{tr}) \neq 0$ then there is a unique solution of (91), $\hat{\pi} = (\Sigma\Sigma^{tr})^{-1}\hat{b}$. Now we can calculate the parameters v_j, r_j and hence apply Proposition 6 to obtain

$$\tilde{W}_j(t) = W_j(t) + \sum_{i=1}^{m}\hat{\pi}_i\sigma_{ij}^W t \qquad (92)$$

which is a Brownian motion under $\tilde{P}$. For the processes

$$\tilde{Q}_{ij}(t) = \left(e^{\sigma_{ij}^N}-1\right)\left(N_j(t) + \lambda_j\left(\sum_{i_1=1}^{m}\hat{\pi}_i\left(e^{\sigma_{i_1j}^N}-1\right)-1\right)t\right) \qquad (93)$$

we know that $N_j(t)$ under $\tilde{P}$ is again a Poisson process with the constant intensity

$$\tilde{\lambda}_j = \left(1 - \sum_{i=1}^{m}\hat{\pi}_i\left(e^{\sigma_{ij}^N}-1\right)\right)\lambda_j.$$

Now we can rewrite (88) as

$$S_i(t) = \exp\left(\tilde{b}_i t + \sum_{j=1}^{d}\sigma_{ij}^W\tilde{W}_j(t) + \sum_{i=1}^{k-d}\sigma_{ij}^N N_j(t)\right). \qquad (94)$$

where

$$\tilde{b}_i = b_i - \sum_{i=1}^{m}\sum_{j=1}^{d}\hat{\pi}_i\sigma_{i_1j}^W\sigma_{ij}^W t,$$

and so the price of European call option $(S_i(T)-K_i)^+$ is:

$$\begin{aligned}
\mathcal{C}_i &= E_{\tilde{P}}(S_i(T)-K_i)^+ \\
&= E_{\tilde{P}}\left(\exp\left(\tilde{b}_i T + \sum_{j=1}^{d}\sigma_{ij}^W\sqrt{T}\frac{\tilde{W}_j(T)}{\sqrt{(T)}} + \sum_{j=1}^{k-d}\sigma_{ij}^N N_j(T)\right) - K_i\right)^+_{.} \qquad (95)
\end{aligned}$$

Since W and N are independent processes we have:

$$\mathcal{C}_i = \int_{A(x_1,..,x_k)} c_i(x_1,..,x_k) \prod_{j=1}^{d} f_W(x_j) \prod_{j=1}^{k-d} f_{N_j}(x_j) \prod_{j=1}^{k} dx_j,$$

where

$$A_i(x_1,..,x_k) = \left\{ x \in \mathbb{R}^k \| \tilde{b}_i T + \sum_{j=1}^{d} \sigma_{ij}^W \sqrt{T} x_j + \sum_{j=1}^{k-d} \sigma_{ij}^N x_j > \log K_i \right\},$$

$$c_i(x_1,..,x_k) = \exp\left(\tilde{b}_i T + \sum_{j=1}^{d} \sigma_{ij}^W \sqrt{T} x_j + \sum_{j=1}^{k-d} \sigma_{ij}^N x_j \right) - K_i,$$

and f_X is the probability density function of X. In our case $f_W = f_{(W(T)/\sqrt{T})}$ is a probability density function of a standard normal variable, and f_{N_j} is a Poisson probability density function with parameter $\tilde{\lambda}_j$. To illustrate further, we take $d = 1$ and $k = 3$, i.e. we have three stocks, one Brownian motion and two Poisson processes.

$$\begin{aligned} \mathcal{C}_i &= \sum_{j=1}^{2} \sum_{n_j=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{A_i(x,n_1,n_2)} (c_i(x,n_1,n_2) - K)\, e^{-\frac{x^2}{2}} dx \prod_{j=1}^{2} \frac{e^{-\tilde{\lambda}_j T} (\tilde{\lambda}_j T)^{n_j}}{n_j!} \\ &= \sum_{j=1}^{2} \sum_{n_j=1}^{\infty} e^{(\tilde{b}_i - \sum_{j=1}^{2} \tilde{\lambda}_j) T + \sum_{j=1}^{2} \sigma_{ij}^N n_j} (1 - \Phi(B_i^1(n_1,n_2))) \prod_{j=1}^{2} \frac{(\tilde{\lambda}_j T)^{n_j}}{n_j!} \\ &\quad - K \sum_{j=1}^{2} \sum_{n_j=1}^{\infty} (1 - \Phi(B_i^2(n_1,n_2))) \prod_{j=1}^{2} \frac{e^{-\tilde{\lambda}_j T} (\tilde{\lambda}_j T)^{n_j}}{n_j!}, \end{aligned} \tag{96}$$

where Φ is the distribution function of the standard normal variable, and

$$B_i^{1,2}(n_1,n_2) = \frac{1}{\sigma_i^W \sqrt{T}} (\log K_i - (\tilde{b}_i T + \sum_{j=1}^{2} \sigma_{ij}^N n_j) \pm (\sigma_i^W)^2 \frac{T}{2}).$$

This provides a straightforward formula easily implemented in software, and comparable to the Black-Scholes formula or similar results of others. This example provides us with a unique EMM if $m = k$ and all the general EMM conditions are fulfilled. This follows from Bardhan and Chao(1995), but also can be easily checked directly.

