ISSN 1064-5624, Doklady Mathematics, 2020, Vol. 101, No. 1, pp. 49–52. © Pleiades Publishing, Ltd., 2020.
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MATHEMATICS

Asymptotic Expansions of Solutions to the Riccati Equation

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Presented by Academician of the RAS B.N. Chetverushkin November 19, 2019

Received November 26, 2019; revised November 27, 2019; accepted December 17, 2019

Abstract—Scalar real Riccati equations with coefficients expandable in convergent power series in a neighborhood of infinity are considered. Extendable solutions to equations of this kind are studied. Methods of power geometry are used to obtain conditions for asymptotic series expansions of these solutions.

Keywords: Riccati equation, extendable solution, power geometry, Newton polygon, asymptotic series

DOI: 10.1134/S1064562420010196

We study the scalar Riccati equation

$$y' + \sum_{i=0}^{2} f_i(x)y^i = 0, \quad y = y(x), \quad x, y \in \mathbb{R}. \tag{1}$$

We assume that the functions $f_i(x)$, $i \in \{0,1,2\}$, are representable as uniformly absolutely convergent real power series

$$f_i(x) = \sum_{j=1}^{\infty} c_{ij}x^{p_{ij}}, \quad c_{ij}, p_{ij} = \text{const} \in \mathbb{R},$$
$$p_{ij+1} < p_{ij}, \quad \lim_{j\to\infty} p_{ij} = -\infty, \quad i \in \{0,1,2\}, \tag{2}$$

in a neighborhood of the point $x = +\infty$. From now on, when power series are mentioned, exponents are assumed to be real numbers, but not necessarily integers, in contrast to traditional power series. In this work, we study arbitrary solutions to Eq. (1) defined in some neighborhood of $x = +\infty$ (solutions of this kind are called extendable).

We assume that Eq. (1) is nonhomogeneous, since the homogeneous equation is a Bernoulli equation, which is directly integrable. Thus, in what follows, we assume that $c_{01}c_{21} \neq 0$ in (2).

In [1] Riccati considered the following equation of form (1):

$$y' + ay^2 + bx^p = 0, \quad a, b, p = \text{const}. \tag{3}$$

The problem of integrating Eq. (3) by quadratures was studied first (see [2]). After many attempts of this kind, Jacob Bernoulli discovered the possibility of expanding the solutions to Eq. (3) in series (see [2] for his correspondence with Leibniz). It turned out later that the solutions to Eq. (3) are expandable in convergent series using Bessel functions (see, for example, [3]). In recent years, power geometry methods were developed that make it possible to obtain series expansions of solutions to differential equations [4].

In what follows, we will use terms accepted in power geometry [4]. The Newton polygon N of Eq. (1) with condition (2) is the closed convex hull of the points $Q = (-1, 1)$, $Q_{ij} = (p_{ij}, i)$, $i \in \{0,1,2\}$, $j \in \{1, 2, \dots\}$. When we study asymptotic expansions of solutions to Eq. (1) in a neighborhood of the point $x = +\infty$, an important role is played by the location of the point Q with respect to the right-hand boundary of the polygon N.

The case when the point Q belongs to the right-hand boundary of the polygon N, that is, the condition

$$\max(p_{01} + p_{21}, 2p_{11}) \le -2 \tag{4}$$

is satisfied, was studied in [5, 6], where it was shown that extendable solutions to Eq. (1) can be represented as series that are absolutely uniformly convergent in a neighborhood of $x = +\infty$. These series were described in detail, and an algorithm to calculate them was given.

In this study, we consider the case when condition (4) is violated, that is, the point Q does not belong to the right-hand boundary of the polygon N, and the inequality

$$\max(p_{01} + p_{21}, 2p_{11}) > -2 \tag{5}$$

holds.

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When $2p_{11} > p_{01} + p_{21}$ in this case, we can calculate two formal power series (formal solutions to Eq. (1)) being asymptotic representations (expansions) of all nontrivial extendable solutions to Eq. (1). These series are generally divergent in a neighborhood of the point $x = +\infty$.

