Upper and lower estimates for rate of convergence in Chernoff's product formula for semigroups of operators

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Abstract

This paper is devoted to the speed (rate) of convergence of Chernoff approximations to strongly continuous one-parameter semigroups. We provide simple natural examples for which this convergence: is arbitrary high; is arbitrary slow; holds in the strong operator topology but does not hold in the norm operator topology. We also prove general theorem that gives estimate from above for the speed of decay of the norm of the residual appearing in Chernoff approximations. We provide also supplementary theorems which makes it easier to check the conditions of the main theorem. This text is a first draft preprint.

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1. Introduction

This is the first draft-preprint version of the text e.g. it lacks up-to-date literature overview. We kindly ask authors of relevant research papers to forgive us. We will fill this gap before submitting to a journal.

This paper is devoted to C_0-semigroups and their approximations. Three standard textbooks on the topic are [1, 2, 3] — of course, this list is incomplete, but each of these books contains enough information to provide necessary background for the paper. Meanwhile, we recall few definitions and facts (following [3]) to fix the notation and make the paper self-contained. In the paper, we do not use any really deep facts of the C_0-semigroup theory and keep all our reasoning very simple and accessible to unprepared reader. However, it appeared that our elementary approach allows to prove a theorem 4.1 that strengthens the result the famous Chernoff theorem on approximations of C_0-semigroups. With this new theorem it is possible to know, what one needs to do to obtain the so-called fast convergent Chernoff approximations and what the speed of convergence can be. Those approximations of C_0-semigroups provide approximations to solutions of Cauchy problem for a large class of partial differential equations: linear evolution equations with variable coefficients (parabolic equations, Schrödinger equations as examples). This provides a flexible and powerful method of construction of new numerical methods for solving Cauchy problem for PDEs. This is why the result presented in the paper is interesting and important.

2. Preliminaries

Definition 2.1. Let $\mathcal{F}$ be a Banach space. Let $\mathscr{L}(\mathcal{F})$ be a set of all bounded linear operators in $\mathcal{F}$. Suppose we have a mapping $V\colon [0,+\infty) \to \mathscr{L}(\mathcal{F})$, i.e. $V(t)$ is a bounded linear operator $V(t)\colon \mathcal{F} \to \mathcal{F}$ for each $t \geq 0$. The mapping V is called [3] a C_0-*semigroup*, or *a strongly continuous one-parameter semigroup* if it satisfies the following conditions:

1) $V(0)$ is the identity operator I, i.e. $\forall \varphi \in \mathcal{F} : V(0)\varphi = \varphi$;

2) V maps the addition of numbers in $[0, +\infty)$ into the composition of operators in $\mathscr{L}(\mathcal{F})$, i.e. $\forall t \geq 0, \forall s \geq 0 : V(t+s) = V(t) \circ V(s)$, where for each $\varphi \in \mathcal{F}$ the notation $(A \circ B)(\varphi) = A(B(\varphi)) = AB\varphi$ is used;

3) V is continuous with respect to the strong operator topology in $\mathscr{L}(\mathcal{F})$, i.e. $\forall \varphi \in \mathcal{F}$ function $t \longmapsto V(t)\varphi$ is continuous as a mapping $[0, +\infty) \to \mathcal{F}$.

Remark 2.1. The definition of a C_0-*group* is obtained by the substitution of $[0, +\infty)$ by $\mathbb{R}$ in the definition above.

Remark 2.2. It is known [3] that if $(V(t))_{t\geq 0}$ is a C_0-semigroup in Banach space $\mathcal{F}$, then the set

$$\left\{ \varphi \in \mathcal{F} : \exists \lim_{t\to+0} \frac{V(t)\varphi - \varphi}{t} \right\} \overset{denote}{=} D(L)$$

is a dense linear subspace in $\mathcal{F}$.

Definition 2.2. The operator $L\colon D(L) \to \mathcal{F}$ defined on the domain $D(L) \subset \mathcal{F}$ by the equality

$$L\varphi = \lim_{t\to+0} \frac{V(t)\varphi - \varphi}{t}$$

is called *an infinitesimal generator* (or just *generator* to make it shorter) of the C_0-semigroup $(V(t))_{t\geq 0}$.

Since the middle of the XX century it is a well known fact that the solution of a well-posed Cauchy problem for a linear evolution partial differential equation (examples: Schödinger equation, parabolic equations) is given by a strongly continuous semigroup of linear bounded operators whose infinitesimal generator is a (usually unbounded) linear operator from the right-hand side of the evolution equation. Let us explain this in more details and introduce some notation which will be useful for the main text.

Let X be an infinite set, and $\mathcal{F}$ be a Banach space of (not necessarily all) number-valued functions on X, and let L be a closed linear operator $L\colon D(L) \to \mathcal{F}$ with the domain $D(L) \subset \mathcal{F}$ dense in $\mathcal{F}$. We consider the Cauchy problem for the evolution equation

$$\begin{cases} u'_t(t,x) = Lu(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $x \in X$, $u_0 \in \mathcal{F}$, $u(t,\cdot) \in \mathcal{F}$ for all $t \geq 0$, and L is, for example, in trivial case the Laplace operator Δ (so $u'_t = Lu$ is the heat equation), or (in nontrivial case) a more sophisticated linear differential operator with variable coefficients that do not depend on t but depend (usually nonlinearly) on x. It is known that, in case of existence of the C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ with the generator $(L, D(L))$ the solution to Cauchy problem (1) exists (in sense that l.h.s. is equal to r.h.s. in $\mathcal{F}$) and is given by the equality $u(t,x) = (e^{tL}u_0)(x)$ for all $t \geq 0$ and $x \in X$. If $u_0 \in D(L)$, then $u(t,\cdot) \in D(L)$ for all $t \geq 0$ and the solution u is a classical solution (in the terminology of [3]), and for arbitrary $u_0 \in \mathcal{F}$ the solution of Cauchy problem exists only as the solution of the corresponding integral equation $u(t,\cdot) = L\int_0^t u(s,\cdot)ds + u_0$. We write sometimes $u(t,x)$ and sometimes $u(t,\cdot)$ assuming that the role of $\mathcal{F}$ can be played by, for example, the $L^p(\mathbb{R})$ space, then (1) holds only for almost all $x \in \mathbb{R}$. We see that in this case the notation $u(t,x)$ is not completely precise because all the versions of the function $x \longmapsto u(t,x)$ correspond to the same vector $u(t,\cdot) \in L^p(\mathbb{R})$, but usually this does not lead to misunderstanding.

Equality $u(t,x) = (e^{tL}u_0)(x)$ shows that finding the semigroup $\left(e^{tL}\right)_{t\geq 0}$ is a hard problem because it is equivalent to solving the Cauchy problem (1) for each $u_0 \in \mathcal{F}$. However, if the so-called Chernoff function is constructed, a function G which satisfies the conditions of the Chernoff theorem (in particular, satisfied $G(t) = I + tL + o(t), t \to +0$), then the semigroup is given by the equality $e^{tL} = \lim_{n\to\infty} G(t/n)^n$. An advantage of this approach arises from the fact that usually it is possible to define G by an explicit and not very long formula which contains coefficients of operator L, thus obtaining approximations to the solution of Cauchy problem (1) converging to the solution in $\mathcal{F}$ as $n \to \infty$. Expressions $G(t/n)^n u_0$ are called Chernoff approximations to the solution of Cauchy problem (1).

Remark 2.3. For linear operator $A\colon D(A) \to \mathcal{F}$ with the domain $D(A) \subset \mathcal{F}$ and $n \in \mathbb{N}$ we define the domain $D(A^n)$ of operator A^n inductively as follows:

$$(f \in D(A^n)) \iff (f \in D(A), Af \in D(A), A^2 f \in D(A), \dots, A^{n-1} f \in D(A)),$$

which implies $D(A) \supset D(A^2) \supset \cdots \supset D(A^n)$.

Remark 2.4. We recall (proposition 1.8 from [3]) that if A is the generator of a C_0-semigroup on $\mathcal{F}$ then $\cap_{n=1}^{\infty} D(A^n)$ is dense in $\mathcal{F}$ and is a core for A, so $D(A^n)$ is also a core for A and dense in $\mathcal{F}$ for all $n \in \mathbb{N}$.

Definition 2.3. (*Introduced in [5]*). Let us say that G is *Chernoff-tangent* to L iff the following conditions of Chernoff tangency (CT) hold:

(CT0). Let $\mathcal{F}$ be a Banach space, and $\mathscr{L}(\mathcal{F})$ be a space of all linear bounded operators in $\mathcal{F}$. Suppose that we have an operator-valued function $G\colon [0, +\infty) \to \mathscr{L}(\mathcal{F})$, or, using other words, we have a family $(G(t))_{t\geq 0}$ of linear bounded operators in $\mathcal{F}$. Closed linear operator $L\colon D(L) \to \mathcal{F}$ is defined on the linear subspace $D(L) \subset \mathcal{F}$ which is dense in $\mathcal{F}$.

(CT1) Function $t \longmapsto G(t)f \in \mathcal{F}$ is continuous for each $f \in \mathcal{F}$.

(CT2) $G(0) = I$, i.e. $G(0)f = f$ for each $f \in \mathcal{F}$.

(CT3) There exists such a dense subspace $\mathcal{D} \subset \mathcal{F}$ that for each $f \in \mathcal{D}$ there exists a limit

$$G'(0)f = \lim_{t\to 0} \frac{G(t)f - f}{t}.$$

(CT4) The closure of the operator $(G'(0), \mathcal{D})$ is equal to $(L, D(L))$.

