Chapter 1
Preliminaries

This introductory chapter includes material needed in what follows yet not belonging to complex analysis proper. To find out how well you are acquainted with the necessary background, you can look at the exercises at the end of the chapter.

If $z = x + iy$ is a complex number ($x, y \in \mathbb{R}$, $i^2 = -1$), then x and y are called the *real* and *imaginary parts* of z, respectively. Notation: $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

A complex number z is represented by the point in the plane with coordinates $(\operatorname{Re} z, \operatorname{Im} z)$; correspondingly, we will identify the set of all complex numbers (denoted by $\mathbb{C}$) with the coordinate plane; the plane whose points are regarded as complex numbers is called the *complex plane*.

If z is a complex number, then its *absolute value* (or *modulus*) is the distance from z (more exactly, from the corresponding point in the complex plane; in what follows, we will no longer make such distinctions) to 0 (i.e., the origin): $|z| = \sqrt{x^2 + y^2}$ where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. The triangle inequality implies that $|z + w| \le |z| + |w|$. The distance between complex numbers z and w is equal to $|z - w|$.

The Ox and Oy axes in the complex plane are called the *real* and *imaginary axes*, respectively. Given $z \in \mathbb{C}$, $z \ne 0$, the *argument* of z is the angle between the vector from 0 to z and the positive real axis (so, for instance, $\pi/4$ is the argument of $1 + i$). The argument of a complex number z is denoted by $\arg z$.

The argument of a complex number is defined up to an integer multiple of 2π: for example, the assertions $\arg(1 + i) = \pi/4$ and $\arg(1 + i) = -7\pi/4$ are equally true. If $|z| = r \ne 0$ and $\arg z = \varphi$, then

$$z = r(\cos\varphi + i\sin\varphi). \tag{1.1}$$

Expression (1.1) is called the *polar form* of the complex number z. To find the product of two complex numbers, we multiply their absolute values and add their arguments.

If $z = x + iy$ with $x, y \in \mathbb{R}$, then the number $x - iy$ is called the *complex conjugate* of z and denoted by $\bar{z}$. The point $\bar{z}$ is the reflection of z in the real axis, and $z\bar{z} = |z|^2$.

Limits of functions of a complex variable and limits of sequences of complex numbers are defined in the same way as in the real case. For example, $\lim_{z\to a} f(z) = b$

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means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - a| < \delta$ implies $|f(z) - b| < \varepsilon$. The theorems on the limit of a sum, difference, product, and quotient of two functions are still valid in the complex case, as well as Cauchy's convergence criterion ("a sequence converges if and only if it is Cauchy").

The notation $\lim\limits_{z \to a} f(z) = \infty$ means that $\lim\limits_{z \to a} |f(z)| = +\infty$ (and similarly for limits of sequences); $\lim\limits_{|z| \to \infty} f(z) = b$ means that "for every $\varepsilon > 0$ there exists $M > 0$ such that $|z| > M$ implies $|f(z) - b| < \varepsilon$."

Derivatives and integrals of functions of a complex variable are a delicate matter (the entire book is devoted to it), but complex-valued functions of a real variable hold no surprises. Namely, if f is a function with complex values defined on an interval of a (real) axis, and if $f(x) = u(x) + iv(x)$ (where u and v are real-valued), then the derivative of f is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = u'(a) + iv'(a);$$

these derivatives enjoy all the elementary properties of derivatives of real-valued functions (the derivative of a sum, product, and difference, the derivative of a composite function if the "inner" function is real-valued), with the same proofs. The integral of such a function f is defined by the formula

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i \int_a^b v(x)\,dx;$$

it can also be defined in terms of Riemann sums (or, for that matter, as a Lebesgue integral). As in the case of real-valued functions, it satisfies the inequality

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx; \tag{1.2}$$

to prove this, it suffices to apply the inequality "the absolute value of a sum is not greater than the sum of the absolute values" to the Riemann sums and take the limit.

1.1 Absolute and Uniform Convergence

Let X be an arbitrary set (you lose nothing by assuming that X is a subset of the complex plane) and $\{f_n\}_{n \in \mathbb{N}}$ be a countable family of *bounded* functions on X with values in $\mathbb{C}$.

Definition 1.1 We say that a series $\sum\limits_{n=1}^{\infty} f_n$ converges *absolutely and uniformly* on X if the series $\sum\limits_{n=1}^{\infty} \sup\limits_{x \in X} |f_n(x)|$ converges.

