Existence of an energy function for three-dimensional chaotic “sink-source” cascades

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M. Barinova, V. Grines, O. Pochinka, and B. Yu

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M. Barinova,[1,a] V. Grines,[1,b] O. Pochinka,[1,c] and B. Yu[2,d]

AFFILIATIONS

[1]Faculty of Informatics, Mathematics, and Computer Science (HSE Nizhny Novgorod)/Department of Fundamental Mathematics, National Research University Higher School of Economics, Nizhny Novgorod 603155, Russia
[2]Department of Mathematics, Tongji University, Shanghai 200092, China

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[a]**Electronic mail:** mkbarinova@yandex.ru
[b]**Electronic mail:** vgrines@yandex.ru
[c]**Author to whom correspondence should be addressed:** olga-pochinka@yandex.ru
[d]**Electronic mail:** binyu1980@gmail.com

ABSTRACT

This paper is a continuation of research in the direction of energy function (a smooth Lyapunov function whose set of critical points coincides with the chain recurrent set of a system) construction for discrete dynamical systems. The authors established the existence of an energy function for any A-diffeomorphism of a three-dimensional closed orientable manifold whose non-wandering set consists of a chaotic one-dimensional canonically embedded surface attractor and repeller.

A Lyapunov function is a continuous function that is constant on the chain components and decreases along trajectories outside them. It is an important characteristic of a dynamical system, initially introduced by A. M. Lyapunov to research the stability of equilibriums of differential equation systems. In modern dynamics, it plays a significant role. The fundamental theorem of dynamical systems, proved by K. Conley, establishes the existence of a Lyapunov function for every dynamical system. If the Lyapunov function is smooth and the set of its critical points coincides with the chain recurrent set of the dynamical system, then it is called an energy function. This function is similar to the energy function associated with a dissipative system in mechanics. Unlike flows, diffeomorphisms do not have an energy function in general. The first such example was constructed by D. Pixton, and it was a Morse–Smale diffeomorphism on a 3-sphere. Necessary and sufficient conditions for the existence of an energy function for an arbitrary Morse–Smale 3-diffeomorphism were found by V. Z. Grines, F. Laudenbach, and O. V. Pochinka. For systems with chaotic dynamics, first, constructions of energy functions were done by M. Barinova, V. Grines, and O. Pochinka for *A*-diffeomorphisms with basic sets of co-dimension one on two- and three-manifolds. This paper is a development of results for co-dimension two basic sets of 3-diffeomorphisms. A class of chaotic dynamical systems with the dynamics "one-dimensional attractor–repeller" is considered in the paper. A construction of an energy function for such a system is provided.

I. INTRODUCTION AND STATEMENT OF RESULTS

Let f be a diffeomorphism given on a smooth closed n-manifold M^n. One says that f *satisfies axiom A* if its non-wandering set is hyperbolic and the periodic points are dense in it. In this case, we say that f is an A-diffeomorphism. For A-diffeomorphisms, the Smale spectral decomposition theorem[1] holds; that is, a non-wandering set is a union of a finite number of pairwise disjoint sets called *basic sets*, each of which is compact, invariant, and topologically transitive.

A basic set Λ is called an *attractor* of an A-diffeomorphism f if it has a compact neighborhood U_Λ such that $f(U_\Lambda) \subset int U_\Lambda$ and $\Lambda = \bigcap_{k\geqslant 0} f^k(U_\Lambda)$. U_Λ is called a *trapping neighborhood* of Λ. A *repeller* is defined as an attractor for f^{-1}. By a *dimension* of an attractor (a repeller), we mean its topological dimension. The set $\bigcup_{k\in\mathbb{Z}} f^k(U_\Lambda)$ is called a *basin* of the attractor Λ.

By Theorem 3 in Ref. 2, each one-dimensional basic set of an A-diffeomorphism on a surface is an attractor or a repeller, and $\dim W^u_x = \dim W^s_x = 1$ for $x \in \Lambda$. The trapping neighborhood for such a basic set is a surface with a boundary (see Sec. II). Therefore, it can be naturally included to a three-dimensional dynamics.

Definition 1 A connected one-dimensional attractor A of A-diffeomorphism $f : M^3 \to M^3$ is called a canonically embedded surface attractor if

- A has a trapping neighborhood U_A of the form $S_A \times [-1, 1]$, where $S_A = S_A \times \{0\}$ is a surface with a boundary and $A \subset int\, S_A$;
- S_A is a trapping neighborhood of A as an attractor for the diffeomorphism $g = f|_{S_A} : S_A \to f(S_A)$; and
- diffeomorphism $f|_{U_A} : U_A \to f(U_A)$ is topologically conjugate to the diffeomorphism $\phi(w, z) = (g(w), z/2) : S_A \times [-1, 1] \to f(S_A) \times [-1/2, 1/2]$.

A one-dimensional repeller is called a *canonically embedded surface repeller* if it is a canonically embedded connected one-dimensional surface attractor for the diffeomorphism f^{-1}.

One naturally expects to construct a diffeomorphism on a closed three-manifold by a gluing dynamics from a canonically embedded surface one-dimensional attractor and repeller. In this paper, the problem was solved (see Sec. IV A) for the attractor and repeller models obtained by DA-surgeries from the same Anosov diffeomorphism. It is clear that the construction admits generalizations to the attractor and repeller obtained by DPA-surgeries from the same pseudo-Anosov diffeomorphism.

Theorem 1 *There are infinitely many pairwise Ω-non-conjugated 3-diffeomorphisms whose non-wandering sets are pairwise homeomorphic, and each of them is a union of a canonically embedded one-dimensional surface attractor and repeller.*

Note that similar "one-dimensional attractor–repeller" dynamics on a surface also always is not structurally stable due to the results by Robinson and Williams.[3] The construction of 3-diffeomorphisms with one-dimensional attractor–repeller (not surface) dynamics first was suggested in Ref. 4, and all examples were also not structurally stable. In Refs. 5 and 6, structurally stable examples with one-dimensional attractor–repeller dynamics were constructed. However, the question whether the basic sets are canonically embedded in surfaces or not remains open because the construction is very different from the one suggested in this paper. The surface dynamics allows us to prove the existence of an energy function for the examples constructed in this paper.

A *Lyapunov function* is a continuous function that is constant on the chain components and decreases along trajectories outside them. It is an important characteristic of a dynamical system, initially introduced by A. M. Lyapunov to research the stability of equilibriums of differential equation systems. In modern dynamics, it plays a significant role. The fundamental theorem of dynamical systems, proved by Conley[7] in 1978, establishes the existence of a Lyapunov function for every dynamical system.

If the Lyapunov function is smooth and the set of its critical points coincides with the chain recurrent set of the dynamical system, then it is called an *energy function*. In this case, many characteristics of a dynamical system directly follow from the properties of its energy function. Unlike flows, diffeomorphisms do not have an energy function in general. Moreover, counterexamples are known even for cascades with a regular dynamics.

