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MATHEMATICS

Application of Special Function Spaces to the Study of Nonlinear Integral Equations Arising in Equilibrium Spatial Logistic Dynamics

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Abstract—In this paper, we study a nonlinear integral equation that arises in a model of spatial logistic dynamics. The solvability of this equation is investigated by introducing special spaces of functions that are integrable up to a constant. Sufficient conditions for the biological characteristics and the parameters of the third spatial moment closure are established that guarantee the existence of the solution of the equation described above in some ball centered at zero. In addition, it is shown that this solution is unique in the considered ball and not zero. This means that, under appropriate conditions, the equilibrium state of the population of a certain species exists and does not coincide with the state of extinction.

Keywords: functional analysis, nonlinear integral equations, mathematical biology

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The subject of this paper is a parametric family of nonlinear integral equations arising in the closure of the third moment in the spatial logistic model developed by Dieckmann and Law [1, 2]. A brief description of this model and the mathematical formulation of the problem under consideration are given in Section 1. In Section 2, we introduce a special function space and present some operators acting in it. In Section 3, the considered operators are used to construct a parametric mapping acting in the above-mentioned function space, whose fixed points coincide with solutions of the studied equations. Additionally, sufficient conditions for the existence of fixed points of this mapping and their uniqueness in some ball for various sets of parameters are indicated.

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1. DESCRIPTION OF THE PROBLEM

1.1. Model of Biological Communities

Consider a population of stationary organisms inhabiting the space $\mathbb{R}^k$. The model is characterized by the following spatially homogeneous biological parameters:

(i) natural death rate ($d \geq 0$),

(ii) aggression of individuals ($d' \geq 0$),

(iii) birth rate ($b > 0$),

(iv) dispersal kernel ($m = m(x)$),

(v) competition kernel ($w = w(x)$).

Here, the dispersal and competition kernels are nonnegative, radially symmetric integrable functions with the L_1-norm equal to 1 that vanish at infinity. The dispersal kernel is the probability density function of a random variable determining the position of descendants with respect to their parents. The competition kernel describes the spatial structure of competition between individuals.

At every time, the state of the population under study is characterized by spatial moments, which are averages of certain statistical characteristics. We consider the first three moments:

(i) $N(t)$ is the mean density of individuals;

(ii) $C(x,t)$ is the mean density of pairs of individuals, where x is the shift of the second individual with respect to the first one;

(iii) $T(x,y,t)$ is the mean density of triplets of individuals, where x and y are the respective shifts of the second and third individuals with respect to the first one.

In this paper, we work with an equilibrium state of the population characterized by a stationary point of the system (thus, the moments are independent of t). It is described by the following system of integral equations (for more details, see [2]):

$$0 = (b-d)N - d'\int_{\mathbb{R}^k} C(x)w(x)dx,$$
$$0 = bm(x)N + \int_{\mathbb{R}^k} bm(y)C(x+y)dy \quad (1)$$
$$-(d + d'w(x))C(x) - \int_{\mathbb{R}^k} d'w(y)T(x,y)dy.$$

1.2. Equilibrium Equation

In this work, we consider a three-parameter family of closures of the third moment:

$$T(x,y) = \frac{\alpha C(x)C(y) + \beta C(x)C(y-x) + \gamma C(y)C(y-x) - \beta N^4}{(\alpha+\gamma)N}, \quad (2)$$

where $\alpha,\beta,\gamma \in \mathbb{R}$ and $\alpha+\gamma \neq 0$. This family is used to reduce the number of unknowns in system (1) (for more details on the closure method, see [3]). Substituting closure (2) into system (1), after some algebra, we obtain

$$Q = \frac{\bar{m} + [\bar{m} * Q] - \bar{w}Q - \frac{\beta}{\alpha+\gamma}Q[\bar{w} * Q] - \frac{\gamma}{\alpha+\gamma}[Q\bar{w} * Q] + \frac{\beta d'}{\alpha+\gamma}N^2}{d + \frac{\alpha(b-d)}{\alpha+\gamma}}, \quad (3)$$

where $Q = C/N$, $\bar{m} = bm$, and $\bar{w} = d'w$. For notational brevity, all arguments of the functions are omitted. Here, the notation $[f * g]$ denotes an integral of the form

$$\int_{\mathbb{R}^k} f(x-y)g(y)dy.$$

Equation (3) is called an equilibrium equation. Note that

$$\lim_{\|x\|_{\mathbb{R}^k}\to+\infty} Q(x) = N, \quad (4)$$

since it was shown in [3] that

$$\lim_{\|x\|_{\mathbb{R}^k}\to+\infty} C(x) = N^2.$$

2. SPACE $\widehat{L_1}(\mathbb{R}^k)$

2.1. Definition

Before beginning a further study of the equilibrium equation, we consider a special function space that contains, as will be shown later, the solution of Eq. (3).

