OPEN r-SPIN THEORY II: THE ANALOGUE OF WITTEN'S CONJECTURE FOR r-SPIN DISKS

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ABSTRACT. We conclude the construction of r-spin theory in genus zero for Riemann surfaces with boundary. In particular, we define open r-spin intersection numbers, and we prove that their generating function is closely related to the wave function of the rth Gelfand–Dickey integrable hierarchy. This provides an analogue of Witten's r-spin conjecture in the open setting and a first step toward the construction of an open version of Fan–Jarvis–Ruan–Witten theory. As an unexpected consequence, we establish a mysterious relationship between open r-spin theory and an extension of Witten's closed theory.

1. INTRODUCTION

In the study of the intersection theory on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$, one of the most important modern results is Witten's conjecture [27], which was proven by Kontsevich [15]. Let $\psi_1, \dots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$ be the first Chern classes of the cotangent line bundles at the n marked points, and let

$$F^c(t_0, t_1, \dots, \varepsilon) := \sum_{\substack{g\geq 0, n\geq 1 \\ 2g-2+n>0}} \sum_{d_1,\dots,d_n\geq 0} \frac{\varepsilon^{2g-2}}{n!} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) t_{d_1} \cdots t_{d_n}$$

be the generating function of their intersection numbers. Here, $\{t_i\}_{i\geq 0}$ and ε are formal variables and the superscript "c", which stands for "closed", is to contrast with the open theory discussed below. Witten's conjecture states that $\exp(F^c)$ is a tau-function of the Korteweg–de Vries (KdV) hierarchy, or equivalently, that $\exp(F^c)$ satisfies a certain collection of linear differential equations known as the Virasoro equations. This uniquely determines all ψ-integrals on $\overline{\mathcal{M}}_{g,n}$.

Witten also proposed a generalization of his conjecture [28], in which the moduli space of curves is enhanced to the moduli space of r-spin structures. On a smooth marked curve $(C; z_1, \dots, z_n)$, an r-spin structure is a line bundle S together with an isomorphism

$$S^{\otimes r} \cong \omega_C \left(-\sum_{i=1}^n a_i [z_i] \right),$$

where $a_i \in \{0, 1, \dots, r-1\}$. There is a natural compactification $\overline{\mathcal{M}}^{1/r}_{g,\{a_1,\dots,a_n\}}$ of the moduli space of r-spin structures on smooth curves, and this space admits a virtual fundamental class c_W known as Witten's class. In genus zero, Witten's class is defined by

$$c_W := e((R^1\pi_*\mathcal{S})^\vee), \tag{1.1}$$

where $\pi : \mathcal{C} \to \overline{\mathcal{M}}^{1/r}_{0,\{a_1,\dots,a_n\}}$ is the universal curve and $\mathcal{S}$ the universal r-spin structure. In higher genus, on the other hand, $R^1\pi_*\mathcal{S}$ may not be a vector bundle, and the definition of Witten's class is much more intricate; see [24, 8, 17, 12, 7] for various constructions.

The *closed r-spin intersection numbers* are defined by

$$\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\frac{1}{r},c} := r^{1-g} \int_{\overline{\mathcal{M}}^{1/r}_{g,\{a_1,\ldots,a_n\}}} c_W \cap \psi_1^{d_1} \cdots \psi_n^{d_n}. \tag{1.2}$$

Witten's r-spin conjecture then states that if t_d^a are formal variables for $0 \leq a \leq r-1$ and $d \geq 0$, and if

$$F^{\frac{1}{r},c}(t_*^*, \varepsilon) := \sum_{\substack{g\geq 0, n\geq 1 \\ 2g-2+n>0}} \sum_{\substack{0\leq a_1,\ldots,a_n\leq r-1 \\ d_1,\ldots,d_n\geq 0}} \frac{\varepsilon^{2g-2}}{n!} \left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\frac{1}{r},c} t_{d_1}^{a_1} \cdots t_{d_n}^{a_n}$$

is the generating function of the r-spin intersection numbers, then $\exp(F^{\frac{1}{r},c})$ becomes, after a simple change of variables, a tau-function of the rth Gelfand–Dickey hierarchy. This result was proven by Faber–Shadrin–Zvonkine [11].

A new direction in the intersection theory of the moduli spaces of curves was initiated by Pandharipande, Solomon, and the third author [23], studying the moduli space of Riemann surfaces with boundary. In that work, a moduli space $\overline{\mathcal{M}}_{0,k,l}$ was considered, which parameterizes tuples $(\Sigma; x_1, \ldots, x_k; z_1, \ldots, z_l)$ in which Σ is a stable disk, $x_i \in \partial\Sigma$ are boundary marked points, and $z_j \in \Sigma \setminus \partial\Sigma$ are internal marked points. Furthermore, intersection numbers on $\overline{\mathcal{M}}_{0,k,l}$, which can be viewed as integrals of ψ-classes at the internal marked points, were constructed and hence a generating function $F_0^o(t_0, t_1, \ldots, s)$ was defined as a direct generalization of the genus-zero part of F^c; the new formal variable s tracks the number of boundary marked points. This construction has been extended in [25] to all genera, and the complete open potential $F^o(t_0, t_1, \ldots, s, \varepsilon)$ has been constructed. Verifying a conjecture of [23], the first and third authors proved that F^o satisfies certain "open Virasoro equations" [6], and moreover, the first author proved that these equations imply that $\exp(F^o)$ is explicitly related to the wave function of the KdV hierarchy [2]. These results provide an open analogue of Witten's conjecture on $\overline{\mathcal{M}}_{g,n}$. An introduction to these results in more physical language can be found in [10].

1.1. **Genus-zero open r-spin intersection theory.** A natural question, then, is what the open analogue of Witten's r-spin conjecture should be. In order to make sense of the conjecture, one first must define an appropriate open r-spin moduli space $\overline{\mathcal{M}}^{1/r}_{g,k,\{a_1,\ldots,a_l\}}$ and an open analogue of Witten's bundle $(R^1\pi_*\mathcal{S})^\vee$. We carried out this construction in genus zero in [4], leading to a definition of the moduli space of *graded r-spin disks* $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ as well as the open Witten bundle $\mathcal{W}$ and cotangent line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_l$ at the internal marked points.

In this work, we build on the foundations of [4] to define *open r-spin intersection numbers*.[1] As in [23], this is a subtle task because the open moduli space has boundary, so the integration of the top Chern classes of the open Witten bundle and the cotangent line bundles over the open moduli space is not well-defined; to define intersection numbers, one must find a canonical way to prescribe the boundary behavior of sections of the bundles. Additionally, one must resolve various orientation issues, constructing orientations for both the moduli space and the Witten bundle such that the Witten bundle becomes canonically relatively oriented.

[1]The construction from [23] is recovered as a special case, when $r = 2$ and all a_i are zero.

After addressing these issues, the resulting intersection numbers are defined by

$$\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_l}^{a_l} \sigma^k \right\rangle_0^{\frac{1}{r},o} := \int_{\overline{\mathcal{PM}}^{1/r}_{0,k,\{a_1,\ldots,a_l\}}} e\left(\mathcal{W} \oplus \bigoplus_{i=1}^{l} \mathbb{L}_i^{\oplus d_i}, s_{\text{canonical}} \right), \tag{1.3}$$

where, on a manifold with boundary M, the notation $e(E,s)$ denotes the Euler class of E relative to the section s of $E|_{\partial M}$. The section $s_{\text{canonical}}$ is any choice of nowhere-vanishing boundary section of $\mathcal{W} \oplus \bigoplus_{i=1}^{l} \mathbb{L}_i^{\oplus d_i}$ that satisfies a "canonicity" condition defined Section 3; Theorem 3.17 shows that such $s_{\text{canonical}}$ indeed exists and that any two choices give rise to the same intersection numbers. Finally, $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ is a subspace of $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ obtained by removing certain boundary strata; the restriction to this subspace makes defining canonicity more convenient without affecting the intersection numbers that result.

Equipped with these numbers, we define an open r-spin potential by

$$F_0^{\frac{1}{r},o}(t_*^*, s) := \sum_{\substack{k,l \geq 0 \\ k+2l>2}} \sum_{\substack{0 \leq a_1,\ldots,a_l \leq r-1 \\ d_1,\ldots,d_l \geq 0}} \frac{1}{k!l!} \left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_l}^{a_l} \sigma^k \right\rangle_0^{\frac{1}{r},o} t_{d_1}^{a_1} \cdots t_{d_l}^{a_l} s^k.$$

Before stating the main theorem regarding $F^{\frac{1}{r},o}$, we must recall some definitions from the theory of the Gelfand–Dickey hierarchy.

1.2. **Gelfand–Dickey hierarchy.** Consider formal variables T_i for $i \geq 1$. A *pseudo-differential operator* A is a Laurent series

$$A = \sum_{n=-\infty}^{m} a_n(T_1, T_2, \ldots, \varepsilon)\partial_x^n,$$

where $m \in \mathbb{Z}$, ∂_x is considered as a formal variable, and $a_n(T_1, T_2, \ldots, \varepsilon) \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_1, T_2, \ldots]]$. The space of such operators is endowed with the structure of a non-commutative associative algebra, in which the multiplication is defined by the formula

$$\partial_x^k \circ f := \sum_{l=0}^{\infty} \frac{k(k-1)\ldots(k-l+1)}{l!} \frac{\partial^l f}{\partial x^l} \partial_x^{k-l},$$

where $k \in \mathbb{Z}$, $f \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_*]]$, and the variable x is identified with T_1. For any $r \geq 2$ and any pseudo-differential operator A of the form

$$A = \partial_x^r + \sum_{n=1}^{\infty} a_n \partial_x^{r-n},$$

there exists a unique pseudo-differential operator $A^{\frac{1}{r}}$ of the form

$$A^{\frac{1}{r}} = \partial_x + \sum_{n=0}^{\infty} \widetilde{a}_n \partial_x^{-n}$$

such that $\left(A^{\frac{1}{r}}\right)^r = A$.

Let $r \geq 2$, and consider the operator

$$L := \partial_x^r + \sum_{i=0}^{r-2} f_i \partial_x^i, \quad f_i \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_*]].$$

It is not hard to check that for any $n \geq 1$, the commutator $[(L^{n/r})_+, L]$ has the form $\sum_{i=0}^{r-2} h_i \partial_x^i$ with $h_i \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_*]]$, where $(\cdot)_+$ denotes the part of a pseudo-differential operator with non-negative powers of ∂_x. The *rth Gelfand–Dickey hierarchy* is the following system of partial differential equations for the functions $f_0, f_1, \ldots, f_{r-2}$:

$$\frac{\partial L}{\partial T_n} = \varepsilon^{n-1}[(L^{n/r})_+, L], \quad n \geq 1. \tag{1.4}$$

Consider the solution L of this hierarchy specified by the initial condition

$$L|_{T_{\geq 2}=0} = \partial_x^r + \varepsilon^{-r} r x. \tag{1.5}$$

The r-spin Witten conjecture states that $\frac{\partial F^{\frac{1}{r},c}}{\partial t_d^{r-1}} = 0$ for $d \geq 0$, and under the change of variables

$$T_k = \frac{1}{(-r)^{\frac{3k}{2(r+1)} - \frac{1}{2} - d} k!_r} t_d^a, \quad 0 \leq a \leq r-2, \quad d \geq 0, \tag{1.6}$$

where $k = a + 1 + rd$ and

$$k!_r := \prod_{i=0}^{d} (a + 1 + ri),$$

we have

$$\operatorname{res} L^{n/r} = \varepsilon^{1-n} \frac{\partial^2 F^{\frac{1}{r},c}}{\partial T_1 \partial T_n}$$

whenever $n \geq 1$ and $r \nmid n$, where $\operatorname{res} L^{n/r}$ denotes the coefficient of ∂_x^{-1} in $L^{n/r}$.

1.3. **Main algebraic results.** With L as above, let $\Phi(T_*, \varepsilon)$ be the solution of the system of equations

$$\frac{\partial \Phi}{\partial T_n} = \varepsilon^{n-1} (L^{n/r})_+ \Phi, \quad n \geq 1, \tag{1.7}$$

that satisfies the initial condition

$$\Phi|_{T_{\geq 2}=0} = 1.$$

Denote $\phi := \log \Phi$ and consider the expansion

$$\phi = \sum_{g \in \mathbb{Z}} \varepsilon^{g-1} \phi_g, \quad \phi_g \in \mathbb{C}[[T_*]].$$

Let

$$T_{mr} = \frac{1}{(-r)^{\frac{m(r-2)}{2(r+1)}} m! r^m} t_{m-1}^{r-1}, \quad m \geq 1. \tag{1.8}$$

The main result of the current paper is the following:

Theorem 1.1. *We have*

$$F_0^{\frac{1}{r},o} = \frac{1}{\sqrt{-r}} \phi_0 \big|_{t_d^{r-1} \mapsto \frac{1}{\sqrt{-r}} (t_d^{r-1} - r\delta_{d,0} s)} - \frac{1}{\sqrt{-r}} \phi_0 \big|_{t_d^{r-1} \mapsto \frac{1}{\sqrt{-r}} t_d^{r-1}}. \tag{1.9}$$

This provides the open r-spin version of Witten's conjecture in genus zero.

In the case $r = 2$, the function Φ was first considered in [2], and the first author found there an explicit formula for the function F^o in terms of the function $\phi = \log \Phi$. Properties of the function Φ for general r were first studied in [1]. In particular, note that the system of differential equations (1.7) for the function Φ coincides with the system of differential equations for the wave function of the KP hierarchy; in [1], the authors found an explicit formula for Φ in terms of the wave function.

The idea of the proof of Theorem 1.1 is to verify that the genus-zero open r-spin intersection numbers satisfy certain geometric recursions (known as topological recursion relations) that allow one to determine all of the open theory using just a handful of numbers that can be explicitly calculated. We then verify that the right-hand side of equation (1.9) satisfies the same recursions with the same initial conditions.

In addition, our calculations yield a simple explicit formula for all genus-zero primary open intersection numbers:

Theorem 1.2. *Suppose $k, l \geq 0$ and $0 \leq a_1, \dots a_l \leq r-1$. Then we have*

(1.10)
$$\left\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = \begin{cases} \dfrac{(k+l-2)!}{(-r)^{l-1}}, & \text{if } k \geq 1 \text{ and } \dfrac{(r-2)(k-1)+2\sum a_i}{r} = 2l+k-3, \\ 0, & \text{otherwise.} \end{cases}$$

It is interesting to compare this result with the explicit formula for genus-zero primary closed r-spin intersection numbers [22]. One can immediately see that the structure of the primary open intersection numbers is much simpler than the structure of the primary closed intersection numbers.

A mirror theoretic interpretation of the invariants constructed in this paper and of formula (1.10) will appear in [13].

1.4. **Open-closed correspondence.** An interesting and unexpected consequence of this work is that it reveals a connection between open r-spin theory and what we refer to as *closed extended r-spin theory*. More specifically, it is possible to enhance the closed theory by allowing exactly one of the twists a_i to be equal to -1; in this case, $R^1\pi_*\mathcal{S}$ is still a bundle in genus zero, so the closed theory can be defined exactly as usual. We define by

$$F_0^{\frac{1}{r},\mathrm{ext}}(t_*^*) := \sum_{n\geq 2} \frac{1}{n!} \sum_{\substack{0\leq a_1,\dots,a_n\leq r-1 \\ d_1,\dots,d_n\geq 0}} \left\langle \tau_0^{-1}\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_0^{\frac{1}{r},\mathrm{ext}} t_{d_1}^{a_1}\cdots t_{d_n}^{a_n}$$

the generating series for closed extended r-spin intersection numbers.

In the companion paper [3] to this work, we prove topological recursion relations for the closed extended theory, and from these one finds that the generating function of genus-zero intersection numbers in that setting is *also* closely related to the wave function of the rth Gelfand–Dickey hierarchy. Using this, we deduce the following theorem:

Theorem 1.3. *The generating series $F_0^{\frac{1}{r},o}$ and $F_0^{\frac{1}{r},ext}$ are related by the following formula:*

$$F_0^{\frac{1}{r},o} = -\frac{1}{r}\, F_0^{\frac{1}{r},ext}\Big|_{t_d^{r-1}\mapsto t_d^{r-1}-r\delta_{d,0}s} + \frac{1}{r} F_0^{\frac{1}{r},ext}.$$

1.5. **Plan of the paper.** The structure of the paper is as follows. Section 2 reviews from [4] the definition and properties of the moduli space of open graded r-spin disks and the Witten bundle $\mathcal{W}$. In Section 3, we define the canonical boundary conditions; this section, and in particular the key Proposition 3.20, is the heart of the geometric construction of the paper. Equipped with this construction, we give the definition of the intersection numbers and several basic sample calculations. We prove topological recursion relations satisfied by the open r-spin intersection numbers in Section 4. Using this result, in Section 5 we prove Theorems 1.1, 1.2, and 1.3, and we derive open string and dilaton equations for the open r-spin intersection numbers. In Section 6, we give a prediction for the behavior of open r-spin theory in higher genus, and provide evidence for its correctness. The last section details the construction of boundary conditions for the definition of the intersection numbers. The definitions and conventions regarding multisections are contained in the appendix.

1.6. **Acknowledgements.** The authors would like to thank M. Gross, J. Guéré, T. Kelly, R. Pandharipande, D. Ross, J. Solomon, E. Witten, and A. Netser Zernik for discussions and suggestions related to this work. A. B. received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 797635 and was also supported by grant ERC-2012-AdG-320368-MCSK in the group of Rahul Pandharipande at ETH Zurich and grant RFFI-16-01-00409. E. C. was supported by NSF DMS grant 1810969. R.T. (incumbent of the Lillian and George Lyttle Career Development Chair) was supported by a research grant from the Center for New Scientists of Weizmann Institute, by Dr. Max Rössler, the Walter Haefner Foundation, and the ETH Zürich Foundation, by the ISF (grant No. 335/19) and partially by ERC-2012-AdG-320368-MCSK.

2. Review of graded r-spin disks

We begin by reviewing the definition of graded r-spin disks, their moduli space, and the relevant bundles on this moduli space. For details, we direct the reader to [4].