The first such example was constructed by Pixton[8] in 1977, and it was a Morse–Smale diffeomorphism on a 3-sphere. Necessary and sufficient conditions for the existence of an energy function for an arbitrary Morse–Smale 3-diffeomorphism were found by V. Z. Grines, F. Laudenbach, and O. V. Pochinka in Ref. 9.

For systems with chaotic dynamics, first, constructions of energy functions were done in Refs. 10–12 for A-diffeomorphisms with basic sets of co-dimension one on two- and three-manifolds. This paper is a development of results for co-dimension one basic sets to co-dimension two basic sets of 3-diffeomorphisms.

Theorem 2 *Every A-diffeomorphism on a closed orientable 3-manifold M^3, whose non-wandering set is a union of a connected canonically embedded one-dimensional surface attractor and repeller, has an energy function.*

II. ONE-DIMENSIONAL BASIC SETS FOR DIFFEOMORPHISMS OF SURFACES

Let M^2 be a closed surface and $f : M^2 \to M^2$ be an A-diffeomorphism. In this section, we give a brief description of one-dimensional basic sets for diffeomorphisms of surfaces. For simplicity everywhere below, we assume that Λ is a connected attractor.

By Theorem 1 in Ref. 2, $\Lambda = \bigcup_{x\in\Lambda} W^u_x$. In addition, by Lemmas 2.1, 2.4, and 2.5 in Ref. 13, at least one of the connected components of the set $W^s_x \setminus \{x\}, x \in \Lambda$ contains a dense set in Λ, and there are a finite number of points $x \in \Lambda$ for which one of the connected components $W^{s\varnothing}_x$ of the set $W^s_x \setminus \{x\}$ does not intersect Λ. Such points are called an *s-boundary*, and their set is not empty and consists of a finite number of periodic points. The set $W^s_\Lambda \setminus \Lambda$ consists of a finite number of path-connected components.

Bunch b of the attractor Λ is the union of the unstable mani-

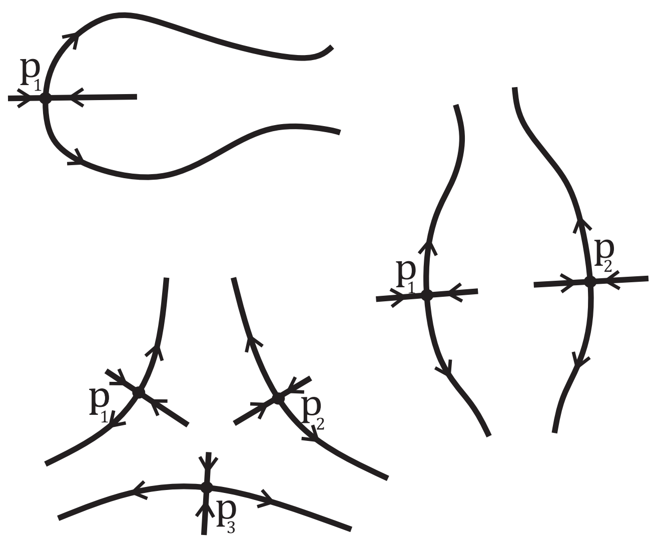

FIG. 1. Bunches of degrees 1, 2, and 3.

folds $W^u_{p_1}, \dots, W^u_{p_{r_b}}$ of s-boundary points $p_1, \dots, p_{r_b}$ of Λ for which $W^{s\varnothing}_{p_1}, \dots, W^{s\varnothing}_{p_{r_b}}$ belong to the same path-connected components of $W^s_\Lambda \setminus \Lambda$. The number r_b is called a *degree of the bunch* (see Fig. 1). Let B_Λ be the set of all bunches of the attractor Λ.

Let us explain how to construct a trapping neighborhood for Λ. A proof of the existence of a neighborhood with properties below is possible to find, for example, in Ref. 14.

The property $\dim W^s_x = \dim W^u_x = 1$, $x \in \Lambda$ allows one to introduce the notation $(y,z)^s$ ($(y,z)^u$) for the arc of a stable (unstable) manifold bounded by points y, z. Then, for the bunch $b \in B_\Lambda$, we choose a sequence of points $x_1, \dots, x_{2r_b}$ such that

- x_{2j-1}, x_{2j} belong to different connected components of the set $W^u_{p_j} \setminus p_j$;
- $x_{2j+1} \in W^s_{x_{2j}}$ (suppose $x_{2r_b+1} = x_1$); and
- $(x_{2j}, x_{2j+1})^s \cap \Lambda = \emptyset, j = 1, \dots, r_b$.

For $j \in \{1, \dots, r_b\}$, we choose a pair of points $\tilde{x}_{2j-1}$, $\tilde{x}_{2j}$ and a simple curve l_j with boundary points $\tilde{x}_{2j-1}$, $\tilde{x}_{2j}$ such that

- $(\tilde{x}_{2j}, \tilde{x}_{2j+1})^s \subset (x_{2j}, x_{2j+1})^s$;
- the curve l_j intersects transversally with a stable manifold of any point of the arc $(x_{2j-1}, x_{2j})^u$ exactly at one point; and
- the curve $L_b = \bigcup_{j=1}^{r_b}[l_j \cup (\tilde{x}_{2j}, \tilde{x}_{2j+1})^s]$ is a simple closed smooth curve and the set $L_\lambda = \bigcup_{b \in B_\lambda} L_b$ has the following properties:
  - $f(L_\lambda) \cap L_\lambda = \emptyset$;
  - for every curve L_b, $b \in B_\lambda$, there exists a curve from the set $f(L_\lambda)$ such that these two curves are the boundary of the two-dimensional annulus K_b; and
  - annulus $\{K_b,\ b \in B_\Lambda\}$ are pairwise disjoint (see Fig. 2).

Let $K_\Lambda = \bigcup_{b \in B_\Lambda} K_b$. It is directly verifiable that a surface with boundary $U_\Lambda = \bigcup_{k \in \mathbb{N}} f^k(K_\Lambda) \cup \Lambda$ is a trapping neighborhood of the attractor Λ.

III. CONSTRUCTION OF A CANONICALLY EMBEDDED SURFACE ATTRACTOR (REPELLER)

A. Anosov diffeomorphism on a 2-torus

Let $C \in SL(2, \mathbb{Z})$ be a hyperbolic matrix with eigenvalues λ_1, λ_2 such that $\lambda = |\lambda_1| > 1$ and $|\lambda_2| = 1/\lambda$. Since the matrix C has a determinant equal to 1, it induces a hyperbolic automorphism $\widehat{C} : \mathbb{T}^2 \to \mathbb{T}^2$ with a fixed point O. This diffeomorphism is an Anosov diffeomorphism possessing two transversal invariant foliations (stable and unstable) whose leaves are irrational windings on the torus.