Consider the set of functions of the form $f = F + n$, where $F \in L_1(\mathbb{R}^k)$ and $n \in \mathbb{R}$. In what follows, F is called the functional part of the element f and is denoted by $\mathcal{F}f$, while n is called the numerical part of f and is denoted by $\mathcal{N}f$. Obviously, this set is linear with respect to addition and multiplication by a number. On the set, we introduce a normed space structure with the norm defined as

$$\|f\|_{\widehat{L_1}} = \|\mathcal{F}f\|_{L_1} + |\mathcal{N}f|.$$

The resulting space is denoted by $\widehat{L_1}(\mathbb{R}^k)$.

Remark 1. Elements f and g of the space $\widehat{L_1}(\mathbb{R}^k)$ are equal if and only if their functional and numerical parts are equal, respectively.

Lemma 1. *The space* $\widehat{L_1}(\mathbb{R}^k)$ *is Banach.*

2.2. Some Operators in $\widehat{L_1}(\mathbb{R}^k)$

Consider some operators acting in $\widehat{L_1}(\mathbb{R}^k)$ that are of greatest interest in this work.

The convolution operator $\mathcal{C}_\varphi$ acting on functions from $\widehat{L_1}(\mathbb{R}^k)$ is defined as

$$\mathcal{C}_\varphi f = [\varphi * f] = \int_{\mathbb{R}^k} \varphi(x-y)f(y)dy,$$

where $\varphi \in L_1(\mathbb{R}^k)$.

Lemma 2. *The operator* $\mathcal{C}_\varphi$ *is a bounded linear operator acting in* $\widehat{L_1}(\mathbb{R}^k)$ *with the norm* $\|\varphi\|_{L_1}$.

We introduce the auxiliary function space

$$BL_1(\mathbb{R}^k) = \{f \in L_1(\mathbb{R}^k) \mid \operatorname*{ess\,sup}_{\mathbb{R}^k}|f| < +\infty\},$$

equipped with the norm

$$\|f\|_{BL_1} = \max\{\|f\|_{L_1}, \operatorname*{ess\,sup}_{\mathbb{R}^k}|f|\}.$$

The self-convolution operator $\mathcal{S}_\varphi$ acting on functions from $\widehat{L_1}(\mathbb{R}^k)$ is defined as

$$\mathcal{S}_\varphi f = [f\varphi * f] = \int_{\mathbb{R}^k} f(x-y)\varphi(x-y)f(y)dy,$$

where $\varphi \in BL_1(\mathbb{R}^k)$. It can be shown that this operator acts in $\widehat{L_1}(\mathbb{R}^k)$. Moreover, the following result is true.

Lemma 3. *For any pair of elements $f, g \in \widehat{L_1}(\mathbb{R}^k)$,*

$$\|\mathcal{S}_\varphi f - \mathcal{S}_\varphi g\|_{\widehat{L_1}} \le \|\varphi\|_{BL_1}(\|f\|_{\widehat{L_1}} + \|g\|_{\widehat{L_1}})\|f-g\|_{\widehat{L_1}}.$$

The convolution product operator $\mathcal{P}_\varphi$ acting from $\widehat{L_1}(\mathbb{R}^k)$ is defined as

$$\mathcal{P}_\varphi f = f[\varphi * f] = f(x)\int_{\mathbb{R}^k} \varphi(x-y)f(y)dy,$$

where $\varphi \in BL_1(\mathbb{R}^k)$. This operator also acts in $\widehat{L_1}(\mathbb{R}^k)$, and the following result is true, which is similar to Lemma 3.