Proposition 1.2 (Weierstrass M-test) *Let* $\sum_{n=1}^{\infty} f_n$ *be a series of bounded functions on* X. *If there exists a positive integer* N *such that* $\sup_{x \in X} |f_n(x)| \leq a_n$ *for all* $n \geq N$, *and if the series* $\sum_{n=1}^{\infty} a_n$ *converges, then the series* $\sum_{n=1}^{\infty} f_n$ *converges absolutely and uniformly.*

See [6, Chap. XVI, Sec. 2].

Proposition 1.3 *If a series of bounded functions* $\sum f_n$ *converges on* X *absolutely and uniformly, then it converges on* X *uniformly. Moreover, the series obtained from* $\sum f_n$ *by any rearrangement of its terms also converges on* X *absolutely and uniformly, and to the same function.*

For the case of series with constant terms, see [5, Chap. V, Sec. 5, Proposition 4]; the modifications necessary for the case of functional series are left to the reader.

Let $\{f_{m,n}\}_{m,n \in \mathbb{N}}$ be a family of bounded functions defined on X and indexed by two positive integers. Then the formal sum $\sum_{m,n \in \mathbb{N}} f_{m,n}$ is called a *double series*. We say that a double series converges absolutely and uniformly if it converges absolutely and uniformly for some (and hence, by Proposition 1.3, any) ordering of its terms.

Proposition 1.4 *Let* $\{f_{m,n}\}_{m,n \in \mathbb{N}}$ *be a family of bounded functions on* X. *Then the following two conditions are equivalent.*

(1) *The double series* $\sum_{m,n} f_{m,n}$ *converges absolutely and uniformly.*

(2) *For every* $m \in \mathbb{N}$ *the series* $\sum_{n=1}^{\infty} f_{m,n}$ *converges absolutely and uniformly, and, denoting the sum of this series by* f_m, *the series* $\sum_{m=1}^{\infty} f_m$ *also converges absolutely and uniformly.*

Moreover, if the equivalent conditions (1) *and* (2) *are satisfied, then the sum of the series* $\sum_{m,n \in \mathbb{N}} f_{m,n}$ *coincides with the sum of the series* $\sum_{m=1}^{\infty} f_m$.

The reader may either prove these statements as an exercise, or find the proofs in the literature, or, finally, appropriately modify the proofs of the corresponding facts for series with constant terms (they are easier to find in textbooks).

1.2 Open, Closed, Compact, Connected Sets

In this section, we deal with subsets of the Euclidean space $\mathbb{R}^n$ for arbitrary n, but our applications mainly involve the case $n = 2$ (the complex plane $\mathbb{C}$ identified with $\mathbb{R}^2$) and $n = 1$ (the real line). If v is a point in $\mathbb{R}^n$, by $|v|$ we denote its Euclidean norm (the square root of the sum of the squares of its coordinates). The distance between points $v_1, v_2 \in \mathbb{R}^n$ is equal to $|v_1 - v_2|$. If $n = 2$ and we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, then $|v|$ is the absolute value of the complex number v.

Recall that the *ε-neighborhood* of a point $a \in \mathbb{R}^n$ is the set

$$\{z \in \mathbb{R}^n : |z - a| < \varepsilon\}$$

(here ε is a positive real number).

Definition 1.5 A subset $U \subset \mathbb{R}^n$ is said to be *open* if for every point $a \in U$ there is an ε-neighborhood of a, for some ε, that is contained in U.

Definition 1.6 A subset $F \subset \mathbb{R}^n$ is said to be *closed* if its complement $\mathbb{R}^n \setminus F$ is open.

Proposition 1.7 (1) *The union of an arbitrary family of open sets is open. The intersection of an arbitrary* finite *family of open sets is open.*

(2) *The intersection of an arbitrary family of closed sets is closed. The union of an arbitrary* finite *family of closed sets is closed.*

Proposition 1.8 *A subset $F \subset \mathbb{R}^n$ is closed if and only if it satisfies the following property: if $\{a_k\}$ is a sequence of points of F and $\lim\limits_{k\to\infty} a_k = a$, then $a \in F$.*

(The limit of a sequence in $\mathbb{R}^n$ is defined in the same way as in $\mathbb{C}$, that is, $a_k \to a$ if $\lim\limits_{k\to\infty} |a_k - a| = 0$.)

Definition 1.9 The *closure* of a subset $X \subset \mathbb{R}^n$ is the intersection of all closed sets containing X.

Proposition 1.10 *The closure of a subset $X \subset \mathbb{R}^n$ coincides with the set of all limits of convergent sequences $\{a_k\}$ with $a_k \in X$.*

The closure of a set X is denoted by $\overline{X}$.

Definition 1.11 The *interior* of a subset $X \subset \mathbb{R}^n$ is the set of all points $a \in X$ such that some ε-neighborhood of a is contained X. The interior of a set X is denoted by $\mathrm{Int}(X)$.

