On positive recurrence of 1D diffusions with switching

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Abstract

Positive recurrence of one-dimensional diffusion with switching with an additive Wiener process and with one recurrent and one transient regime is established under suitable conditions on the drift in both regimes and on the intensities of switching.

Keywords: Switching diffusion, Positive recurrence, Stationarity

1. Introduction

On a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a one-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\ge 0}$ on it, a one-dimensional SDE with switching is considered,

$$dX_t = b(X_t, Z_t)\,dt + dW_t, \quad t \ge 0, \quad X_0 = x,\ Z_0 = z,$$

where Z_t is a continuous-time Markov process on a finite state space $S = \{0, 1\}$ with (positive) intensities of respective transitions $\lambda_{01} =: \lambda_0$, & $\lambda_{10} =: \lambda_1$; the process Z is assumed to be independent of W and adapted to the filtration $(\mathcal{F}_t)$. In the first instant we assume throughout the paper that these intensities are constants; this may be relaxed. Under the regime $Z = 0$ the process X is assumed positive recurrent, while under the regime $Z = 1$ its modulus may increase "in square mean" with the rate comparable to the decrease rate under the regime $Z = 0$. This intuitive wording will be specified in the assumptions. Denote

$$b(x, 0) = b_-(x), \quad b(x, 1) = b_+(x).$$

The problem addressed in this note is to find sufficient conditions for the positive recurrence (and, hence, for convergence to the stationary regime) for solutions of SDEs with switching in the case where not for all values of the modulating process the SDE is recurrent, and where it is recurrent, this property is "not very strong". Earlier such a problem was tackled in [3] in the exponential recurrent case; its method apparently does not work for the weaker polynomial recurrence. A new approach is offered. Other SDEs with switching were considered in [1, 5, 6, 7], see also the references therein. Neither of these works address exactly the problem which is attacked in this paper: some of them tackled an exponential recurrence, some other just a recurrence versus a transience.

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2. Main result: positive recurrence

Since the drift b is Borel measurable in x and bounded, a strong solution of the system exists and is pathwise unique, see [9]. Also, conditions sufficient for positive recurrence and convergence for one of the frozen values of Z will be assumed. As it was said, Z takes just two possible values 0 and 1, with constant intensities of transitions between them. Under the state $Z = 0$ the process X is assumed to be positive recurrent (see [8]), while for $Z = 1$ the opposite – i.e., transient – *with a comparable rate,* see below. The goal is to establish the overall positive recurrence, under certain additional combined assumptions on the intensities of Z and on the drift of X in both regimes.

We will use a "Lyapunov function" x^2, but the reader should have in mind that, in fact, it serves as a "fair" Lyapunov function – that is, decreases on average while away from some neighbourhood of the origin – only under the regime $Z = 0$. So, it could be called a partial Lyapunov function. However, under the regime $Z = 1$ the same function helps evaluate and keep under control the averaged increase of the value of the second moment of the process.

Theorem 1. *Let the drift b be bounded and let there exist $r_-, r_+, M > 0$ such that*

$$xb_-(x) \le -r_-, \quad xb_+(x) \le +r_+, \quad \forall\, |x| \ge M, \tag{1}$$

and

$$2r_- > 1 \quad \& \quad \kappa_1^{-1} := \frac{\lambda_0(2r_+ + 1)}{\lambda_1(2r_- - 1)} < \frac{1}{2}. \tag{2}$$

Then the process is positive recurrent; in particular, there exists $C > 0$ such that for all M_1 large enough and all $x \in \mathbb{R}$

$$\mathbb{E}_x \tau_{M_1} \le C(x^2 + 1), \tag{3}$$

where

$$\tau_{M_1} := \inf(t \ge 0 : |X_t| \le M_1).$$

Moreover, the process (X_t, Z_t) has a unique invariant measure, and for each nonrandom initial condition x, z there is a convergence to this measure in total variation when $t \to \infty$.

Note that the constant C here in (3) is computable. The meaning of the assumption on κ_1 in (2) is that the process should spend more time in the "negative" regime than in the "positive" one, with a correction due to the bounds r_+ and r_-; the condition $2r_- > 1$ is necessary for the approach: it serves for the existence of the invariant measure. In fact, for the claim of the theorem it suffices to assume

$$2r_- > 1 \quad \& \quad \kappa_1^{-1} := \frac{\lambda_0(2r_+ + 1)}{\lambda_1(2r_- - 1)} < 1,$$

although, it would change the constant C in the right hand side of the bound (3). However, with the bound $1/2$ instead of 1 the calculus – severely truncated in this short presentation – seems more explicit, so we have assumed (2). Let $M_1 \gg M$; let us define the sequence of stopping times

$$T_0 := \inf(t \ge 0 : Z_t = 0),$$

and further

$$0 \le T_0 < T_1 < T_2 < \dots,$$

where each T_n is the next moment of switch of Z, and denote

$$\tau := \inf(T_n \ge 0 : |X_{T_n}| \le M_1),$$

where the choice of M_1 will be specified later. Note that $\tau_{M_1} \le \tau$, so it suffices to evaluate $\mathbb{E}_x \tau$. In what follows it will be chosen $\epsilon > 0$ such that $2\lambda_0(2r_+ + 1 + \epsilon) = \lambda_1(2r_- - 1 - \epsilon)$ (see (2)).