When $2p_{11} \le p_{01} + p_{21}$, we additionally need to analyze the coefficients c_{i1} of the first approximations in the expansions of the functions $f_i(x)$. If $\tilde{c}_{11}^2 - 4c_{01}c_{21} < 0$, then there are no extendable solutions to Eq. (1) (here, $\tilde{c}_{11} = c_{11}$ when $2p_{11} = p_{01} + p_{21}$ and $\tilde{c}_{11} = 0$ when $2p_{11} < p_{01} + p_{21}$). If $\tilde{c}_{11}^2 - 4c_{01}c_{21} > 0$, then there are extendable solutions, and a power transformation reduces this situation to the above main case when $2p_{11} > p_{01} + p_{21}$. When condition (5) and the equality $\tilde{c}_{11}^2 - 4c_{01}c_{21} = 0$ hold, a power transformation reduces Eq. (1) to either a form for which condition (4) is satisfied or a form for which $\tilde{c}_{11}^2 - 4c_{01}c_{21} \ne 0$.

1. STATEMENT OF THE MAIN RESULTS

Remark 1. To state the results, it is convenient to write $f_1(x)$ in another form.

If condition (5) holds, then we assume, without loss of generality, that the series expansion $f_1(x) = \sum_{j=1}^{\infty} c_{1j}x^{p_{1j}}$ is such that $p_{11} \ge 0.5(p_{01} + p_{21})$ (if $p_{11} = 0.5(p_{01} + p_{21})$, then the coefficient c_{11} can be zero; if $p_{11} > 0.5(p_{01} + p_{21})$, then $c_{11} \ne 0$).

We recall the notion of an asymptotic expansion (representation) of a function.

Definition. A power series $\sum_{i=1}^{\infty} c_i x^{\alpha_i}$, $\alpha_{i+1} < \alpha_i$, $\lim_{i\to\infty}\alpha_i = -\infty$, is said to be an *asymptotic expansion* (*representation*) of a function $f(x)$ as $x \to +\infty$ if

$$\lim_{x\to+\infty} x^{-\alpha_m}\left(f(x) - \sum_{i=1}^{m} c_i x^{\alpha_i}\right) = 0$$

for all $m \ge 1$.

Theorem 1. *If conditions* (2) *and* (5) *and one of the two conditions*

(1) $p_{01} + p_{21} < 2p_{11}$,

(2) $p_{01} + p_{21} = 2p_{11}$, $c_{11}^2 - 4c_{01}c_{21} > 0$

hold, *then there are two formal series*

$$\begin{gathered} z_j(x) = \sum_{i=1}^{\infty} a_{ji}x^{s_{ji}}, \quad j = 1, 2, \\ s_{11} = p_{01} - p_{11}, \quad s_{21} = p_{11} - p_{21}, \quad s_{ji+1} < s_{ji}, \\ \lim_{i\to\infty} s_{ji} = -\infty, \quad a_{ji} = \text{const}, \end{gathered} \tag{6}$$

that formally satisfy Eq. (1). *The set M of extendable solutions to Eq.* (1) *is the union of two nonempty sets* $M = M_1 \cup M_2$, *where elements of the sets* M_j, $j = 1, 2$, *are extendable solutions whose asymptotic expansions as* $x \to +\infty$ *are* $z_j(x)$. *In* (6), *when condition* (1) *is satisfied*, *we have*

$$a_{11} = -c_{01}(c_{11})^{-1}, \quad a_{21} = -c_{11}(c_{21})^{-1};$$

when condition (2) *is satisfied*, *we have*

$$a_{11} = (2c_{21})^{-1}(-c_{11} + b),$$
$$a_{21} = (2c_{21})^{-1}(-c_{11} - b), \quad b = \sqrt{c_{11}^2 - 4c_{01}c_{21}}.$$

The meaning of the inequality in condition (2) in Theorem 1 is explained in Theorems 2 and 3.

Theorem 2. *If conditions* (2) *and* (5) *and the equality* $p_{01} + p_{21} = 2p_{11}$ *hold*, *then there are no extendable solution to Eq.* (1) *when* $c_{11}^2 - 4c_{01}c_{21} < 0$.