The interior of a set X coincides with the union of all open sets contained in X, and also with the complement to the closure of the set $\mathbb{R}^n \setminus X$.

Definition 1.12 A subset $K \subset \mathbb{R}^n$ is said to be *compact* if it is closed and bounded.

Proposition 1.13 *The following three conditions are equivalent*:

(1) *a subset $K \subset \mathbb{R}^n$ is compact*;

(2) *for every sequence $a_m \in K$ there exists a subsequence $\{a_{m_k}\}$ such that the limit $\lim\limits_{k\to\infty} a_{m_k}$ exists and lies in K*;

(3) *for every family of open sets $\{U_\alpha\}_{\alpha\in I}$ satisfying the property $K \subset \bigcup_\alpha U_\alpha$ there exists a finite collection $\alpha_1, \ldots, \alpha_l \in I$ such that*

$$K \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_l}.$$

A family of sets $\{U_\alpha\}$ satisfying condition (3) of the proposition is called an *open cover* of the set K, and condition (3) itself can be briefly stated as "every open cover has a finite subcover."

For all of the above, see [5, Chap. 7, Sec. 1].

Proposition 1.14 *If $K \subset \mathbb{R}^n$ is a compact set, then every continuous function $f\colon K \to \mathbb{R}$ attains its maximum and minimum on K.*

See [5, Chap. 7, Sec. 1].

Finally, we need the notion of connectedness. To save space, we define it only for open sets (it will not be encountered in other situations).

Definition 1.15 An open subset $U \subset \mathbb{R}^n$ is said to be *disconnected* if it can be represented as the union of two disjoint nonempty open sets.

An open subset $U \subset \mathbb{R}^n$ is said to be *connected* if it is not disconnected.

Proposition 1.16 *An open subset in $\mathbb{R}$ is connected if and only if it is an interval $(a; b)$ where a and b are real numbers, $+\infty$, or $-\infty$.*

Connected open subsets in $\mathbb{R}^n$ with $n > 1$ have no such simple characterization, but we will state and prove one important connectedness criterion.

Proposition 1.17 *An open subset $U \subset \mathbb{R}^n$ is connected if and only if for any two points $a, b \in U$ there exists a continuous map $\gamma\colon [0; 1] \to U$ such that*

$$\gamma(0) = a, \quad \gamma(1) = b.$$

In other words, an open set in the plane is connected if and only if any two its points can be joined by a curve.

Proof Assume that any two points of an open set $U \subset \mathbb{R}^n$ can be joined by a curve; we will prove that U is connected. Assume to the contrary that $U = U_1 \cup U_2$ where U_1 and U_2 are open, nonempty, and disjoint. Pick points $a \in U_1$, $b \in U_2$, and let $\gamma\colon [0; 1] \to U$ be a curve joining a and b (with $\gamma(0) = a$, $\gamma(1) = b$). Define a function $f\colon [0; 1] \to \mathbb{R}$ as follows: $f(t) = 1$ if $\gamma(t) \in U_1$, and $f(t) = 2$ if $\gamma(t) \in U_2$. We claim that f is continuous. Indeed, if, say, $\gamma(t) \in U_1$, then for some $\eta > 0$ there is an η-neighborhood $V \ni \gamma(t)$ lying in U_1 (see Definition 1.5); hence, since γ is continuous, there exists $\delta > 0$ such that $|t'-t| < \delta$ implies $\gamma(t') \in V \subset U_1$. Therefore, $f(t') = f(t) = 1$; in particular, for every $\varepsilon > 0$ we have

$$|t' - t| < \delta \Rightarrow \gamma(t') = \gamma(t) = 1 \Rightarrow |f(t') - f(t)| = 0 < \varepsilon,$$

which proves that f is continuous. Since a function defined on an interval that takes only two values 1 and 2 cannot be continuous, we arrive at a contradiction.

Conversely, let U be connected; we will show that any two points of U can be joined by a curve. Pick an arbitrary point $a \in U$; it suffices to show that it can be joined by a curve to every point $b \in U$. To this end, set

$$U_1 = \{z \in U\colon a \text{ and } z \text{ can be joined by a curve}\},$$
$$U_2 = U \setminus U_1.$$