With this definition, classical Chernoff theorem may be stated in the following wording.

Theorem 2.1. *(*P. R. Chernoff (1968)*, see [3, 4]). Let $\mathcal{F}$ and $\mathscr{L}(\mathcal{F})$ be as above. Suppose that the operator $L\colon \mathcal{F} \supset Dom(L) \to \mathcal{F}$ is linear and closed, and function G takes values in $\mathscr{L}(\mathcal{F})$. Suppose that these assumptions are fulfilled:*

(E) There exists a C_0-semigroup $(e^{tL})_{t\geq 0}$ with the infenitesimal generator $(L, Dom(L))$.

(CT) G is Chernoff-tangent to $(L, Dom(L))$.

(N) There exists such a number $\omega \in \mathbb{R}$, that $\|G(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

Then for each $f \in \mathcal{F}$ we have $(G(t/n))^n f \to e^{tL} f$ as $n \to \infty$ with respect to norm in $\mathcal{F}$ uniformly with respect to $t \in [0, T]$ for each $T > 0$, i.e.

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| e^{tL} f - (G(t/n))^n f \right\| = 0.$$

We work with arbitrary Banach space $\mathcal{F}$ which we assume to be over fields $\mathbb{R}$ or $\mathbb{C}$, all the results in the paper are true for both cases.

This communication is dedicated to the study (for fixed $t > 0$) of the speed of decreasing (as a function of $f \in \mathcal{F}$) of the norm of the difference between approximate and exact solution $\|G(t/n)^n f - e^{tL} f\|$ as $n \to \infty$. The main result of the paper is theorem 4.1.

3. Examples of arbitrary slow and arbitrary fast convergence

Let us first provide examples of arbitrary high and arbitrary slow convergence. Our first examples of such kind we proposed in [6], and now we develop them. There are also some examples and results obtained by other researchers, we will add references here in the final version of the text.

The following fact should be well known, but clear short proof is better than reference. The C_0-(semi)group of translations will be basic for (counter)examples provided in this section.

Lemma 3.1. *(On the group of translations.) Consider the space $\mathcal{F} = UC_b(\mathbb{R})$ of all uniformly continuous bounded functions $f\colon \mathbb{R} \to \mathbb{R}$ with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Consider $(Q(t)f)(x) = f(x+t)$. Then:*

1. $(Q(t))_{t\geq 0}$ is a C_0-semigroup in $UC_b(\mathbb{R})$; so we can use notation $Q(t) = e^{tL}$.

2. The generator $(L, D(L))$ is given by $L = [f \mapsto f']$, i.e. $(Lf)(x) = f'(x)$ and

$$D(L) = UC_b^1(\mathbb{R}) \overset{define}{=} \{f | f, f' \in UC_b(\mathbb{R})\}.$$

3. $D(L^n) = UC_b^n(\mathbb{R}) \overset{define}{=} \{f | f, f', \ldots, f^{(n)} \in UC_b(\mathbb{R})\}$.

4. $(Q(t))_{t\in\mathbb{R}}$ is a C_0-group in $UC_b(\mathbb{R})$.

5. Operator $(f \mapsto f', UC_b^1(\mathbb{R}))$ is closed.

6. Each of the spaces $UC_b^n(\mathbb{R})$ is dense in $UC_b(\mathbb{R})$ and is a core for $(f \mapsto f', UC_b^1(\mathbb{R}))$.

Proof. 1. Conditions $Q(0)f = f$ and $Q(t_1)Q(t_2)f = Q(t_1+t_2)f$ follow directly from the formula $(Q(t)f)(x) = f(x+t)$. Condition $\lim_{t\to 0} \|Q(t)f - f\| = 0$ for each $f \in UC_b(\mathbb{R})$ follows from the fact that f is uniformly continuous. Indeed, for fixed $f \in UC_b(\mathbb{R})$ and $\varepsilon > 0$ there exists $\delta > 0$ such that inequality $|t| < \delta$ implies $|f(x+t) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$, so $\|Q(t)f - f\| = \sup_{x\in\mathbb{R}} |f(x+t) - f(x)| \leq \varepsilon$.

2a. Let us prove that $D(L) \subset UC_b^1(\mathbb{R})$. Suppose $f \in D(L) \subset UC_b(\mathbb{R})$, then by definition of a generator $f \in D(L) \iff 0 = \lim_{t\to 0} \frac{1}{t}\|e^{tL}f - f - tLf\| = \lim_{t\to 0} \frac{1}{t}\sup_{x\in\mathbb{R}} |f(x+t) - f(x) - t(Lf)(x)|$, so $\frac{1}{t}(f(x+t) - f(x)) \to (Lf)(x)$ uniformly (hence pointwise) as $t \to 0$. Pointwise convergence implies that for each $x \in \mathbb{R}$ function f is differentiable and $f'(x) = (Lf)(x)$. Also function Lf is uniformly continuous because functions $\frac{1}{t}(f(x+t)-f(x))$ are uniformly continuous ($f \in UC_b(\mathbb{R})$) and converge to Lf uniformly. Finally, function Lf is bounded because function $x \longmapsto \frac{1}{t}(f(x+t)-f(x))$ is bounded for each t because f is bounded, and for some small t we have $|\frac{1}{t}(f(x+t) - f(x)) - Lf(x)| < 1$ for all $x \in \mathbb{R}$. Summing up, we proved that function $Lf = f'$ is uniformly continuous and bounded, so $f \in UC_b^1(\mathbb{R})$.

2b. Let us now prove that $UC_b^1(\mathbb{R}) \subset D(L)$. To do that we need to take $f \in UC_b^1(\mathbb{R})$ and prove that $\frac{1}{t}(f(x+t)-f(x)) \to f'(x)$ uniformly in $x \in \mathbb{R}$ as $t \to 0$. Let us prove by contradiction: suppose there exists such $\varepsilon_0 > 0$ that for each $t > 0$ there exists such $x_t \in \mathbb{R}$ that $|\frac{1}{t}(f(x_t+t)-f(x_t))-f'(x_t)| \geq \varepsilon_0$. As f' exists, by Lagrange's theorem there exists $\xi \in (x_t, x_t + t)$ such that $\frac{1}{t}(f(x_t + t) - f(x_t)) = f'(\xi)$ so $|f'(\xi) - f'(x_t)| \geq \varepsilon_0$. But $|\xi - x_t| < t$ and $t > 0$ is arbitrary so we have a contradiction with the fact that f' is uniformly continuous. Then $f \in D(L)$.

3. Directly follows from item 2 and Remark 2.3.

4. Proofs 1 and 2a do not depend on the sign of t, proof 2b goes on for $t < 0$ with a change of sign where needed.

5. Assume that $f_n \in UC_b^1(\mathbb{R})$ and $f_n \to f$, assume that there exists $g \in UC_b(\mathbb{R})$ such that $f_n' \to g$. We need to prove that $f \in UC_b^1(\mathbb{R})$ and $f' = g$. This all follows from theorems of calculus on differentiation under the limit sign. Indeed, if f_n converges to f uniformly, and f_n' converges to g uniformly then g is differentiable and $f' = g$. The condition $f \in UC_b^1(\mathbb{R})$ follows from the fact that $f_n' \in UC_b(\mathbb{R})$ and $\sup_{x\in\mathbb{R}} |f_n'(x) - f'(x)| \to 0$.

6. The density follows from the fact that $C_b^\infty(\mathbb{R}) \subset UC_b^n(\mathbb{R})$ and $C_b^\infty(\mathbb{R})$ is dense in $UC_b(\mathbb{R})$ due to lemma 1 in [9]. The fact that the closure of $(f \mapsto f', UC_b^n(\mathbb{R}))$ is $(f \mapsto f', UC_b^1(\mathbb{R}))$ is shown exactly by the reasoning that we used in the proof of item 5. □

Later we will work with the notion of modulus of continuity. For fixing notation and details we recall (following [8], pp. 168-174) the definition and some simple facts concerning this notion.

Definition 3.1. For uniformly continuous function $f \colon \mathbb{R} \to \mathbb{R}$ its modulus of continuity is a function $\omega_f\colon [0,+\infty) \to [0,+\infty)$ defined by the equality

$$\omega_f(x) = \sup_{|x_1 - x_2| \le x} |f(x_1) - f(x_2)|.$$

Proposition 3.1. *Function $m\colon [0,+\infty) \to [0,+\infty)$ is a modulus of continuity for some uniformly continuous function $f\colon \mathbb{R} \to \mathbb{R}$ iff the conditions i)-iv) hold. Moreover, m is a modulus of continuity for itself, i.e. if we set $f(x) = m(x)$ for $x \ge 0$ and $f(x) = 0$ for $x < 0$, then $\omega_f(x) = m(x)$.*

i) $m(0) = 0$;

ii) m is non-decreasing: $x_1 > x_2$ implies $m(x_1) \ge m(x_2)$;

iii) m is continuous;

iv) m is semiadditive in the sense that for all $x_1 \ge 0$, $x_2 \ge 0$ we have $m(x_1 + x_2) \le m(x_1) + m(x_2)$.