Remark 2. Condition (2) in this theorem is excessive. For the theorem to be valid, it suffices that the functions $f_i(x)$, $i \in \{0, 1, 2\}$ have power asymptotics as $x \to +\infty$:

$$f_i(x) = c_{i1}x^{p_{i1}}(1 + o(x^{-\varepsilon})),$$
$$\varepsilon = \text{const} > 0, \quad i \in \{0, 1, 2\}.$$

We now consider the case when conditions (2) and (5) and the equalities $p_{01} + p_{21} = 2p_{11}$ and $c_{11}^2 - 4c_{01}c_{21} = 0$ hold. In this case, we call Eq. (1) degenerate; otherwise, it is called nondegenerate.

Theorem 3. *If Eq.* (1) *is degenerate*, *then there exists a transformation of the form*

$$\begin{gathered} y = z + h(x), \\ h(x) = \sum_{i=1}^{m} h_i x^{r_i}, \quad h_i, r_i = \text{const}, \\ r_{i+1} < r_i, \quad r_1 = p_{01} - p_{11}, \quad h_1 = -(2c_{21})^{-1}c_{11}, \end{gathered} \tag{7}$$

that makes Eq. (1) *nondegenerate.*

Remark 3. If transformation (7) reduces Eq. (1) to the form

$$\begin{gathered} z' + \sum_{i=0}^{2} g_i(x)z^i = 0, \\ g_i(x) = \sum_{j=1}^{\infty} \tilde{c}_{ij}x^{\tilde{p}_{ij}}, \quad \tilde{p}_{ij+1} < \tilde{p}_{ij}, \end{gathered} \tag{8}$$

such that

$$\max(\tilde{p}_{01} + \tilde{p}_{21}, 2\tilde{p}_{11}) \le -2, \tag{9}$$

then the existing extendable solutions to Eq. (8) are representable as series absolutely uniformly convergent in a neighborhood of $x = +\infty$ (see [5, 6]). The situation when condition (9) does not hold for nondegenerate equation (8) is described above in Theorems 1 and 2.

2. EXAMPLES

Example 1. We consider the equation

$$y' + y^2 - xy + 1 = 0, \qquad (10)$$

for which the assumptions of Theorem 1 are satisfied. We show that one of the series in (6) is convergent in this case, while the other is divergent. To calculate these series, we consider truncated equations and the corresponding transformations of the equation under consideration (see [4]). Figure 1 shows the Newton polygon N of Eq. (10).

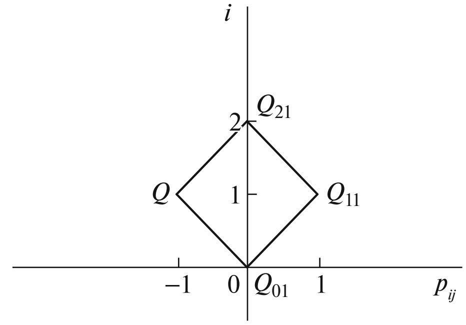

Fig. 1.

The function $z_1(x) = u_1(x) = x^{-1}$ is a solution to the truncated equation $-xu_1 + 1 = 0$, which corresponds to the right-hand lower edge of the polygon N; simultaneously, this function is a solution to Eq. (10). Calculating terms of the series $z_2(x) = \sum_{i=1}^{\infty} a_{2i}x^{s_{2i}}$ yields the following results. The first term $v_1 = a_{21}x^{s_{21}} = x$ is a solution to the truncated equation $v_1^2 - xv_1 = 0$, which corresponds to the right-hand upper edge of the Newton polygon of Eq. (10). Upon the change $y = y_1 + x$, Eq. (10) takes the form

$$y_1' + y_1^2 + xy_1 + 2 = 0. \qquad (11)$$

The next term $v_2 = a_{22}x^{s_{22}} = -2x^{-1}$ of the series $z_2(x)$ is a solution to the truncated equation $xv_2 + 2 = 0$, which corresponds to the right-hand lower edge of the Newton polygon of Eq. (11). By performing the change $y_1 = y_2 - 2x^{-1}$, we transform Eq. (11), after which the truncated equation corresponding to the right-hand lower edge of the Newton polygon of the transformed equation is again considered. We continue this process to calculate the next terms of the formal series $z_2(x)$. As a result of the calculations, we can straightforwardly see that the estimate $v_k(x) = a_kx^{-2k+3}$, $a_k \le -k!$, $k \ge 2$, holds, which implies that the series $z_2(x) = \sum_{k=1}^{\infty} v_k(x)$ diverges in any neighborhood of the point $x = +\infty$.