We claim that U_1 and U_2 are open. Indeed, if $z \in U_1$, i.e., a can be joined by a curve to z, then a can be joined by a curve to every point from an ε-neighborhood of z contained in U (Fig. 1.1), so this neighborhood is contained in U_1; we have shown that U_1 is open. If, on the other hand, $z \in U_2$, i.e., z cannot be joined by a curve to a, then every point from an ε-neighborhood $V \ni z$ contained in U cannot be joined by a curve to a: otherwise, the curve joining a to a point $z' \in V$ could be extended by the line segment between z' and z. Therefore, $V \subset U_2$ and U_2 is also open. It remains to observe that $U_1 \ni a$ (the point a can be joined to itself by a "curve," namely, the constant map), so U_1 is nonempty; since U is connected, we obtain $U_2 = \varnothing$, $U = U_1$, and the point a can be joined by a curve to every point from U, as required. □

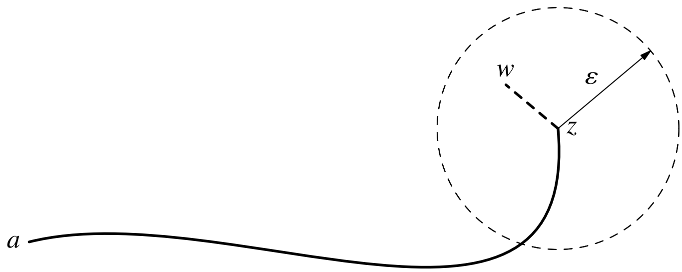

Fig. 1.1 For the proof of Proposition 1.17

Note also that in complex analysis, connected open subsets in $\mathbb{C}$ are often called *domains*.

1.3 Power Series

Consider a power series

$$c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots \tag{1.3}$$

(all coefficients c_j and the number a are complex numbers, the variable z is also assumed to be complex).

Proposition 1.18 (1) *There exists $R \in [0; +\infty]$ such that the series* (1.3) *converges absolutely for $|z-a| < R$ and diverges* (*its terms do not tend to zero*) *for $|z-a| > R$.*

(2) *We have*

$$R = 1/\overline{\lim}\, \sqrt[n]{|c_n|}. \tag{1.4}$$

In (1.4) it is meant that $1/(+\infty) = 0$, $1/0 = +\infty$. This formula is called the *Cauchy–Hadamard theorem*, and the "number" R is called the *radius of convergence* of the series (1.3). (The word "number" is in quotation marks because R can be infinite).

For the Cauchy–Hadamard theorem, see [5, Chap. V, Sec. 5].

If R is the radius of convergence of a series (1.3), then the set

$$\{z \in \mathbb{C} \colon |z - a| < R\}$$

is called the *disk of convergence* of this series (if $R = +\infty$, then the disk of convergence coincides with the entire plane).

Proposition 1.19 *A series* (1.3) *converges absolutely and uniformly on every compact subset of its disk of convergence.*

Note that in many interesting cases, a power series does not converge uniformly on the whole disk of convergence. Actually, uniform convergence on every compact subset of a given open set U (but not necessarily on the whole set U) is a typical situation in complex analysis.

Proof Let the radius of convergence be equal to $R > 0$, and let K be a compact subset of the disk of convergence. The continuous function $z \mapsto |z - a|$ attains its maximum on K; denote this maximum by r. We have $r < R$ and

$$K \subset \bar{D}_r = \{z \colon |z - a| \le r\}.$$

Pick a real number r' such that $r < r' < R$ and a number z_0 such that $|z_0 - a| = r'$. By Proposition 1.18, the series (1.3) converges absolutely for $z = z_0$, i.e., the series $\sum |c_n| r'^n$ converges; in particular, all terms of this series are bounded, i.e., there is a constant $C > 0$ such that

$$|c_n| r'^n \le C \iff |c_n| \le \frac{C}{r'^n} \quad \text{for all } n.$$

If now $z \in \bar{D}_r$, i.e., $|z - a| \le r$, then

$$|c_n (z - a)^n| \le |c_n| r^n = |c_n| r'^n \left(\frac{r}{r'}\right)^n \le C \left(\frac{r}{r'}\right)^n.$$

Therefore, on the set $\bar{D}_r$ the terms of the series (1.3) are uniformly bounded by the terms of the convergent geometric series $C \cdot \sum (r/r')^m$, so on $\bar{D}_r$ our power series converges uniformly by the Weierstrass M-test (Proposition 1.2). □

Corollary 1.20 *The sum of a power series is a continuous function of z on its disk of convergence.*

Proof Indeed, in terms of the previous proof, it suffices to check that the function is continuous on every set $\bar{D}_r$ for $0 < r < R$, and this is obvious by the uniform convergence. □

1.4 The Exponential Function

Proposition-Definition 1.21 The series

$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \tag{1.5}$$

converges absolutely for every $z \in \mathbb{C}$. Its sum is denoted by e^z or $\exp(z)$, and the function $z \mapsto e^z$ is called the *exponential function*, or *exponential*.