2.1. **The moduli space of graded r-spin disks.** A *nodal marked disk* is defined as a tuple

$$(C, \phi, \Sigma, \{z_i\}_{i\in I}, x_{j\,j\in B}, m^I, m^B),$$

in which

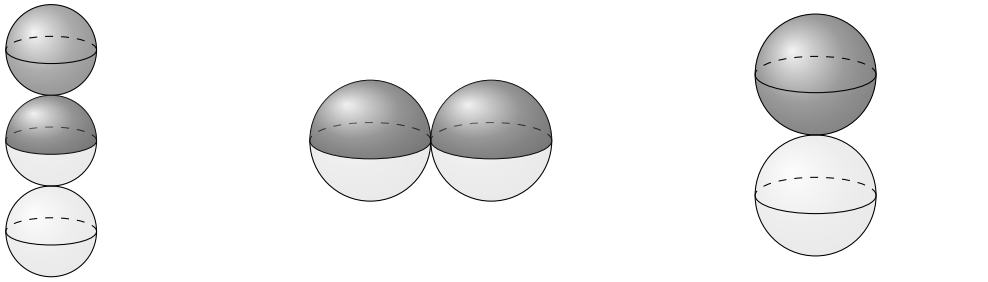
- C is a nodal, possibly disconnected, with each connected component having genus zero, orbifold Riemann surface with isotropy only at special points;
- $\phi : C \to C$ is an anti-holomorphic involution that realizes the coarse underlying Riemann surface $|C|$ topologically as the union of two Riemann surfaces, Σ and $\overline{\Sigma} = \phi(\Sigma)$, glued along the common subset $\mathrm{Fix}(|\phi|)$;
- $z_i \in C$ are a collection of distinct *internal marked points* whose images in $|C|$ lie in $\Sigma \setminus \mathrm{Fix}(|\phi|)$, with *conjugate marked points* $\overline{z_i} := \phi(z_i)$;
- $x_j \in \mathrm{Fix}(\phi)$ are a collection of distinct *boundary marked points* whose images in $|C|$ lie in $\partial\Sigma$;
- m^I and m^B are markings of I and B, respectively, for which no internal marked point on a connected component $C' \subseteq C$ with $C' \cap \phi(C') \neq \emptyset$ is marked 0.

Here, a *marking* of a set A is a function

$$m : A \to \{0\} \cup 2^{\mathbb{N}}_*$$

such that, for all $a \neq a' \in m^{-1}(0)$, the intersection $m(a) \cap m(a')$ is empty, where $2^{\mathbb{N}}_*$ denotes the set of all nonempty subsets of $\mathbb{N}$. We say that a nodal marked disk is *stable* if each irreducible component has at least three special points.

Nodal marked disks can have three types of nodes, illustrated below by shading $\Sigma \subseteq |C|$ in each case:



(A) Internal node (B) Boundary node (C) Contracted boundary

Note that $\partial\Sigma \subseteq \mathrm{Fix}(|\phi|)$ and $\mathrm{Fix}(|\phi|) \setminus \partial\Sigma$ is exactly the union of contracted boundaries.

Note that $\partial\Sigma$, the boundary of Σ, is equipped with a natural orientation, dictated by the choice of the preferred half $\Sigma \subseteq |C|$. This, in turn, induces a notion of positivity for ϕ-invariant sections of $\omega_{|C|}$ over $\partial\Sigma$: a section s is said to be *positive* if, for any point p and any tangent vector $v \in T_p(\partial\Sigma)$ in the direction of orientation, we have $\langle s(p), v\rangle > 0$, where $\langle -, -\rangle$ denotes the natural pairing between cotangent and tangent vectors.

A *twisted r-spin structure* on a nodal marked disk is a complex line bundle S on C whose coarse underlying bundle $|S|$ satisfies

$$|S|^{\otimes r} \cong \omega_{|C|} \otimes \mathcal{O}\left(-\sum_{i\in I_0} a_i[z_i] - \sum_{i\in I} a_i[\overline{z_i}] - \sum_{j\in B} b_j[x_j]\right),$$

where

$$a_i \in \{-1, 0, 1, \ldots, r-1\} \quad \text{and } b_j \in \{0, 1, \ldots, r-1\}$$

together with an involution $\widetilde{\phi} : S \to S$ lifting ϕ. We insist that connected components of C that meet $\mathrm{Fix}(\phi)$ contain no z_i with $a_i = -1$ and that connected components of C that do not meet $\mathrm{Fix}(\phi)$ contain at most one such z_i. The numbers a_i and b_j are referred to as the internal and boundary *twists*, respectively.

Observation 2.1. There exists a twisted r-spin structure with twists a_i, b_j on a connected nodal marked disk C if and only if

$$\frac{2\sum a_i + \sum b_j - (r-2)}{r} \in \mathbb{Z}. \tag{2.1}$$

The analogue in the closed case is more well-known [28]: a twisted r-spin structure with twists a_i exists on a (closed) connected nodal marked genus-zero curve if and only if

$$\frac{\sum a_i - (r-2)}{r} \in \mathbb{Z}. \tag{2.2}$$

To extend the definition of twists to nodes, let $n : \widehat{C} \to C$ be the normalization morphism. Then n^*S might not be a twisted r-spin structure, because its connected components might

contain too many marked points of twist -1. However, there is a canonical way to choose a minimal subset $\mathcal{R}$ of the half-nodes of C making

$$\widehat{S} := n^*S \otimes \mathcal{O}\left(-\sum_{q\in\mathcal{R}} r[q]\right) \tag{2.3}$$

a twisted r-spin structure; see [4, Section 2.3] for the details. In particular, then, for each irreducible component C_l of $\widehat{C}$ with internal marked points $\{z_i\}_{i\in I_l}$, boundary marked points $\{x_j\}_{j\in B_l}$ and half-nodes $\{p_h\}_{h\in N_l}$, we have

$$\left(|\widehat{S}|\big|_{|C_l|}\right)^{\otimes r} \cong \omega_{|C_l|} \otimes \mathcal{O}\left(-\sum_{i\in I_l} a_i[z_i] - \sum_{i\in I_l} a_i[\overline{z_i}] - \sum_{j\in B_l} b_j[x_j] - \sum_{h\in N_l} c_j[p_h]\right)$$

for numbers

$$a_i, c_h \in \{-1, 0, 1, \ldots, r-1\}, \quad b_j \in \{0, 1, \ldots, r-1\}$$

satisfying the same conditions as above. The numbers c_h are called the *twists* of S at its half-nodes. If p and p' are the two branches of a node in C, then

$$c_p + c_{p'} \equiv r-2 \mod r. \tag{2.4}$$

The node is said to be *Ramond* if one (hence both) of its branches satisfy $c_p \equiv -1 \mod r$, and it is said to be *Neveu–Schwarz* otherwise. The set $\mathcal{R}$ is chosen in a way that guarantees that each internal Ramond node has precisely one half edge in $\mathcal{R}$.

Associated to each twisted r-spin structure S, we define

$$J := S^\vee \otimes \omega_C.$$

This bundle admits an involution, as well, induced by the involutions on C and S; abusing notation slightly, we denote the involution on J also by $\widetilde{\phi}$.

Assume that C has no contracted boundary node, and let A be the complement of the special points in $\partial\Sigma$. We say that a twisted r-spin structure on such C is *compatible* if there exists a $\widetilde{\phi}$-invariant section

$$v \in \Gamma\left(A, |S|^{\widetilde{\phi}}\right)$$

(called a *lifting* of S) such that the image of $v^{\otimes r}$ under the map on sections induced by the inclusion

$$|S|^{\otimes r} \to \omega_{|C|}$$

is positive. Such a v always admits a companion lifting

$$w \in \Gamma\left(A, |J|^{\widetilde{\phi}}\right)$$

of J for which $\langle w, v\rangle \in \Gamma(A, \omega_{|C|})$ is positive, where $\langle -, -\rangle$ denotes the natural pairing between $|S|^\vee$ and $|S|$. This w is uniquely determined by v up to multiplication by a continuous $\mathbb{R}^+$-valued function.

Given a lifting of a twisted r-spin structure, we say that a boundary marked point or boundary half-node x_j is *legal*, or that the lifting *alternates* at x_j, if w cannot be continuously extended to x_j without vanishing; note that this definition depends only on v, not on the specific companion lifting w chosen. We say that w is a *grading* if every boundary marked point is legal and furthermore, for every Neveu–Schwarz boundary node, one of the two half-nodes is legal and the other is illegal.

If C has a contracted boundary node, then the definition of compatibility and grading must be adapted. In this case, we say that a twisted r-spin structure on C is *compatible* if the contracted boundary node q is Ramond and there exists a $\widetilde{\phi}$-invariant element

$$v \in |S|\big|_q$$

such that the image of $v^{\otimes r}$ under the map

$$|S|^{\otimes r}\big|_q \to \omega_{|C|}\big|_q$$

is positive imaginary under the canonical identification of $\omega_{|C|}\big|_q$ with $\mathbb{C}$ given by the residue. See [4, Definition 2.8] for more details. As in the case without a contracted boundary node, such a v admits a $\widetilde{\phi}$-invariant

$$w \in |J|\big|_q$$

such that $\langle v, w\rangle$ is positive imaginary. We refer to this w as a *grading*.

We say that two gradings are *equivalent* if they differ by a continuous positive function $A \to \mathbb{R}^+$ (in the case without a contracted boundary node) or multiplication by a positive real number (in the case with a contracted boundary node).

The relation between the twists and legality, and the obstructions to having a grading, are summarized in the following proposition (which summarizes [4, Proposition 2.5 and Observation 2.13] and the behavior of a Neveu-Schwarz node in a graded structure):

Proposition 2.2. *(1) When r is odd, any twisted r-spin structure is compatible, and there is a unique equivalence class of liftings.*

(2) Suppose r is odd and v is a lifting over a punctured neighborhood of a boundary marked point x_j. Then x_j is legal if and only if its twist is odd.

(3) When r is even, the boundary twists b_j in a compatible twisted r-spin structure must be even. In this case, there is a unique equivalence class of compatible structures.

(4) Suppose r is even. There exists a lifting over $\partial\Sigma \setminus \{x_j\}_{j\in B}$ that alternates precisely at a subset $D \subseteq \{x_j\}_{j\in B}$ if and only if

$$\frac{2\sum a_i + \sum b_j + 2}{r} \equiv |D| \mod 2. \tag{2.5}$$

(5) Ramond boundary nodes can appear in a graded structure only when r is odd, and in this case, their half-nodes are illegal with twists $r-1$.

(6) In a graded spin structure, any Neveu-Schwarz boundary node has one legal half-node and one illegal half-node.

In the case $\mathrm{Fix}(\phi) = \emptyset$, our notion of grading is vacuous, but we replace it with an additional datum (see [4, Definition 2.8] for an equivalent definition):

Definition 2.3. A *smooth graded r-spin sphere* is a nodal marked disk, where $\mathrm{Fix}(\phi) = \emptyset$ and Σ is a smooth sphere, together with

(1) a twisted r-spin structure S,
(2) a choice of one distinguished internal marked point z_i, called the *anchor* and satisfying $m^I(z_i) = 0$, which is the marked point of twist -1 if one exists.

If the twist a_i of the anchor is $r-1$, there is a map

$$\tau' : (|S| \otimes \mathcal{O}([z_i]))^{\otimes r}\big|_{z_i} \to \omega_{|C|}([z_i])\big|_{z_i} \cong \mathbb{C},$$

where the second identification is the residue map. In this case, we also fix

(3) an involution $\widetilde{\phi}$ on the fiber $(|S| \otimes \mathcal{O}([z_i]))_{z_i}$ such that

$$\tau'(\widetilde{\phi}(v)^{\otimes r}) = -\overline{\tau'(v^{\otimes r})}$$

for all $v \in (|S| \otimes \mathcal{O}([z_i]))_{z_i}$, where $w \mapsto \overline{w}$ is the standard conjugation, and

$$\left\{\tau'(v^{\otimes r}) \mid v \in (|S| \otimes \mathcal{O}([z_i]))^{\widetilde{\phi}}_{z_i}\right\} \supseteq i\mathbb{R}_+,$$

where i is the root of -1 in the upper half-plane;

(4) a connected component V of $(|S| \otimes \mathcal{O}([z_i]))^{\widetilde{\phi}}_{z_i} \setminus \{0\}$, called the *positive direction*, such that $\tau'(v^{\otimes r}) \in i\mathbb{R}_+$ for any $v \in V$.

Our primary objects of interest are the following:

Definition 2.4. A *stable graded r-spin disk* is a stable nodal marked disk, together with

(1) a compatible twisted r-spin structure S in which all boundary marked points have twist $r-2$ and all contracted boundary nodes are Ramond;
(2) an equivalence class of gradings;
(3) a choice of one distinguished special point (called the *anchor* and marked zero) in each connected component C' of C that is either disjoint from the set $\mathrm{Fix}(\phi)$ or meets the set $\mathrm{Fix}(\phi)$ in a single contracted boundary node. We require that the anchor is the contracted boundary node, if one exists, or the unique marked point in C' of twist -1, if one exists; otherwise, the anchor can be any marked point. Finally, we require that the collection of anchors is ϕ-invariant;
(4) an involution $\widetilde{\phi}$ on the fiber $(|S| \otimes \mathcal{O}([z_i]))_{z_i}$ over any anchor z_i of twist $r-1$, and an orientation of the $\widetilde{\phi}$-fixed subspace of $(|S| \otimes \mathcal{O}([z_i]))_{z_i}$ for any such z_i, as in Definition 2.3.

In [4], we showed that there exists a moduli space of stable graded r-spin disks:

Theorem 2.5. [4, Theorem 3.4] *There is a moduli space $\overline{\mathcal{M}}^{1/r}_{0,B,I}$ of connected stable graded r-spin disks with boundary marked points marked by B and internal marked points marked by I. It is a compact smooth orientable orbifold with corners of real dimension $k+2l-3$, equipped with a universal bundle and a universal grading.*

In case $B = \{1, \ldots, k\}$ and $I = \{1, \ldots, l\}$, we denote this moduli space by $\overline{\mathcal{M}}^{1/r}_{0,k,l}$. There is a decomposition of $\overline{\mathcal{M}}^{1/r}_{0,B,I}$ into open-and-closed suborbifolds with corners

$$\overline{\mathcal{M}}^{1/r}_{0,k,\vec{a}},$$

where $\vec{a} = \{a_i\}_{i \in I}$ and $\overline{\mathcal{M}}^{1/r}_{0,k,\vec{a}}$ denotes the suborbifold of $\overline{\mathcal{M}}^{1/r}_{0,B,I}$ consisting of those disks for which the ith internal marked point has twist a_i. This suborbifold is nonempty if and only if the twists satisfy

$$\frac{2\sum a_i + (k-1)(r-2)}{r} \in \mathbb{Z} \quad \text{and} \quad \frac{2\sum a_i + (k-1)(r-2)}{r} \equiv k+1 \mod 2.$$

2.2. **Stable graded r-spin graphs.** Analogously to the more familiar setting of the moduli space of curves, $\overline{\mathcal{M}}^{1/r}_{0,B,I}$ can be stratified according to decorated dual graphs. The relevant dual graphs Γ consist of

(i) a vertex set V, decomposed into *open* and *closed* vertices $V = V^O \sqcup V^C$, and
(ii) a half-edge set H, decomposed into *boundary* and *internal* half-edges $H = H^B \sqcup H^I$.

The half-edge set is equipped with an involution σ_1, which we view as reversing the two halves of an edge. The fixed points of σ_1 are the *tails* T of Γ, and we decompose T into $T^B := H^B \cap T$ and $T^I := H^I \cap T$. The two-element orbits of σ_1 are the *edges* E of Γ, which can also be decomposed as $E = E^B \sqcup E^I$. Furthermore, the data of Γ includes a subset

$$H^{CB} \subseteq T^I$$

of *contracted boundary tails*, as well as a marking m^B on T^B and a marking m^I on $T^I \setminus H^{\mathrm{CB}}$. We say Γ is *closed* if $V^O = \emptyset$, and we say Γ is *smooth* if $E = H^{CB} = \emptyset$. See [4, Section 3.1] for details.

In particular, each element of $\overline{\mathcal{M}}^{1/r}_{0,k,l}$ has a corresponding dual graph Γ, and this Γ is equipped with three additional decorations: a map

$$\mathrm{tw} : H \to \{-1, 0, 1, \ldots, r-1\}$$

encoding the twist of S at each marked point and half-node, a map

$$\mathrm{alt} : H^B \to \mathbb{Z}/2\mathbb{Z}$$

given by $\mathrm{alt}(h) = 0$ if the half-node given by h is illegal and $\mathrm{alt}(h) = 1$ otherwise, and a subset

$$T^* \subseteq T^I$$

given by the anchors. These additional decorations satisfy some relations that implement in the graph level the requirement that boundary markings have twist $r-2$, together with properties of anchors and the constraints of (2.1), (2.2), (2.4) and Proposition 2.2.

A *genus-zero graded r-spin dual graph* is a (possibly disconnected) graph, each of whose connected components is the dual graph of an element of some moduli space $\overline{\mathcal{M}}^{1/r}_{0,k,l}$. In particular, given a connected such Γ, there is a closed suborbifold with corners

$$\overline{\mathcal{M}}^{1/r}_{\Gamma} \subseteq \overline{\mathcal{M}}^{1/r}_{0,B,I}$$

whose general point is a graded r-spin disk with dual graph Γ. We write $\mathcal{M}^{1/r}_{\Gamma}$ for the open substack of $\overline{\mathcal{M}}^{1/r}_{\Gamma}$ consisting of curves whose dual graph is exactly Γ. If Γ is disconnected, we define $\overline{\mathcal{M}}^{1/r}_{\Gamma}$ as the product of the moduli spaces associated to its connected components.

There are various forgetful maps between these moduli spaces: we denote by

$$\mathrm{For}_{\mathrm{spin}} : \overline{\mathcal{M}}^{1/r}_{0,k,l} \to \overline{\mathcal{M}}_{0,k,l}$$

the map that forgets the spin structure, and for $B', I' \subseteq \mathbb{Z}$, we denote by

$$\mathrm{For}_{B',I'} : \overline{\mathcal{M}}^{1/r}_{\Gamma} \to \overline{\mathcal{M}}^{1/r}_{\Gamma'}$$

the map that forgets all twist-zero internal marked points marked by I' and all twist-zero illegal boundary marked points marked by B'. Note that the latter procedure may create unstable components, which must be contracted, so we denote by $\Gamma' = \mathrm{for}_{B',I'}(\Gamma)$ the new graph that results.