Lemma 4. *For any pair of elements $f, g \in \widehat{L_1}(\mathbb{R}^k)$,*

$$\|\mathcal{P}_\varphi f - \mathcal{P}_\varphi g\|_{\widehat{L_1}} \le \|\varphi\|_{BL_1}(\|f\|_{\widehat{L_1}} + \|g\|_{\widehat{L_1}})\|f-g\|_{\widehat{L_1}}.$$

3. EXISTENCE AND UNIQUENESS OF A SOLUTION TO THE EQUILIBRIUM EQUATION

3.1. Equilibrium Operator

Now we are ready for a further study of equilibrium equation (3). In what follows, the birth and competition kernels are additionally assumed to be almost everywhere bounded. In this case, they belong to the class $BL_1(\mathbb{R}^k)$.

A solution of Eq. (3) is sought in the space $\widehat{L_1}(\mathbb{R}^k)$. In this case, in view of condition (4), we can conclude that $N = \mathcal{N}Q$. Taking this into account, we rewrite the equation in the operator form

$$Q = \mathcal{A}Q, \tag{5}$$

where the operator $\mathcal{A}$ acts from $\widehat{L_1}(\mathbb{R}^k)$ according to the rule

$$\mathcal{A}f = \frac{\bar{m} + [\bar{m} * f] - \bar{w}f - \dfrac{\beta}{\alpha+\gamma}f[\bar{w} * f] - \dfrac{\gamma}{\alpha+\gamma}[f\bar{w} * f] + \dfrac{\beta d'}{\alpha+\gamma}(\mathcal{N}f)^2}{d + \dfrac{\alpha(b-d)}{\alpha+\gamma}}. \tag{6}$$

With the help of the operators introduced earlier, representation (6) can be rewritten in the form

$$\mathcal{A}f = \frac{\bar{m} + \mathcal{C}_{\bar{m}}f - \bar{w}f - \dfrac{\beta}{\alpha+\gamma}\mathcal{P}_{\bar{w}}f - \dfrac{\gamma}{\alpha+\gamma}\mathcal{S}_{\bar{w}}f + \dfrac{\beta d'}{\alpha+\gamma}(\mathcal{N}f)^2}{d + \dfrac{\alpha(b-d)}{\alpha+\gamma}}. \tag{7}$$

Note that $\mathcal{A}$ acts in $\widehat{L_1}(\mathbb{R}^k)$, since the operators $\mathcal{C}_{\bar{m}}$, $\mathcal{P}_{\bar{w}}$, and $\mathcal{S}_{\bar{w}}$ act in this space. In fact, the problem of solving Eq. (3) has been reduced to the problem of finding a fixed point of operator (7). This operator will be called the equilibrium operator.

3.2. Fixed Point of the Equilibrium Operator

Taking into account Lemmas 2–4, we can estimate how strongly the distance between two elements of the space $\widehat{L_1}(\mathbb{R}^k)$ changes under the action of the operators $\mathcal{C}_{\bar{m}}$, $\mathcal{S}_{\bar{w}}$, and $\mathcal{P}_{\bar{w}}$. With the use of this result, it is easy to find sufficient conditions for the contraction of the operator $\mathcal{A}$ in some ball B of the space $\widehat{L_1}(\mathbb{R}^k)$. By estimating $\|\mathcal{A}\|_{\widehat{L_1}}$ for $f \in B$, it is possible to find a closed ball $B' \subset B$ that is invariant under the equilibrium operator. However, a closed ball of a complete metric space is itself a complete metric space. In view of this, by using the Banach principle, we can prove that the operator $\mathcal{A}$ has a unique fixed point in B'.

This argument leads to the following result.

Theorem 1. *If*

$$\gamma < 0,$$
$$\alpha b + \gamma d > 0,$$
$$2\beta - \gamma > 0,$$

$$b - d > \frac{\alpha + \gamma}{-\gamma}\|\overline{w}\|_{BL_1},$$

$$b - d > \frac{4b(2\beta - \gamma)}{-\gamma},$$

and a positive number R satisfies the system of inequalities

$$\begin{aligned} &-\frac{\alpha + \gamma}{4\beta - 2\gamma} \le R < -\frac{\gamma(b-d)}{\|\overline{w}\|_{BL_1}(4\beta - 2\gamma)} - \frac{\alpha + \gamma}{4\beta - 2\gamma}, \\ &\frac{\gamma(b-d) - \sqrt{D}}{2\|\overline{w}\|_{BL_1}(2\beta - \gamma)} \le R \le \frac{\gamma(b-d) + \sqrt{D}}{2\|\overline{w}\|_{BL_1}(2\beta - \gamma)}, \end{aligned} \tag{8}$$

where

$$D = \gamma^2(b-d)^2 - 4b(\alpha + \gamma)(2\beta - \gamma)\|\overline{w}\|_{BL_1},$$

then equilibrium operator (6) *has a unique fixed point in the ball of radius R centered at the origin*.