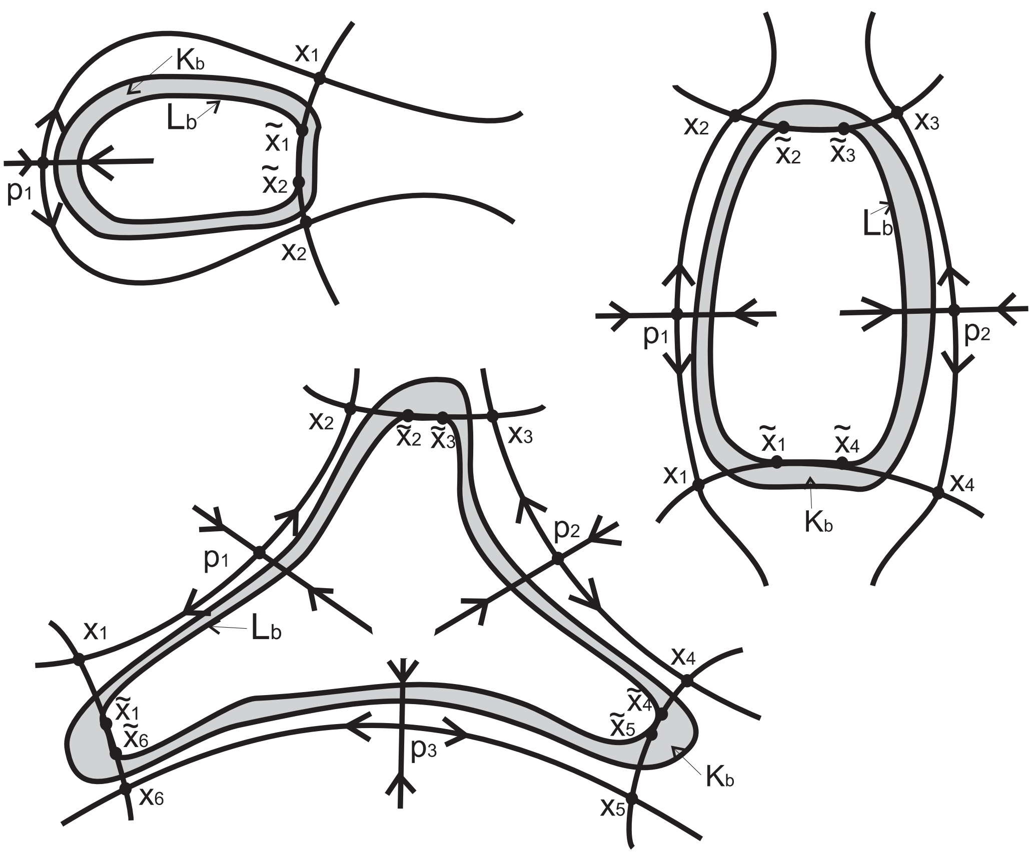

FIG. 2. Construction of a trapping neighborhood for Λ.

In addition, the set of periodic points of the diffeomorphism $\widehat{C}$ is dense on $\mathbb{T}^2$.

B. Smale surgery

S. Smale proposed the so-called "surgery" for obtaining a diffeomorphism with a one-dimensional attractor and a fixed source from the Anosov diffeomorphism. The idea is to take a composition of $\widehat{C}$ with an expansion near its fixed point, continuing by the identical map out of its small neighborhood. In this section, we make such a surgery with some specific properties to use them for the construction in Sec. IV A.

Let (x, y) be local coordinates in a neighborhood $U(O)$ of O on $\mathbb{T}^2$ such that the diffeomorphism $\widehat{C}$ in these coordinates has the form

$$\widehat{C}(x, y) = \left(x/\lambda, \lambda y\right).$$

Then, $Ox \subset W^s_O$ and $Oy \subset W^u_O$, as well as $\{y = const\}$ and $\{x = const\}$ are stable and unstable foliations of $\widehat{C}$. Let

$$V(O) = \{(x, y) \in U(O) : x^2 + y^2 \leqslant \lambda^{-6}\}.$$

Lemma 1 *There exists a diffeomorphism $\widehat{B}_R : \mathbb{T}^2 \to \mathbb{T}^2$ with the following properties:*

- *$\widehat{B}_R$ is identical on $\mathbb{T}^2 \setminus U(O)$;*
- *$\widehat{B}_R$ preserves the unstable foliation of $\widehat{C}$; and*
- *$\widehat{B}_R \circ \widehat{C}(x, y) = \left(\frac{x}{\lambda}, \frac{y}{\lambda}\right)$ on $V(O)$.*

Proof. Let $\sigma : \mathbb{R} \to [0, 1]$ be a sigmoid defined by the formula

$$\sigma(x) = \begin{cases} 0, & x \leqslant 0, \\ \dfrac{1}{1+\exp\left(\frac{\frac{1}{2}-x}{x^2(x-1)^2}\right)}, & 0 < x < 1, \\ 1, & x \geqslant 1 \end{cases}$$

and $\beta_{a,b} : \mathbb{R} \to \mathbb{R}$ be a linear function $\beta_{a,b}(x) = \frac{x-a}{b-a}$ for real numbers $a < b$. With the composition of this function

$$\mathcal{T}_{a,b} = \sigma \circ \beta_{a,b} : \mathbb{R} \to \mathbb{R},$$

which sends $[a, b]$ to $[0, 1]$, we will use for further construction (see Fig. 3 for $a = \lambda^{-3}$, $b = 1$). Let $\mu(x) = \mathcal{T}_{\lambda^{-3},1}(x)$. Define a diffeomorphism $\nu_R : \mathbb{R} \to \mathbb{R}$ (see Fig. 4) by the formula

$$\nu_R(x) = \lambda^{-2(1-\mu(|x|))}x.$$

Let $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 4\}$. Define a diffeomorphism $B_R : \mathbb{D}^2 \to \mathbb{D}^2$ by the formula

$$B_R(x, y) = (x, \nu_R(y)).$$

By construction, $B_R(x, y) = \left(x, \lambda^{-2}y\right)$ if $x^2 + y^2 \leqslant \lambda^{-6}$ and is identical on $\partial\mathbb{D}^2$. Let $\widehat{B}_R : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism that is B_R in $U(O)$ and is identical outside $U(O)$. □

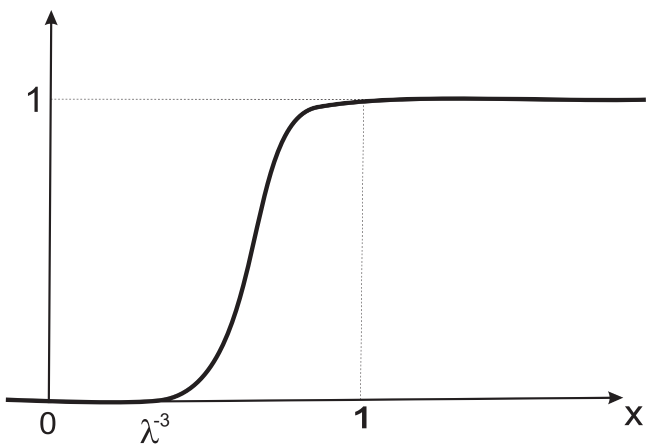

FIG. 3. Graph of the function $\mu(x)$.