2.3. **The Witten bundle and the cotangent line bundles.** The crucial bundle for the definition of r-spin theory is the *Witten bundle* on the moduli space. Roughly speaking, if $\pi : \mathcal{C} \to \overline{\mathcal{M}}^{1/r}_{0,k,l}$ is the universal curve and $\mathcal{S} \to \mathcal{C}$ is the universal spin bundle with companion bundle $\mathcal{J} := \mathcal{S}^\vee \otimes \omega_\pi$, then we define

$$\mathcal{W} := (R^0\pi_*\mathcal{J})_+ = (R^1\pi_*\mathcal{S})^\vee_-, \tag{2.6}$$

where the subscripts $+$ and $-$ denote invariant or anti-invariant sections under the universal involution $\widetilde{\phi} : \mathcal{J} \to \mathcal{J}$ or $\widetilde{\phi} : \mathcal{S} \to \mathcal{S}$. To be more precise, defining $\mathcal{W}$ by (2.6) would require one to deal with derived pushforward in the orbifold-with-corners context, so to avoid this technicality, we define $\mathcal{W}$ by pullback of the analogous bundle from a subset of the closed moduli space $\overline{\mathcal{M}}^{1/r}_{0,k+2l}$; see [4, Section 4.1].

Remark 2.6. The Witten bundles are orbifold vector bundles. The action of the isotropy groups of the moduli space is the one induced from lifting the fiberwise action on J (which, in turn, is induced from the action on the spin bundle) to the space of global sections of J. In particular, the group acting on a generic fiber in the open case is trivial, while in the closed connected case it is the group of rth roots of unity that acts by scaling.

The real rank of the Witten bundle, on a component of the moduli space with internal twists $\{a_i\}$ and boundary twists $\{b_j\}$, is

$$\frac{2\sum a_i + \sum b_j - (r-2)}{r}. \tag{2.7}$$

Furthermore, in [4, Theorem 5.2], we prove that in the graded case, where all $b_j = r-2$, $\mathcal{W}$ is canonically relatively oriented relative to the moduli space. More precisely, for any set $\vec{a} = \{a_i\}_{i\in I}$ of internal twists, we construct an orientation of $\overline{\mathcal{M}}^{1/r}_{0,B,\vec{a}}$ satisfying certain properties [4, Proposition 3.12] and a compatible orientation of $T\overline{\mathcal{M}}^{1/r}_{0,B,\vec{a}} \oplus \mathcal{W}$. Although both orientations depend on choices, the orientation that they induce on the total space is canonical and independent of choices.

In addition, we prove in [4] that the Witten bundle satisfies certain decomposition properties along nodes. We reiterate these properties here as they are needed in what follows, but we direct the reader to [4, Section 4.2] for further deatils.

To set the stage, let Γ be a genus-zero graded r-spin dual graph, and let e be an edge of Γ. Define

$$\mathrm{detach}_e(\Gamma)$$

to be the "detaching" of Γ at e, which is the disconnected graph obtained by cutting the edge e. This creates new tails, so we must extend the marking and (possibly) the anchors. To do so, if e is a boundary edge, we mark both newly-created tails 0 and add no new anchor. If e is an internal edge, then exactly one of the connected components of $\mathrm{detach}_e(\Gamma)$ is closed (that is, has no open vertices) and unanchored; let h denote the half-edge of e in this component, and let h' denote the other half-edge of e. In this case, we mark h by 0, and declare it to be an anchor, and we mark h' by the union of the markings of the internal tails $h'' \neq h$ in the same component as h.

Analogously, we can detach Γ along a contracted boundary tail $t \in H^{\mathrm{CB}}$. Namely, we define

$$\mathrm{detach}_t(\Gamma)$$

to be the graph that agrees with Γ except that t is no longer considered an element of H^{CB}. All other decorations (including the fact that t is the anchor) remain the same.

Now, given a genus-zero graded r-spin dual graph Γ, let $\widehat{\Gamma}$ be obtained by detaching either an edge or a contracted boundary tail. Then there are morphisms

$$\overline{\mathcal{M}}^{1/r}_{\widehat{\Gamma}} \xleftarrow{q} \overline{\mathcal{M}}_{\widehat{\Gamma}} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}^{1/r}_{\Gamma} \xrightarrow{\mu} \overline{\mathcal{M}}^{1/r}_{\Gamma} \xrightarrow{i_\Gamma} \overline{\mathcal{M}}^{1/r}_{0,k,l}, \tag{2.8}$$

where $\overline{\mathcal{M}}_\Gamma \subseteq \overline{\mathcal{M}}_{0,k,l}$ is the moduli space of marked disks (without r-spin structure) specified by the dual graph Γ. The morphism q is defined by sending the spin structure S to the spin structure $\widehat{S}$ defined by (2.3); it has degree one but is not an isomorphism because it does not induce an isomorphism on isotropy groups. The morphism μ is the projection from the fiber product; it is an isomorphism, but we distinguish between its domain and codomain because they have different universal objects. Finally, the morphism i_Γ is the inclusion.

There are Witten bundles $\mathcal{W}$ on $\overline{\mathcal{M}}^{1/r}_{0,k,l}$ and $\widehat{\mathcal{W}}$ on $\overline{\mathcal{M}}^{1/r}_{\widehat{\Gamma}}$, and the decomposition properties are stated in terms of how these bundles are related under pullback via the morphisms (2.8).

Proposition 2.7. [4, Proposition 4.7] *Let Γ be a stable genus-zero graded r-spin dual graph with a single edge e, and let $\widehat{\Gamma}$ be the detaching of Γ along e. Then the Witten bundle decomposes as follows:*

(i) If e is a Neveu–Schwarz edge, then $\mu^* i_\Gamma^* \mathcal{W} = q^* \widehat{\mathcal{W}}$.

(ii) If e is a Ramond boundary edge, then there is an exact sequence

$$0 \to \mu^* i_\Gamma^* \mathcal{W} \to q^* \widehat{\mathcal{W}} \to \mathcal{T}_+ \to 0, \tag{2.9}$$

where $\mathcal{T}_+$ is a trivial real line bundle.

(iii) If e is a Ramond internal edge connecting two closed vertices, write $q^*\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_1 \boxplus \widehat{\mathcal{W}}_2$, in which $\widehat{\mathcal{W}}_1$ is the Witten bundle on the component containing the anchor of Γ and $\widehat{\mathcal{W}}_2$ is the Witten bundle on the other component. Then there is an exact sequence

$$0 \to \widehat{\mathcal{W}}_2 \to \mu^* i_\Gamma^* \mathcal{W} \to \widehat{\mathcal{W}}_1 \to 0. \tag{2.10}$$

Furthermore, if $\widehat{\Gamma}'$ is defined to agree with $\widehat{\Gamma}$ except that the twist at each Ramond tail is $r-1$, and $q' : \overline{\mathcal{M}}_{\widehat{\Gamma}} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}^{1/r}_{\Gamma} \to \overline{\mathcal{M}}^{1/r}_{\widehat{\Gamma}'}$ is defined analogously to q, then there is an exact sequence

$$0 \to \mu^* i_\Gamma^* \mathcal{W} \to (q')^* \widehat{\mathcal{W}}' \to \mathcal{T} \to 0, \tag{2.11}$$

where $\widehat{\mathcal{W}}'$ is the Witten bundle on $\overline{\mathcal{M}}^{1/r}_{\widehat{\Gamma}'}$ and $\mathcal{T}$ is a line bundle whose rth power is trivial.

(iv) If e is a Ramond internal edge connecting an open vertex to a closed vertex, write $q^*\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_1 \boxplus \widehat{\mathcal{W}}_2$, in which $\widehat{\mathcal{W}}_1$ is the Witten bundle on the open component (defined via $\widehat{\mathcal{S}}|_{\mathcal{C}_1}$) and $\widehat{\mathcal{W}}_2$ is the Witten bundle on the closed component. Then the exact sequences (2.10) and (2.11) both hold.

Analogously, if Γ has a single vertex, no edges, and a contracted boundary tail t, and $\widehat{\Gamma}$ is the detaching of Γ along t, then there is a decomposition property:

(v) If $\mathcal{W}$ and $\widehat{\mathcal{W}}$ denote the Witten bundles on $\overline{\mathcal{M}}^{1/r}_{0,k,l}$ and $\overline{\mathcal{M}}^{1/r}_{\widehat{\Gamma}}$, respectively, then the sequence (2.9) holds.

Remark 2.8. If the edge e in Proposition 2.7 is a boundary edge, then the map q is an isomorphism, and in this case, the proposition says that the Witten bundle pulls back under the gluing morphism $\overline{\mathcal{M}}^{1/r}_{\widehat{\Gamma}} \to \overline{\mathcal{M}}^{1/r}_{0,k,l}$.

In the case of an internal Neveu-Schwarz edge, although q is not an isomorphism, a multisection of $q^*\widehat{\mathcal{W}}$ canonically and uniquely induces a multisection of $\mu^* i_\Gamma^* \mathcal{W}$, since the orbits of the action of the automorphisms on the fibers of the latter are contained in the corresponding orbits of the former. Indeed, on $\widehat{\mathcal{W}}$, the group acts by independent scalings of its two summands by roots of unity, while on $\mathcal{W}$, the action scales the two components by roots of unity that differ by some ξ^d, where ξ is an rth root of unity, d is a multiple of $\gcd(m,r)$, and m is the multiplicity at one of the half-nodes.

The last important line bundles for the calculations that follow are the cotangent line bundles at internal marked points. These line bundles have already been defined on the moduli space $\overline{\mathcal{M}}_{0,k,l}$ of stable marked disks (without spin structure), as the line bundles with fiber $T^*_{z_i}\Sigma$, or equivalently, the pullback of the usual cotangent line bundles under the doubling map $\overline{\mathcal{M}}_{0,k,l} \to \overline{\mathcal{M}}_{0,k+2l}$ that sends Σ to C. The bundle $\mathbb{L}_i$ on $\overline{\mathcal{M}}^{1/r}_{0,B,I}$ is the pullback of this cotangent line bundle on $\overline{\mathcal{M}}_{0,k,l}$ under the morphism that forgets the spin structure. Because $\mathbb{L}_i$ is a complex line bundle, it carries a canonical orientation.

3. The definition of open correlators

Equipped with the bundles $\mathcal{W}$ and $\mathbb{L}_i$, our goal is to define intersection numbers by integrating the top Chern class of a direct sum of these bundles over the moduli space—in other words, by counting the (weighted) number of zeroes of a section. However, since the moduli space has boundary, such a count is only well-defined after prescribing the boundary behavior of the section. Our aim, then, is to define canonical boundary conditions for sections of $\mathcal{W}$ or $\mathbb{L}_i$.

In particular, we will define what it means for a smooth section s of $\mathcal{W}$ or $\mathbb{L}_i$ to be *canonical* at any boundary stratum $\mathcal{M}^{1/r}_\Gamma$ associated to a graph $\Gamma \in \partial\Gamma_{0,k,\vec{a}}$ with a single boundary edge e or a single contracted boundary tail. In the case where Γ has an edge e and the illegal half-edge of e has twist zero, the relevant idea has already appeared in [23]: we define s to be canonical if it is pulled back along the forgetful map $F_\Gamma : \mathcal{M}^{1/r}_\Gamma \to \mathcal{M}^{1/r}_{\mathcal{B}\Gamma}$, where $\mathcal{B}\Gamma$ is obtained by detaching e and forgetting the illegal side. This forgetting procedure is no longer defined, however, if the illegal side of e has positive twist. Instead, in this case, we define a section s of $\mathcal{W}$ to be canonical at $\mathcal{M}^{1/r}_\Gamma$ if the evaluation of s at the illegal half-node is "positive", meaning that it is a nonzero element of the fiber of J in the direction of the grading. A similar definition applies in the case where Γ has a contracted boundary tail.

The existence of a section that is canonical at $\mathcal{M}^{1/r}_\Gamma$ uses the fact that, when all boundary twists are $r-2$, the evaluation map is surjective unless the Witten bundle has rank zero (and in the latter case, the first type of boundary conditions are sufficient). However, since these boundary conditions are defined stratum-wise on the moduli space, we must also check that they can be defined consistently – that is, we must ensure the existence of global canonical sections. Indeed, this is already an issue for boundary conditions of the first type: a section s may be canonical for a particular $\mathcal{M}^{1/r}_\Gamma$, and yet it may be impossible to extend s to a global smooth section without vanishing at some boundary points of the moduli space. Even

when $r = 2$, this situation arises, and it was resolved in [23] by working with multi-valued sections rather than sections. See [23, Figure 2] for an example.

For boundary conditions of the second type, the situation is even more complicated. In fact, as illustrated in Example 3.23, it may happen that for a boundary point p in the moduli space and a neighborhood U of p, one cannot find a section of $\mathcal{W}$ of U satisfying the positivity constraints at all boundary strata intersecting U. We solve this problem by slightly modifying the moduli space, replacing it by the subspace

$$\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \subseteq \overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$$

obtained by excluding strata $\mathcal{M}^{1/r}_\Gamma$ with at least one illegal half-edge of positive twist. We then define the positivity constraint for a section s in a neighborhood U of such strata by requiring that, for $\Sigma \in U \cap \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$, the evaluation of s_Σ at certain intervals $I_{h,\Sigma} \subseteq \partial\Sigma$ is positive, where, for each $h \in H^B(\Gamma)$, $I_{h,\Sigma}$ converges in the universal curve to a neighborhood of the boundary node corresponding to h. We prove in Proposition 3.20, which is the key geometric idea of this work and the place where the twist $r-2$ at boundary marked points is required, that this requirement is consistent for higher codimension corners of the moduli space.

Although $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ is noncompact, and the boundary conditions depend on choices, the resulting intersection numbers are independent of choices. This is the content of the main theorem of this section, Theorem 3.17.

3.1. **Canonical multisections.** First, some notation for graph operations.

Definition 3.1. Given a graded r-spin dual graph Γ, and given an edge e of Γ connecting vertices v_1 and v_2, the *smoothing* of Γ along e is the graph $d_e\Gamma$ obtained by contracting e and replacing the two vertices v_1 and v_2 with a single vertex v_{12} that is declared to be closed if and only if both v_1 and v_2 are closed. Similarly, the *smoothing* of Γ along a contracted boundary tail $h \in H^{CB}$ is the graph $d_h\Gamma$ obtained by erasing h and moving the vertex v from which h emanates from V^C to V^O.

Remark 3.2. If Λ is a smoothing of Γ, then each (half-)edge h of Λ corresponds to a unique (half-)edge $\iota_{\Lambda,\Gamma}(h)$ of Γ.

For a set S of edges and contracted boundary tails, one can perform a sequence of smoothings, and the graph obtained is independent of the order in which those smoothings are performed; denote the result by $d_S\Gamma$. Let

$$\begin{aligned} \partial^!\Gamma &= \{\Lambda \mid \Gamma = d_S\Lambda \text{ for some } S\}, \\ \partial\Gamma &= \partial^!\Gamma \setminus \{\Gamma\}, \\ \partial^B\Gamma &= \{\Lambda \in \partial\Gamma \mid E^B(\Lambda) \cup H^{CB}(\Lambda) \neq \emptyset\}. \end{aligned} \tag{3.1}$$

We refer to a boundary half-edge h with $\mathrm{alt}(h) = 0$ and $\mathrm{tw}(h) > 0$ (or the corresponding half-node) as *positive*. We write $H^+(\Lambda)$ for the set of half-edges h of a graph Λ such that either h or $\sigma_1(h)$ is positive. We denote by

$$\partial^+\Gamma \subseteq \partial^!\Gamma$$

the graphs with at least one positive half-edge, and we write

$$\partial^0\Gamma = \partial^B\Gamma \setminus \partial^+\Gamma.$$

Given this notation, let $\overline{\mathcal{PM}}_\Gamma^{1/r}$ be the orbifold with corners defined by

$$\overline{\mathcal{PM}}_\Gamma^{1/r} = \overline{\mathcal{M}}_\Gamma^{1/r} \setminus \left(\coprod_{\Lambda \in \partial^+ \Gamma} \mathcal{M}_\Lambda^{1/r} \right),$$

and let $\partial^+ \overline{\mathcal{M}}_\Gamma^{1/r}$ be the orbifold with corners defined by

$$\partial^+ \overline{\mathcal{M}}_\Gamma^{1/r} = \coprod_{\Lambda \in \partial^+_\Gamma} \mathcal{M}_\Lambda^{1/r}.$$

Note that the boundary of $\overline{\mathcal{PM}}_\Gamma^{1/r}$, which we will denote by $\partial^0 \overline{\mathcal{M}}_\Gamma$ in keeping with the notation $\partial^0 \Gamma$ of the previous paragraph, contains only strata corresponding to graphs without positive half-edges.

Our next goal is to define what it means for vectors in the fiber of the Witten bundle to "evaluate positively," and to specify where such positive evaluation is required. First, the definition of evaluation:

Definition 3.3. Given a vector v in the fiber $\mathcal{W}_C$ of the Witten bundle over a graded r-spin disk C, and a special point q of C, the *evaluation* of v at q is $\mathrm{ev}_q(v) := v(q) \in J_q$.

Definition 3.4. We say that a marked orbifold Riemann surface with boundary is *partially stable* if it has exactly one internal marked point and no boundary marked points.

Given a graded r-spin disk, let $n : \widehat{C} \to C$ be the normalization map. A connected component of $\widehat{C}$ is said to be *strongly positive* if it contains either (a) a partially stable component, (b) a contracted boundary node, or (c) a component that becomes partially stable after forgetting all illegal half-nodes of twist zero. A dual graph that represents a surface with strongly positive components is said to be strongly positive .

A vector $v \in \mathcal{W}_C$ *evaluates positively* at a component D of C if the evaluation of $n^* v$ at each boundary point of D is positive, with respect to the grading of the boundary or the contracted boundary of D. A vector that evaluates positively at any strongly positive component of C is said to be *positive on strongly positive components*.