Proposition 3.2. *If $m\colon [0,+\infty) \to [0,+\infty)$ and function $x \longmapsto \frac{m(x)}{x}$ is non-increasing for $x > 0$, then m is semiadditive, i.e. iv) holds.*

Remark 3.1. If for function $f\colon \mathbb{R} \to \mathbb{R}$ we have $\omega_f(h) = o(h)$ as $h \to 0$, then $f(x) \equiv \text{const}$. Indeed, for each $x \in \mathbb{R}$ and $h \neq 0$ we have

$$0 \le |f(x+h) - f(x)| \le \sup_{|x_1 - x_2| \le |h|} |f(x_1) - f(x_2)| = \omega_f(|h|),$$

$$0 \le \frac{|f(x+h) - f(x)|}{|h|} \le \frac{\omega_f(|h|)}{|h|},$$

so $\lim\limits_{h\to 0} \left| \frac{f(x+h)-f(x)}{h} \right| = 0$ hence $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$ i.e. $f'(x) = 0$ for each $x \in \mathbb{R}$. As f is differentiable then for all $x_1, x_2 \in \mathbb{R}$ by Lagrange's theorem $f(x_1) - f(x_2) = (x_1 - x_2) f'(\xi(x_1, x_2)) = (x_1 - x_2) \cdot 0 = 0$ so $f(x) \equiv \text{const}$. This is why for non-constant uniformly continuous function f cases e.g. $\omega_f(h) = \sqrt{h}$ and $\omega_f(h) = 2h$ are possible but the case e.g. $\omega_f(h) = 2h\sqrt{h}$ is not possible.

Now let us provide a family of Chernoff functions for the (semi)group of translations. Function v serves as a parameter in this family and determines the speed of convergence of the Chernoff approximations.

Theorem 3.1. *Consider $\mathcal{F} = UC_b(\mathbb{R})$ – the space of all uniformly continuous bounded functions $f\colon \mathbb{R} \to \mathbb{R}$ with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Consider the group of translations $(e^{tL} f)(x) = f(x+t)$ in $UC_b(\mathbb{R})$. Suppose* *that function* $v\colon [0,+\infty) \to \mathbb{R}$ *satisfies conditions* $\lim_{x\to+\infty} v(x) = 0$ *and* $v(x) \geq 0$. *For each* $f \in UC_b(\mathbb{R})$ *define* $G(0)f = f$ *and*

$$(G(t)f)(x) = f(x + t + tv(1/t)) \text{ for all } x \in \mathbb{R}, t > 0. \tag{1}$$

Then: 1. For all $t \geq 0$ *we have* $\|e^{tL}\| = \|G(t)\| = 1$.

2. G *is a Chernoff function for* $(e^{tL})_{t\geq 0}$. *For all* $T > 0$ *we have*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL} f\| = 0 \text{ for each } f \in UC_b(\mathbb{R}).$$

3. If, additionally, function v *is continuous and non-increasing everywhere on* $[0,+\infty)$, *then for all* $f \in UC_b(\mathbb{R})$ *and all* $T > 0$ *we have*

$$\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL} f\| = \omega_f(Tv(n/T)) \text{ for each } n = 1, 2, 3, \ldots \tag{2}$$

where ω_f *is the modulus of continuity of the function* f.

Proof. Before enjoying the proof please see lemma 3.1 for the properties of the (semi)group of translations.

1. We have $\|e^{tL}f\| = \sup_{x\in\mathbb{R}} |f(x+t)| = \sup_{x\in\mathbb{R}} |f(x)| = \|f\|$ so $\|e^{tL}\| = 1$ for all $t \geq 0$. It is clear that $G(t)$ is a linear bounded operator for each $t \geq 0$. For fixed t we have $\|G(t)f\| = \sup_{x\in\mathbb{R}} |f(x + t + tv(1/t))| = \sup_{y\in\mathbb{R}} |f(y)| = \|f\|$ so $\|G(t)\| = 1$ for all $t \geq 0$.

2. Let us prove that G is a Chernoff function for the group of translations, i.e. let us check the conditions (CT0)-(CT4) for G defined above and L defined on $D(L) = UC_b^1(\mathbb{R})$ by equality $(Lf)(x) = f'(x)$.

Operator $(L, D(L))$ is closed because due to lemma 3.1 it is the generator of the C_0-group of translations. This is a standard fact that $\mathcal{F} = UC_b(\mathbb{R})$ is a Banach space. We have proved condition (CT0).

Let us prove (CT1) i.e. prove that for all $f \in \mathcal{F}$ we have $\lim_{t\to 0} \|G(t)f - f\| = 0$. As each $f \in UC_b(\mathbb{R})$ is uniformly continuous then for all $\varepsilon > 0$ exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \varepsilon$. For $t \to 0$ we have $v(1/t) \to 0$ by conditions of the lemma, hence $t + 2tv(1/t) \to 0$. So there exists $t_0 > 0$ such that condition $0 < t < t_0$ implies $t + tv(1/t) < \delta$. Then (setting $x_1 = x+t+tv(1/t)$, $x_2 = x$) we obtain $|f(x+t+tv(1/t)) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$ and all $t \in (0, t_0)$, so $\|G(t)f - f\| = \sup_{x\in\mathbb{R}} |f(x+t+tv(1/t)) - f(x)| \leq \varepsilon$. We proved that (CT1) also holds.

(CT2) holds by definition of G.

Let us check (CT3) and (CT4). Take any φ from the set $\mathcal{D} = UC_b^2(\mathbb{R})$ of all functions $\mathbb{R} \to \mathbb{R}$ that are uniformly continuous and bounded with the first and second derivative. Then by Taylor's expansion in powers of a small number $t + tv(1/t)$ with remainder in Lagrange's form we conclude that for each $x \in \mathbb{R}$ and $t > 0$ there exists such a number $\xi_{x,t}$ between x and $x + t + tv(1/t)$ that

$$(G(t)\varphi)(x) = \varphi(x + t + tv(1/t)) =$$
$$= \varphi(x) + (t + tv(1/t))\varphi'(x) + \frac{1}{2}(t + tv(1/t))^2\varphi''(\xi_{x,t}), \text{ so}$$

$$\frac{1}{t}\|G(t)\varphi - \varphi - t\varphi'\| = \frac{1}{t}\sup_{x\in\mathbb{R}}|\varphi(x + t + tv(1/t)) - \varphi(x) - t\varphi'(x)| =$$
$$= \sup_{x\in\mathbb{R}}\left|v(1/t)\varphi'(x) + \frac{t}{2}(1 + v(1/t))^2\varphi''(\xi_{x,t})\right| \leq$$
$$\leq v(1/t)\sup_{x\in\mathbb{R}}|\varphi'(x)| + \frac{t}{2}(1 + v(1/t))^2\sup_{x\in\mathbb{R}}|\varphi''(\xi_{x,t})| =$$
$$= o(1)\cdot\text{const} + o(1)\cdot\text{const} = o(1) \text{ as } t \to 0$$

because $\lim_{t\to 0} v(1/t) = 0$ and functions φ', φ'' are bounded. So we have $G(t)\varphi = \varphi + tL\varphi + o(t)$ for all $\varphi \in \mathcal{D}$ so so (CT3) is true. But $\mathcal{D} = D(L^2)$ and due to remark 2.4 it is a core for L, so (CT4) is also true.

Condition (N) is true because $\|G(t)\| = 1 = e^{0\cdot t}$, so thanks to the Chernoff theorem we have $\lim_{n\to\infty}\sup_{t\in[0,T]}\|G(t/n)^n f - e^{tL}f\| = 0$. Item 2 is proved.

3. It follows from the definition of function G that $(G(t/n)f)(x) = f(x + t/n + (t/n)v(n/t))$ and $(G(t/n)^n f)(x) = f(x + t + tv(n/t))$, so

$$\sup_{t\in[0,T]}\|e^{tL}f - G(t/n)^n f\| = \max\left(\|e^{0L}f - G(0/n)^n f\|, \sup_{t\in(0,T]}\|e^{tL}f - G(t/n)^n f\|\right) =$$

$$\max\left(0, \sup_{t\in(0,T]}\|e^{tL}f - G(t/n)^n f\|\right) = \sup_{t\in(0,T]}\sup_{x\in\mathbb{R}}|(e^{tL}f)(x) - (G(t/n)^n f)(x)| =$$

$$\sup_{t\in(0,T]}\sup_{x\in\mathbb{R}}|f(x+t) - f(x+t+tv(n/t))| = \sup_{(t,x)\in(0,T]\times\mathbb{R}}|f(x+t) - f(x+t+tv(n/t))| =$$

[change of variables $t_1 = t$, $x_1 = x + t$ maps the set $(0, T] \times \mathbb{R}$ onto itself]

$$= \sup_{(t_1,x_1)\in(0,T]\times\mathbb{R}}|f(x_1) - f(x_1 + t_1 v(n/t_1))| =$$

[A short interlude. Recall that function $[0,+\infty) \ni y \longmapsto v(y)$ is non-increasing and continuous thanks to conditions in the item 3 of the theorem, so function $(0,T] \ni t_1 \mapsto v(n/t_1)$ is non-decreasing and continuous, and function $(0,T] \ni t_1 \mapsto t_1 v(n/t_1)$ is increasing and continuous hence it maps $(0,T]$ onto $(0, Tv(n/T)$. Making change of variable $\tau = t_1 v(n/t_1)$ we continue our chain of equalities.]

$$= \sup_{(\tau,x_1)\in(0,Tv(n/T)]\times\mathbb{R}} |f(x_1) - f(x_1+\tau)| = \sup_{0<|x_2-x_1|\le Tv(n/T)} |f(x_1) - f(x_2)| =$$

$$= \sup_{|x_2-x_1|\le Tv(n/T)} |f(x_1) - f(x_2)| = \omega_f(Tv(n/T)),$$

where ω_f is the modulus of continuity of function f. Recall that ω_f is well-defined because f is uniformly continuous. Item 3 is proved. □

With the above theorem we can provide examples powerful enough to answer rather general questions.