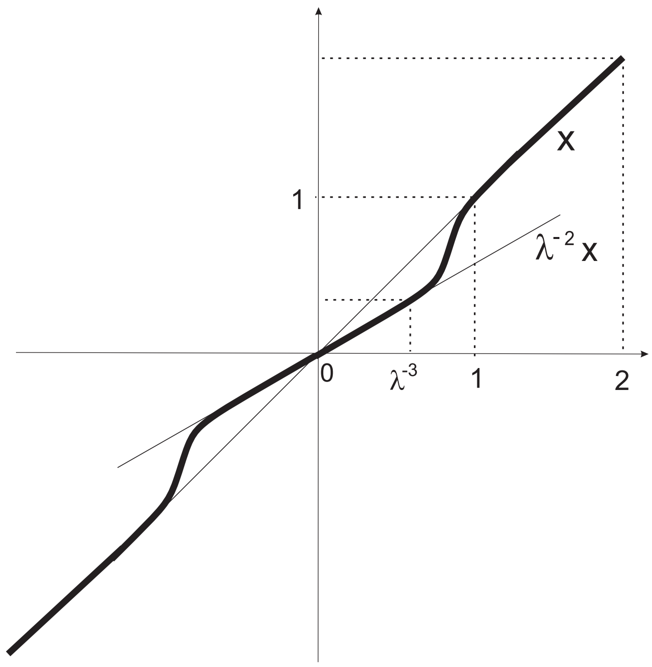

FIG. 4. Graph of the function $\nu_R(x)$.

According to Ref. 15, diffeomorphism

$$\widehat{\Psi}_R = \widehat{B}_R \circ \widehat{C}$$

is a *DA*-diffeomorphism whose non-wandering set consists of a one-dimensional repeller R with a unique bunch of degree 2 and a sink fixed point at O.

Lemma 2 *There exists a diffeomorphism $\widehat{B}_A : \mathbb{T}^2 \to \mathbb{T}^2$ with the following properties:*

- *$\widehat{B}_A$ is identical on $\mathbb{T}^2 \setminus U(O)$;*
- *$\widehat{B}_A$ preserves the stable foliation of $\widehat{C}$; and*
- *$\widehat{B}_A \circ \widehat{C}(x, y) = (\lambda x, \lambda y)$ on $V(O)$.*

Proof. Let $\nu_A : \mathbb{R} \to \mathbb{R}$ be the inverse function for ν_R. Define a diffeomorphism $B_A : \mathbb{D}^2 \to \mathbb{D}^2$ by the formula

$$B_A(x, y) = (\nu_A(x), y).$$

By construction, $B_A(x, y) = (\lambda^2 x, y)$ if $x^2 + y^2 \leqslant \lambda^{-6}$ and is identical on $\partial\mathbb{D}^2$. Define a diffeomorphism $\widehat{B}_A : \mathbb{T}^2 \to \mathbb{T}^2$, which is B_A in $U(O)$ and is identical outside $U(O)$. □

According to Ref. 15, diffeomorphism

$$\widehat{\Psi}_A = \widehat{B}_A \circ \widehat{C}$$

is a *DA*-diffeomorphism whose non-wandering set consists of a one-dimensional attractor with a unique bunch of degree 2 and a source fixed point at O.

C. Diffeomorphism on $\mathbb{T}^2 \times \mathbb{R}$ with a one-dimensional canonically embedded surface attractor (repeller)

Consider a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by the formula $\varphi(z) = \frac{z}{\lambda}$. Define a diffeomorphism Φ_A on $\mathbb{T}^2 \times \mathbb{R}$ by the formula

$$\Phi_A(w, z) = (\widehat{\Psi}_A(w), \varphi(z)).$$

The diffeomorphism Φ_A is an A-diffeomorphism whose non-wandering set consists of one fixed saddle point $O \times \{0\}$ and a one-dimensional connected attractor A located on the torus $\mathbb{T}^2 \times \{0\}$ such that $W^s_A = (\mathbb{T}^2 \setminus O) \times \mathbb{R}$. Let us show that the surface attractor A is canonically embedded.

In $U(O)$, let $W(O) = \{(x, y) | x^2 + y^2 \leqslant \lambda^{-8}\}$, $S_A = \mathbb{T}^2 \setminus int\, W(O)$, and $U_A = S_A \times [-\lambda^{-3}, \lambda^{-3}]$. Then, $\Phi_A(U_A) = (\mathbb{T}^2 \setminus int\, V(O)) \times [-\lambda^{-4}, \lambda^4]$, and therefore, $\Phi_A(U_A) \subset int\, U_A$ and $\bigcap_{n\in\mathbb{N}} \Phi^n_A(U_A) = A$. This means that U_A is a trapping neighborhood for A (see Fig. 5).

Consider a diffeomorphism $\Phi_R : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ given by the formula

$$\Phi_R(w, z) = (\widehat{\Psi}_R(w), \varphi^{-1}(z)).$$

Then, Φ_R is an A-diffeomorphism whose non-wandering set consists of a saddle point $O \times \{0\}$ and a one-dimensional surface canonically embedded repeller R located on the 2-torus $\mathbb{T}^2 \times \{0\}$ such that $W^u_A = (\mathbb{T}^2 \setminus O) \times \mathbb{R}$. The trapping neighborhoods U_R of the repeller R are exactly the same as for A, and hence, the surface repeller R is canonically embedded.

IV. PROOF OF THEOREM 1

We will realize the following scheme of the diffeomorphism f construction.

- for the closed orientable surface G of genus 2, we construct a smooth subset $K \cong G \times [0, 1]$ in $(\mathbb{T}^2 \setminus O) \times \mathbb{R}$ such that $\Phi_A(G \times \{0\}) = G \times \{1\}$, $\Phi_R(G \times \{1\}) = G \times \{0\}$ and K is a fundamental domain of the restriction of the diffeomorphism Φ_A to $W^s_A \setminus A$ and a fundamental domain of the restriction of the diffeomorphism Φ_R to $W^u_R \setminus R$ simultaneously;
- construct a diffeomorphism $H : K \to K$ such that $H(G \times \{0\}) = G \times \{1\}$, $H(G \times \{1\}) = G \times \{0\}$,

$$H \circ \Phi_A|_{G\times\{0\}} = \Phi_R \circ H|_{G\times\{0\}},$$

and H can be extended up to a diffeomorphism $H : W^s_A \setminus A \to W^u_R \setminus R$, which conjugates Φ_A with Φ_R; and
- denote by M^3 a three-dimensional orientable closed manifold, which is a result of gluing of W^s_A and W^u_R with respect to H and by $p : W^s_A \sqcup W^u_R \to M^3$ the natural projection. Then, the desired diffeomorphism $f : M^3 \to M^3$ coincides with the diffeomorphism $p\Phi_A p^{-1}|_{p(W^s_A)}$ on $p(W^s_A)$ and with the diffeomorphism $p\Phi_R p^{-1}|_{p(W^u_R)}$ on $p(W^u_R)$.