Positive evaluation will also be required on certain "intervals" in $\partial\Sigma$, and we define the relevant intervals now; they should be viewed as a smoothly-varying family of intervals that approximates a family of neighborhoods of the nodes in a nodal surface.

Definition 3.5. Let Γ be a graded graph and consider $u \in \mathcal{M}_\Lambda \subseteq \partial^+ \overline{\mathcal{M}}_\Gamma^{1/r}$. A Λ-*set with respect to* Γ is an open set $U \subseteq \overline{\mathcal{M}}_\Gamma^{1/r}$ whose closure intersects precisely those $\mathcal{M}_\Xi^{1/r}$ with $\Lambda \in \partial^! \Xi$. A Λ-*neighborhood with respect to* Γ of $u \in \overline{\mathcal{M}}_\Gamma^{1/r}$ is a neighborhood $U \subseteq \overline{\mathcal{M}}_\Gamma^{1/r}$ of U that is a Λ-set with respect to Γ. Since there is a unique smooth graph Γ with $\Lambda \in \partial^! \Gamma$, we refer to a Λ-neighborhood with respect to a smooth Γ simply as a Λ-neighborhood.

Let C be a graded marked disk. An *interval* is a connected open set $I \subseteq \partial\Sigma$ such that, if we write $\partial\Sigma = S/\sim$ for a space S homeomorphic to S^1, the preimage of I under the quotient map $q : S \to \partial\Sigma$ is the union of an open set with a finite number of isolated points.

Suppose $C \in \mathcal{M}_\Lambda^{1/r}$, and let n_h be a boundary half-node corresponding to a half-edge $h \in H^B(\Lambda)$. We say that n_h *belongs* to the interval I if the corresponding node n_{h/σ_1} lies in I. In this case, again denoting by $n : \widehat{C} \to C$ the normalization map and denoting

FIGURE 2. The intervals I_h and $I_{h'}$, corresponding to half-nodes n_h and $n_{h'}$, are drawn as thicker lines over the thinner lines that denote the boundary. The picture on the right represents a nearby point in the moduli space to the picture on the left, but where the node has been smoothed.

$\widehat{\Sigma} = n^{-1}(\Sigma)$, we write $n^{-1}(I) = I_1 \cup I_2$ in which I is a half-open interval with endpoint n_h and I_2 is a half-open interval with endpoint $n_{\sigma_1(h)}$, under the canonical orientation of $\partial\widehat{\Sigma}$.

Suppose U is contained in some Λ-set. Then a Λ-*family of intervals* for U is a family $\{I_h(u)\}_{h\in H^+(\Lambda),u\in U}$ such that:

(1) Each $I_h(u)$ is an interval in $\partial\Sigma_u$, where $\Sigma_u \subseteq \pi^{-1}(u)$ is the preferred half in the fiber of the universal curve $\pi : \mathcal{C} \to U$, and the endpoints of each $I_h(u)$ vary smoothly with respect to the smooth structure of the universal curve restricted to U.
(2) For $h \notin \{h', \sigma_1(h')\}$, we have $I_h(u') \cap I'_{h'}(u') = \emptyset$. If $h = \sigma_1(h')$, we have $I_h(u') \cap I'_{h'}(u') \neq \emptyset$ precisely if $u' \in \mathcal{M}_\Xi^{1/r}$ and $h \in H^+(\Xi)$ under the locally-defined injection $\iota : H(\Xi) \to H(\Lambda)$. In this case, $I_h(u') \cap I_{\sigma_1(h)}(u')$ consists exactly of the node n_{h/σ_1}.
(3) There are no marked points in $I_h(u')$, and the only half-node that belongs to $I_h(u')$ is n_h, which is defined precisely when $u' \in \mathcal{M}_\Xi^{1/r}$ and $h \in H^+(\Xi)$.

When the point u corresponds to a stable graded disk C we sometimes write $I_h(C)$ for $I_h(u)$. See Figure 2 for the local picture of intervals in smooth and nodal disks.

Definition 3.6. Let C be a graded r-spin disk, and let $A \subseteq \partial\Sigma$ be a subset without legal special points. Then an element $w \in \mathcal{W}_\Sigma$ *evaluates positively at* A if $\mathrm{ev}_x(w)$ is positive with respect to the grading for every $x \in A$.

Let Γ be a graded graph, let U be a Λ-set with respect to Γ, and let I_h be a Λ-family of intervals for U. Given a multisection s of $\mathcal{W}$ defined in a subset of $\overline{\mathcal{PM}}_\Gamma^{1/r}$ that includes $U \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$, we say s is (U, I)-*positive* (with respect to Γ) if for any $u' \in U \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$, any local branch $s_i(u')$ evaluates positively at each $I_h(u')$.

A multisection s defined in $W \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$, where W is a neighborhood of $u \in \overline{\mathcal{M}}_\Lambda^{1/r}$, is *positive near* u (with respect to Γ) if there exists a Λ-neighborhood $U \subseteq W$ of u and a Λ-family of intervals $I_*(-)$ for U such that s is (U, I)-positive.

If W is a neighborhood of $\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}$, then a multisection s defined in a set containing

$$U_+ = \left(W \cap \overline{\mathcal{PM}}_\Gamma^{1/r}\right) \cup \bigcup_{\Lambda \text{ strongly positive}} \overline{\mathcal{PM}}_\Lambda^{1/r} \tag{3.2}$$

is *positive* (with respect to Γ) if it is positive near each point of $\partial^+\overline{\mathcal{M}}^{1/r}_\Gamma$ and at strongly positive components. As above, we omit the phrase "with respect to Γ" whenever Γ is smooth.

The notion of canonicity of a multisection of $\mathcal{W}$ over (an open subset of) $\overline{\mathcal{PM}}^{1/r}_\Gamma$ has two parts: a positivity constraint, in terms of the above definitions, and a requirement that the multisection be pulled back from a smaller graph called the "base" of Γ. To define the base, we first define $\mathrm{for}_{\mathrm{illegal}}(\Gamma)$ to be the graph obtained by forgetting all illegal, twist-zero boundary tails with marking zero, and we define $E^0(\Gamma) \subseteq E^B(\Gamma)$ to be the set of boundary edges with one illegal side of twist zero. For a set N of edges, we denote by $\mathrm{detach}_N(\Gamma)$ the graph obtained from Γ by detaching all edges in N (according to the procedure detach_e explained in Section 2.3) above.

Definition 3.7. The *base* of a graded graph Γ is defined as

$$\mathcal{B}\Gamma = \mathrm{for}_{\mathrm{illegal}}(\mathrm{detach}_{E^0(\Gamma)}(\Gamma)).$$

The moduli space $\overline{\mathcal{M}}^{1/r}_{\mathcal{B}\Gamma}$ is referred to as the *base moduli* of $\overline{\mathcal{M}}^{1/r}_\Gamma$. It admits a map

$$F_\Gamma : \overline{\mathcal{M}}^{1/r}_\Gamma \to \overline{\mathcal{M}}^{1/r}_{\mathcal{B}\Gamma} \tag{3.3}$$

that associates to a moduli point the result of the normalization at all boundary nodes *which correspond to edges of* Γ, and forgets the resulting illegal boundary points which have twist zero.

Notation 3.8. Let

$$f_\Gamma : \partial^!\Gamma \to \partial^!\mathcal{B}\Gamma$$

be defined by $f_\Gamma(\Gamma') = \Gamma''$ whenever F_Γ maps $\mathcal{M}^{1/r}_{\Gamma'}$ to $\mathcal{M}^{1/r}_{\Gamma''}$. Explicitly,

$$\Gamma'' = \mathrm{for}_{\mathrm{illegal}}(\mathrm{detach}_{\iota_{\Gamma,\Gamma'}(E^0(\Gamma))}(\Gamma')$$

(c.f. Remark 3.2). Observe that any half-edge h of Γ' whose vertex is not removed by the $\mathrm{for}_{\mathrm{illegal}}$ operation (though maybe contracted) naturally corresponds to an edge of Γ''; abusing notation, we denote this edge by $f_\Gamma(h)$. Note that, because of contracted components, there may be $h \neq h' \in H(\Gamma')$ with $f_\Gamma(h) = f_\Gamma(h')$.

Remark 3.9. We denote by $\mathcal{B}C$ the graded surface that corresponds to the moduli point $F_{\Gamma(C)}(C)$, or by $\mathcal{B}\Sigma$ its preferred half. Let $\widehat{C}'$ be the subsurface of the normalization of C obtained by removing components that are neither stable nor partially stable, which admits a natural map

$$\phi_\Gamma : \widehat{C}' \to \mathcal{B}C.$$

The following observations are straightforward but useful for later. First, analogously to [23, Observations 3.14 and 3.28], we have the following, which serves as a compatibility relation for boundary conditions:

Observation 3.10. Let Γ be a graded graph and let $\Gamma' \in \partial\Gamma$. Then F_Γ takes $\overline{\mathcal{M}}^{1/r}_{\Gamma'}$ to $\overline{\mathcal{M}}^{1/r}_{f_\Gamma(\Gamma')}$, and $\mathcal{B}f_\Gamma(\Gamma') = \mathcal{B}\Gamma'$. Moreover, we have

$$F_{f_\Gamma(\Gamma')} \circ F_\Gamma|_{\overline{\mathcal{M}}^{1/r}_{\Gamma'}} = F_{\Gamma'}.$$

Next, analogously to [23, Observation 3.15], we have the following observation, which plays a key role in proving the independence of the intersection numbers of choices:

Observation 3.11. Let Γ be a smooth graded graph and $\Lambda \in \partial^0\Gamma$ be such that $\mathcal{B}\Lambda$ has no partially stable components. Then

$$\dim_{\mathbb{R}} \mathcal{M}^{1/r}_{\mathcal{B}\Lambda} \leq \dim_{\mathbb{R}} \mathcal{M}_\Gamma - 2.$$

The behavior of the Witten bundle and the tautological lines under forgetful maps shows the following:

Observation 3.12. Because $\mathcal{W} = \mathrm{For}^*_{B',I'}\mathcal{W}$ for all $B', I' \subseteq \mathbb{Z}$ (see [4, Equation (4.3)]), there is a canonical isomorphism between the Witten bundle $\mathcal{W}$ on $\overline{\mathcal{M}}^{1/r}_\Gamma$ and the pullback $F^*_\Gamma\mathcal{W}$ of the Witten bundle from $\overline{\mathcal{M}}^{1/r}_{\mathcal{B}\Gamma}$.

Observation 3.13. If the component that contains the internal tail labeled i in $\mathcal{B}\Gamma$ is stable, then $\mathbb{L}_i \to \overline{\mathcal{M}}^{1/r}_\Gamma$ is canonically identified with $F^*_\Gamma(\mathbb{L}_i \to \overline{\mathcal{M}}^{1/r}_{\mathcal{B}\Gamma})$.

Indeed, $\mathbb{L}_i$ is pulled back along the maps appearing in the definition F_Γ, unless the stable component containing the internal tail i becomes unstable along the way. This can happen only if, in the graph Γ, the vertex containing the ith tail has a unique internal half-edge and all boundary half-edges are illegal of twist zero.

We are now ready to define the canonical boundary conditions we seek. In what follows, for an orbifold vector bundle E over an orbifold with corners M, we denote by $C^\infty_m(E)$ the space of smooth multisection; see Appendix A for our conventions and notation regarding multisections.

Definition 3.14. Let Γ be a smooth graded graph with only open vertices, let U_+ be as in (3.2), and let U be a set containing $\partial^0\overline{\mathcal{PM}}^{1/r}_\Gamma \cup U_+$. Then a smooth multisection s of $\mathcal{W}$ over U is *canonical* if

(1) s is positive, and
(2) s is *pulled back from the base*, in the sense that for any $\Lambda \in \partial^0\Gamma$, we have

$$s|_{\mathcal{M}^{1/r}_\Lambda} = F^*_\Lambda s^{\mathcal{B}\Lambda} \tag{3.4}$$

for some $s^{\mathcal{B}\Lambda} \in C^\infty_m(\mathcal{M}^{1/r}_{\mathcal{B}\Lambda}, \mathcal{W})$.

In case $U = \overline{\mathcal{PM}}^{1/r}_\Gamma$, we say that s is a *global canonical multisection*. A section that is positive and satisfies the second item of the definition for Λ in some set $A \subseteq \partial^0\Gamma$ is said to be *canonical with respect to* $\bigcup_{\Lambda\in A}\mathcal{M}^{1/r}_\Lambda$.

Definition 3.15. A smooth multisection s of $\mathbb{L}_i \to \partial\overline{\mathcal{PM}}^{1/r}_\Gamma$, for Γ smooth and graded, is called *canonical* if for every $\Lambda \in \partial^0\Gamma$ with $\mathcal{B}\Lambda$ stable, we have

$$s|_{\mathcal{M}^{1/r}_\Lambda} = F^*_\Lambda s^{\mathcal{B}\Lambda} \tag{3.5}$$

for some $s^{\mathcal{B}\Lambda} \in C^\infty_m(\overline{\mathcal{PM}}^{1/r}_{\mathcal{B}\Lambda}, \mathcal{W})$. A global multisection is canonical if its restriction to $\partial\overline{\mathcal{PM}}^{1/r}_\Gamma$ is canonical. Again, we say that s is *canonical with respect to* $\bigcup_{\Lambda\in A}\mathcal{M}^{1/r}_\Lambda$, where $A \subseteq \partial^0\Gamma$, if (3.5) holds for any $\Lambda \in A$ with a stable base.

Throughout what follows, we consider the (uniquely determined) multisections $\{s^{\mathcal{B}\Lambda}\}_{\Lambda\in\partial^0\Gamma}$ in the definitions above as part of the information of a canonical multisection, both in the $\mathbb{L}_i$ case and the $\mathcal{W}$ case. For a bundle obtained as a direct sum of the Witten bundle and cotangent line bundles, we define a canonical multisection to be a direct sum of canonical multisections of the corresponding bundles.

Observation 3.16. Sums of canonical multisections of $\mathcal{W} \to \overline{\mathcal{PM}}_\Gamma^{1/r}$ or $\mathbb{L}_i$, or multiples of canonical multisections by a positive scalar, are canonical.

This is easy to see for multisections of $\mathbb{L}_i$ (and in fact, we may allow the scalars to be negative or complex). The proof for multisections of $\mathcal{W}$ is also straightforward, except for verifying that the sum of positive multisections is positive. For this, suppose that s_1 and s_2 are positive. For any $u \in \mathcal{M}_\Lambda^{1/r} \subseteq \partial^+ \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$, there are open Λ-neighborhoods U_i and families of intervals $I_{i,h}$ for U_i on which s_i is positive. For each h, the intersection $I_{1,h}(u) \cap I_{2,h}(u)$ contains an interval around the node n_h. Thus, in some open subset $V \subseteq U_1 \cap U_2$, which is itself a Λ-neighborhood, the collection $\{\hat{I}_h = I_{1,h}(-) \cap I_{2,h}(-)\}$ forms a Λ-family of intervals for V. The sum $s_1 + s_2$ is then $(V, \hat{I})$-positive.

3.2. **Definition of intersection numbers.** We refer the reader to Appendix A for our definition of relative Euler class and of the weighted cardinality of the zero locus of a multisection. Our first main theorem is the following:

Theorem 3.17. *Let $E \to \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$ be the bundle*

$$E := \bigoplus_{i \in [l]} \mathbb{L}_i^{\oplus d_i} \oplus \mathcal{W},$$

and assume that rank$(E) = \dim \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$. Then there exists $U_+ = U \cap \overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}}$, where U is a neighborhood of $\partial^+ \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$ such that $\overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}} \setminus U$ is a compact orbifold with corners, and a nowhere-vanishing canonical multisection $\mathbf{s} \in C_m^\infty(U_+ \cup \partial^0 \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}, E)$.

Thus, one can define, using the canonical relative orientation of $\mathcal{W}$ specified in [4, Theorem 5.2]*, the Euler number*

$$\int_{\overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}}} e(E; \mathbf{s}) \in \mathbb{Q}.$$

Moreover, the resulting number does not depend on the choice of U or the canonical multisection $\mathbf{s}$.

An equivalent formulation is that transverse global canonical multisections exist, and that the weighted number of zeroes of such a multisection is independent of the choice of the transverse global canonical multisection.

Definition 3.18. With the above notation, when the condition rank$(E) = \dim \overline{\mathcal{M}}_\Gamma^{1/r}$ holds, we define the *open r-spin intersection numbers* (or *correlators*) by

$$\left\langle \prod_{i \in [l]} \tau_{d_i}^{a_i} \sigma^k \right\rangle^{1/r,o} = \int_{\overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}}} e(E; \mathbf{s}) \in \mathbb{Q},$$

for any canonical multisection $\mathbf{s}$ that does not vanish at $\partial^0 \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}} \cup U_+$, where U_+ is as above. When the above rank condition does not hold, the integral is defined to be zero. If $d_i = 0$ for some i, we typically omit the subscript 0 in the symbols τ_0^a.

In the case $k = 0$ without boundary marked points, one can say more.

Proposition 3.19. *Suppose $k = 0$. Then there exists a global canonical nowhere-vanishing multisection s of $\mathcal{W} \to \overline{\mathcal{PM}}^{1/r}_{0,0,\vec{a}}$. In particular, all open r-spin intersection numbers on the moduli space of disks without boundary marked points vanish.*

The proof of the theorem and the proposition are relegated to Section 7 below. However, we do prove here the key geometric proposition required for the construction, in order to illuminate the necessity of working with $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ rather than $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ and of constraining all boundary marked points to have twist $r-2$:

Proposition 3.20. *The following positivity claims hold:*

1. *Suppose C is a graded r-spin disk that is either stable or partially stable. Then there exists an element $w \in \mathcal{W}_C$ that evaluates positively at each strongly positive component of C.*
2. *Let U be a contractible Λ-neighborhood of $\Sigma \in \partial^+\overline{\mathcal{M}}_\Gamma$, where Γ is connected, smooth, and graded and Λ has no internal edges or illegal boundary half-edges of twist zero. Let $(I_h)_{h\in H^+(\Lambda)}$ be a Λ-family of intervals for U. Then there exists a (U, I)-positive section s.*

Our basic tool for constructing positive sections is the following lemma:

Lemma 3.21. *Suppose C is a smooth pre-stable r-spin disk with a lifting, or a smooth pre-stable graded $r-$spin sphere. Let $h = deg(|J|)$, and let a and b be non-negative integers with $a + 2b \leq h + 1$. Then for any distinct points $p_1, \ldots, p_{a+b} \in \Sigma \subseteq C$, where the first a are boundary points and the rest are internal points, the total evaluation map*

$$\bigoplus_i ev_{p_i} : \mathcal{W}_C \to \bigoplus_{i=1}^{a} |J|^{\tilde{\phi}}_{p_i} \oplus \bigoplus_{i=a+1}^{a+b} |J|_{p_i}$$

is surjective; here we use the canonical identification $H^0(J) = H^0(|J|)$ to identify the fiber of $\mathcal{W}$ with $H^0(|J|)$. In particular, for any a distinct boundary points and b distinct internal points, with $a + 2b = h$, there exists a nonzero section of the Witten bundle that vanishes at all of them. The section is unique up to multiplication by a real non zero scalar.