Proposition 3.3. *There exists a Banach space $\mathcal{F}$, C_0-semigroup $(e^{tL})_{t\ge 0}$ in $\mathcal{F}$ with generator $(L, D(L))$, and Chernoff function G for operator $(L, D(L))$ such that Chernoff approximations converge on each vector but do not converge in operator norm. More precisely:*

1. $\lim_{n\to\infty} \|G(t/n)^n f - e^{tL} f\| = 0$ *for all* $f \in \mathcal{F}$,

2. $\|e^{tL}\| = \|G(t)\| = 1$,

3. for each $t > 0$ *and each* $n \in \mathbb{N}$ *there exists* $f_n \in \mathcal{F}$ *such that* $\|f_n\| = 1$ *and* $\|G(t/n)^n f_n - e^{tL} f_n\| \ge \|f_n\|$ *so* $\|G(t/n)^n - e^{tL}\| \ge 1 \not\to 0$ *as* $n \to \infty$.

Proof. Indeed, consider $\mathcal{F} = UC_b(\mathbb{R})$, $(e^{tL}f)(x) = f(x+t)$ set $v(t) = 1/t$ in theorem 3.1, then $(G(t)f)(x) = f(x + t + tv(1/t))$ becomes $(G(t)f)(x) = f(x+t+t^2)$ and item 2 of theorem 3.1 says that $\lim_{n\to\infty} \|G(t/n)^n f - e^{tL} f\| = 0$ for all $f \in \mathcal{F}$. Item 1 is proved. Item 2 directly follows from definition of the norm in $\mathcal{F}$ and formulas for e^{tL} and $G(t)$.

Consider

$$f_n(x) = \begin{cases} 0 & \text{for } \ x \le 0, \\ \frac{n}{t^2}x & \text{for } \ 0 < x < t^2/n, \\ 1 & \text{for } \ x \ge t^2/n. \end{cases}$$

$$(e^{tL} f_n)(x) = f_n(x+t) = \begin{cases} 0 & \text{for } \ x+t \le 0, \\ \frac{n}{t^2}(x+t) & \text{for } \ 0 < x+t < t^2/n, \\ 1 & \text{for } \ x+t \ge t^2/n. \end{cases}$$

$$
(e^{tL}f_n)(x) = f_n(x+t) = \begin{cases} 0 & \text{for } \ x \leq -t, \\ \frac{n}{t^2}(x+t) & \text{for } \ -t < x < t^2/n - t, \\ 1 & \text{for } \ x \geq t^2/n - t. \end{cases}
$$

It directly follows from $(G(t)f)(x) = f(x+t+t^2)$ that $(G(t/n)f)(x) = f(x+t/n+(t/n)^2)$ and $(G(t/n)^n f)(x) = f(x+t+t^2/n)$ for all $f \in \mathcal{F}$. So

$$
(G(t/n)^n f_n)(x) = \begin{cases} 0 & \text{for } \ x \leq -t - t^2/n, \\ \frac{n}{t^2}(x+t+t^2/n) & \text{for } \ -t - t^2/n < x < t^2/n - t - t^2/n, \\ 1 & \text{for } \ x \geq t^2/n - t - t^2/n. \end{cases}
$$

$$
(G(t/n)^n f_n)(x) = \begin{cases} 0 & \text{for } \ x \leq -t - t^2/n, \\ \frac{n}{t^2}x + n/t + 1 & \text{for } \ -t - t^2/n < x < -t, \\ 1 & \text{for } \ x \geq -t. \end{cases}
$$

Then for $x = -1$ we have $(e^{tL}f_n)(-t) = 0$ and $(G(t/n)^n f_n)(-t) = 1$ so $\|e^{tL}f_n - G(t/n)^n f_n\| = \sup_{x\in\mathbb{R}} |(e^{tL}f_n)(x) - (G(t/n)^n f_n)(x)| \geq |0-1| = 1$. Item 3 is proved. □

Example 3.1. *C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space $\mathcal{F}$, Chernoff function G and vector $f \in \mathcal{F}$ such that $f \notin D(L)$ but the speed of convergence is arbitrary high. More precisely: for arbitrary chosen non-increasing continuous function $v\colon [0,+\infty) \to [0,+\infty)$ vanishing at infinity at arbitrary high rate (e.g. $v(x) = (1+x)^{-k}$, $v(x) = e^{-x}$, $v(x) = e^{-e^x}$) and all $T > 0$ we have $\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL}f\| = Tv(n/T)$ for all $n = 1, 2, 3, \dots$ such that $Tv(n/T) \leq 1$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = \|f\| = 1$.*

Indeed, set $\mathcal{F}$, $(e^{tL})_{t\geq 0}$, G as in theorem 3.1 and define $f(x) = \max(0, \min(x, 1))$. Then f is not differentiable at 0 so $f \notin D(L)$, and $\omega_f(x) = x$ for $x \in [0,1]$ hence example is correct thanks to item 3 of the theorem 3.1.

Remark 3.2. It is possible to show that in the above setting there exists also vector g on which the convergence is arbitrary slow.

Example 3.2. *C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space $\mathcal{F}$, Chernoff function G and vector $f \in \mathcal{F}$ such that $f \in \cap_{j=1}^{\infty} D(L^j)$ but the speed of convergence is arbitrary low, i.e. for arbitrary chosen non-increasing continuous function $u\colon [0,+\infty) \to [0,+\infty)$ vanishing at infinity at arbitrary low rate (e.g. $v(x) = (1+x)^{-1/k}$, $v(x) = 1/\log(x+e)$, $v(x) = 1/\log(\log(x+e^e))$) and all $T > 0$ we have $\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL}f\| = Tv(n/T)$ for all $n = 1, 2, 3, \dots$ such that $Tv(n/T) \leq 1/2$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = \|f\| = 1$.*

Indeed, set $\mathcal{F}$, $(e^{tL})_{t\geq 0}$, G as in theorem 3.1 and define

$$f(x) = \begin{cases} x & \text{for } x \in [0, 1/2], \\ 1 & \text{for } x \geq 1, \\ C^\infty\text{-continued with } 0 \leq f'(x) \leq 1 & \text{for } x \in [1/2, 1], \\ -f(-x) & \text{for } x < 0. \end{cases}$$

Each derivative of function f is continuous on $\mathbb{R}$ and vanishes outside $[-1, 1]$, so it is bounded hence $f \in \cap_{j=1}^{\infty} D(L^j)$. Also $\omega_f(x) = x$ for $x \in [0, 1/2]$ hence example is correct thanks to item 3 of the theorem 3.1.

Remark 3.3. The above given examples show that in a very natural and simple setting we should not expect the convergence in the operator norm $\|G(t/n)^n - e^{tL}\| \to 0$ as $n \to \infty$. Instead, in general setting we have only convergence on every vector $\|G(t/n)^n f - e^{tL} f\| \to 0$ as $n \to \infty$, and this convergence may be arbitrary slow, so we need to choose a vector and a Chernoff function wisely if we want to have fast convergence.

In the next section we provide conditions that guarantee high speed of convergence. Simplifying a lot, one can see that we show that if $G'(0) = L, G''(0) = L^2, \ldots, G^{(m)}(0) = L^m$ and the difference $G(t) - \sum_{k=0}^{m} t^k A^k/k!$ is estimated properly on a suitable set of vectors, then the speed of convergence on this set of vectors is not lower than $1/n^m$ or close to this.

4. Fast convergence

We start from the simple, purely algebraic lemma that establishes decomposition that is basic for our approach.

Lemma 4.1. *Let Z and Y be elements of a ring with associative (but maybe non-commutative) multiplication with unity (e.g. Z and Y may be linear, everywhere defined operators mapping some linear space into itself). Then the following equality holds:*

$$Z^n - Y^n = \sum_{k=0}^{n-1} Z^{n-k-1}(Z - Y)Y^k. \tag{3}$$

Proof.

$$R.h.s. = \sum_{k=0}^{n-1} Z^{n-k-1}(Z - Y)Y^k = \sum_{k=0}^{n-1} Z^{n-k}Y^k - \sum_{k=0}^{n-1} Z^{n-k-1}Y^{k+1} =$$

$$= \left(Z^n Y^0 + \sum_{k=1}^{n-1} Z^{n-k} Y^k \right) - \left(\sum_{j=0}^{n-2} Z^{n-j-1} Y^{j+1} + Z^{n-(n-1)-1} Y^{n-1+1} \right) \overset{j=k-1}{=}$$

$$= Z^n + \sum_{k=1}^{n-1} Z^{n-k} Y^k - \sum_{k=1}^{n-1} Z^{n-k} Y^k - Y^n = Z^n - Y^n = L.h.s.$$

□

This lemma has the following corollary that says something on the high speed of convergence of Chernoff approximations.