3. Sketch of the proof

Let $x > M$ for definiteness: the case $x < -M$ is fully similar.

Lemma 1. *Let the assumptions of the theorem hold true. For any $\delta > 0$ there exists M_1 such that*

$$\sup_{|x|>M_1} \mathbb{E}_x \left(\int_0^{T_1} 1(\inf_{0\le s\le t} |X_s| \le M) dt | Z_0 = 0 \right) + \sup_{|x|>M_1} \mathbb{E}_x \left(\int_0^{T_0} 1(\inf_{0\le s\le t} |X_s| \le M) dt | Z_0 = 1 \right) < \delta. \quad (4)$$

Proof. Let X_t^i, $i = 0, 1$ denote the solution of the equation

$$dX_t^i = b(X_t^i, i)\, dt + dW_t, \quad t \ge 0, \quad X_0^i = x.$$

Let $Z_0 = 0$, then $T_0 = 0$. Due to uniqueness, the processes X and X^0 coincide a.s. on $[0, T_1]$, and due to the independence of Z and W, and, hence, of Z and X^0, we have

$$\mathbb{E}_x \left(\int_0^{T_1} 1(\inf_{0\le s\le t} |X_s| \le M) dt | Z_0 = 0 \right) = \mathbb{E}_x \int_0^{T_1} 1(\inf_{0\le s\le t} |X_s^0| \le M) dt$$
$$= \mathbb{E}_x \int_0^{\infty} 1(t < T_1) 1(\inf_{0\le s\le t} |X_s^0| \le M) dt = \int_0^{\infty} \mathbb{E}_x 1(t < T_1) \mathbb{P}(\inf_{0\le s\le t} |X_s^0| \le M) dt$$
$$= \int_0^{\infty} \exp(-\lambda_0 t) \mathbb{P}(\inf_{0\le s\le t} |X_s^0| \le M) dt.$$

So, it suffices to take t such that

$$\int_t^{\infty} e^{-\lambda_0 s} ds < \delta/4,$$

and then, by virtue of the boundedness of b, to choose $M_1 > M$ such that for this value of t

$$t \mathbb{P}_x(\inf_{0\le s\le t} |X_s^0| \le M) < \delta/4.$$

The bound for the second term in (4) follows by using the process X^1 and the intensity λ_1. QED

Lemma 2. *Let the assumptions of the theorem hold true. If M_1 is large enough, then for any $|x| > M_1$*

$$\left(\mathbb{E}_x(X^2_{T_1\wedge\tau} | Z_0 = 0) + \lambda_0^{-1}((2r_- - 1) - \epsilon)\right) \vee \left(\mathbb{E}_x(X^2_{T_0\wedge\tau} | Z_0 = 1) - \lambda_1^{-1}((2r_- + 1) + \epsilon)\right) \le x^2. \quad (5)$$

Proof. Recall that $T_0 = 0$ under the condition $Z_0 = 0$. By Ito's formula

$$dX_t^2 - 2X_t dW_t = 2X_t b(X_t) dt + dt \le (-2r_- + 1) dt,$$

where the latter inequality holds on the set $(|X_t| > M)$ due to the assumptions (1). Further (here $\|b\| = \sup_x |b(x)|$), since $1(|X_t| > M) = 1 - 1(|X_t| \le M)$, we obtain

$$\int_0^{T_1\wedge\tau} 2X_t b(X_t) dt = \int_0^{T_1\wedge\tau} 2X_t b(X_t) 1(|X_t| > M) dt + \int_0^{T_1\wedge\tau} 2X_t b(X_t) 1(|X_t| \le M) dt$$
$$\le -2r_- \int_0^{T_1\wedge\tau} 1(|X_t| > M) dt + \int_0^{T_1\wedge\tau} 2M\|b\| 1(|X_t| \le M) dt$$
$$= -2r_- \int_0^{T_1\wedge\tau} 1 dt + \int_0^{T_1\wedge\tau} (2M\|b\| + 2r_-) 1(|X_t| \le M) dt$$
$$\le -2r_- \int_0^{T_1\wedge\tau} 1 dt + (2M\|b\| + 2r_-) \int_0^{T_1\wedge\tau} 1(|X_t| \le M) dt.$$