If C has a Ramond marked point and no point with -1 twist, and in particular if it has a contracted boundary component, then the evaluation map at this Ramond marked point is surjective.

Proof. Let $n := a + 2b$, and let $\{q_1, \ldots, q_n\} \subseteq C$ consist of the points p_i together with the conjugates of $p_{a+1}, \ldots, p_{a+b}$. Then the short exact sequence

$$0 \to |J|\left(-\sum_{i=1}^{n}[q_i]\right) \to |J| \to \bigoplus_i |J|_{q_i} \to 0$$

induces the long exact sequence

$$\ldots \to H^0(|J|) \to \bigoplus_{i=1}^{n} |J|_{q_i} \to H^1(|J|(-\sum[q_i])) \to \ldots, \tag{3.6}$$

and $\oplus_i \mathrm{ev}_{q_i}$ is obtained from the first map in (3.6) by taking $\tilde{\phi}$-invariant parts. We have

$$h^1\left(|J|\left(-\sum q_i\right)\right) = h^0\left(|J|^\vee\left(\sum q_i\right) \otimes \omega_{|C|}\right) = 0,$$

since the latter has negative degree. This proves the first part of the claim, where the uniqueness of the nonzero section follows from the fact that, if v_1 and v_2 both vanish at $p_1, \dots, p_{a+b}$, then some linear combination $\lambda v_1 + \mu v_2$ with $\lambda, \mu \in \mathbb{R}^*$ would have $\deg(|J|) + 1$ zeroes, which is impossible for a nonzero section. The proof of the second claim is similar. $\square$

Using this, we are prepared to prove the key proposition:

Proof of Proposition 3.20. For the first item, note that each connected component of C may have at most one strongly positive component. Since the Witten bundle decomposes as a direct sum over different connected components, it is enough to prove the claim for a connected Σ that is strongly positive We use Lemma 3.21. If C is closed and has a contracted boundary node, we choose w to be an arbitrary vector that evaluates positively at that node. If C is open and it has a single internal Ramond marked point and no boundary marked points, then $\mathcal{W}_C$ is a real line and we choose w to be any nonzero element that evaluates positively at one boundary point (hence at any boundary point, since $\deg(|J|_C) = 0$ in this case). Note that w is $\mathrm{Aut}(C)$-invariant.

For the second item, assume first that $\mathrm{Aut}(\Lambda)$ is trivial.

Fix a point in U corresponding to a smooth, open C. Note that if $w \in \mathcal{W}_C$ is a nonzero vector that is positive at some boundary point p and negative at another boundary point q, then the total number of zeroes of w (counted with multiplicity) in the boundary arc from p to q, plus the total number of boundary marked points along this interval, must be odd. Parity considerations show that we could have equivalently stated the same criterion but with the arc from q to p.

Our strategy will thus be to show that one can choose a set $Z = Z_C$ of z boundary points in $A = \partial\Sigma \setminus \big(\sqcup_{h \in H^+(\Lambda)} I_h(C)\big)$, where

$$z \leq \operatorname{rank} \mathcal{W} - 1, \qquad z \equiv \operatorname{rank}(\mathcal{W}) - 1 \mod 2,$$

so that in any connected component K of A (which is an interval), the total number of boundary markings in K plus $|Z \cap K|$ is even. We shall then choose an arbitrary set Z^c of $\frac{\operatorname{rank}(\mathcal{W})-1-z}{2}$ internal points. By Lemma 3.21, one can find a nonzero vector $w \in \mathcal{W}_C$ with simple zeroes at $Z \cup Z^c$, and, possibly by replacing w with $-w$, this vector satisfies the positivity constraints we want.

Since U is contractible, we can then construct a diffeomorphism $\phi : \mathcal{C}(U') \to U' \times C$ between the universal curve restricted to $U' = U \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$ and $U' \times \Sigma$, which takes the fiber over the moduli point $[C']$ to $([C'], \Sigma)$ and satisfies $\phi(I(C')_h) = ([C'], I(C)_h)$. We then define the section at $[C']$ as the unique section, up to rescaling by a nonzero real scalar, whose zeroes are $\phi^{-1}([C'], Z \cup Z^c)$. The scalar will be chosen so that the resulting section s is smooth and equals w at $[C]$. Thus, the section will be (U, I)-positive, if we can construct w.

The first step in the construction is the following combinatorial claim:

Lemma 3.22. *For any vertex v of Λ, write $\deg(v) = \operatorname{rank}(\mathcal{W}_v) + m_v - 1$, where m_v is the number of legal half-edges of v with twist smaller than $r - 2$.*

If $\operatorname{rank}(\mathcal{W}_v) = 0$, then $\deg(v)$ is non-negative and even. Furthermore, we have

$$\operatorname{rank}(\mathcal{W}) - 1 \geq \sum_{v \mid \operatorname{rank}(\mathcal{W}_v) > 0} \deg(v) \tag{3.7}$$

and

$$\operatorname{rank}(\mathcal{W}) - 1 = \sum_{v \mid \operatorname{rank}(\mathcal{W}_v)>0} \deg(v) \mod 2.$$

Proof. For the first claim, let v be a vertex with $\operatorname{rank}(\mathcal{W}_v) = 0$. When r is even, by (2.7) and (2.5), the number of legal (boundary) half-edges at v is congruent modulo 2 to

$$\frac{2\sum a_i + \sum b_j - (r-2)}{r} + 1 = \operatorname{rank}(\mathcal{W}_v) + 1 = 1.$$

Furthermore, the twists of the legal half edges are even (again by Proposition 2.2), and none can equal $r-2$ because for an illegal half edge with twist $r-2$ the second half edge is legal with twist 0, contradicting our assumptions on Λ. Thus, all of the legal half-edges have twist smaller than $r-2$, and the number m_v of such half-edges is odd; this proves that $\deg(v)$ is non-negative and even.

If r is odd, then Proposition 2.2 implies that the legal half-edges are precisely those whose twist is odd. Since

$$\operatorname{rank}(\mathcal{W}_v) = \frac{2\sum a_i + \sum b_j - (r-2)}{r} = 0,$$

by (2.7), it is easy to see that the number of odd-twist half-edges must be odd. None of these can have twist $r-1$ or $r-2$, so m_v is also odd, and we again conclude that $\deg(v)$ is non-negative and even.

For the second claim, we use the decomposition properties of the Witten bundle to conclude that

$$\operatorname{rank}(\mathcal{W}) - 1 = \sum_{v \in V(\Lambda)} (\operatorname{rank}(\mathcal{W}_v) - 1) + |E \setminus E^{\mathrm{Ramond}}| = \sum_{v \in V(\Lambda)} \deg(v),$$

where E^{Ramond} are the Ramond boundary edges and the right equality is a consequence of the fact that any boundary edge that is not Ramond has exactly one legal half-edge, whose twist is less than $r-2$ by the assumption on Λ. Therefore, by the first claim,

$$\sum_{v \in V(\Lambda)} \deg(v) \geq \sum_{v \mid \operatorname{rank}(\mathcal{W}_v)>0} \deg(v)$$

and

$$\sum_{v \in V(\Lambda)} \deg(v) = \sum_{v \mid \operatorname{rank}(\mathcal{W}_v)>0} \deg(v) \mod 2,$$

completing the proof of Lemma 3.22. □

With this lemma in hand, we can describe the strategy for constructing the vector w. It is enough to define Z by its intersections $Z \cap K$ with each connected component K of A. We choose Z_K to be an arbitrary set of $k_K \in \{0,1\}$ points in K, where k_K is defined by

$$k_K \equiv |K \cap \{x_i\}_{i \in B}| \mod 2.$$

Thus, in order to finish, we must show that

$$z := \sum_{K \text{ a connected component of } A} k_K \leq \operatorname{rank}(\mathcal{W}) - 1, \tag{3.8}$$

and

$$z \equiv \operatorname{rank}(\mathcal{W}) - 1 \mod 2 \tag{3.9}$$

Now, any interval K is associated to some vertex v of Λ, the vertex that corresponds to the component C_v of the normalization of C containing K. We denote this relation by $K \to v$. Thanks to the previous lemma, (3.8) and (3.9) will follow if we can show, for all v, that

$$\sum_{K\to v} k_K \leq \deg(v) \tag{3.10}$$

and

$$\sum_{K\to v} k_K = \deg(v) \mod 2. \tag{3.11}$$

Note that for v with $\operatorname{rank}(\mathcal{W}_v) = 0$ (which, as noted above, have no legal half-edges of twist $r-2$), we have

$$\sum_{K\to v} k_K = 0,$$

and by Lemma 3.22, equations (3.10) and (3.11) hold in this case.

We now turn to the case $\operatorname{rank}(\mathcal{W}_v) > 0$, in which the fact that all boundary twists are all $r-2$ plays a crucial role. Write k_v, m_v, and n_v for the numbers of legal boundary points with twist $r-2$, legal boundary points with twists less than $r-2$, and illegal boundary points of C_v, respectively. By assumption, the twists of the n_v illegal points are nonzero, and hence are at least two. Thus,

$$\begin{aligned} \deg(v) &= \operatorname{rank}(\mathcal{W}_{C_v}) - 1 + m_v \\ &\geq \frac{2n_v + (r-2)k_v - (r-2)}{r} - 1 + m_v \\ &\geq m_v - 1 + \min(k_v, n_v) \\ &= \min(k_v + m_v - 1, n_v + m_v - 1), \end{aligned} \tag{3.12}$$

where we have used the integrality of $\deg(v)$. In addition,

$$\sum_{K\to v} k_K \equiv k_v \equiv \deg(v) \mod 2, \tag{3.13}$$

again using the parity constraints of 2.2. The number of intervals that may contain a boundary marked point of twist $r-2$ is, of course, no more than k_v.

There now are several cases to check in order to prove (3.10) and (3.11). Suppose, first, that $n_v \geq k_v$. If $m_v \geq 1$ then

$$\deg(v) \geq k_v \geq \sum_{K\to v} k_K,$$

proving the claim for v. If $m_v = 0$, then $\deg(v) \geq k_v - 1$, but then, by (3.13), we again have $\deg(v) \geq k_v$, so the claim still holds.

Assume, now, that $k_v \geq n_v + 1$. If $k_v \geq n_v + 2$, then (3.12) can be strengthened to give the bound

$$\deg(v) \geq \frac{2n_v + (r-2)(n_v+2) - (r-2)}{r} - 1 + m_v \geq m_v - 1 + n_v + 1 = m_v + n_v,$$

where again we have used the integrality of $\deg(v)$. But, since

$$n_v + m_v = |\{K | K \to v\}| \geq \sum_{K\to v} k_K,$$

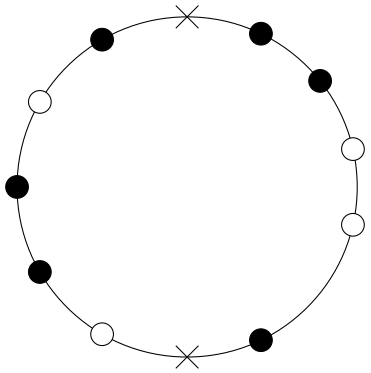


FIGURE 3. The scheme by which we create a positive section, for simplicity in the case $m_v = 0$, using the choice of zeroes. Filled circles denote legal points (of twist $r-2$), unfilled circles denote illegal points, and $\times$s denote the placement of zeroes.

equations (3.10) and (3.11) hold for v in this case as well. Note that exactly the same argument would work if $k_v = n_v + 1$, as long as at least one of the m_v legal points or one of the internal points has a positive twist.

Thus, suppose that $k_v = n_v + 1$, all internal points have twist zero, and all m_v legal points of twist less than $r-2$ have twist zero. In this case, $\deg(v) = n_v + m_v - 1$, so again, we are done if $m_v \geq 2$, as this implies that $\deg(v) \geq k_v$. We are left with the cases $m_v = 0$ or $m_v = 1$. If $m_v = 0$, then $\deg(v) = n_v - 1 = k_v - 2$. But in this case, there must be at least one interval between illegal boundary points that contains more than one legal marking (of twist $r-2$), and hence,

$$\sum_{K \to v} k_K \leq k_v - 2 = \deg(v)$$

and again we are done. If $m_v = 1$, then $\deg(v) = n_v = k_v - 1$, but this contradicts (3.13).

Equations (3.10) and (3.11) are thus fully proven, and the proposition follows in the case $|\mathrm{Aut}(\Lambda) = 1|$. If $|\mathrm{Aut}(\Lambda)| > 1$, then by our assumptions on Λ and Γ and internal labelings, it must hold that $k > 1$ and some boundary points have the same label. But then, let $\widetilde{U}$ be a cover of U obtained by changing the labels to injective labels, so that $|\mathrm{Aut}(\widetilde{U})| = 1$ and

$$\widetilde{U}/G \cong U$$

for a finite group G. Construct, on $\widetilde{U}$, a section $\tilde{s}$ that satisfies the requirements of the proposition. Then, if $\tilde{q} : \widetilde{U}/G \to U$ is the quotient map, the section s defined as the averaging

$$s(u) = \frac{\sum_{v \in \tilde{q}^{-1}(u)} \tilde{s}(v)}{|G|}$$

satisfies the requirements. This completes the proof of Proposition 3.20. □

To justify the need for working with the space $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ in the above construction, note that there is a naïve way one might try to define the positivity boundary conditions: one could require that, for any $C \in \partial^+ \overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$, the evaluation of w_C is positive near each illegal boundary half-node of positive twist (or at least, its evaluation is positive near a nonempty subset of such half-nodes,and is non-negative around the rest). Indeed, this naïve idea is the

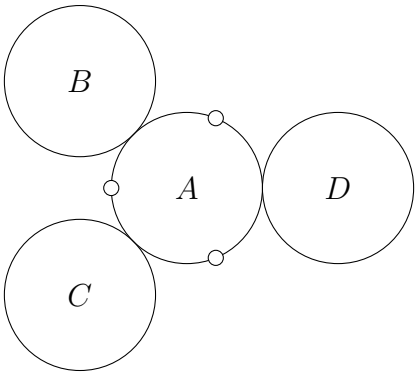


FIGURE 4. A configuration for which the naïve notion of positivity fails.

motivation for our definition of positivity, and it works for corners of the moduli space of low codimension. However, the next example illustrates why this simpler approach fails in general.

Example 3.23. In Figure 4, a configuration of a nodal disk is shown. All four components are disks meeting at boundary nodes. The component labeled A contains three illegal boundary marked points with the same positive twist. (The figure should be viewed as arising in a partial normalization of a nodal graded r-spin disk, where the labeled illegal points are half-nodes.) Suppose that the nodes of B, C, and D are Neveu–Schwarz, that their legal sides are in A, and that their legal sides all have the same twist $a < r-2$. Internal marked points and twists can then be chosen so that the ranks of the Witten bundles on the components A, B, C, and D are 0, 1, 1, and 1, respectively, and moreover, one can choose these twists in the same way for each of the components B, C, and D.

If one tries, in this example, to implement the naïve boundary conditions on $\partial^+\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ by requiring the evaluation maps at neighborhoods of illegal half-nodes to be positive, we would in particular require the evaluations at the illegal half-nodes of B, C, and D to be positive. In addition, after smoothing the nodes of B, C, and D, the evaluations at the illegal marked points of A would also necessarily be positive. But then, suppose we smooth the node where D meets A. The Witten bundle for the smooth component has rank one and degree zero, and hence the restriction of the section to this component either vanishes nowhere or is identically zero. If it is nonzero, then it is easy to see that it must evaluate negatively at one of the illegal points of the smoothed component; thus, the restriction of the section to the smoothed component must be zero. But this implies, in the nodal limit and using continuity, that it must vanish on D. Similar considerations show that it must vanish on B and C, as well. Thus, a section that is positive on B, C, and D cannot be extended to a section that is positive, in the naïve sense, for all possible smoothings of this disk.

In light of this example, one can see that the naïve approach fails because, for nodal disks, there are in general "fewer zeroes in hand" than there are in the smoothing. We work in the smaller space $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ to avoid this problem.

3.3. **Examples of calculations.** We conclude this section with some explicit examples of calculations.

Example 3.24. Consider the real one-dimensional integral $\langle \tau^a \tau^b \rangle_0^{1/r,o}$, where $a+b=r-1$. In this case, one can actually extend canonical sections to the boundary in a non-vanishing way, so we can work on $\overline{\mathcal{M}}^{1/r}_{0,0,\{a,b\}}$ (which is a real closed interval) rather than on its subset $\overline{\mathcal{PM}}^{1/r}_{0,0,\{a,b\}}$ (which is an open interval).

One end of the interval $\overline{\mathcal{M}}^{1/r}_{0,0,\{a,b\}}$ corresponds to the closed surface obtained by contracting the boundary of the disk, while the other end corresponds to the nodal surface composed of two disks, connected by a boundary node, each of which contains one of the internal marked points. The Witten bundle $\mathcal{W}$ is of real rank one and degree zero, which means that, at least for points $C \in \mathcal{M}^{\frac{1}{r}}_{0,0,\{a,b\}}$, any nonzero element $s_\Sigma \in \mathcal{W}_C$ is nowhere-vanishing on $\partial\Sigma \subseteq C$ when considered as a global section over C.