Below, we give details of the gluing.

A. Gluing of the punctured basins

Lemma 3 *There is a smooth submanifold K of $(\mathbb{T}^2 \setminus O) \times \mathbb{R}$ and a diffeomorphism $H : K \to K$ such that*

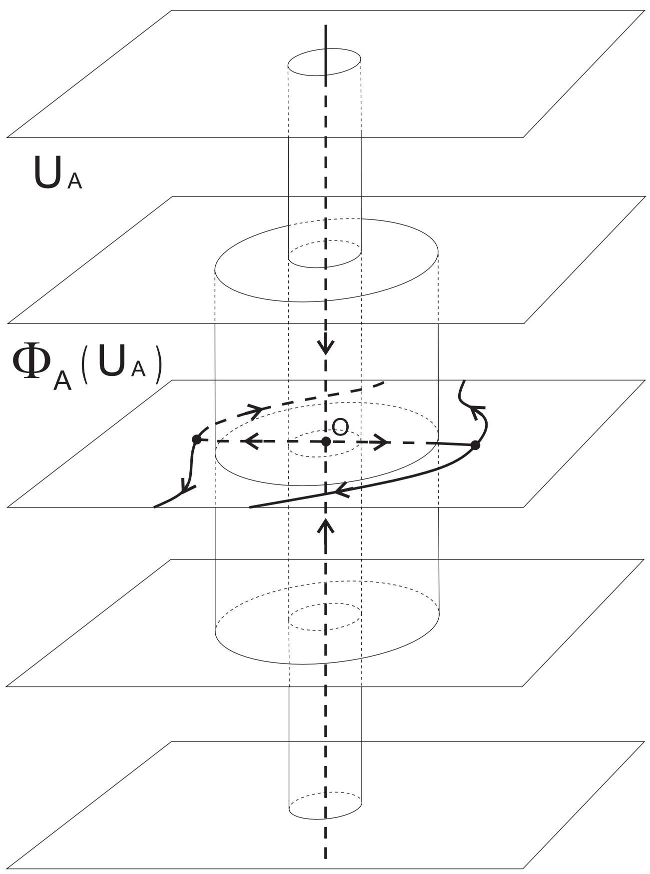

FIG. 5. Trapping neighborhood for A.

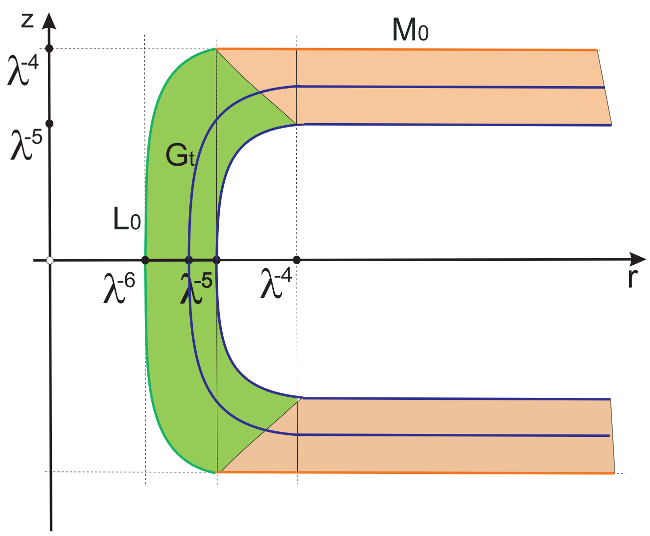

FIG. 6. Construction of the domain K and the diffeomorphism H.

- $K \cong G \times [0,1]$ *for the closed orientable surface G of genus 2, $\Phi_A(G\times\{0\}) = G\times\{1\}$, $\Phi_R(G\times\{1\}) = G\times\{0\}$, and K is a fundamental domain of the restriction of the diffeomorphism Φ_A to $W^s_A \setminus A$ and a fundamental domain of the restriction of the diffeomorphism Φ_R to $W^u_R \setminus R$ simultaneously.*
- *$H(G\times\{0\}) = G\times\{1\}$, $H(G\times\{1\}) = G\times\{0\}$, and*
$$H\circ\Phi_A|_{G\times\{0\}} = \Phi_R \circ H|_{G\times\{0\}}.$$
- *H can be extended to a diffeomorphism $H: W^s_A \setminus A \to W^u_R \setminus R$, which conjugates Φ_A with Φ_R.*

Proof. In the local coordinates (x,y) of the neighborhood $U(O)$, let $r = \sqrt{x^2+y^2}$. Let $a_0 = \lambda^{-4}$, $b_0 = \lambda^{-5} - \lambda^{-6}$, and

$$L_0 = \left\{(r,z) : r = \lambda^{-5} - \frac{b_0}{a_0}\sqrt{a_0^2 - z^2}, \lambda^{-6} \leqslant r \leqslant \lambda^{-5}\right\}.$$

Let $G_0 \subset (\mathbb{T}^2 \setminus O)\times\mathbb{R}$ coincide with L_0 for $\lambda^{-6}\leqslant r\leqslant\lambda^{-5}$ and coincide with $M_0 = \mathbb{T}^2\times\{\lambda^{-4},\lambda^{-4}\}$ outside the set $\{(r,z) : r < \lambda^{-5}\}$ (see Fig. 6). By the construction, $G_0 \cong G$ for the closed orientable surface G of genus 2.

Define a function $\hat\mu : \mathbb{T}^2 \to [0,1]$ by the formula $\hat\mu(w) = \mu(r)$ in $U(O)$ and $\hat\mu(w)$ equals 1 outside $U(O)$. For every $t\in[0,1]$, let $E_t : \mathbb{T}^2\times\mathbb{R} \to \mathbb{T}^2\times\mathbb{R}$ be a diffeomorphism given by the formula

$$E_t(w,z) = \left(\lambda^{t(1-\hat\mu(w))}w, \lambda^{-t}z\right).$$

By the construction, $\{G_t = E_t(G_0),\ t\in[0,1]\}$ is a family of pairwise disjoint surfaces, each of which is diffeomorphic to G. Then, $K = \bigcup_{t\in[0,1]} G_t$ is diffeomorphic to $G\times[0,1]$. As $\Phi_A(x,y,z) = \left(\lambda x, \lambda y, \frac{z}{\lambda}\right)$ for $(x,y,z)\in L_0$, then $\Phi_A(L_0) = E_1(L_0)$. As $\Phi_A(w,z) = \left(\widehat\Psi_A(w), \frac{z}{\lambda}\right)$ for $(w,z)\in M_0$, then $\Phi_A(M_0) = E_1(M_0)$. Thus, $\Phi_A(G_0) = G_1$. Moreover, $\Phi_A(int\,K)\cap K = \emptyset$ and $\bigcup_{n\in\mathbb{Z}}\Phi_A^n(K) = W^s_A\setminus A$. Then, K is a fundamental domain of the restriction of the diffeomorphism Φ_A to $W^s_A\setminus A$.