The absolute convergence of the series (1.5) for every z is well known (it follows, for example, from d'Alembert's ratio test). Since the series has infinite radius of convergence, the exponential is continuous on the entire complex plane $\mathbb{C}$. If $z \in \mathbb{R}$, then, of course, the exponential of z is equal to e^z in the usual sense.

Note that at the moment we do not define the function $z \mapsto a^z$ for any a different from e.

Setting $z = i\varphi$ with $\varphi \in \mathbb{R}$ in (1.5), we obtain the well-known Euler's formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi; \tag{1.6}$$

therefore, $e^{i\varphi}$ has absolute value 1 and argument φ, while a complex number with absolute value r and argument φ can be written as $re^{i\varphi}$.

Substituting $-\varphi$ for φ in (1.6) and then summing and subtracting the resulting equations, we arrive at the well-known formulas for sines and cosines:

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \quad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}. \tag{1.7}$$

Here is the main property of the exponential.

Proposition 1.22 *For any $z, w \in \mathbb{C}$ we have $e^{z+w} = e^z e^w$.*

Sketch of the proof It is well known (see [5, Chap. V, Sec. 5, Proposition 5]) that if $a_0 + a_1 + \ldots + a_n + \ldots$ and $b_0 + b_1 + \ldots + b_n + \ldots$ are absolutely convergent series with sums A and B, respectively, then the series

$$a_0b_0 + (a_0b_1 + a_1b_0) + \ldots + (a_0b_n + a_1b_{n-1} + \ldots + a_nb_0) + \ldots$$

is also absolutely convergent and its sum is equal to AB. Applying this to the series (1.5) for e^z and e^w, we see that the proposition will follow from the equation

$$1 \cdot \frac{w^n}{n!} + \frac{z}{1!} \cdot \frac{w^{n-1}}{(n-1)!} + \frac{z^2}{2!} \cdot \frac{w^{n-2}}{(n-2)!} + \ldots + \frac{z^n}{n!} \cdot 1 = \frac{(z+w)^n}{n!},$$

which is nothing else than the binomial theorem for $(z+w)^n$. □

Euler's formula implies that $e^{2\pi i} = 1$; then it follows from Proposition 1.22 that $e^{z+2\pi i} = e^z$ for all z. In other words, the exponential is a periodic function with period $2\pi i$.

1.5 Necessary Background From Multivariable Analysis

Let $U \subset \mathbb{R}^n$ be an open set. A map $F \colon U \to \mathbb{R}^m$ is said to be *differentiable*, or, for clarity, *real differentiable*, at a point $a \in U$ if there exists a linear map $L \colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$F(a+h) - F(a) = L(h) + \varphi(h) \quad \text{where } \lim_{h\to 0} \frac{|\varphi(h)|}{|h|} = 0.$$

See [5, Chap. VIII, Sec. 2].

The map L is called the *derivative* (or sometimes, for clarity, *real derivative*) of the map F at the point a (in [5, 6], the term "differential" is used).

If F is given by the formula

$$(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

and if all the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist and are continuous on the whole set U (in this case, F is said to be *of class* C^1), then F is real differentiable at every point $a \in U$ and its derivative, as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, is represented by the matrix $\left(\frac{\partial f_j}{\partial x_i}\right)$, called the *Jacobian matrix* of F; if $m = n$ (i.e., the Jacobian matrix is square), then its determinant is called the *Jacobian* of F (at the given point).

In most of the book, the above-mentioned results will be applied in the case where $m = n = 2$, $\mathbb{R}^m = \mathbb{R}^n = \mathbb{C}$, and the Euclidean norm is nothing else than the absolute value of a complex number.

We will also need some information on the area (or measure) of open sets in the plane. Since the reader is not assumed to be familiar with Lebesgue measure and integral, we adopt the following approach. Recall that the *support* of a function $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ is the closure of the set of all points where it does not vanish. If $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function with compact support, then by $\int\limits_{\mathbb{R}^n} \varphi \, dx$ we mean the integral of φ over a parallelepiped containing its support (see [6, Chap. XI, Sec. 1, Definition 7]).

Definition 1.23 Let $U \subset \mathbb{R}^n$ be an open set. A *partition of unity with compact supports* on U is a countable family of continuous functions $\{\varphi_i \colon \mathbb{R}^n \to [0; +\infty)\}$ satisfying the following properties:

(1) the support of every function φ_i is compact and lies in U;

(2) every point $x \in U$ has an ε-neighborhood $V \ni x$, $V \subset U$ such that the supports of all but finitely many functions φ_i are disjoint with V;

(3) for every point $x \in U$ we have $\sum\limits_i \varphi_i(x) = 1$ (this sum is finite by condition (2)).