Now, take any such section $s \in C^\infty(\mathcal{W} \to \mathcal{M}^{\frac{1}{r}}_{0,0,\{a,b\}})$ such that, for any $C \in \mathcal{M}^{\frac{1}{r}}_{0,0,\{a,b\}}$, the restriction of s_C to $\partial\Sigma \subseteq C$ is positive. By performing a proper rescaling, s can be extended to $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,0,\{a,b\}}$, and by continuity, it also agrees with the grading at the endpoints of the moduli space. This precisely implies that s satisfies the boundary conditions necessary to be a canonical section. Since s is nowhere-vanishing, we have

$$\langle \tau^a \tau^b \rangle_0^{1/r,o} = 0.$$

We shall later see, either as a consequence of the topological recursions or as a direct implication of Proposition 3.19, that when $k=0$, all intersection numbers are zero.

Example 3.25. Next, consider the real one-dimensional integral $\langle \tau^1 \sigma^2 \rangle_0^{\frac{1}{r},o}$. Again, we extend the section from $\overline{\mathcal{PM}}^{1/r}_{0,2,\{1\}}$ to $\overline{\mathcal{M}}^{1/r}_{0,2,\{1\}}$, which is again a real closed interval. Each endpoint corresponds to a nodal surface composed of two disks meeting at a boundary node, in which one disk contains the boundary marked points and a half-node n_- and the other disk contains the internal marked point and a half-node n_+; here, n_- is the illegal half-node. The difference between the two endpoints is that in one of them, the cyclic order of the boundary marked points on the n_- component is x_1, x_2, n_-, while in the other, it is x_2, x_1, n_-. Call these two surfaces C_1 and C_2, respectively.

Again, for $C \in \mathcal{M}^{\frac{1}{r}}_{0,2,\{1\}}$, an element $s_C \in \mathcal{W}_\Sigma \setminus \{0\}$ does not vanish on $\partial\Sigma \subseteq C$, so its direction agrees with the grading in exactly one of the two boundary arcs $x_1 \to x_2$ or $x_2 \to x_1$. The canonical boundary conditions for $\mathcal{W}$ dictate that at n_-, the section s_{Σ_i} agrees with the grading. Thus, s_{Σ_1} agrees with the grading along the arc $x_2 \to x_1$ that contains n_-, and s_{Σ_2} agrees with the grading along the other arc. Hence, a canonical section must vanish at some point of $\mathcal{M}^{\frac{1}{r}}_{0,2,\{1\}}$, and we have $\langle \tau^1 \sigma^2 \rangle_0^{1/r,o} = \pm 1$.

In fact,

$$\langle \tau^1 \sigma^2 \rangle_0^{1/r,o} = 1.$$

To see this, suppose $\mathcal{M}^{\frac{1}{r}}_{0,2,\{1\}}$ is oriented from C_1 to C_2; in the language of [4, Section 3.3], this is the orientation $\tilde{\mathfrak{o}}^\pi$ for the order $\pi = (1,2)$. Then $\mathcal{W}$ has a canonical relative orientation, which is given by the following real section on $\mathcal{M}^{\frac{1}{r}}_{0,2,\{1\}}$:

$$\sigma = \left((-1)^r \frac{i(\bar{z}_1 - z_1)dw}{(w-z_1)(w-\bar{z}_1)} \left(\frac{(x_2-x_1)dw}{(w-x_1)(w-x_2)} \right)^{r-2} \right)^{\frac{1}{r}};$$

see [4, Definition 5.10 and Construction/Notation 5.7]. Note that $\sigma^{\otimes r}$ is positive with respect to the canonical orientation of $T^*\partial\Sigma$ when w is in the arc $x_1 \to x_2$, and when r is even, we take the real rth root that is positive with respect to the grading for w in the arc $x_1 \to x_2$. By taking the limit in which $C \to C_i$, we see that canonical boundary conditions in this case correspond to a negative vector in $\mathcal{W}_{C_1}$ and to a positive one in $\mathcal{W}_{C_2}$. Therefore, the intersection number is $+1$.

Example 3.26. When $r = 2$, a dimension count shows that $\langle \sigma^3 \rangle_0^{\frac{1}{2},o} = \pm 1$. The definition of orientation then shows that $\langle \sigma^3 \rangle_0^{\frac{1}{2},o} = -1$. (Note that this is a different convention than the one used in [23].)

4. Open topological recursion relations

The open r-spin intersection numbers satisfy topological recursion relations (TRRs) that allow all genus-zero intersection numbers to be computed. These relations involve closed r-spin intersection numbers with (at most) one marked point of twist -1, which we studied in detail in [3]. In this situation (as in the closed situation with only non-negative twists), $R^1\pi_*\mathcal{S}$ is a bundle in genus zero, so Witten's class c_W can be defined by the formula (1.1). We define *closed extended r-spin intersection numbers* by the formula

$$\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_0^{\frac{1}{r},\mathrm{ext}} := r \int_{\overline{\mathcal{M}}^{1/r}_{0,\{a_1,\ldots,a_n\}}} c_W \cap \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

Equipped with this definition, the topological recursion relations for the open r-spin intersection numbers are as follows:

Theorem 4.1. (a) *(Boundary marked point TRR) Suppose $l, k \geq 1$. Then*

$$\left\langle \tau_{d_1+1}^{a_1} \prod_{i=2}^{l} \tau_{d_i}^{a_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = \sum_{a=-1}^{r-2} \sum_{S \coprod R = \{2,\ldots,l\}} \left\langle \tau_0^a \tau_{d_1}^{a_1} \prod_{i\in S} \tau_{d_i}^{a_i} \right\rangle_0^{\frac{1}{r},ext} \left\langle \tau_0^{r-2-a} \prod_{i\in R} \tau_{d_i}^{a_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} +$$
$$+ \sum_{\substack{S \coprod R = \{2,\ldots,l\} \\ k_1+k_2=k-1}} \binom{k-1}{k_1} \left\langle \tau_{d_1}^{a_1} \prod_{i\in S} \tau_{d_i}^{a_i} \sigma^{k_1} \right\rangle_0^{\frac{1}{r},o} \left\langle \prod_{i\in R} \tau_{d_i}^{a_i} \sigma^{k_2+2} \right\rangle_0^{\frac{1}{r},o}.$$

(b) *(Internal marked point TRR) Suppose $l \geq 2$. Then*

$$\left\langle \tau_{d_1+1}^{a_1} \prod_{i=2}^{l} \tau_{d_i}^{a_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = \sum_{a=-1}^{r-2} \sum_{S \coprod R = \{3,\ldots,l\}} \left\langle \tau_0^a \tau_{d_1}^{a_1} \prod_{i\in S} \tau_{d_i}^{a_i} \right\rangle_0^{\frac{1}{r},ext} \left\langle \tau_0^{r-2-a} \tau_{d_2}^{a_2} \prod_{i\in R} \tau_{d_i}^{a_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} +$$
$$+ \sum_{\substack{S \coprod R = \{3,\ldots,l\} \\ k_1+k_2=k}} \binom{k}{k_1} \left\langle \tau_{d_1}^{a_1} \prod_{i\in S} \tau_{d_i}^{a_i} \sigma^{k_1} \right\rangle_0^{\frac{1}{r},o} \left\langle \tau_{d_2}^{a_2} \prod_{i\in R} \tau_{d_i}^{a_i} \sigma^{k_2+1} \right\rangle_0^{\frac{1}{r},o}.$$

The proof of this theorem is the content of the current section. Before giving the proof, we first must construct canonical multisections with certain special properties; this is carried out in the next section, and in Section 4.2 we prove Theorem 4.1.

4.1. **Special canonical multisections.**

4.1.1. *Coherent multisections.* We work with the notation of Appendix A for multisections.

Consider a (possibly disconnected) graded stable r-spin dual graph Γ. For any twisted r-spin structure, we set

$$J' := J \otimes \mathcal{O}\left(\sum_t r[z_t]\right),$$

where the sum is over all anchors t with $\mathrm{tw}(t) = -1$ and z_t is the special point specified by t. Similarly, we set

$$\mathcal{J}' := \mathcal{J} \otimes \mathcal{O}\left(\sum_t r\Delta_{z_t}\right),$$

where Δ_{z_t} is the divisor in the universal curve corresponding to z_t. Define an orbifold vector bundle $\overline{\mathcal{R}}_\Gamma$ on $\overline{\mathcal{M}}_\Gamma^{1/r}$ by

$$\overline{\mathcal{R}}_\Gamma = \bigoplus_t \sigma_{z_t}^* \mathcal{J}',$$

where the sum is again over all anchors t with $\mathrm{tw}(t) = -1$ and σ_{z_t} is the corresponding section of the universal curve.[2] We use the notation $\overline{\mathcal{R}}_\Gamma$ both for the bundle and for its total space, and we denote by

$$\wp : \overline{\mathcal{R}}_\Gamma \to \overline{\mathcal{M}}_\Gamma^{1/r}$$

the projection. In case $\Gamma' \in \partial\Gamma$, there is a map $\overline{\mathcal{R}}_{\Gamma'} \to \overline{\mathcal{R}}_\Gamma$ between total spaces, which is an embedding on the coarse underlying level.

We denote by $\mathcal{W}'$ the bundle

$$\mathcal{W}' = (R^0\pi_*\mathcal{J}')_+$$

on $\overline{\mathcal{M}}_\Gamma$, and by $\mathcal{W}$ the analogous bundle $(R^0\pi_*\mathcal{J}')_+$ on $\overline{\mathcal{R}}_\Gamma$. (It is possible that Γ has closed components with no contracted boundary tails, whose corresponding moduli space consists of closed r-spin structures with no involution; in this case, the notation $(R^0\pi_*\mathcal{J}')_+$ should be interpreted simply as $R^0\pi_*\mathcal{J}'$.)

Definition 4.2. Let Γ be a connected graded r-spin dual graph, and let s be a section of $\mathcal{W}$ over a subset $U \subseteq \overline{\mathcal{R}}_\Gamma$. We say that s is *coherent* if either the anchor of Γ is not twisted -1, or the following holds: for any point $\zeta = (C, u_t) \in U$, corresponding to a graded r-spin disk C and an element $u_t \in J'_{z_t}$, the element $s(\zeta) \in H^0(J')$ satisfies

$$\mathrm{ev}_{z_t} s(\zeta) = u_t.$$

A coherent multisection s is defined as a multisection each of whose local branches is coherent.

When Γ is disconnected, we say that a (multi)section s of $\mathcal{W}$ over $U \subseteq \overline{\mathcal{R}}_\Gamma$ is coherent if it can be written as the restriction to U of

$$\boxplus_{\Lambda \in \mathrm{Conn}(\Gamma)} s_\Lambda,$$

where $\mathrm{Conn}(\Gamma)$ denotes the set of connected components of Γ and each s_Λ is a coherent (multi)section of $\mathcal{W}_\Lambda \to U_\Lambda$ for some $U_\Lambda \subseteq \overline{\mathcal{R}}_\Lambda$.

[2] In the special case that all anchors are either Neveu–Schwarz or twisted $r-1$, we have $\overline{\mathcal{R}}_\Gamma = \overline{\mathcal{M}}_\Gamma^{1/r}$. In general, $\overline{\mathcal{R}}_\Gamma$ is the parameter space of nodal graded genus-zero surfaces together with an element $u_t \in J'_{z_t}$ for each anchor whose twist is -1. Throughout what follows (except in the current section), the notation $\overline{\mathcal{R}}_\Gamma$ will not be used for graphs that are not closed, to avoid confusion.

Note that, if s is a coherent multisection of $\mathcal{W} \to \overline{\mathcal{R}}_\Gamma$ and $\zeta \in \overline{\mathcal{M}}_\Gamma^{1/r} \hookrightarrow \overline{\mathcal{R}}_\Gamma$, the evaluation $\mathrm{ev}_{z_t}(s(\zeta))$ is equal to zero whenever z_t is an anchor with twist -1, and thus $s(\zeta)$ is induced by a multisection of J. In other words, the restriction of a coherent multisection s to $\overline{\mathcal{M}}_\Gamma^{1/r}$ is canonically identified with a multisection of $\mathcal{W} \to \overline{\mathcal{M}}_\Gamma^{1/r}$; we denote this induced section by $\overline{s}$ in what follows. In case Γ has no anchor of twist -1, we have $\overline{s} = s$.

Note, also, that adding a coherent multisection of $\mathcal{W} \to \overline{\mathcal{R}}_\Gamma$ to a multisection of $\wp^*(\mathcal{W} \to \overline{\mathcal{M}}_\Gamma^{1/r})$ yields another coherent multisection.

4.1.2. *An example of coherent sections.* Let Γ be a connected closed graded r-spin dual graph such that the anchor, denoted t, has twist -1. Then the fiber of J' at the anchor can be identified with the trivial line $\mathbb{C}$, canonically up to a global choice of a root of unity. Let $\overline{\mathcal{M}}_\Gamma^{1/r,\mathrm{rigid}}$ be the degree-r cover of $\overline{\mathcal{M}}_\Gamma^{1/r}$ classifying pairs (C, ρ), where $\rho : J'_{z_t} \xrightarrow{\sim} \mathbb{C}$ is one of the r identifications. Denote by $\mathcal{W}' \to \overline{\mathcal{M}}_\Gamma^{1/r,\mathrm{rigid}}$ the pullback of the Witten bundle. Similarly, define $\overline{\mathcal{R}}_\Gamma^{\mathrm{rigid}}$ as the degree-r cover of $\overline{\mathcal{R}}_\Gamma$ defined by adding an identification of the fiber at the anchor with $\mathbb{C}$, and $\mathcal{W} \to \overline{\mathcal{R}}_\Gamma^{\mathrm{rigid}}$ the pullback of $\mathcal{W} \to \overline{\mathcal{R}}_\Gamma$ to this cover. We again denote by $\wp$ the projection $\overline{\mathcal{R}}_\Gamma^{\mathrm{rigid}} \to \overline{\mathcal{M}}_\Gamma^{1/r,\mathrm{rigid}}$.

For $\Lambda \in \partial^! \Gamma$, let $E_* \subseteq E(\Lambda)$ be the minimal collection of Neveu–Schwarz edges such that, after detaching E_*, the connected component of the anchor has no Neveu–Schwarz edges. Write

$$\mathrm{detach}_{E_*}\Lambda = \Lambda_0 \sqcup \Lambda_*,$$

where Λ_0 is the connected component of the anchor and Λ_* is the remaining (possibly disconnected) graph. Then the bundle $\mathcal{W}' \to \overline{\mathcal{M}}_\Lambda^{1/r,\mathrm{rigid}}$ naturally decomposes as a direct sum $\mathcal{W}'_{\Lambda_0} \boxplus \mathcal{W}_{\Lambda_*}$ (up to the isotropy groups actions) by Item (i) of Proposition 2.7. Moreover, using Remark 2.8, a section s^{Λ_0} of $\mathcal{W}'_{\Lambda_0}$ gives rise to a section $s^{\Lambda_0} \boxplus 0$ of $\mathcal{W}'_\Lambda$.

Assume, now, that Γ is smooth, and let $A_0(\Gamma) = \{\Lambda_0 \mid \Lambda \in \partial^! \Gamma\}$. Note that $\Gamma \in A_0(\Gamma)$ and, furthermore, $A_0(\Xi) \subseteq A_0(\Gamma)$ for any $\Xi \in A_0(\Gamma)$. We construct a family

$$(s^\Xi \in \Gamma(\overline{\mathcal{M}}_\Xi^{1/r,\mathrm{rigid}}, \mathcal{W}'_\Xi))_{\Xi \in A_0(\Gamma)}$$

of sections that evaluate to 1 at the anchor of any moduli point, and that satisfy

$$s^\Xi|_{\overline{\mathcal{M}}_\Lambda^{1/r,\mathrm{rigid}}} = s^{\Lambda_0} \boxplus 0 \tag{4.1}$$

for any $\Lambda \in \partial\Xi$ with a Neveu–Schwarz edge.

The construction is by induction on the dimension of $\overline{\mathcal{M}}_\Xi$. When the dimension is negative, there is nothing to prove. Suppose, then, that we have constructed the requisite family of sections for all Ξ' such that $\dim(\overline{\mathcal{M}}_{\Xi'}) < n$, and let $\Xi \in A_0(\Gamma)$ be such that $\dim(\overline{\mathcal{M}}_\Xi) = n$. Write

$$N = \bigsqcup_{\substack{\Lambda \in \partial\Xi, \\ \Lambda \text{ has a Neveu-Schwarz edge}}} \overline{\mathcal{M}}_\Lambda^{1/r,\mathrm{rigid}}.$$

We define $s^\Xi|_N$ by (4.1); note that this definition makes sense, since if $\Lambda \in \partial\Xi$ and $\Lambda' \in \partial\Lambda$, we have

$$(s^\Xi|_{\overline{\mathcal{M}}_\Lambda^{1/r,\mathrm{rigid}}})|_{\overline{\mathcal{M}}_{\Lambda'_0}^{1/r,\mathrm{rigid}}} = (s^{\Lambda_0} \boxplus 0)|_{\overline{\mathcal{M}}_{\Lambda'_0}^{1/r,\mathrm{rigid}}} = s^{\Lambda'_0} \boxplus 0 = s^\Xi|_{\overline{\mathcal{M}}_{\Lambda'}^{1/r,\mathrm{rigid}}}.$$

For $(C,\rho) \in \overline{\mathcal{M}}_\Xi^{1/r,\mathrm{rigid}} \setminus N$, there exists an element $u_C \in \mathcal{W}'_{(C,\rho)}$ with $\rho(\mathrm{ev}_{z_t}(u_{(C,\rho)})) = 1$ by Lemma 3.21; extend u_C to a smooth section $s'_{(C,\rho)} \in \Gamma(U,\mathcal{W}')$ in a neighborhood U of (C,ρ) not intersecting N. For $(C,\rho) \in N$, define $s'_{(C,\rho)}$ as an extension of the already-defined $s^\Xi_{(C,\rho)}|_N$ to a small neighborhood U. In both cases, we can choose the neighborhoods U to be small enough so that $v(C',\rho) := \rho(\mathrm{ev}_{z_t}(s'_{(C,\rho)}(C',\rho))) \neq 0$ for any $(C',\rho) \in U$. Define $s_{(C,\rho)} = s'_{(C,\rho)}/v \in \Gamma(U,\mathcal{W}')$. This section evaluates to 1 at the anchor and agrees with the previous definition on $U \cap N$.