By the same way, it can be shown that $\Phi_R(G_1) = G_0$ and K is a fundamental domain of the restriction of the diffeomorphism Φ_R to $W^u_R\setminus R$.

Now, let us construct the required diffeomorphism $H: K\to K$. Let $\widehat B_t : \mathbb{T}^2\to\mathbb{T}^2$, $t\in[0,1]$, coincide with

$$B_t(x,y) = (1-\sigma(t))B_A(x,y) + \sigma(t)B_R(x,y)$$

in $U(0)$ and is identical outside $U(0)$. Let

$$\widehat\Psi_t = \widehat B_t \circ \widehat C : \mathbb{T}^2\to\mathbb{T}^2.$$

Then, $\widehat\Psi_0 = \widehat\Psi_A$ and $\widehat\Psi_1 = \widehat\Psi_R$. Define a diffeomorphism $H: G\times[0,1]\to G\times[0,1]$ by the formula

$$H(w,z,t) = \left(\widehat\Psi_t(w), \lambda^{2\sigma(t)-1}z, 1-\sigma(t)\right).$$

Notice that $H|_{G\times\{0\}} = \Phi_A|_{G\times\{0\}}$ and $H|_{G\times\{1\}} = \Phi_R|_{G\times\{1\}}$, then $H(G\times\{0\}) = G\times\{1\}$, $H(G\times\{1\}) = G\times\{0\}$, and $H\circ\Phi_A|_{G\times\{0\}} = \Phi_R\circ H|_{G\times\{0\}}$.

As $B_1 = B_R B_0 B_A^{-1}$, then H can be extended up to the diffeomorphism $H: W^s_A\setminus A\to W^u_R\setminus R$ by the formula $H(w,z) = \Phi_R^k H \Phi_A^{-k}(w,z)$, where $(w,z)\in W^s_A\setminus A$ and $\Phi_A^{-k}(w,z)\in K$. Then, H conjugates Φ_A with Φ_R. □

B. Behavior of invariant two-dimensional manifolds of the diffeomorphism f

To represent a behavior of invariant two-dimensional manifolds of the diffeomorphism f, we set $L = \bigcup_{t\in[0,1]} L^t$. Denote by L_+ a quarter of the domain L, defined by the condition $x, y \geqslant 0$. Figure 7 shows two-dimensional foliations of the attractor (green) and the repeller (red) of the diffeomorphism f in the domain $p(L_+)$. The front and back boundaries of the cube are $p(L_+\cap Oxz)$ and $p(L_+\cap Oyz)$, respectively. The two-dimensional foliations in the domain have a unique tangent curve. By virtue of symmetry, the diffeomorphism f has exactly four curves along which two-dimensional invariant foliations have a tangency. Thus, the constructed diffeomorphism is not structurally stable.

Moreover, authors think that the following hypothesis is true.

Hypothesis 1 There are no structural stable 3-diffeomorphisms whose non-wandering set is a union of canonically embedded one-dimensional surface attractor and repeller.

C. Why there are infinitely many pairwise Ω-non-conjugated diffeomorphisms

Let f be an A-diffeomorphism on a closed orientable 3-manifold such that $NW(f) = A\cup R$, where A and R are a connected canonically embedded one-dimensional surface attractor and repeller. The following lemma implies the statement of Theorem 1.

Lemma 4 *Let Λ be a connected non-trivial basic set of an A-diffeomorphism f, then Λ is a basic set for a diffeomorphism f^m for every $m\in\mathbb{N}$. Moreover, there exists a sequence of natural numbers $m_1 < m_2 < \cdots < m_n < \cdots$ such that diffeomorphisms $f^{m_1}, f^{m_2}, \ldots, f^{m_n}, \ldots$ are pairwise Ω-non-conjugated.*

Proof. As $\Lambda = W^u_\Lambda \cap W^s_\Lambda$ and $f(W^u_\Lambda) = W^u_\Lambda$, $f(W^s_\Lambda) = W^s_\Lambda$, then Λ is a basic set for a diffeomorphism f^m for every $m\in\mathbb{N}$. Moreover, by A-axiom, periodic points of $f^m|_\Lambda$ are dense in Λ, and, hence, there are infinitely many. However, among them, there are only finitely many points of a given period. Indeed, suppose the contrary: $f^n|_\Lambda$ has an infinite number of points of period k. Then, due to compactness of Λ, $f^{nk}|_\Lambda$ has an infinite number of fixed points, which form a sequence $\{p_n\} \subset \Lambda$ converging to a fixed point $p \in \Lambda$. It contradicts to the absence of fixed points different from p in a neighborhood of fixed hyperbolic point p.

Thus, f^n has periodic points of an infinite number of different periods and only finitely many points of a given period. Denote by $r_m \geqslant 0$ a number of fixed points of f^m. Let $m_1 = 1$ and $k_2 > 1$ be a period of some periodic point of f^{m_1} and $m_2 = m_1 k_2$. Then, $r_{m_2} > r_{m_1}$. Applying a similar idea for the diffeomorphism f^{m_2}, we get m_3 such that $r_{m_3} > r_{m_2}$. Thus, we get a sequence of natural numbers $m_1 < m_2 < \cdots < m_n < \cdots$ such that $r_{m_1} < r_{m_2} < \cdots < r_{m_n} < \cdots$, and, hence, diffeomorphisms $f^{m_1}|\Lambda, f^{m_2}|_\Lambda, \ldots, f^{m_n}|_\Lambda, \ldots$ are nonconjugated. □

V. PROOF OF THEOREM 2

Let f be an A-diffeomorphism on a closed orientable 3-manifold M^3, whose non-wandering set is a union of connected canonically embedded one-dimensional surface attractor A and repeller R. By the definition of a canonically embedded attractor, there exists a trapping neighborhood U_A such that $U_A \setminus int(f(U_A)) = S_g \times [0,1] = K^A$ and K^A is the fundamental domain of the attractor basin. Since the non-wandering set of the diffeomorphism f consists only of the attractor A and the repeller R and they are canonically embedded, the wandering set is representable in the form

$$M^3 \setminus (A \cup R) = \bigcup_{n=-\infty}^{+\infty} f^n(K^A) = S_g \times (-\infty, +\infty)$$

so that $\lim_{t\to-\infty} S_g \times \{t\} = R$ and $\lim_{t\to+\infty} S_g \times \{t\} = A$. That is, the wandering set is foliated by the surface S_g.

Then, we define the function $\varphi : M^3 \to [0,1]$ as follows:

$$\varphi(w) = \begin{cases} \frac{1}{\pi}\text{arctg}\, t + \frac{1}{2} & \text{if } w \in S_g \times \{t\}, \\ 1 & \text{if } w \in R, \\ 0 & \text{if } w \in A. \end{cases}$$

By the construction, the function φ is a continuous Lyapunov function for f. The desired energy function is the result of smoothing the function φ on A and R due to the following lemma.