Cf. [6, Chap. XV, Sec. 2, Definition 18].

Definition 1.24 Let $U \subset \mathbb{R}^n$ be an open set and $h : U \to [0; +\infty)$ be a continuous function. The *integral* of h *over* U is the number

$$\sum_i \int_{\mathbb{R}^n} h \cdot \varphi_i \, dx_1 \dots dx_n \tag{1.8}$$

where $\{\varphi_i\}$ is a partition of unity with compact supports on U (if the series in the left-hand side of (1.8) diverges, then the integral is defined to be $+\infty$).

One can (easily) check that the integral does not depend on the choice of a partition of unity.

Definition 1.25 Let $U \subset \mathbb{R}^n$ be an open set. The *measure* of U is the integral $\mu(U) = \int_U 1 \, dx_1 \dots dx_n$ (a nonnegative number or $+\infty$).

If U is a bounded set, then its measure is finite.

Let U and V be open subsets in $\mathbb{R}^n$; a *diffeomorphism* between U and V is a bijective map $F\colon U \to V$ of class C^1 whose inverse is also of class C^1.

Proposition 1.26 *Let $U, V \subset \mathbb{R}^n$ be open sets and $F\colon U \to V$ be a diffeomorphism of class C^1. Then*

$$\mu(V) = \int_U |J(F)(x_1, \dots, x_n)| \, dx_1 \dots dx_n$$

where $|J(F)(x_1, \dots, x_n)|$ is the determinant of the Jacobian matrix of F at the point $x_1, \dots, x_n$.

This proposition can be deduced from the change of variable formula [6, Chap. XI, Sec. 5, Theorem 1].

1.6 Linear Fractional Transformations

Definition 1.27 Let a, b, c, d be complex numbers such that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nondegenerate. Then the map from $\mathbb{C}$ to $\mathbb{C}$ given by the formula

$$z \mapsto \frac{az + b}{cz + d} \tag{1.9}$$

is called a *linear fractional map*, or *linear fractional transformation*.

The nondegeneracy condition guarantees that the numerator is not proportional to the denominator, i.e., the map is not constant.

Definition 1.27 is stated with a (deliberate) carelessness: if $c \neq 0$, then for $z = -d/c$ the denominator vanishes and the map (1.9) is not defined, so, strictly speaking, it cannot be called a map from $\mathbb{C}$ to $\mathbb{C}$. To avoid repeating this caveat, it is convenient to proceed as follows. Consider the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where ∞ is a symbol (called, sure enough, "infinity") to be dealt with according to the following rules.

First, we set $a \pm \infty = \infty$ and $a/\infty = 0$ for every complex number a, and also $a \cdot \infty = \infty$ and $a/0 = \infty$ for every *nonzero* complex number a (the expressions $0 \cdot \infty$, $\infty \pm \infty$, and $0/0$ are still not defined).

Second, if $f(z) = (az + b)/(cz + d)$ is a linear fractional map, we set

$$f(\infty) = \lim_{|z|\to\infty} f(z) = \frac{a}{c}.$$

With these conventions, every linear fractional transformation becomes a one-to-one map from $\overline{\mathbb{C}}$ onto itself.

The set $\overline{\mathbb{C}}$ is called the *extended complex plane*, or *Riemann sphere*.

Sets of the form $\{z \in \mathbb{C} \colon |z| > R\}$ will be regarded as "punctured neighborhoods of infinity" on the Riemann sphere; and the same sets with the point ∞ added, as neighborhoods of infinity (without puncture). For the reader familiar with the corresponding definitions, I should mention that this definition of neighborhoods of infinity endows $\overline{\mathbb{C}}$ with the structure of a topological space homeomorphic to the two-dimensional sphere (see Chap. 13).

The main property of linear fractional transformations is that they take lines and circles to lines and circles. More exactly, for every line $\ell \subset \mathbb{C}$ the set $\ell \cup \{\infty\} \subset \overline{\mathbb{C}}$ will be called a "line on the Riemann sphere" (it is the closure of the set $\ell \subset \mathbb{C}$ with respect to the above-mentioned topology on $\overline{\mathbb{C}}$). By a circle on the Riemann sphere we will mean a usual circle in $\mathbb{C}$ (a circle is a bounded set, it "does not go to infinity," so there is no need to add ∞). Now we introduce the following term.

Definition 1.28 A *generalized circle* is a subset in $\overline{\mathbb{C}}$ that is either a line on the Riemann sphere or a circle.