Lemma 4.2. *Suppose that number* $T > 0$ *is given, and* C_0*-semigroup* $(e^{tA})_{t\geqslant 0}$ *with generator* $(A, D(A))$ *in Banach space* $\mathcal{F}$ *satisfies for some* $M_1 \geq 1$ *and* $w \geq 0$ *the condition* $\|e^{tA}\| \leqslant M_1 e^{wt}$ *for all* $t \in [0, T]$*. Suppose that there is a mapping* $S\colon (0, T] \to \mathscr{L}(\mathcal{F})$*, i.e.* $S(t)\colon \mathcal{F} \to \mathcal{F}$ *is a linear bounded operator for each* $t \in (0, T]$*. Suppose that there exists some constant* $M_2 \geq 1$ *such that* $\|S(t)^k\| \leqslant M_2 e^{kwt}$ *for all* $t \in (0, T]$ *and all* $k = 1, 2, 3, \ldots$ *Suppose that numbers* $m \in \{0, 1, 2, \ldots\}$ *and* $p \in \{1, 2, 3, \ldots\}$ *are fixed. Suppose that there is a* $(e^{tA})_{t\geqslant 0}$*-invariant subspace* $\mathcal{D} \subset D(A^{m+p}) \subset \mathcal{F}$*, i.e.* $(e^{tA})(\mathcal{D}) \subset \mathcal{D}$ *for any* $t \geq 0$*. Suppose that there are such functions* $C_j\colon (0, T] \to [0, +\infty)$*,* $j = 0, 1, \ldots, m + p$ *that for all* $t \in (0, T]$ *and all* $f \in \mathcal{D}$ *we have*

$$\left\| S(t)f - e^{tA} f \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} C_j(t) \|A^j f\|. \tag{4}$$

Then for all $t > 0$*, all integer* $n \geq t/T$ *and all* $f \in \mathcal{D}$ *the following estimate is true:*

$$\|S(t/n)^n f - e^{tA} f\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} e^{-wt/n} C_j(t/n) \|A^j f\|. \tag{5}$$

Proof. Setting $Z = S(t/n), Y = e^{(t/n)A}$ in formula (3) we obtain

$$\|S(t/n)^n f - e^{tA} f\| \overset{by\ (3)}{=} \left\| \sum_{k=0}^{n-1} S(t/n)^{n-k-1} \left(S(t/n) - e^{(t/n)A} \right) \left(e^{(t/n)A} \right)^k f \right\| \leq$$

$$\leq \sum_{k=0}^{n-1} \left\| S(t/n)^{n-k-1} \right\| \cdot \left\| \left(S(t/n) - e^{(t/n)A} \right) \left(e^{(t/n)A} \right)^k f \right\| \overset{by\ (4)}{\leq}$$

[here we put $\left(e^{(t/n)A}\right)^k f$ in the place of f in (4)]

$$\leq \sum_{k=0}^{n-1} \left\|S(t/n)^{n-k-1}\right\| \cdot \frac{t^{m+1}}{n^{m+1}} \sum_{j=0}^{m+p} C_j(t/n) \left\|A^j \left(e^{(t/n)A}\right)^k f\right\| =$$

[here we use the fact that C_0-semigroup $(e^{tA})_{t\geq 0}$ maps $\mathcal{D}$ into $\mathcal{D}$ and commutes with A^j]

$$= \sum_{k=0}^{n-1} \left\|S(t/n)^{n-k-1}\right\| \cdot \frac{t^{m+1}}{n^{m+1}} \sum_{j=0}^{m+p} C_j(t/n) \left\|\left(e^{(t/n)A}\right)^k A^j f\right\| \leq$$

$$\leq \sum_{k=0}^{n-1} \left\|S(t/n)^{n-k-1}\right\| \cdot \frac{t^{m+1}}{n^{m+1}} \|e^{(kt/n)A}\| \sum_{j=0}^{m+p} C_j(t/n) \|A^j f\| \leq$$

$$\leq \sum_{k=0}^{n-1} M_2 e^{(n-k-1)wt/n} \cdot \frac{t^{m+1}}{n^{m+1}} M_1 e^{w(kt/n)} \sum_{j=0}^{m+p} C_j(t/n) \|A^j f\| =$$

$$= \sum_{k=0}^{n-1} M_1 M_2 \frac{t^{m+1}}{n^{m+1}} e^{wt(n-1)/n} \sum_{j=0}^{m+p} C_j(t/n) \|A^j f\| =$$

$$= M_1 M_2 \frac{t^{m+1}}{n^m} e^{wt} \sum_{j=0}^{m+p} e^{-wt/n} C_j(t/n) \|A^j f\|.$$

□

Usually a priori estimate in the form (4) is not known. To overcome this problem we recall the following fact which most likely is known but a short proof is better than reference. Using the following lemma, one can obtain (4) studying only the norm of difference between $S(t)$ and its Taylor's polynomial because, as we will see now, e^{tA} can also be approximated by (the same!) Taylor's polynomial.

Lemma 4.3. *Let $\mathcal{F}$ be Banach space, let $(e^{tA})_{t\geqslant 0}$ be a C_0-semigroup in $\mathcal{F}$ with generator $(A, D(A))$. Then for all $t \geqslant 0$, all $m = 0, 1, 2, \ldots$ and all $f \in D(A^{m+1})$ we have the following formulas, where the integral is understood in Bochner's sense:*

$$e^{tA} f = \sum_{k=0}^{m} \frac{t^k A^k f}{k!} + \int_0^t \frac{(t-s)^m}{m!} e^{sA} A^{m+1} f \, ds, \tag{6}$$

$$\left\| e^{tA}f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant \frac{t^{m+1}}{(m+1)!} \|A^{m+1} f\| \cdot \sup_{s\in[0,t]} \left\| e^{sA} \right\|. \tag{7}$$

Proof. Denote $Q(t) = e^{tA}$. By definition of the generator of a C_0-semigroup (see definition 2.2), function $t \mapsto Q(t)f$ is differentiable at $t = 0$ iff $f \in D(A)$, and $Q'(0)f = Af$. By the semigroup composition property (see definition 2.1) this implies the differentiability of this function at all $t \in [0, +\infty)$. The derivative in arbitrary time $t \geq 0$ can be (using only above-mentioned definitions) found as follows:

$$Q'(t)f = \lim_{h\to 0} \frac{1}{h}(Q(t+h)f - Q(t)f) = \lim_{h\to+0} \frac{1}{h}(Q(t)Q(h)f - Q(t)f) =$$
$$= Q(t) \lim_{h\to+0} \frac{1}{h}(Q(h)f - f) = Q(t)Q'(0)f = Q(t)Af.$$

So derivative is expressed in terms of semigroup, and for $f \in D(A)$ function $t \mapsto Q'(t)f = Q(t)Af$ is differentiable at $t \in [0, +\infty)$ iff $Af \in D(A)$ which is equivalent to $f \in D(A^2)$; moreover, $Q''(0)f = AAf = A^2 f$. Repeating this argument we see that function $t \mapsto Q(t)f$ is n times differentiable at $t \in [0, +\infty)$ iff $f \in D(A^n)$; for such f we have $Q^{(n)}(t)f = Q(t)A^n f$.

General Taylor's formula (th. 12.4.4 in [7]) after rescaling reads as

$$F(t) = F(0) + F'(0)t + \cdots + \frac{t^m}{m!} F^{(m)}(0) + \frac{1}{m!} \int_0^t (t-s)^m F^{(m+1)}(s) ds,$$

and for $F(t) = Q(t)f = e^{tA}f$, $F^{(n)}(t)f = e^{tA}A^n f$ becomes (6).

Formula (7) is a simple corollary of (6). □

Now we are ready to state and prove the main result of the paper.

Theorem 4.1. *Suppose that*

1. *Number $T > 0$ is given, and C_0-semigroup $(e^{tA})_{t\geqslant 0}$ with generator $(A, D(A))$ in Banach space $\mathcal{F}$ satisfies for some $M_1 \geq 1$ and $w \geq 0$ the condition $\|e^{tA}\| \leqslant M_1 e^{wt}$ for all $t \in [0, T]$.*
2. *There is a mapping $S\colon (0, T] \to \mathscr{L}(\mathcal{F})$, i.e. $S(t)\colon \mathcal{F} \to \mathcal{F}$ is a linear bounded operator for each $t \in (0, T]$. There exists some constant $M_2 \geq 1$ that $\|S(t)^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k = 1, 2, 3, \ldots$*
3. *Numbers $m \in \{0, 1, 2, \ldots\}$ and $p \in \{1, 2, 3, \ldots\}$ are fixed. There is a $(e^{tA})_{t\geqslant 0}$-invariant subspace $\mathcal{D} \subset D(A^{m+p}) \subset \mathcal{F}$, i.e. $(e^{tA})(\mathcal{D}) \subset \mathcal{D}$ for any $t \geq 0$ (for example $\mathcal{D}$ may be equal to $D(A^{m+p})$).*

4. *There exist such functions* $K_j\colon (0,T] \to [0,+\infty)$, $j = 0,1,\ldots,m+p$ *that for all* $t \in (0,T]$ *and all* $f \in \mathcal{D}$ *we have*

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t) \|A^j f\|. \tag{8}$$

Then:

1. *For all* $t > 0$, *all integer* $n \geq t/T$ *and all* $f \in \mathcal{D}$ *we have*

$$\|S(t/n)^n f - e^{tA} f\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n) \|A^j f\|, \tag{9}$$

where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$ *and* $C_j(t) = K_j(t)e^{-wt}$ *for* $j \neq m+1$.

2. *If* $\mathcal{D}$ *is dense in* $\mathcal{F}$ *and for all* $j = 0,1,\ldots,m+p$ *we have* $K_j(t) = o(t^{-m})$ *when* $t \to +0$, *then for all* $g \in \mathcal{F}$ *and* $\mathcal{T} > 0$ *the following equality is true:*

$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t \in (0,\mathcal{T}]} \left\| S(t/n)^n g - e^{tA} g \right\| = 0. \tag{10}$$

Proof. 1. With the help of estimate (8) and lemma 4.3 for each $f \in D(A^{m+p})$ and each $t \in (0,T]$ we have

$$\|S(t)f - e^{tA}f\| \leq \left\| S(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| + \left\| \sum_{k=0}^{m} \frac{t^k A^k f}{k!} - e^{tA} f \right\| \leq$$

$$\leq t^{m+1} \sum_{j=0}^{m+p} K_j(t) \|A^j f\| + \frac{t^{m+1}}{(m+1)!} \|A^{m+1} f\| \cdot \sup_{s \in [0,t]} \left\| e^{sA} \right\| \leq$$

$$\leq t^{m+1} \left(\sum_{j=0}^{m+p} K_j(t) \|A^j f\| + \frac{M_1 e^{wt}}{(m+1)!} \|A^{m+1} f\| \right) = t^{m+1} \sum_{j=0}^{m+p} e^{wt} C_j(t) \|A^j f\|,$$

where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$ and $C_j(t) = K_j(t)e^{-wt}$ for $j \neq m+1$. Now we see that conditions of lemma 4.2 are satisfied, so (5) is true with $C_j(t/n)$ replaced by $e^{wt/n} C_j(t/n)$. Then (9) follows from (5). Item 1 is proved.