Lemma 5 *Let M be a smooth compact n-manifold, $K \subset M$ be a closed subset of M, and U be some closed neighborhood of K such that $K \subset int\, U$. Let a continuous surjective function $\varphi : U \to [0;1]$ be C^1-smooth on $U \setminus K$ and $\varphi^{-1}(0) = K$. Then, for every $\delta \in (0,1)$, there exists a C^1-smooth function $g : [0;1] \to [0;1]$ satisfying the following properties:*

- $g'(0) = 0$, $g'(c) > 0$, $\forall c \in (0;1)$;
- $g(c) = c$, $\forall c \in [\delta;1]$; *and*
- $\psi = g \circ \varphi$ *is C^1-smooth on the set U.*

Proof. Let d be a Riemannian metric on the manifold M. For $c \in (0,1]$, let $\alpha(c) = \min\{1, d(\varphi^{-1}([c,1]), K)\}$ and $\beta(c) = \max\{1, \max_{x\in\varphi^{-1}([c,1])} |grad\varphi(x)|\}$. By construction, $\alpha(c)$ is non-decreasing on $(0;1]$, $\lim_{c\to0}\alpha(c) = 0$, and $\beta(c)$ is non-increasing. Then, the function $\gamma(c) = \frac{\alpha(c)}{\beta(c)}$ is a non-decreasing function, $\gamma(c) > 0$ on the half-interval $(0;1]$ and $\lim_{c\to0}\gamma(c) = 0$ since $\gamma(c) \leqslant \alpha(c)$.

For $\delta \in (0,1)$, we first construct a C^1-smooth function $g : [0;1] \to [0;1]$ such that

(a) $g'(c) > 0$ for any $c \in (0;1)$,
(b) $g(c) \leqslant c\gamma(c)$ for any $c \in (0;\frac{\delta}{4})$,
(c) $g'(c) \leqslant \gamma(c)$ for any $c \in (0;\frac{\delta}{4})$, and
(d) $g(c) = c$ for any $c \in [\delta;1]$.

After that, we show that such a function satisfies the properties of lemma.

To construct a function with properties (a)–(d), let us take a cover of the half-interval $(0,1]$ by the sets $(a_1,b_1] = (\delta,1]$, $(a_2,b_2] = \left(\frac{\delta}{2},1\right]$, $(a_i,b_i) = \left(\frac{\delta}{2^{i-1}},\frac{\delta}{2^{i-3}}\right)$, $i = 3,4,\ldots$. Let $\{\sigma_i\}$ be an arbitrary partition of a unity subordinate to this covering; that is, $\sigma_i : (0,1] \to [0,1]$ is a smooth function with $supp(\sigma_i) \subset (a_i,b_i)$ and $\sum_{i\in\mathbb{N}}\sigma_i(c) = 1$ for any $c \in (0,1]$. We are going to construct first $g'(c)$ as a sum $\sum_{i=1}^{\infty}\varepsilon_i\sigma_i(c)$ for a sequence of non-negative ε_i.

Let $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_i = \gamma\,(a_i)$ for all $i = 4,5,\ldots$, and ε_3 will be determined later to satisfy condition (d). Since each point $c \in (0,1]$ belongs to the support of less than four maps from the partition, the sum $S(x) = \sum_{i=1}^{\infty}\varepsilon_i\sigma_i(x)$ is a smooth function on $(0,1]$. It can be extended continuously by $S(0) = 0$ as $\lim_{c\to0}\gamma(c) = 0$. Define a function $g : [0,1] \to [0,1]$ by the formula $g(c) = \int_0^c S(x)\,dx$. Let us show that it satisfies (a)–(d).

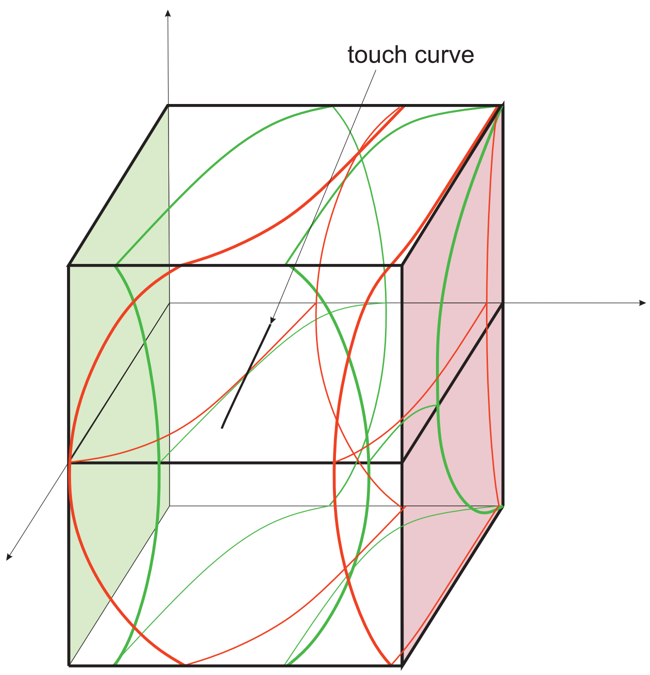

FIG. 7. Leaves of two-dimensional foliations of the diffeomorphism f.

(a) Since $g'(c) = S(c) = \sum_{i=1}^{\infty} \varepsilon_i \sigma_i(c)$, then $g'(c) > 0$ for any $c \in (0;1)$.

(b) For $i = 4, 5, \ldots$, the sequence $\{\varepsilon_i\}$ is non-increasing. Notice that for any $c \in (0;1]$, there is a unique number i^* such that $c \in (a_{i^*}, a_{i^*-1}]$. Then, $\sigma_{i^*}(c) \neq 0$ and $\sigma_i(c) = 0$ for all $i \notin \{i^*, i^*+1\}$. From the choice of the parameters ε_i for $c \in (0, \frac{\delta}{4})$, we obtain the chain of inequalities $g(c) = \int_0^c S(x)\,dx = \int_0^c \left(\sum_{i=1}^{\infty} \varepsilon_i \sigma_i(x)\right) dx = \int_0^c \left(\sum_{i=i^*}^{\infty} \varepsilon_i \sigma_i(x)\right) dx \leqslant \int_0^c \left(\sum_{i=i^*}^{\infty} \varepsilon_{i^*} \sigma_i(x)\right) dx = \varepsilon_{i^*} \int_0^c \left(\sum_{i=i^*}^{\infty} \sigma_i(x)\right) dx \leqslant \varepsilon_{i^*} \int_0^c \left(\sum_{i=1}^{\infty} \sigma_i(x)\right) dx = \varepsilon_{i^*} \int_0^c 1\, dx = \varepsilon_{i^*} c = c\gamma\,(a_{i^*}) \leqslant c\gamma(c)$.