By compactness, we can cover $\overline{\mathcal{M}}_\Xi^{1/r,\mathrm{rigid}}$ by finitely many open sets U_i on which there exist sections $s_i \in \Gamma(U_i,\mathcal{W}')$ as above. Using a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$, we can construct a global section $s^\Xi = \sum \rho_i s_i \in \Gamma(\overline{\mathcal{M}}_\Xi^{1/r,\mathrm{rigid}},\mathcal{W}')$ that evaluates to 1 at the anchor of any moduli point and satisfies (4.1). The induction follows.

We now have a section s^Γ that evaluates to 1 at the anchor. Define the section $u^{\mathrm{rigid}} \in \Gamma(\overline{\mathcal{R}}_\Gamma^{\mathrm{rigid}},\mathcal{W})$, for a point $(\lambda,C,\rho) \in \overline{\mathcal{R}}_\Gamma$ with $(C,\rho) \in \overline{\mathcal{M}}_\Gamma^{1/r,\mathrm{rigid}}$ and $\lambda \in (\mathcal{J}'_C)_{z_t}$, by

$$u^{\mathrm{rigid}}(\lambda,C,\rho) = \rho(\lambda)\wp^* s^\Gamma(C,\rho).$$

Finally, define $u \in \Gamma(\overline{\mathcal{R}}_\Gamma,\mathcal{W})$ by the averaging

$$u(\lambda,C,\rho) = \frac{1}{r}\sum_{\rho:(J'_\Sigma)_{z_t}\simeq\mathbb{C}} u^{\mathrm{rigid}}(\lambda,C,\rho)$$

for $C \in \overline{\mathcal{M}}_\Gamma^{1/r}$ and $\lambda \in (\mathcal{J}'_C)_{z_t}$, where we use the canonical identification between the fibers $\mathcal{W}_{(\lambda,C)}$ and $\mathcal{W}_{(\lambda,C,\rho)}$. The section u is coherent and vanishes precisely on $\overline{\mathcal{M}}_\Gamma^{1/r} \hookrightarrow \overline{\mathcal{R}}_\Gamma$.

4.1.3. *The assembling operation* ✠. Given a graded r-spin dual graph Γ and a subset $E' \subseteq E(\Gamma)$ of internal edges, there is an "assembling" procedure by which a coherent multisection s' of $\mathcal{W} \to \overline{\mathcal{R}}_{\mathrm{detach}_{E'}(\Gamma)}$ is glued to give a coherent multisection $✠_{\Gamma,E'}(s')$ of $\mathcal{W} \to \overline{\mathcal{R}}_\Gamma$; this is important later both in the proof of the TRRs and in the construction of boundary conditions. We explain the procedure in the case where Γ is connected, $E' = \{e\}$ consists of a single edge, and s' is a coherent section, for simplicity of notation, but the generalization to a set of edges is straightforward.

Let $\widehat{\Gamma} := \mathrm{detach}_{E'}(\Gamma)$, and let

$$\mathrm{Detach}_{E'} : \overline{\mathcal{M}}_\Gamma^{1/r} \to \overline{\mathcal{M}}_{\widehat{\Gamma}}^{1/r}$$

be the map $q \circ \mu^{-1}$, in the notation of (2.8). This map of degree one, and if e is Neveu–Schwarz, there is an analogous map

$$\mathrm{Detach}_{E'} : \overline{\mathcal{R}}_\Gamma \to \overline{\mathcal{R}}_{\widehat{\Gamma}},$$

also of degree one. Proposition 2.7 extends to the Witten bundles over $\overline{\mathcal{R}}_\Gamma$ and $\overline{\mathcal{R}}_{\widehat{\Gamma}}$; in particular, item (i) of this proposition shows that, up to the automorphism group actions, $\mathrm{Detach}_{E'}^*\widehat{\mathcal{W}}$ and $\mathcal{W}$ are naturally isomorphic. Moreover, Remark 2.8 extends as well and allows us to identify multisections of $\mathrm{Detach}_{E'}^*\widehat{\mathcal{W}}$ as multisection of $\mathcal{W}$. With this identification, we set

$$✠_{\Gamma,E'}(s') = \mathrm{Detach}_{E'}^*(s') = \mathrm{Detach}_{E'}^*(s'_1 \boxplus s'_2),$$

where s'_1 and s'_2 are the coherent multisections for the connected components of $\widehat{\Gamma}$.

If e is Ramond, then the fact that Γ is anchored implies that exactly one of its half-edges has twist -1 after detaching. Denote this half-edge by h_2, and let Γ_2 be its connected component in $\widehat{\Gamma}$; let h_1 be the other half-edge of e', with connected component Γ_1. Choose an identification $\rho : J'_{z_{h_1}} \to J'_{z_{h_2}}$, between the fibers of J' at the half nodes z_{h_i}; there are r possible identifications, which differ by rth roots of unity from one another. By definition, a point in $\overline{\mathcal{R}}_{\widehat{\Gamma}}$ consists of (ξ_1, ξ_2, ν), where $\xi_1 \in \overline{\mathcal{R}}_{\Gamma_1}$, $\xi_2 \in \overline{\mathcal{M}}^{1/r}_{\Gamma_2}$, and $\nu \in J'_{z_{h_2}}$. A coherent multisection s' associates to each such tuple an element

$$s'(\xi_1, \xi_2, \nu) = s'_1 \boxtimes s'_2 \in H^0(J')$$

such that $\mathrm{ev}_{z_{h_2}} s'_2 = \nu$ and s'_1 is coherent. We write $s'_1 = s'_1(\xi_1)$ and $s'_2 = s'_2(\xi_2, \nu)$, and consider[3]

$$s'_1(\xi_1) \boxtimes s'_2(\xi_2, \rho(\mathrm{ev}_{z_1}(s'_1(\xi_1)))). \tag{4.2}$$

This is a multisection of a bundle $J'_1 \boxtimes J'_2$ on a disconnected curve together with an identification ρ of the fibers of J'_1 and J'_2 over the half-nodes z_{h_i}, so it gives rise to a function on the fiber of $(q')^*\widehat{\mathcal{W}}$. Moreover, the multisection is constructed so that its values on the two-half nodes agree under ρ, or in the language of Proposition 2.7, its image in $\mathcal{T}$ vanishes. It therefore induces a function on the fiber of $\mu^* i_\Gamma^* \mathcal{W}$, which is a function on $H^0(J')$ on the connected curve; the induced multisection of $\mathcal{W} \to \overline{\mathcal{R}}_\Gamma$ is $\maltese_{\Gamma,E}(s)(\zeta_1, \zeta_2)$.

It is straightforward to see that $\maltese_{\Gamma,E}(s)$ is coherent and independent of the choice of ρ. We refer to $\maltese$ as the *assembling operator*.

Observation 4.3. Let Γ_1 be a graded r-spin dual graph. Suppose that Γ_2 is obtained from Γ_1 by detaching edges E_1, and Γ_3 is obtained from Γ_2 by detaching edges E_2. If s is a coherent multisection of $\mathcal{W} \to \overline{\mathcal{R}}_{\Gamma_3}$, then

$$\maltese_{\Gamma_1,E_1}(\maltese_{\Gamma_2,E_2}(s)) = \maltese_{\Gamma_1,E_1\cup E_2}(s).$$

4.1.4. *Special canonical multisections and transversality.*

Definition 4.4. Let $\mathcal{V}(\Gamma)$ be the collection of dual graphs with one vertex that may appear as connected components of

$$\mathrm{for}_{\mathrm{marking}}(\mathrm{for}_{\mathrm{illegal}}(\mathrm{detach}_{E(\Lambda)}(\Lambda))), \quad \Lambda \in \partial^{\mathrm{I}}\Gamma \setminus \partial^{+}\Gamma,$$

where $\mathrm{for}_{\mathrm{marking}}$ is the map that changes all boundary markings to zero. We call the elements of $\mathcal{V}(\Gamma)$ *abstract vertices*. For $i \neq 0$, let $\mathcal{V}^i(\Gamma) \subseteq \mathcal{V}(\Gamma)$ be the collection of abstract vertices that have an internal tail labeled i. For an internal marking $i \neq 0$ and a graph Γ with $i \in I(\Gamma)$, denote by $v_i^*(\Gamma)$ the (single-vertex) connected component of

$$\mathrm{for}_{\mathrm{marking}}(\mathrm{for}_{\mathrm{illegal}}(\mathrm{detach}_{E(\Gamma)}(\Gamma)))$$

that contains i as an internal marking.

Denote by $\Phi_{\Gamma,v} : \overline{\mathcal{M}}^{1/r}_\Gamma \to \overline{\mathcal{M}}^{1/r}_v$ the natural map. Namely, if $\mathrm{For}_{\mathrm{marking}}$ is the map (on the moduli level) that changes all boundary markings to zero and $\mathrm{For}_{\mathrm{illegal}}$ is the map that forgets all illegal, twist-zero boundary marked points with marking zero, then $\Phi_{\Gamma,v}$ is the composition of

$$\mathrm{For}_{\mathrm{illegal}} \circ \mathrm{For}_{\mathrm{marking}} \circ \mathrm{Detach}_{E(\Gamma)}$$

[3]When, as in (4.1.3), we substitute a multisection t_1 into another multisection t_2, the result is the multisection whose local branches are $t_2^i(t_1^j)$, where (t_2^i, t_1^j) are all possible pairs of local branches. The weight (see Definition A.1) of the local branch $t_2^i(t_1^j)$ is $\mu_2^i \mu_1^j$, where μ_*^* denotes the weight of the local branch t_*^*.

with the projection to $\overline{\mathcal{M}}_v^{1/r}$. Write $\Phi_{\Gamma,i}$ for $\Phi_{\Gamma,v_i^*(\Gamma)}$.

Analogously to Observation 3.13 we have

Observation 4.5. The map $\Phi_{\Gamma,i}|_{\overline{\mathcal{M}}_\Gamma^{1/r}}$ factors through F_Γ. If $v_i^*(\mathcal{B}\Gamma)$ is stable, then $\mathbb{L}_i \to \overline{\mathcal{M}}_\Gamma^{1/r}$ is canonically identified with $\Phi_{\Gamma,i}^*(\mathbb{L}_i \to \overline{\mathcal{M}}_{v_i^*(\Gamma)}^{1/r})$.

Definition 4.6. A canonical multisection $s \in C_m^\infty(\overline{\mathcal{M}}_\Gamma^{1/r}, \mathbb{L}_i)$ is *special canonical* if for every stable $v \in \mathcal{V}^i(\Gamma)$, there exists a canonical multisection s^v of $\mathbb{L}_i \to \overline{\mathcal{M}}_v$ such that for every $\Lambda \in \partial\Gamma$, we have

$$s|_{\mathcal{M}_\Lambda^{1/r}} = \Phi_{\Lambda,i}^* s^{v_i^*(\Lambda)}. \tag{4.3}$$

Note that this equality uses Observation 4.5. Write $\mathcal{S}_i = \mathcal{S}_i(\Gamma)$ for the vector space of special canonical multisections of $\mathbb{L}_i$.

Definition 4.7. A canonical multisection $s \in C_m^\infty(\overline{\mathcal{PM}}_\Gamma^{1/r}, \mathcal{W})$ is *special canonical* if the for any abstract vertex $v \in \mathcal{V}(\Gamma)$, there is a coherent multisection $s^v \in C_m^\infty(\overline{\mathcal{R}}_v, \mathcal{W})$, and those multisections satisfy

(1) if v is closed and the anchor t has twist $r-1$ then $\mathrm{ev}_t(s^v)$ is always positive with respect to the orientation of Definition 2.4, Item (4);
(2) for any $\Lambda \in \partial^! \Gamma \setminus \partial^+\Gamma$,

$$s|_{\mathcal{M}_\Lambda^{1/r}} = F_\Lambda^*(\maltese_{\mathcal{B}\Lambda, E(\mathcal{B}\Lambda)}((s^v)_{v \in \mathrm{Conn}(\mathrm{detach}(\mathcal{B}\Lambda))})) \tag{4.4}$$

where $\mathrm{Conn}(-)$ denotes the set of connected components of a graph and $\mathrm{detach}(-)$ denotes the detaching of a graph along all of its edges.

We consider s^v for $v \in \mathcal{V}(\Gamma)$ or $v \in \mathcal{V}^i(\Gamma)$ as part of the information of the special canonical multisection.

Intuitively, while canonical multisections are positive multisections that are pulled back from their base moduli, special canonical multisections satisfy the more restrictive property that their $\mathcal{W}_v$ component, for an abstract vertex v, is pulled back from the corresponding moduli $\mathcal{M}_v^{1/r}$. Because there are cases (when v is closed and the anchor is Ramond) that the Witten bundle does not decompose as a direct sum over irreducible components, we must use coherent multisections and the map $\maltese$ instead of usual multisections and direct sums.

Since s, s^v are global multisections, continuity considerations give:

Observation 4.8. Equations (4.3) and (4.4) still hold if we replace $s|_{\mathcal{M}_\Lambda^{1/r}}$ by $s|_{\overline{\mathcal{PM}}_\Lambda^{1/r}}$.

Remark 4.9. The requirement in the definitions of special canonical multisections that s^v be canonical (and in the case of $\mathbb{L}_i$, the requirement that s be canonical) is actually redundant. In fact, each s^v for open v is even special canonical. We do not need this fact, but we sketch the idea.

Let s be a special canonical multisection of the Witten bundle. If v is an abstract vertex appearing in $\mathrm{detach}_{E(\Lambda)}(\Lambda)$, if $\Xi \in \partial v$, and if $\Lambda' \in \partial\Lambda$ is a the graph obtained from Λ by replacing v in Ξ in the obvious way, then from (4.4) applied to Λ and extended to the boundary, we have

$$s|_{\mathcal{M}_{\Lambda'}^{1/r}} = F_\Lambda^*(\maltese_{\mathcal{B}\Lambda, E(\mathcal{B}\Lambda)}((s^u|_{\mathcal{M}_u^{1/r}})_{u \in \mathrm{Conn}(\mathrm{detach}(\mathcal{B}\Lambda))\setminus\{v\}}, s^v|_{\mathcal{M}_\Xi^{1/r}})).$$

On the other hand, by applying directly to Λ', we get

$$s|_{\mathcal{M}^{1/r}_{\Lambda'}} = F^*_{\Lambda'}(\maltese_{\mathcal{B}\Lambda',E(\mathcal{B}\Lambda')}((s^u|_{\mathcal{M}^{1/r}_u})_{u\in\mathrm{Conn}(\mathrm{detach}(\mathcal{B}\Lambda'))})).$$

Comparing the two equations using Observations 3.10 and 4.3 and the definition of $\maltese_*$, we see

$$s^v|_{\mathcal{M}^{1/r}_\Xi} = F^*_{\Lambda}(\maltese_{\mathcal{B}\Xi,E(\mathcal{B}\Xi)}((s^u|_{\mathcal{M}^{1/r}_u})_{u\in\mathrm{Conn}(\mathrm{detach}(\mathcal{B}\Xi))})).$$

The positivity for s^v is easily deduced from the positivity of s. A similar argument works for special canonical multisections of $\mathbb{L}_i$.

For the proof of the TRRs, we need special canonical multisections with strong transversality properties, which are stated in the lemma below. The proof of the existence of such multisections is given in Section 7.2.

Lemma 4.10. *Choose $k, l \geq 0$, and for each $i \in \{1,\dots,l\}$, let $a_i \in \{1,\dots,r-1\}$ be such that $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \neq \emptyset$. Let $\Gamma_{0,k,\vec{a}}$ denote the graded graph with a single vertex v, boundary tails marked by $\{1,\dots,k\}$ and internal tails marked $\{1,\dots,l\}$, such that the ith internal tail has twist a_i. Then, for any $d_1,\dots,d_l \geq 0$, there exist global special canonical multisections*

$$s \in C^\infty_m(\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}, \mathcal{W}),\ s_{ij} \in \mathcal{S}_i,\ 1 \leq i \leq l,\ 1 \leq j \leq d_i,$$

with the following transversality property: For any vertex $v \in \mathcal{V}(\Gamma_{0,k,\vec{a}})$ and any

$$K \subseteq \bigcup_{i\in I(v)} \{i\} \times \{1,\dots,d_i\},$$

one has

$$\hat{s}^v \oplus \bigoplus_{ab\in K} s^v_{ab} \pitchfork 0,$$

where $\hat{s}^v = s^v$ if v is open and otherwise $\hat{s}^v = \bar{s}^v$ (which also equals s^v if the anchor of v is not twisted -1). In particular, if

$$\frac{2\sum_{i=1}^l a_i + (k-1)(r-2)}{r} + 2\sum_{i=1}^l d_i = 2l + k - 3, \tag{4.5}$$

then the multisections can be chosen so that $\mathbf{s} = s \oplus \bigoplus_{i,j} s_{ij}$ is nowhere-vanishing on $\partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$.

4.2. **Proof of Theorem 4.1.** The basic idea of the proof of Theorem 4.1, which occupies the remainder of this section, is similar to the proof of Theorem 1.5 in [23]. There are two main conceptual differences, however: the usage of positivity and the contribution from internal Ramond nodes. (The latter is a possibly surprising feature of the theory, and it is also the place where the closed extended r-spin theory enters the picture.) There are also certain technical differences, mainly related to orientations, multiplicities of zeroes, and the fact that we work on $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ rather than on $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$.

We require the following two lemmas concerning homotopies.

Lemma 4.11. *Let E_1 and E_2 be bundles on $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ of the form*

$$E_j = \mathcal{W}^{\varepsilon^j} \oplus \bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus d_i^j}$$

where $d_i^j \geq 0$ *and* $\epsilon^j \in \{0,1\}$ *satisfy* $\epsilon^1 + \epsilon^2 = 1$. *Let* $d_i = d_i^1 + d_i^2$ *for each* i, *and define*

$$E = E_1 \oplus E_2 = \mathcal{W} \oplus \bigoplus_{i \in [l]} \mathbb{L}_i^{\oplus d_i}.$$

Assume that the d_i^j *are chosen so that* $\operatorname{rank}_{\mathbb{R}}(E) = k + 2l - 3$.