6.3 A non-Levy example.

We consider our model with two assets $S_i, i = 1, 2$. These assets are driven by one Brownian motion W and two marked point processes N_1 and N_2. We assume that

$$\begin{aligned}
b_i(t) &= b_i, \quad i = 1, 2\,; \\
\sigma_{11}^N(t,z) &= \sigma_{11}^N(s) z^2; \\
\sigma_{12}^N(t,z) &= \sigma_{12}^N(s) z; \\
\sigma_{21}^N(t,z) &= \sigma_{21}^N(s) z; \\
\sigma_{22}^N(t,z) &= \sigma_{22}^N(s) z^2; \\
M_1 &= \int_0^m \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-z_1)^2}{2}\right) dz; \\
M_2 &= \int_0^m \frac{1}{z_2} \exp\left(-\frac{z}{z_2}\right) dz; \\
\phi_1(t,z) &= \frac{1}{M_1} \exp\left(-\frac{(z-z_1)^2}{2}\right),\ z \in [0,m]; \\
\phi_2(t,z) &= \frac{1}{M_2} \exp\left(-\frac{z}{z_2}\right),\ z \in [0,m].
\end{aligned} \tag{97}$$

In other words the sizes of jumps of process N_1 are distributed as a truncated normal with mean z_1 and the sizes of jumps of process N_2 are distributed as a truncated exponential with parameter z_2. We can see that $\sigma_{ij}^N(t,z)$ is an invertible strictly monotone function of z on the interval $[0,m]$. The other jump-diffusion conditions are also fulfilled.

Hence the logarithms of prices are given by

$$\begin{aligned}
\log(S_1(t)) &= b_1 t + \int_0^t \sigma_1^W(s) dW(s) \\
&\quad + \int_0^t \sigma_{11}^N(s) \int_0^m z^2 dN_1(s,z) \\
&\quad + \int_0^t \sigma_{12}^N(s) \int_0^m z dN_2(s,z), \\
\log(S_2(t)) &= b_2 t + \int_0^t \sigma_2^W(s) dW(s) \\
&\quad + \int_0^t \sigma_{21}^N(s) \int_0^m z dN_1(s,z) \\
&\quad + \int_0^t \sigma_{22}^N(s) \int_0^m z^2 dN_2(s,z)
\end{aligned} \tag{98}$$

Now we calculate $\hat{b}_i$ using Proposition 2 and keeping in mind that sizes of jumps are bounded.

$$\begin{aligned}\hat{b}_1(t) &= b_1 t + \frac{1}{2}(\sigma_1^W(t))^2 \\ &\quad + \lambda_1(t) \int_0^m \left(e^{\sigma_{11}^N(t) z^2} - 1 - \sigma_{11}^N(t) z^2 \right) \frac{1}{M_1} e^{-\frac{(z-z_1)^2}{2}} dz \\ &\quad + \lambda_2(t) \int_0^m \left(e^{\sigma_{12}^N(t) z} - 1 - \sigma_{12}^N(t) z \right) \frac{1}{M_2} e^{-\frac{z}{z_2}} dz; \qquad (99) \\ \hat{b}_2(t) &= b_2 t + \frac{1}{2}(\sigma_2^W(t))^2 \\ &\quad + \lambda_2(t) \int_0^m \left(e^{\sigma_{21}^N(t) z} - 1 - \sigma_{21}^N(t) z \right) \frac{1}{M_1} e^{-\frac{(z-z_1)^2}{2}} dz \\ &\quad + \lambda_2(t) \int_0^m \left(e^{\sigma_{22}^N(t) z^2} - 1 - \sigma_{22}^N(t) z^2 \right) \frac{1}{M_2} e^{-\frac{z}{z_2}} dz. \qquad (100)\end{aligned}$$

The above market (98) is not complete. To construct the minimal martingale measure here, we rewrite condition (71) for the parameter $\hat{\ell}$, replacing $\hat{\ell}$ by $-\hat{\pi}$, to obtain a system of linear equations in the $\hat{\ell}$ which must have a unique solution for $\hat{\ell}_1(t)$ and $\hat{\ell}_2(t)$ if the corresponding determinant is not equal to zero. The integrals in equations (99) and (100) are of the form

$$\int_0^m (c_1 z^2 + c_2 z + c_3) e^{-a_1 z^2 + a_2 z + a_3} dz,$$

which we can rewrite as

$$\exp\left(a_3 + \frac{a_2^2}{4a_1}\right) \int_0^m (c_1 z^2 + c_2 z + c_3) \exp\left(- \frac{\left(z - \frac{a_2}{2a_1}\right)^2}{2\left(\frac{1}{2a_1}\right)} \right) dz. \qquad (101)$$

If we also require that $a_1 > 0$ in all these integrals, then the above integral can be calculated using the formulas for the first two moments of the truncated normal distribution in terms of standard normal distribution. See for example Johnson and Kotz (1970).

After calculation of the integrals we can solve the equations for $\hat{\pi}$. To obtain minimal martingale measure we also have to verify the condition (74) for every t. Then using (76) we obtain the exact formulas for the minimal martingale measure parameters. Thus we can obtain the density process Z using (47) and finally the European option price will be given by

$$\mathcal{C}_i = EZ(S_i - K_i)^+.$$

7 Conclusion

This paper develops explicit and computable formulas for equivalent martingale measures that are valid for a broad class of jump-diffusion models of market behaviour. The best transformation is obtained under a number of common optimality criteria and option pricing formulae are provided that generalise those of Black and Scholes.

There are a number of techniques for obtaining an optimal martingale measure that are not covered here. We did not consider super-hedging, quantile hedging, Hellinger process minimization and other procedures for derivative pricing in incomplete markets. It may be interesting to compare the results obtained by these techniques to ours for our market model.

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