(c) For $g'(c)$, $c \in \left(0, \frac{\delta}{4}\right)$, the following estimate holds: $g'(c) = \sum_{i=i^*}^{\infty} \varepsilon_i \sigma_i(c) \leqslant \varepsilon_{i^*} \sum_{i=i^*}^{\infty} \sigma_i(c) = \varepsilon_{i^*} \leqslant \gamma(c)$.

(d) For $c \in [\delta; 1]$, the following chain of the equalities holds: $g(c) = \int_0^c \left(\sum_{i=1}^{\infty} \varepsilon_i \sigma_i(x)\right) dx = \int_0^{\delta} \left(\sum_{i=4}^{\infty} \varepsilon_i \sigma_i(x)\right) dx + \int_0^{\delta} \varepsilon_3 \sigma_3(x)\, dx + \int_0^{\delta} \varepsilon_2 \sigma_2(x)\, dx + \int_{\delta}^{c} (\varepsilon_1 \sigma_1(x) + \varepsilon_2 \sigma_2(x))\, dx = \int_0^{\delta} \left(\sum_{i=4}^{\infty} \varepsilon_i \sigma_i(x)\right) dx + \varepsilon_3 \int_0^{\delta} \sigma_3(x)\, dx + \int_0^{\delta} \sigma_2(x)\, dx + \int_{\delta}^{c} (\sigma_1(x) + \sigma_2(x))\, dx = \int_0^{\delta} \left(\sum_{i=4}^{\infty} \varepsilon_i \sigma_i(x)\right) dx + \varepsilon_3 \int_0^{\delta} \sigma_3(x)\, dx + \int_0^{\delta} \sigma_2(x)\, dx + (c - \delta)$. Let $\varepsilon_3 = \frac{\delta - \int_0^{\delta} \left(\sum_{i=4}^{\infty} \varepsilon_i \sigma_i(x)\right) dx - \int_0^{\delta} \sigma_2(x)\, dx}{\int_0^{\delta} \sigma_3(x)\, dx}$, then $g(c) = c$ for $c \in [\delta; 1]$.

Now, let us show that g satisfies the properties of the lemma.

Since $0 < g(c) \leqslant \gamma(c)$ for all $c \in \left(0, \frac{\delta}{4}\right)$, then $\lim_{c \to 0} g(c) = 0 = g(0)$ and $g(c)$ is continuous function on $[0, 1]$. Also, $0 < g'(c) \leqslant \gamma(c)$ for all $c \in \left(0, \frac{\delta}{4}\right)$, then $g'(0) = 0$ and $g(c)$ is C^1-smooth on $[0, 1]$. It remains to show that the superposition $\psi = g \circ \varphi$ is a C^1-smooth function on U.

Notice that $grad\psi = g' \cdot grad\varphi$. Since the function ψ is C^1-smooth on the set $U \setminus K$ as a superposition of C^1-smooth functions, it remains to show that the function ψ is C^1-smooth on the set K.

Consider any point $a \in K$ and a local chart (U_a, h_a), where the neighborhood is chosen in such a way that $\varphi(w) < \frac{\delta}{4}$ for all $w \in U_a$. First, we show differentiability. If the function $\psi_a = \psi(h_a^{-1}(x))$ is differentiable at O, then the function ψ is differentiable at a. Moreover, the function ψ_a is differentiable at the point O and has partial derivatives equal to zero at this point if and only if $\lim_{s \to O} \frac{\psi_a(s)}{\rho(s,O)} = 0$, where $s(x_1, \ldots, x_n) \in \mathbb{R}^n$ and ρ the Euclidean metric in $\mathbb{R}^n$, defined by the formula $\rho(s^1, s^2) = \sqrt{\sum_{i=1}^{n} (x_i^1 - x_i^2)^2}$ for $s^1(x_1^1, \ldots, x_n^1), s_2(x_1^2, \ldots, x_n^2) \in \mathbb{R}^n$. A verification of the equality $\lim_{s \to O} \frac{\psi_a(s)}{\rho(s,O)} = 0$ completes the proof of differentiability.

We introduce a metric d_a on $\mathbb{R}^n$ by the formula $d_a(s^1, s^2) = d(h_a^{-1}(s^1), h_a^{-1}(s^2))$ for $s^1, s^2 \in \mathbb{R}^n$. By Ref. 16 (lecture 15), the metrics ρ and d_a are equivalent in some compact neighborhood $U(O)$ of the point O; that is, there exist constants $0 < c_1 \leqslant c_2$ such that

$$\forall s^1, s^2 \in U(O) : c_1 d_a(s^1 s^2) \leqslant \rho(s^1, s^2) \leqslant c_2 d_a(s^1, s^2).$$

For $s \in U(O)$, we set $w = h_a^{-1}(s)$ and $c = \varphi(h_a^{-1}(s)) = \varphi(w)$. Then, $\lim_{s \to O} \frac{\psi_a(s)}{\rho(s,O)} = \lim_{s \to O} \frac{\psi(h_a^{-1}(s))}{c_1 d(h_a^{-1}(s),a)} = \lim_{w \to a} \frac{\psi(w)}{c_1 d(w,a)} = \lim_{w \to a} \frac{g(\varphi(w))}{c_1 d(w,a)} = \lim_{w \to a} \frac{g(c)}{c_1 d(w,a)} \leqslant \lim_{w \to a} \frac{c\alpha(c)}{\beta(c) c_1 d(w,a)} \leqslant \lim_{w \to a} \frac{\varphi(w) d(w,a)}{c_1 d(w,a)} = \lim_{w \to a} \frac{\varphi(w)}{c_1} = 0$.

Now, let us show that the partial derivatives $(\psi_a)'_{x_i}$, $i \in \{1, \ldots, n\}$ are continuous at O; that is, $\lim_{s \to O} (\psi_a)'_{x_i}(s) = 0$, which is equivalent to $\lim_{s \to O} |grad\psi_a(s)| = 0$. Denote by $J_{h_a^{-1}}$ the Jacobian of the map h_a^{-1} by $||J_{h_a^{-1}}||$ its norm subordinated to the Euclidean norm of the vector in $\mathbb{R}^n$ and by B a constant such that $||J_{h_a^{-1}}(s)|| \leqslant B$ for all points s in some neighborhood of the point O. Then, $\lim_{s \to O} |grad\ \psi_a(s)| = \lim_{s \to O} |J_{h_a^{-1}}(s) \cdot g'(c) \cdot grad\ \ \varphi(w)| \leqslant \lim_{s \to O} ||J_{h_a^{-1}}(s)|| \cdot |g'(c)| \cdot |grad\ \ \varphi(w)| \leqslant \lim_{s \to O} B \cdot \frac{\alpha(c)}{\beta(c)} \cdot |grad\ \ \varphi(w)| \leqslant \lim_{w \to a} B \cdot \frac{d(w,a)}{|grad\ \varphi(w)|} \cdot |grad\varphi(w)| \leqslant \lim_{w \to a} B \cdot d(w, a) = 0$.

Thus, the function ψ is C^1-smooth on U. □

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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