2. Suppose $g \in \mathcal{F}$, $\mathcal{T} > 0$ and $\varepsilon > 0$ are given. It is sufficient to find such integer $n_0 \geq \mathcal{T}/T$ that for all $n > n_0$ and all $t \in (0,\mathcal{T}]$ we have $\|S(t/n)^n g - e^{tA} g\| < \varepsilon$.

The set $\mathcal{D}$ is dense in $\mathcal{F}$ so for any $\delta > 0$ there exists such $f \in \mathcal{D}$ that $\|f - g\| < \delta$. Then, using inequality (9) from item 1 proven above we obtain:

$$\|S(t/n)^n g - e^{tA} g\| \le$$
$$\le \|S(t/n)^n g - S(t/n)^n f\| + \|S(t/n)^n f - e^{tA} f\| + \|e^{tA} f - e^{tA} g\| \le$$
$$\le \|S(t/n)^n\| \cdot \|g - f\| + \|S(t/n)^n f - e^{tA} f\| + \|e^{tA}\| \cdot \|f - g\| \le$$
$$\le M_2 e^{nwt/n}\delta + \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|A^j f\| + M_1 e^{wt}\delta \le$$
$$\le (M_1 + M_2)e^{w\mathcal{T}}\delta + M_1 M_2 \mathcal{T} e^{w\mathcal{T}} \sum_{j=0}^{m+p} (t/n)^m C_j(t/n)\|A^j f\|.$$

Taking $\delta = \varepsilon e^{-w\mathcal{T}}/(2M_1 + 2M_2)$ and $n_0 \ge \mathcal{T}/T$ such that

$$M_1 M_2 \mathcal{T} e^{w\mathcal{T}} \sum_{j=0}^{m+p} (t/n)^m C_j(t/n)\|A^j f\| < \varepsilon/2$$

for all $n \ge n_0$ and $t \in (0, \mathcal{T}]$ (such n_0 exists due to $\lim_{n\to\infty} C_j(t/n)(t/n)^m = 0$ thanks to condition $K_j(t) = o(t^{-m})$ when $t \to +0$ for all $j = 0, 1, \ldots, m + p$) we get: $\|S(t/n)^n g - e^{tA} g\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for any $n \ge n_0$ and $t \in (0, \mathcal{T}]$. $\square$

Remark 4.1. Condition $\|S(t)^k\| \leqslant M_2 e^{kwt}$ may seem difficult to obtain, but if we have estimate $\|S(t)\| \leqslant e^{wt}$ then $\|S(t)^k\| \leqslant M_2 e^{kwt}$ is true for $M_2 = 1$.

Example 4.1. *Let us consider particular modeling example. Suppose* $\|e^{tA}\| \le e^t$, $\|S(t)\| \le e^t$, $\|S(t)f - f - tAf - \frac{1}{2}t^2 A^2 f\| \le t^2\sqrt{t}\|A^3 f\|$ *for all* $f \in D(A^3)$ *and* $t \in (0; 1]$. *Then* $\mathcal{D} = D(A^3)$, $m = 2$, $M_1 = M_2 = w = 1$, $K_0(t) = K_1(t) = 0$, $K_2(t) = 1/\sqrt{t}$ *for any* $t \in (0; 1]$. *So estimate (9) of theorem 4.1 states that for any fixed* $t > 0$ *the following estimate is true for all* $f \in D(A^3)$ *and integer* $n \ge t$, *having the following asymptotic behaviour as* $n \to \infty$:

$$\|S(t/n)^n f - e^{tA} f\| \le \frac{t^3 e^t}{n^2}\left(\frac{1}{\sqrt{t/n}} + \frac{e^{t/n}}{3!}\right)\|A^3 f\| =$$
$$= e^t\left(\frac{t^2\sqrt{t}}{n\sqrt{n}} + \frac{e^{t/n} t^3}{6n^2}\right)\|A^3 f\| = \frac{t^2\sqrt{t}e^t}{n\sqrt{n}}\|A^3 f\| + O\left(\frac{1}{n^2}\right).$$

A more meaningful example of the usage of the theorem 4.1 can be found in the proof of theorem 5.2 in the next section.

5. Approximation of solutions of second-order parabolic equations

5.1. Estimation of the norms of derivatives of a function via the norms of degrees of a second-order differential operator

First, we prove theorem 5.1 on estimation of the norms of derivatives of a function via the norms of degrees of a second-order differential operator. To do this, we need the following two lemmas.

Lemma 5.1. *For each twice differentiable function $u\colon \mathbb{R} \to \mathbb{R}$ and any $h > 0$, the inequality holds*

$$\sup_{x\in\mathbb{R}} |u'(x)| \leqslant h \cdot \sup_{x\in\mathbb{R}} |u''(x)| + \frac{1}{h} \cdot \sup_{x\in\mathbb{R}} |u(x)|. \tag{11}$$

Proof. Let us expand the function u using the first-order Taylor formula at the point $x \in \mathbb{R}$ for the increment $2h$ with remainder in Lagrange form: $u(x+2h) = u(x)+u'(x)\cdot 2h+u''(\xi)\cdot(2h)^2/2$, where $\xi \in (x, x+2h)$. Express the derivative from this formula: $u'(x) = -u''(\xi) \cdot h + (u(x + 2h) - u(x))/(2h)$. Taking the supremums of the modules by $x \in \mathbb{R}$, we get from here the necessary estimate (11). ☐

Lemma 5.2. *Let $q \in \{1, 2, 3, \ldots\}$, the functions $a, b, c\colon \mathbb{R} \to \mathbb{R}$ be differentiable $(2q - 2)$ times, the operator L maps every twice differentiable function $u\colon \mathbb{R} \to \mathbb{R}$ to the function $Lu = au'' + bu' + cu$, and the function $v\colon \mathbb{R} \to \mathbb{R}$ be differentiable $2q$ times. Then the following three statements are true:*

a) the function $L^q v$ can be written as

$$L^q v = a^q v^{(2q)} + \sum_{i=0}^{2q-1} p_i \cdot v^{(i)}, \tag{12}$$

where $p_0, \ldots, p_{2q-1}$ are some functions that are homogeneous polynomials of degree q of the functions a, b, c and their derivatives of order no higher than $2(q - 1)$;

b) the following inequality holds:

$$\|L^q v\| \le \sum_{i=0}^{2q} C_i \cdot \|v^{(i)}\|, \tag{13}$$

where $C_i = \|p_i\|$ for $i = 0, \ldots, 2q - 1$, and $C_{2q} = \|a\|^q$;

c) in the case $\inf_{x\in\mathbb{R}} |a(x)| > 0$ *the following estimate is correct:*

$$\|v^{(2q)}\| \leqslant \left\|\frac{1}{a^q}\right\| \cdot \|L^q v\| + \sum_{i=0}^{2q-1} \left\|\frac{p_i}{a^q}\right\| \cdot \|v^{(i)}\|. \tag{14}$$

Proof. a) The equality (12) will be proved by mathematical induction with respect to the parameter q.

Base of induction: $q = 1$. In this case, $L^q v = Lv = av'' + bv' + cv$, so $p_0 = c$ and $p_1 = b$ in (12).

Induction step: $q \to q + 1$. Let us assume that the statement a) of the lemma is true for the number $q \in \{1, 2, 3, \ldots\}$ and show that it remains true when replacing q with $q + 1$.

Substituting Lv instead of function v in (12), we get:

$$L^{q+1}v = L^q(Lv) = a^q \cdot (Lv)^{(2q)} + \sum_{i=0}^{2q-1} p_i \cdot (Lv)^{(i)} =$$

$$= a^q \cdot \Big((av'')^{(2q)} + (bv')^{(2q)} + (cv)^{(2q)}\Big) + \sum_{i=0}^{2q-1} p_i \cdot \Big((av'')^{(i)} + (bv')^{(i)} + (cv)^{(i)}\Big).$$

Next, using the Leibniz formula $(uv)^{(i)} = \sum_{j=0}^{i} C_i^j u^{(i-j)} v^{(j)}$ in each term of the right part and selecting separately first term, we find:

$$L^{q+1}v = a^{q+1}v^{(2q+2)} + \sum_{j=0}^{2q-1} a^q C_{2q}^j a^{(2q-j)} v^{(j+2)} +$$
$$+ \sum_{j=0}^{2q} a^q C_{2q}^j \cdot \Big(b^{(2q-j)} v^{(j+1)} + c^{(2q-j)} v^{(j)}\Big) +$$
$$+ \sum_{i=0}^{2q-1} p_i \sum_{j=0}^{i} C_i^j \cdot \Big(a^{(i-j)} v^{(j+2)} + b^{(i-j)} v^{(j+1)} + c^{(i-j)} v^{(j)}\Big).$$