Remark 2. Under the conditions of Theorem 1, the fixed point of the equilibrium operator is nonzero, since the image of the zero element of $\widehat{L_1}(\mathbb{R}^k)$ under the operator $\mathcal{A}$ is nonzero.

Below is an example of model parameters satisfying the conditions of Theorem 1. Suppose that $\alpha = 1/2$, $\beta = -7/16$, and $\gamma = -1$. Then

$$\gamma = -1 < 0,$$

$$2\beta - \gamma = \frac{1}{8} > 0.$$

If we choose $b = 1$ and $d = 1/10$, then

$$\alpha b + \gamma d = \frac{1}{2} - \frac{1}{10} = \frac{2}{5} > 0,$$

$$b - d = \frac{9}{10} > \frac{1}{2} = \frac{4b(2\beta - \gamma)}{-\gamma}.$$

The competition kernel is chosen in the form of the normal distribution density with zero expectation and standard deviation equal to $(10\sqrt{2\pi})^{-1}$, i.e.,

$$w(x) = 10e^{-100\pi x^2},$$

Additionally, we use $s = 1$. Then $\|\overline{w}\|_{BL_1} = 10$. Therefore,

$$b - d = \frac{9}{10} > -5 = \frac{\alpha + \gamma}{-\gamma}\|\overline{w}\|_{BL_1}.$$

With this choice, D is equal to 331/100; hence,

$$\frac{\gamma(b-d) + \sqrt{D}}{\|\overline{w}\|_{BL_1}(2\beta - \gamma)} = \frac{-1 \cdot 9/10 + \sqrt{331/100}}{10 \cdot 1/8}$$

$$= \frac{-72 + 8\sqrt{331}}{100} > \frac{-72 + 8 \cdot 18}{100} = \frac{36}{50}.$$

Moreover,

$$-\frac{\alpha + \gamma}{4\beta - 2\gamma} = \frac{2}{5} < \frac{36}{50},$$

i.e.,

$$-\frac{\alpha + \gamma}{4\beta - 2\gamma} < \frac{\gamma(b-d) + \sqrt{D}}{\|\overline{w}\|_{BL_1}(2\beta - \gamma)}. \tag{9}$$

On the other hand,

$$\frac{\gamma(b-d) - \sqrt{D}}{2\|\overline{w}\|_{BL_1}(2\beta - \gamma)} = \frac{-72 - 8\sqrt{331}}{100} < 0,$$

and

$$-\frac{\gamma(b-d)}{\|\overline{w}\|_{BL_1}(4\beta - 2\gamma)} - \frac{\alpha + \gamma}{4\beta - 2\gamma} = \frac{72}{100} + \frac{2}{5} > 0;$$

therefore,

$$\frac{\gamma(b-d) - \sqrt{D}}{2\|\overline{w}\|_{BL_1}(2\beta - \gamma)} < -\frac{\gamma(b-d)}{\|\overline{w}\|_{BL_1}(4\beta - 2\gamma)} - \frac{\alpha + \gamma}{4\beta - 2\gamma}. \tag{10}$$

It follows from (9) and (10) that system (8) is solvable. Now if R is set equal, for example, to 2/5, then all the conditions of Theorem 1 are satisfied.

4. CONCLUSIONS

We have analyzed the existence and uniqueness of a solution to the problem of finding an equilibrium state of a population of organisms. It was shown that a solution of system (1) can sought in the form of a fixed point of some operator acting in a special function space. With the help of the Banach principle, we found sufficient conditions imposed on the biological parameters of the models and parameters of the closure of the third spatial moment that guarantee the existence and uniqueness of a fixed point of this operator in a ball centered at the origin. Moreover, it was shown that this state is nontrivial.

Previously, nonlinear integral equations obtained by closing the third moment were analyzed only numerically. An analytical study of such equations under the three-parameter closure (2) in the case of nonzero α, β, and γ has been performed for the first time. Additionally, the results of this paper illustrate the importance of choosing suitable parameters of closure (2), since it was shown in [4, 5] that, for $\alpha = 1$ and $\beta = a = 0$, a nontrivial equilibrium state exists only for $d = 0$.

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