Thus, always for $x > M_1$ (recall that $x > M$, so the condition $|x| > M_1$ coincides with $x > M_1$),

$$\mathbb{E}_x \int_0^{T_1\wedge\tau} 2X_t b(X_t)dt \le -2r_- E \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)E_x \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt$$

$$= -2r_-\mathbb{E} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_x \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt$$

$$\le -2r_-\mathbb{E} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_x \int_0^{T_1} 1(|X_t| \le M)dt$$

$$\le -2r_-\mathbb{E} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\delta.$$

For our fixed $\epsilon > 0$ let us choose $\delta = \lambda_0^{-1}\epsilon/(2M\|b\|+2r_-)$. Then, since $x > M_1$ implies $T_1 \wedge \tau = T_1$,

$$\mathbb{E}_x X^2_{T_1\wedge\tau} - x^2 \le -(2r_- - 1)\mathbb{E}_x \int_0^{T_1} dt + \lambda_0^{-1}\epsilon = -\lambda_0^{-1}((2r_- - 1) - \epsilon).$$

Hence, the first part of the bound (5) follows. The second one can be proved similarly. QED

Lemma 3. *Let the assumptions of the theorem hold true. Then, if M_1 is large enough,*

$$\mathbb{E}_x(X^2_{T_2\wedge\tau}|Z_0 = 0) \le \mathbb{E}_x(X^2_{T_1\wedge\tau}|Z_0 = 0) + \lambda_1^{-1}((2r_+ + 1) + \epsilon)), \tag{6}$$

$$\mathbb{E}_x(X^2_{T_1\wedge\tau}|Z_0 = 1) \le \mathbb{E}_x(X^2_{T_0\wedge\tau}|Z_0 = 1) - \lambda_0^{-1}((2r_+ - 1) - \epsilon)). \tag{7}$$

Proof. Let $Z_0 = 0$; recall that it implies $T_0 = 0$. If $\tau \le T_1$, then (6) is trivial. Let $\tau > T_1$. Similarly to the above, but using now the solution X^1_t of the equation

$$dX^1_t = b(X^1_t, 1)\,dt + dW_t, \quad t \ge T_1, \quad X^1_{T_1} = X_{T_1}$$

instead of X^0, by choosing M_1 large enough, due to the assumptions (1) we guarantee the bound

$$1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \le 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1}}(T_2 - T_1)((2r_+ + 1) + \epsilon))$$

$$= +1(|X_{T_1}| > M_1)(\lambda_1^{-1}((2r_+ + 1) + \epsilon)).$$

It was used that $|x| \wedge |X_{T_1}| > M_1$ implies $T_2 \le \tau$. In particular, it follows that for $|x| > M_1$

$$(\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \le 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1}}(T_2 - T_1)((2r_+ + 1) + \epsilon))$$

$$= +1(|X_{T_1}| > M_1)(\lambda_1^{-1}((2r_+ + 1) + \epsilon)),$$

because $|X_{T_1}| \le M_1$ implies $\tau \le T_1$ and $\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau} = 0$. So, still for $|x| > M_1$,

$$\mathbb{E}_x(\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \le \mathbb{E}_x 1(|X_{T_1}| > M_1)(\lambda_1^{-1}((2r_+ + 1) + \epsilon)) \le \lambda_1^{-1}((2r_+ + 1) + \epsilon).$$

The inequality (6) follows. For $Z_0 = 1$ we have $T_0 > 0$, and the bound (7) follows similarly. QED

In fact, the bound (6) is analogous to the second part of (5), while (7) to the first part of (5). Let us return to the theorem. Bounds similar to (5) and (6)–(7) can be established for all T_n with values n odd and even, respectively. Summing up, and using Fatou's lemma, we evaluate each positive summand from above via the modulus of the respective preceding negative term using the assumption (2), which leads to the estimate

$$(2r_+ + 1 + \epsilon)\mathbb{E}_x \tau \le x^2$$

for $Z_0 = 0$. In the case of $Z_0 = 1$ we get

$$(2r_+ + 1 + \epsilon)\mathbb{E}_x \tau \le x^2 + C$$

with $C = \lambda_1^{-1}((2r_+ + 1 + \epsilon)$, since in the latter case the first summand is positive. In both cases the bound (3) follows. This implies existence of the invariant measure, see [2, Theorem 6.1], [4, Section 4.4]. Convergence to it holds due to the coupling method. Thus, this measure is unique. QED

4. Conclusion

The novelty is the analysis of positive recurrence for the process with two regimes, only in one of which is positive recurrent, while the other is transient. The method is based on stopping times where the component Z switches, and on Lyapunov functions as auxiliary tools. The approach to convergence and mixing based on stopping times was offered in the papers by the author in a general context; now it is applied to a new class of "partially recurrent" processes. The method is applicable in dimensions $d > 1$ with variable diffusion and likely also with variable intensities of Z depending on the component X.

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