This shows that the function $L^{q+1}v$ can be written in the form similar to (12):

$$L^{q+1}v = a^{q+1}v^{(2q+2)} + \sum_{i=0}^{2q+1} r_i \cdot v^{(i)},$$

where $r_0, \ldots, r_{2q+1}$ are some homogeneous polynomials of degree $q+1$ of the functions a, b, c and their derivatives of order no higher than $2q$. Thus, the induction step is completed and the statement of point a) of the theorem is proved.

b) Inequality (13) immediately follows from the formula (12).

c) Expressing the function $v^{(2q)}$ from the equality (12) and evaluating its norm, we obtain the required inequality (14). □

Example 5.1. *For $q = 2$, the decomposition* (12) *has the following form:*

$$\begin{aligned} L^q v = L^2 v &= a \cdot (av'' + bv' + cv)'' + b \cdot (av'' + bv' + cv)' + c \cdot (av'' + bv' + cv) = \\ &= a^2 v^{IV} + (2aa' + 2ab) \cdot v''' + (aa'' + a'b + b^2 + 2ab' + 2ac) \cdot v'' + \\ &+ (ab'' + bb' + 2ac' + 2bc) \cdot v' + (ac'' + bc' + c^2) \cdot v. \end{aligned}$$

Theorem 5.1. *Suppose $n \in \{0, 1, 2, \ldots\}$, the functions $a, b, c \colon \mathbb{R} \to \mathbb{R}$ are differentiable $2\lfloor (n-1)/2 \rfloor$ times and the inequality $\inf_{x \in \mathbb{R}} |a(x)| > 0$ holds. Suppose, in addition, the operator L maps each doubly differentiable function $u \colon \mathbb{R} \to \mathbb{R}$ to the function $Lu = au'' + bu' + cu$. Then there are nonnegative constants $C_0, C_1, \ldots, C_{\lfloor (n+1)/2 \rfloor}$, such that for any $2\lfloor (n+1)/2 \rfloor$ times differentiable function $v \colon \mathbb{R} \to \mathbb{R}$, the following inequality is true:*

$$\|v^{(n)}\| \leqslant \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} C_k \|L^k v\|. \tag{15}$$

Proof. We apply the induction by parameter n.

1) The base of induction: $n = 0$. In this case (15) has the form $\|v\| \leq C_0 \|v\|$, so we can take $C_0 = 1$.

2) The induction step. Let the statement of the theorem be proved for all $n \leqslant m - 1$. We must prove it for $n = m$.

Consider two possible cases: m is odd and m is even.

a) Let m be odd. Then by lemma 5.1 for $u = v^{(m-1)}$ and for any $h > 0$ we have:

$$\|v^{(m)}\| \leqslant h \|v^{(m+1)}\| + \frac{1}{h} \|v^{(m-1)}\|. \tag{16}$$

According to point c) of the lemma 5.2 for $q = (m+1)/2$ and for some nonnegative constants $A_0, \ldots, A_m$ the next inequality is satisfied:

$$\|v^{(m+1)}\| \leqslant \left\| \frac{1}{a^{(m+1)/2}} \right\| \cdot \|L^{(m+1)/2} v\| + \sum_{i=0}^{m} A_i \|v^{(i)}\|.$$

Substituting it into (16), we get:

$$\|v^{(m)}\| \leqslant h\Big\|\frac{1}{a^{(m+1)/2}}\Big\| \cdot \|L^{(m+1)/2}v\| + \sum_{i=0}^{m-2} hA_i\|v^{(i)}\| + \\ + \Big(hA_{m-1} + \frac{1}{h}\Big)\|v^{(m-1)}\| + hA_m\|v^{(m)}\|.$$

From here we have:

$$(1 - hA_m)\|v^{(m)}\| \leqslant h\Big\|\frac{1}{a^{(m+1)/2}}\Big\| \cdot \|L^{(m+1)/2}v\| + \sum_{i=0}^{m-2} hA_i\|v^{(i)}\| + \\ + \Big(hA_{m-1} + \frac{1}{h}\Big)\|v^{(m-1)}\|. \tag{17}$$

Choose $h > 0$ so that $1 - hA_m > 0$ (we can take $h = 1$ when $A_m = 0$, and $h = 1/(2A_m)$ when $A_m > 0$). Then, expressing $\|v^{(m)}\|$ from (17), we get that for some nonnegative constants $B_0, \dots, B_{m-1}$ the estimate is correct

$$\|v^{(m)}\| \leqslant B_m\|L^{(m+1)/2}v\| + \sum_{i=0}^{m-1} B_i\|v^{(i)}\|. \tag{18}$$

By virtue of the induction assumption, all expressions $\|v^{(i)}\|$, $i = 0, \dots, m-1$ included in the right-hand side of (18) are evaluated in terms of linear combinations of the values $\|L^k v\|$, $k = 0, \dots, (m-1)/2$. So, from (18) it follows that for some nonnegative constants $C_0, \dots, C_{(m+1)/2}$ the following estimate of the type (15) is true, that we need:

$$\|v^{(m)}\| \leqslant \sum_{k=0}^{(m+1)/2} C_k\|L^k v\|.$$

b) Let m be even. Then, according to point c) of the lemma 5.2 for $q = m/2$ and some nonnegative constants $A_0, \dots, A_{m-1}$, we have:

$$\|v^{(m)}\| \leqslant \Big\|\frac{1}{a^{m/2}}\Big\| \cdot \|L^{m/2}v\| + \sum_{i=0}^{m-1} A_i\|v^{(i)}\|. \tag{19}$$

By virtue of the induction assumption, all expressions $\|v^{(i)}\|$, $i = 0, \dots, m-1$ included in the right-hand side of (19), can be evaluated by linear combinations of the values $\|L^k v\|$, $k = 0, \dots, m/2$. So, from (19) it follows that for some nonnegative constants $C_0, \dots, C_{m/2}$ the following estimate of the type (15) is true, that we need:

$$\|v^{(m)}\| \leqslant \sum_{k=0}^{m/2} C_k \|L^k v\|.$$

So the induction step is done. The inequality (15), and with it the theorem 5.1 are proved. $\square$

5.2. Usage of estimates of the norms of derivatives via the norms of degrees of a second-order differential operator

Now, using theorems 4.1 and 5.1, as well as the results of the book [10], we prove a theorem on the approximation of solutions of second-order parabolic equations via Chernoff function.

Let denote by $HC_b(\mathbb{R})$ the space of all Hölder continuous functions $u\colon \mathbb{R} \to \mathbb{R}$, and for any $n \in \{1, 2, 3, \dots\}$ let denote by $HC_b^n(\mathbb{R})$ the space of all such functions $u \in HC_b(\mathbb{R})$, that $u', \dots, u^{(n)} \in HC_b(\mathbb{R})$. It is clear that $HC_b^n(\mathbb{R}) \subset UC_b(\mathbb{R})$ and $HC_b^n(\mathbb{R})$ is dense in $UC_b(\mathbb{R})$ for any $n \in \{1, 2, 3, \dots\}$. Similarly, the space $UC_b^n(\mathbb{R})$ is defined.

Theorem 5.2. *Suppose that*

1. *Numbers $m, q \in \{1, 2, 3, \dots\}$ are fixed, and $\hat{q} = 2\lfloor (q+1)/2 \rfloor$. Functions a, b, c from the class $HC_b^{2m+\hat{q}-2}(\mathbb{R})$ are given, and $\inf_{x\in\mathbb{R}} a(x) > 0$. Operator L on $UC_b(\mathbb{R})$ with domain $D(L) = HC_b^2(\mathbb{R})$ is defined by the formula $Lu = au'' + bu' + cu$.*
2. *Numbers $T > 0$, $M \geq 1$ and $\sigma \geq 0$ are given. For any $t \in (0, T]$ a bounded linear operator $S(t)$ on $UC_b(\mathbb{R})$ is defined such that $\|S(t)^k\| \leqslant Me^{k\sigma t}$ for any $k = 1, 2, 3, \dots$.*
3. *There exist nonnegative constants $K_0, K_1, \dots, K_{2m+q}$ such that for all $t \in (0, T]$ and all $f \in UC_b^{2m+q}(\mathbb{R})$ we have*

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k L^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{2m+q} K_j \|f^{(j)}\|. \tag{20}$$

Then:

1. *The closure $\overline{L}$ of operator L is a generator of C_0-semigroup $(e^{t\overline{L}})_{t\geqslant 0}$ in Banach space $UC_b(\mathbb{R})$, and the condition $\|e^{t\overline{L}}\| \leqslant e^{\gamma t}$ for all $t \geq 0$ is satisfied, where $\gamma = \max(0, \sup_{x\in\mathbb{R}} c(x))$.*

2. *For all $t > 0$, all integer $n \geq t/T$ and all $f \in UC_b^{2m+\hat{q}}(\mathbb{R})$ we have*

$$\|S(t/n)^n f - e^{t\overline{L}} f\| \leq \frac{Mt^{m+1}e^{wt}}{n^m} \sum_{j=0}^{2m+\hat{q}} C_j \|f^{(j)}\|, \tag{21}$$

where $w = max(\sigma, \gamma)$, $\hat{q} = 2\lfloor (q+1)/2 \rfloor$ and $C_0, C_1, \ldots, C_{2m+\hat{q}}$ are nonnegative constants that are independent of t and n.

3. *For all $g \in UC_b(\mathbb{R})$ and all $\mathcal{T} > 0$ the following equality is true:*

$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t \in (0, \mathcal{T}]} \left\| S(t/n)^n g - e^{t\overline{L}} g \right\| = 0. \tag{22}$$

Proof.