Proposition 1.29 *Every linear fractional transformation takes generalized circles to generalized circles.*

Proof Every linear fractional transformation $z \mapsto (az + b)/(cz + d)$ is easily seen to be a composition of transformations of the form $z \mapsto Az$ $(A \neq 0)$, $z \mapsto z + B$, and $z \mapsto 1/z$ (to prove this, it suffices to divide the polynomial $az + b$ by $cz + d$ with remainder). Transformations of the first two types (dilations and translations) take lines to lines and circles to circles, so it remains to consider only the case of $1/z$.

Lemma 1.30 *The equation of any generalized circle has the form*

$$pz\bar{z} + Az + \bar{A}\bar{z} + q = 0 \quad \textit{where } A \in \mathbb{C},\ p, q \in \mathbb{R}. \tag{1.10}$$

Proof Let $\ell \subset \mathbb{C}$ be a line given by an equation $px + qy + r = 0$ $(p, q, r \in \mathbb{R})$. Substituting $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$, we see that the equation of ℓ has the form

$$\frac{p - iq}{2} z + \frac{p + iq}{2} \bar{z} + r = 0,$$

or, denoting $(p - iq)/2 = A$,

$$Az + \bar{A}\bar{z} + r = 0 \quad \text{where } A \in \mathbb{C},\ r \in \mathbb{R}.$$

In a similar way, the equation of any circle in the plane has the form

$$p(x^2 + y^2) + qx + ry + s = 0 \quad \text{where } p, q, r, s \in \mathbb{R}.$$

Substituting $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$, we obtain the equation

$$pz\bar{z} + Bz + \bar{B}\bar{z} + s \quad (B \in \mathbb{C}, s \in \mathbb{R}),$$

which is again an equation of the form (1.10). □

Returning to the action of the transformation $z \mapsto 1/z$ on generalized circles, observe that if $w = 1/z$, then $z = 1/w$; substituting $1/w$ for z in (1.10) yields an equation for w which also has the form (1.10), and we are done. □

Proposition 1.31 *Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be two triples of distinct points on the Riemann sphere. Then there is a unique linear fractional transformation that sends each a_i to the corresponding b_i.*

Sketch of the proof We verify this for the case where $b_1 = 0$, $b_2 = 1$, $b_3 = \infty$, and all a_j are finite (leaving the rest as an exercise for the reader).

If a linear fractional transformation $z \mapsto (pz + q)/(rz + s)$ sends a_1 to 0 and a_3 to ∞, then $pz + q$ must vanish for $z = a_1$ and $rz + s$ must vanish for $z = a_3$. Therefore, the map can be rewritten as

$$f(z) = c\frac{z - a_1}{z - a_3}, \quad c \neq 0;$$

the condition $f(a_2) = 1$ fixes the coefficient c. □

An important property of linear fractional maps is that they are *conformal*: linear fractional transformations preserve angles between curves. Later in this section, we will need this property for angles between (generalized) circles, which can be proved by quite elementary methods; however, we postpone the proof until Chap. 2, where this property will be deduced from a more general fact, and also the general definition of a conformal map will be given.

We also need the notion of symmetry (reflection) with respect to a circle.

Definition 1.32 Let $C \subset \overline{\mathbb{C}}$ be a generalized circle. Points $p_1, p_2 \in \overline{\mathbb{C}}$ are said to be *symmetric* with respect to C if they do not lie on C but every generalized circle that passes through p_1 and p_2 is orthogonal to C.

If p is a point on C, then it is symmetric to itself.

If C is a line, then symmetry with respect to C in the sense of Definition 1.32 is equivalent to symmetry in the ordinary sense (see Fig. 1.2).

Proposition 1.33 *For every generalized circle C and every point $p \in \overline{\mathbb{C}}$ there is a unique point symmetric to p with respect to C.*

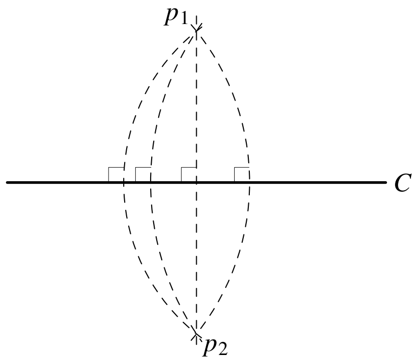

Fig. 1.2 Symmetry with respect to a line from the point of view of Definition 1.32

Proof If p and p' are symmetric points with respect to C and $A: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a linear fractional transformation, then (in view of Definition 1.32 and the preservation of angles) $A(p)$ and $A(p')$ are symmetric points with respect to $A(C)$, and vice versa. Applying a linear fractional transformation that takes C to a line reduces the question to the unique existence of a point symmetric to a given point with respect to a line. □