We now give an example illustrating Theorem 3.

Example 2. We consider the Riccati equation

$$y' + xy^2 + (2x - 2)y + x - 2 + x^{-1} + x^{-2} + x^{-4} = 0. \qquad (12)$$

This equation is degenerate, since conditions (2) and (5) and the equalities $p_{01} + p_{21} = 2p_{11}$ and $c_{11}^2 - 4c_{01}c_{21} = 0$ hold. Figure 2 shows the Newton polygon N of Eq. (12).

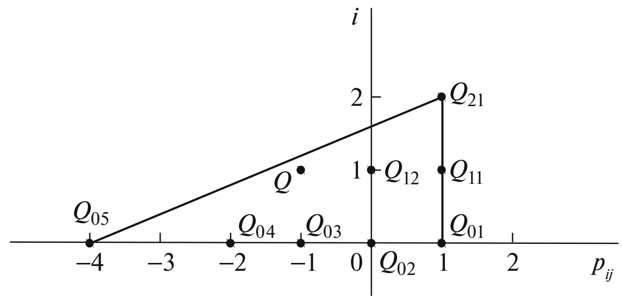

Fig. 2.

The function $u_1(x) = -1$ is a solution to the truncated equation $xu_1^2 + 2u_1x + x = 0$, which corresponds to the right-hand vertical edge of the polygon N. Upon the change $y = y_1 - 1$, Eq. (12) takes the form

$$y_1' + xy_1^2 - 2y_1 + x^{-1} + x^{-2} + x^{-4} = 0. \qquad (13)$$

In this case, condition (5) and the equalities $p_{01} + p_{21} = 2p_{11}$ and $c_{11}^2 - 4c_{01}c_{21} = 0$ hold once again, that is, Eq. (13) is degenerate.

The function $u_2(x) = x^{-1}$ is a solution to the truncated equation $xu_2^2 - 2u_2 + x^{-1} = 0$, which corresponds to the right-hand edge of the Newton polygon of Eq. (13). Upon the change $y_1 = y_2 + x^{-1}$, Eq. (13) takes the form

$$y_2' + xy_2^2 + x^{-4} = 0. \qquad (14)$$

This equation satisfies condition (4) and is thus nondegenerate. The results obtained in [5, 6] are applicable to this equation. The function $u_3(x) = \frac{1}{3}x^{-3}$ is a solution to the truncated equation $u_3' + x^{-4} = 0$, which corresponds to the right lower edge of the Newton polygon of Eq. (14). By continuing the calculations, we conclude that there is a solution to Eq. (14) presented by the power series $y_2 = \frac{1}{3}x^{-3} + \ldots$, which absolutely uniformly converges in a neighborhood of the point $x = +\infty$. Equation (14) is associated with the truncated equation $v' + xv^2 = 0$, which is solved by the function $v(x) = 2x^{-2}$. By continuing the calculations according to [5, 6], we derive a solution to Eq. (14) presented by the absolutely uniformly convergent series $y_2 = 2x^{-2} - x^{-3} - x^{-4}(\ln x + a) + \dots$ in a neighborhood of $x = +\infty$, whose terms are the products of decreasing powers of x and polynomials in $\ln x$. Thus, we deduce that the family of extendable solutions to Eq. (12) includes the power series $y = -1 + x^{-1} + \frac{1}{3}x^{-3} + \dots$ and the family of series $y = -1 + x^{-1} + 2x^{-2} - x^{-3} - x^{-4}(\ln x + a) + \dots$, where a is an arbitrary constant. All the series absolutely uniformly converge in a neighborhood of the point $x = +\infty$. We can show that there are no other extendable solutions to Eq. (12).

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Translated by N. Berestova