1. Using theorem 8.2.1 on p. 111 and corollary 8.3.1 on p. 114 of the book [10] for linear operator $(L-\gamma)u = au'' + bu' + (c-\gamma)u$, we see that the closure $\overline{L} - \gamma$ of operator $L - \gamma$ is a generator of C_0-semigroup $(e^{t(\overline{L}-\gamma)})_{t \geqslant 0}$ in Banach space $UC_b(\mathbb{R})$, and the condition $\|e^{t(\overline{L}-\gamma)}\| \leqslant 1$ for all $t \geq 0$ is satisfied. Hence the first statement of the theorem follows.
2. It follows from theorem 5.1, that for any $j = 0, \ldots, 2m+q$ there are nonnegative constants $C_{j,0}, C_{j,1}, \ldots, C_{j,\lfloor (j+1)/2 \rfloor}$, such that for any $f \in UC_b^{2m+\hat{q}}(\mathbb{R})$ we have

$$\|f^{(j)}\| \leqslant \sum_{k=0}^{\lfloor (j+1)/2 \rfloor} C_{j,k} \|L^k f\|.$$

From here and from (20), the relations follow

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k L^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{2m+q} \sum_{k=0}^{\lfloor (j+1)/2 \rfloor} K_j C_{j,k} \|L^k f\| =$$

$$= t^{m+1} \sum_{k=0}^{m+\lfloor (q+1)/2 \rfloor} A_k \|L^k f\|,$$

where $A_0, A_1, \ldots, A_{m+\lfloor (q+1)/2 \rfloor}$ are nonnegative constants.
So, taking into account point 2 of condition and proved point 1 of this theorem, we see that all the conditions of theorem 4.1 are met with

$D = UC_b^{2m+\hat{q}}(\mathbb{R})$. Then, it follows from inequality (9) of theorem 4.1 that

$$\|S(t/n)^n f - e^{t\overline{L}} f\| \leq \frac{Mt^{m+1}e^{wt}}{n^m} \sum_{k=0}^{m+\lfloor (q+1)/2 \rfloor} B_k(t/n)\|L^j f\|,$$

for any $t > 0$, $n \geq t/T$ and $f \in UC_b^{2m+\hat{q}}$, where $B_k(t) = A_k e^{-wt} \leq A_k$ for $k \neq m+1$ and $B_{m+1}(t) = A_{m+1}e^{-wt} + 1/(m+1)! \leq A_{m+1} + 1/(m+1)!$. From here and from point b) of lemma 5.2, it follows that for some nonnegative constants $C_0, C_1, \ldots, C_{2m+\hat{q}}$ independing of t and n, the inequality (21) holds, that we are proving:

$$\|S(t/n)^n f - e^{t\overline{L}} f\| \leq \frac{Mt^{m+1}e^{wt}}{n^m} \sum_{j=0}^{2m+\hat{q}} C_j \|f^{(j)}\|.$$

3. Equality (22) follows from the reasoning of the previous point of the proof, as well as from the theorems 4.1 and 5.1.

□

Here is an example of using theorem 5.2.

Example 5.2. *Let $a, b, c \in HC_b^2(\mathbb{R})$ and $\inf_{x\in\mathbb{R}} a(x) > 0$. For each $u \in UC_b^2(\mathbb{R})$ set*

$$Lu = au'' + bu' + cu \tag{23}$$

and for each $t \geq 0$, each $f \in UC_b(\mathbb{R})$ and each $x \in \mathbb{R}$ set

$$\begin{aligned}(S(t)f)(x) = \frac{1}{4} f\big(x + 2\sqrt{a(x)t}\big) + \frac{1}{4} f\big(x - 2\sqrt{a(x)t}\big) + \\ + \frac{1}{2} f\big(x + 2b(x)t\big) + tc(x)f(x).\end{aligned} \tag{24}$$

Then there exist nonnegative constants $C_0, C_1, \ldots, C_4$ such that for any $n \in \mathbb{N}$, $t > 0$ and any $f \in UC_b^4(\mathbb{R})$ the following inequality holds:

$$\begin{aligned}&\|[S(t/n)]^n f - e^{t\overline{L}} f\| \leq \\ &\leq \frac{t^2 e^{\|c\|t}}{n} (C_0\|f\| + C_1\|f'\| + C_2\|f''\| + C_3\|f'''\| + C_4\|f^{(IV)}\|).\end{aligned}$$

Proof. 1) Set $m = 1$, $q = 2$. Then $\hat{q} = 2$ and item 1 of the condition of the theorem 5.2 is met.

2) Let us estimate the norm $\|S(t)f\|$ for any $t > 0$ and $f \in UC_b(\mathbb{R})$ using (24):

$$\|S(t)f\| \le \frac{1}{4}\sup_{x\in\mathbb{R}}|f\big(x+2\sqrt{a(x)t}\big)| + \frac{1}{4}\sup_{x\in\mathbb{R}}|f\big(x-2\sqrt{a(x)t}\big)| +$$

$$+\frac{1}{2}\sup_{x\in\mathbb{R}}|f\big(x+2b(x)t\big)| + t\sup_{x\in\mathbb{R}}|c(x)|\cdot\sup_{x\in\mathbb{R}}|f(x)| =$$

$$= \frac{1}{4}\|f\| + \frac{1}{4}\|f\| + \frac{1}{2}\|f\| + t\|c\|\|f\| = (1+t\|c\|)\|f\| \le e^{\|c\|t}\|f\|.$$

So $\|S(t)\| \le e^{\|c\|t}$ and $\|S(t)^k\| \le e^{k\|c\|t}$ for any $t > 0$ and $k \in \{1, 2, 3, \ldots\}$. Then item 2 of the condition of the theorem 5.2 is met with $M = 1$, $\sigma = \|c\|$ and any $T > 0$.

3) Let us take any function $f \in UC_b^4(\mathbb{R})$ and expand $[S(t)f](x)$ in powers of $t > 0$, using Taylor's formula with remainders in Lagrange's form.

We have:

$$f(x+2\sqrt{a(x)t}) = f(x) + f'(x)\cdot 2\sqrt{a(x)t} + \frac{1}{2}f''(x)\cdot(2\sqrt{a(x)t})^2 +$$

$$+\frac{1}{6}f'''(x)\cdot(2\sqrt{a(x)t})^3 + \frac{1}{24}f^{IV}(\xi_1)\cdot(2\sqrt{a(x)t})^4;$$

$$f(x-2\sqrt{a(x)t}) = f(x) - f'(x)\cdot 2\sqrt{a(x)t} + \frac{1}{2}f''(x)\cdot(2\sqrt{a(x)t})^2 -$$

$$-\frac{1}{6}f'''(x)\cdot(2\sqrt{a(x)t})^3 + \frac{1}{24}f^{IV}(\xi_2)\cdot(2\sqrt{a(x)t})^4;$$

$$f(x+2b(x)t) = f(x) + f'(x)\cdot 2b(x)t + \frac{1}{2}f''(\xi_3)\cdot(2b(x)t)^2.$$

Then, using this three equalities together with (24) we get the following expression for $[S(t)f](x)$:

$$[S(t)f](x) = f(x) + t[a(x)f''(x) + b(x)f'(x) + c(x)f(x)] +$$
$$+t^2\Big[\frac{(a(x))^2}{6}(f^{IV}(\xi_1) + f^{IV}(\xi_2)) + (b(x))^2 f''(\xi_3)\Big]. \tag{25}$$

So, taking into account the formula (23), we come to the inequality

$$\|S(t)f - (f + tLf)\| \le t^2\Big(\frac{\|a\|^2}{3}\|f^{IV}\| + \|b\|^2\|f''\|\Big).$$

Then last item (item 3) of the condition of the theorem 5.2 is met.

4) Further, using item 2 of the theorem 5.2, we get that the estimate

$$\|[S(t/n)]^n f - e^{t\overline{L}} f\| \leq$$
$$\leq \frac{t^2 e^{\|c\| t}}{n}(C_0\|f\| + C_1\|f'\| + C_2\|f''\| + C_3\|f'''\| + C_4\|f^{(IV)}\|),$$

is true for some nonnegative constants C_0, C_1, C_2, C_3, C_4. □

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References

[1] E. Hille, R. S. Phillips: Functional Analysis and Semi-Groups. — American Mathematical Society, 1975.

[2] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations. — Springer-Verlag, New York, 1983.

[3] K.-J. Engel, R. Nagel. One-Parameter Semigroups for Linear Evolution Equations. — Springer, 2000.

[4] Paul R. Chernoff, Note on product formulas for operator semigroups. — J. Funct. Anal. 2:2 (1968) 238-242.

[5] I.D. Remizov. Quasi-Feynman formulas — a method of obtaining the evolution operator for the Schrödinger equation. — Journal of Functional Analysis 270:12 (2016) 4540-4557

[6] A. V. Vedenin, V. S. Voevodkin, V. D. Galkin, E. Yu. Karatetskaya, I. D. Remizov. Speed of Convergence of Chernoff Approximations to Solutions of Evolution Equations. — Math. Notes, 108:3 (2020), 451-456

[7] V.I. Bogachev, O.G. Smolyanov. Real and Functional Analysis. — Springer, 2020.

[8] V.K. Dzyadyk, I.A. Shevchuk. Theory of Uniform Approximation of Functions by Polynomials. — De Gruyter, 2008.

[9] I.D. Remizov. Solution-giving formula to Cauchy problem for multidimensional parabolic equation with variable coefficients. — Journal of Mathematical Physics, 60:7 (2019), 071505.

[10] N.V. Krylov. Lectures on Elliptic and Parabolic Equations in Holder Spaces. — Graduate Texts in Mathematics. Vol. 12. (American Mathematical Society, 1996).