Exercises

1.1. Write the number $-1 + i\sqrt{3}$ in polar form.
1.2. Simplify the expression $(-\sqrt{3} - i)^{2017}$.
1.3. Let $\zeta_1, \ldots, \zeta_n$ be all roots of the equation $z^n = 1$ where n is a positive integer. Find the sum
$$\zeta_1^k + \zeta_2^k + \ldots + \zeta_n^k$$
for every $k \in \mathbb{Z}$.
1.4. (a) Find (draw) the images of the lines $\operatorname{Re} z = 1/2$, $\operatorname{Re} z = 1$, and $\operatorname{Re} z = 3/2$ under the map $z \mapsto z^2$; what are these curves called?
(b) The same question for the lines $\operatorname{Re} z = -1/2$, $\operatorname{Re} z = -1$, and $\operatorname{Re} z = -3/2$ (before you start calculating, consider whether this is worth doing).
(c) The same question for the lines $\operatorname{Im} z = 1/2$, $\operatorname{Im} z = 1$, and $\operatorname{Im} z = 3/2$ (and the same warning).
1.5. Give an "epsilon-delta" proof that the relation $\lim_{n\to\infty} z_n = a$ is equivalent to $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} a$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} a$.
1.6. For each of the following subsets in $\mathbb{C}$, determine whether it is open, or closed, or neither.
(a) $\{z\colon \operatorname{Re}(z) > 2, \operatorname{Im}(z) \le 1\}$.
(b) $\{z = x + iy\colon \sin x + \cos y > 2017\}$.
(c) $\{z\colon \operatorname{Im}(z) \ge 1, \operatorname{Re}(z) \ge -1\}$.
(d) The set of $z \in \mathbb{C}$ such that either $\operatorname{Im} z \ne 0$, or $\operatorname{Im} z = 0$ and $\operatorname{Re} z$ is rational.

1.7. Let U, V be open subsets in $\mathbb{C}$ and $f\colon U \to V$ be a map. Show that f is continuous in the "epsilon-delta" sense if and only if for every open subset $V_1 \subset V$ the set $f^{-1}(V_1)$ is also open.

1.8. Let U, V be open subsets in $\mathbb{C}$ and $f\colon U \to V$ be a continuous bijective map for which the inverse map $f^{-1}\colon V \to U$ is also continuous (such maps are called *homeomorphisms*). Let $b \in V$, and let φ be a function defined on a punctured neighborhood of b. Show that

$$\lim_{w \to b} \varphi(w) = \lim_{z \to f^{-1}(b)} \varphi(f(z)).$$

1.9. Show that every open set $U \subset \mathbb{C}$ can be represented as a union of pairwise disjoint connected open subsets. (*Hint.* Call two points equivalent if they can be joined by a path.)

These open subsets are called the *connected components* of U.

1.10. Let $[a; b] \subset \mathbb{R}$ be a closed interval contained in a (possibly infinite) union of open intervals $I_j \subset \mathbb{R}$. Show that there exist $a = a_0 < a_1 < a_2 < \ldots < a_n = b$ such that every closed interval $[a_k; a_{k+1}]$ is contained in at least one of I_j.

1.11. Let $U \subset \mathbb{C}$ be an open set and $K \subset U$ be a compact subset of U. Show that

$$\inf_{\substack{z \in K \\ w \in \mathbb{C} \setminus U}} |z - w| > 0.$$

1.12. Find the radius of convergence of the series $\sum_{n=1}^{\infty} n! x^{n!}$.

1.13. The series

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

converges for every $x \in \mathbb{R}$. Is this convergence uniform on $\mathbb{R}$?

1.14. Let $e^{z+T} = e^z$ for all $z \in \mathbb{C}$. Show that $T = 2\pi i n$ for some integer n.

1.15. Find the images of the vertical and horizontal lines under the map $z \mapsto e^z$.

1.16. Let $f\colon z \mapsto \frac{a_1 z + b_1}{c_1 z + d_1}$ and $g\colon z \mapsto \frac{a_2 z + b_2}{c_2 z + d_2}$ be linear fractional transformations. Show that their composition $g \circ f$ is a linear fractional transformation of the form $z \mapsto \frac{a_3 + b_3}{c_3 z + d_3}$ where

$$\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

1.17. Fill in the gaps in the proof of Proposition 1.31.

1.18. Show that every sequence of points on the Riemann sphere contains a subsequence converging to a complex number or to ∞.

1.19. Let C be the circle of radius R centered at the origin. Show that the point symmetric to a point z with respect to C is $R^2/\bar{z}$.

1.20. Show that the composition of two reflections with respect to generalized circles is a linear fractional transformation.