Let $A \subseteq \partial^0 \Gamma_{0,k,\vec{a}}$ *and set*

$$C = \coprod_{\Gamma \in A} \mathcal{M}_\Gamma^{1/r} \subseteq \partial^0 \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}.$$

Finally, let $\mathbf{s}$ *and* $\mathbf{r}$ *be two multisections of* $E|_{\partial^0 \overline{\mathcal{PM}}_\Gamma^{1/r} \cup V_+}$, *where*

$$V_+ = \left(W \cap \overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}} \right) \cup \bigcup_{\Lambda \text{ strongly positive}} \overline{\mathcal{PM}}_\Lambda^{1/r} \tag{4.6}$$

for some neighborhood W *of* $\partial^+ \overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$. *Assume that* $\mathbf{s}$ *and* $\mathbf{r}$ *satisfy:*

(1) $\mathbf{s}|_{C \cup V_+}$ *and* $\mathbf{r}|_{C \cup V_+}$ *are canonical;*

(2) the projections of $\mathbf{s}$ *and* $\mathbf{r}$ *to* E_1 *are identical and transverse to the zero-section.*

Then there is a homotopy H *between* $\mathbf{s}$ *and* $\mathbf{r}$, *of the form*

$$H(p,t) = (1-t)\mathbf{s}(p) + t\mathbf{r}(p) + t(1-t)w(p), \tag{4.7}$$

where $w(p) \in C_m^\infty(\partial^0 \overline{\mathcal{PM}}_\Gamma^{1/r} \cup V_+, E)$ *is a canonical multisection, such that* $H|_{\mathcal{M}_\Lambda^{1/r} \times [0,1]}$ *is transverse to the zero-section for any stratum* $\mathcal{M}_\Lambda^{1/r}$, *has constant-in-t projection to* E_1, *and vanishes nowhere on* $\left(C \cup V_+'\right) \times [0,1]$, *where* $V_+' \subseteq V_+$ *is also of the form* (4.6).

This lemma is proven in Section 7.3 below.

Lemma 4.12. *Let* $E \to M$ *be an orbifold vector bundle over an orbifold with corners, with* $\operatorname{rank}(E) = \dim(M)$, *and let* s_0 *and* s_1 *be nowhere-vanishing smooth multisections. Denote by* $p : [0,1] \times M \to M$ *the projection, and, for any open set* $U \subseteq M$ *with* $M \setminus U$ *compact, let*

$$H \in C_m^\infty((U \cup \partial M) \times [0,1], p^* E)$$

be such that $H(x,i) = s_i(x)$ *for* $i = 1,2$. *Suppose that* $H \pitchfork 0$ *and that* $H(u,t) \neq 0$ *for all* $t \in [0,1]$ *and* $u \in U$. *Then*

$$\int_M e(E; s_1) - \int_M e(E; s_0) = \#Z(H) = \#Z(H|_{\partial M \times [0,1]}).$$

This lemma is the noncompact orbifold analogue of [23, Lemma 3.55], and the proof is analogous as well.

Proof of Theorem 4.1. Define a section $t \in C^\infty(\overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}}, \mathbb{L}_1)$ as follows. For a smooth graded r-spin disk C, identify the preferred half Σ with the upper half-plane and set

$$t(C) = dz \left(\frac{1}{z - x_1} - \frac{1}{z - \bar{z}_1} \right) \Bigg|_{z = z_1} \in T_{z_1}^* \Sigma = T_{z_1}^* C. \tag{4.8}$$

This section, which is in fact pulled back from the zero-dimensional moduli space $\overline{\mathcal{M}}_{0,1,1}$ (and hence also from $\overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$), extends to a smooth global section t over $\overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}}$. For a graded r-spin disk C that is not necessarily smooth, set $t(C)$ to be the evaluation at $z_1 \in C$ of the unique meromorphic differential φ_C on C with simple poles at $\bar{z}_1$ and x_1 and at no other

marked points or smooth points, such that the residue at x_1 is 1 and at $\bar{z}_1$ is -1, and the residues at every pair of half-nodes sum to zero.

Let $U_{oc} \subseteq \partial\Gamma_{0,k,\vec{a}}$ be the collection of graphs Γ consisting of an open vertex v^o_Γ and a closed vertex v^c_Γ joined by a unique edge e, where $1 \in I(v^c_\Gamma)$. In what follows, we use the same notation v^o_Γ (respectively, v^c_Γ) both for the vertex and for the corresponding graph that is a connected component of v^o_Γ (respectively, v^c_Γ) in $\mathrm{detach}(\Gamma)$. For $\Gamma \in U_{oc}$, let

$$\mathrm{Detach}_e : \overline{\mathcal{PM}}^{1/r}_\Gamma \to \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma} \times \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$$

be the restriction to $\overline{\mathcal{PM}}^{1/r}_\Gamma$ of the composition $q \circ \mu^{-1}$, where q and μ are again as in (2.8). As above, we note that Detach_e is generically one-to-one.

The moduli space $\overline{\mathcal{M}}^{1/r}_{v^c}$ and its Witten bundle both carry a canonical complex orientation. Write $\tilde{\mathfrak{o}}_{v^c}$, $\mathfrak{o}_{v^c}$, and o_{v^c} for the complex orientation of the Witten bundle on $\overline{\mathcal{M}}^{1/r}_{v^c}$, the complex orientation of $\overline{\mathcal{M}}^{1/r}_{v^c}$, and the complex relative orientation described in [4, Theorem 5.2], respectively. To briefly describe the latter, let π be an arbitrary order of the boundary markings, and let $\hat{\pi}$ be the induced cyclic order, which indexes a connected component $\overline{\mathcal{M}}^{1/r,\hat{\pi}}_{v^o}$ of $\overline{\mathcal{M}}^{1/r}_{v^o}$. In [4, Notation 3.14], an orientation $\tilde{\mathfrak{o}}^\pi_{v^o_\Gamma}$ of $\overline{\mathcal{M}}^{1/r,\hat{\pi}}_{v^o}$ is given, while in [4, Definition 5.15], a corresponding orientation $\mathfrak{o}^\pi_{v^o_\Gamma}$ of the Witten bundle over $\overline{\mathcal{M}}^{1/r,\hat{\pi}}_{v^o}$ is defined. The canonical relative orientation $o_{v^o_\Gamma}$ is defined using them by [4, Equation (5.8)], and it is independent of the choice of π. See [4] for the details.

Write $\mathfrak{o}^\pi_\Gamma = \mathrm{Detach}^*_e(\mathfrak{o}_{v^c_\Gamma} \boxtimes \mathfrak{o}^\pi_{v^o_\Gamma})$ and $\tilde{\mathfrak{o}}^\pi_\Gamma = \mathrm{Detach}^*_e(\tilde{\mathfrak{o}}_{v^c_\Gamma} \boxtimes \tilde{\mathfrak{o}}^\pi_{v^o_\Gamma})$. The first key lemma is the following.

Lemma 4.13. *The zero locus of t is $\bigcup_{\Gamma\in U_{oc}} \overline{\mathcal{PM}}^{1/r}_\Gamma$. The multiplicity of $\overline{\mathcal{PM}}^{1/r}_\Gamma$ in the zero locus is r.*

If we orient by $\tilde{\mathfrak{o}}^\pi_{0,k,\vec{a}}$ the connected component $\overline{\mathcal{PM}}^{\frac{1}{r},\hat{\pi}}_{0,k,\vec{a}}$ of $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ in which the boundary marked points have cyclic order $\hat{\pi}$, then t induces on its zero locus the orientation $\tilde{\mathfrak{o}}^\pi_\Gamma$. Hence, if $\mathcal{W} \to \overline{\mathcal{M}}^{1/r}_{0,k\{a_i\}_{i\in[l]}}$ is canonically relatively oriented, then any $\mathcal{W} \to \overline{\mathcal{PM}}^{1/r}_\Gamma$ appears in the zero locus inside the total space with the relative orientation $o_\Gamma = \mathit{Detach}^*_e(o_{v^c_\Gamma} \boxtimes o_{v^o_\Gamma})$.

Proof. The orientation of $\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ is defined as the pullback of the orientation $\tilde{\mathfrak{o}}^\pi_{0,k,l}$ of $\overline{\mathcal{M}}^{\hat{\pi}}_{0,k,l}$ by the map $\mathrm{For}_{\mathrm{spin}}$. The section t is also pulled back from the same moduli space, and we use the same notation for the corresponding section over $\overline{\mathcal{M}}_{0,k,l}$.

In order to understand the vanishing locus setwise, both for the spin moduli and for the moduli without spin, note that, by definition, on a component of the closed curve C containing x_1 or $\bar{z}_1$, the differential φ_C is nowhere-vanishing. Similarly, φ_C vanishes nowhere on components whose removal disconnects x_1 from $\bar{z}_1$. On other components, it vanishes identically. Thus, t vanishes exactly on stable disks C such that the component containing z_1 is not on the path of components between the components of $\bar{z}_1$ and x_1. This happens exactly when z_1 belongs to a sphere component of Σ. Thus, $\bigcup_{\Gamma\in U_{oc}} \overline{\mathcal{M}}_\Gamma$ (or its spin analogue) is indeed the vanishing locus of t.

Let $\mathrm{for}_{\mathrm{spin}}(\Gamma)$ be the stable graph obtained from $\Gamma \in U_{oc}$ by forgetting the extra structure, and let $\tilde{\mathfrak{o}}^{\hat{\pi}}_{\mathrm{for}_{\mathrm{spin}}(\Gamma)}$ be the orientation on

$$\overline{\mathcal{M}}^{\hat{\pi}}_{\mathrm{for}_{\mathrm{spin}}(\Gamma)} \cong \overline{\mathcal{M}}_{\mathrm{for}_{\mathrm{spin}}(v^c_\Gamma)} \times \overline{\mathcal{M}}^{\hat{\pi}}_{\mathrm{for}_{\mathrm{spin}}(v^o_\Gamma)}$$

given as the product of the complex orientation with the canonical orientation of [4]. We have $\tilde{\mathfrak{o}}^{\hat{\pi}}_{\Gamma} = \mathrm{For}^*_{\mathrm{spin}}\tilde{\mathfrak{o}}^{\hat{\pi}}_{\mathrm{for}_{\mathrm{spin}}(\Gamma)}$.

The vanishing of the section t on $\overline{\mathcal{M}}^{\hat{\pi}}_{0,k,l}$ at $\overline{\mathcal{M}}^{\hat{\pi}}_{\mathrm{for}_{\mathrm{spin}}(\Gamma)}$ is transversal, and the induced orientation as (a part of) the zero locus is $\tilde{\mathfrak{o}}^{\hat{\pi}}_{\mathrm{for}_{\mathrm{spin}}(\Gamma)}$. Indeed, the transversality is proven similarly to Lemma 3.43 of [23], while the orientation claim is easily verified in the two-dimensional case, and the general case follows by an inductive argument, identical to the argument given in the proof of [4, Lemma 3.15].

Returning to the spin case, there is additional r-fold isotropy present in $\mathcal{M}^{1/r}_{\Gamma}$ that was not present in the non-spin case, so the pulled-back section has order of vanishing r in the orbifold sense.

As $\mathcal{W} \to \overline{\mathcal{M}}^{1/r}_{0,k,\vec{a}}$ is canonically relatively oriented, and we orient $\overline{\mathcal{M}}^{\hat{\pi}}_{0,k,\vec{a}}$ by $\tilde{\mathfrak{o}}^{\pi}_{0,k,\vec{a}}$, the definition of the canonical relative orientation implies that $\mathcal{W}$ is oriented by $\mathfrak{o}^{\pi}_{0,k,\vec{a}}$. Thus, by the definition of $\mathfrak{o}^{\pi}_{v^o_\Gamma}$,

$$\mathfrak{o}^{\pi}_{0,k,\vec{a}}\Big|_{\overline{\mathcal{M}}^{1/r\,\hat{\pi}}_{\Gamma}} = \mathfrak{o}^{\pi}_{v^o} \boxtimes \mathfrak{o}_{v^c}.$$

Combining with the induced orientation of the zero locus, we see that the induced relative orientation is as claimed, and Lemma 4.13 is proved. □

Write $n = d_1$, and let

$$E = \mathcal{W} \oplus \bigoplus_{i=1}^{l} \mathbb{L}_i^{\oplus d_i} \to \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}},$$

$$E_1 = \mathcal{W} \oplus \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i=2}^{l} \mathbb{L}_i^{\oplus d_i} \to \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}.$$

Take

$$\mathbf{s} = s \oplus \bigoplus_{\substack{1\le i\le l\\ 1\le j\le d_i}} s_{ij} \tag{4.9}$$

with s a global special canonical multisection of $\mathcal{W}$ and $s_{ij} \in \mathcal{S}_i$, so that $\mathbf{s}$ satisfies the refined transversality of Lemma 4.10. Thus, by definition (and Theorem A.5), the number of zeroes of $\mathbf{s}$ is

$$\#Z(\mathbf{s}) = \left\langle \prod_{i=1}^{l} \tau_{d_i}^{a_i}\sigma^k \right\rangle_0^{\frac{1}{r},o}.$$

Put

$$\mathbf{s}_1 = s \oplus \bigoplus_{\substack{1\le i\le l\\ 1\le j\le d_i\\ (i,j)\neq(1,1)}} s_{ij},$$

and set $\mathbf{t} = t \oplus \mathbf{s}_1$. Let $U_+ = U \cap \partial^0\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ where U is a neighborhood of $\partial^+\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ in which s is nowhere vanishing; positivity implies that such a neighborhood exists.

The zero locus $Z\left(t|_{\partial^0\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}}\right)$ consists of boundary strata of real codimension at least three in $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$. Thus, $\mathbf{t}$ does not vanish on $\partial^0\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$. The refined transversality of Lemma 4.10 guarantees that on such boundary strata, $\mathbf{s}_1$ does not vanish. By the refined

transversality of $\mathbf{s}_1$ again, its zero locus is transverse to $\overline{\mathcal{PM}}^{1/r}_\Gamma$ for $\Gamma \in U_{oc}$ and does not intersect $\partial\overline{\mathcal{PM}}^{1/r}_\Gamma$ for $\Gamma \in U_{oc}$. Hence, by Lemma 4.13 and the definition of weighted cardinality in Notation A.4, $\mathbf{t}$ has isolated zeroes and one has

$$\#Z(\mathbf{t}) = \#Z(t \oplus \mathbf{s}_1) = \sum_{\Gamma \in U_{oc}} r \#Z(\mathbf{s}_1|_{\overline{\mathcal{PM}}^{1/r}_\Gamma}). \tag{4.10}$$

In addition, using Theorem A.5, the integral

$$\int_{\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}} e\left(E; \mathbf{t}|_{\partial^0\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \cup U_+}\right)$$

is defined and equals $\#Z(\mathbf{t})$.

Lemma 4.14.

$$\#Z(\mathbf{t}) = \sum_{a\in\{-1,\ldots,r-2\}} \sum_{A \coprod B = \{2,\ldots,l\}} \left\langle \tau_0^a \tau_{n-1}^{a_1} \prod_{i\in A} \tau_{d_i}^{a_i} \right\rangle_0^{\frac{1}{r},ext} \left\langle \tau_0^{r-2-a} \prod_{i\in B} \tau_{d_i}^{a_i} \sigma^k \right\rangle_0^{\frac{1}{r},o}.$$

Proof. For any $\Gamma \in U_{oc}$, we have a degree-1 morphism

$$\mathrm{Detach}_e : \overline{\mathcal{PM}}^{1/r}_\Gamma \longrightarrow \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma} \times \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}.$$

Let

$$\mathrm{Proj}_o : \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma} \times \overline{\mathcal{M}}_{v^c_\Gamma} \longrightarrow \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma}, \qquad \mathrm{Proj}_c : \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma} \times \overline{\mathcal{M}}_{v^c_\Gamma} \longrightarrow \overline{\mathcal{M}}_{v^c_\Gamma}$$

be the projection maps. Using the notation $\overline{\mathcal{R}}_v$ of Section 4.1.1, we write

$$\mathcal{W}^c = \mathcal{W} \to \overline{\mathcal{R}}_{v^c_\Gamma}, \qquad \mathcal{W}^o = \mathcal{W} \to \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma},$$
$$E^c_\Gamma = \mathcal{W}^c \oplus \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i\in I(v^c_\Gamma)\setminus\{1\}} \mathbb{L}_i^{\oplus d_i} \longrightarrow \overline{\mathcal{R}}_{v^c_\Gamma},$$
$$E^o_\Gamma = \mathcal{W}^o \oplus \bigoplus_{i\in I(v^o_\Gamma)} \mathbb{L}_i^{\oplus d_i} \longrightarrow \overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma},$$

where we abuse notation, in the definition of E^c_Γ, by letting $\mathbb{L}_i$ denote the pullback of the tautological line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$ to $\overline{\mathcal{R}}_{v^c_\Gamma}$ under the natural map $\wp : \overline{\mathcal{R}}_{v^c_\Gamma} \to \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$. Recall that if e is Neveu–Schwarz, then $\overline{\mathcal{R}}_{v^c_\Gamma} = \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$.

By the definition of canonical multisections and Observation 4.8, the E^o_Γ-component of $\mathbf{s}$ may be written as

$$(\mathrm{Proj}_o \circ \mathrm{Detach}_e)^* \mathbf{s}^o_\Gamma,$$

where

$$\mathbf{s}^o_\Gamma = s^o \oplus \bigoplus_{i\in I(v^o), j\in[d_i]} s_{ij}^{v^o}$$

for canonical $s_{ij}^{v^o} \in C^\infty_m(\overline{\mathcal{PM}}^{1/r}_{v^o}, \mathbb{L}_i)$ and $s^o \in C^\infty_m(\overline{\mathcal{PM}}^{1/r}_{v^o}, \mathcal{W})$. Moreover, $\mathbf{s}^o_\Gamma$ is transverse, since it satisfies the refined transversality of Lemma 4.10.

Similarly, we have multisections

$$\mathbf{s}^c_\Gamma \in C^\infty_m(\overline{\mathcal{R}}_{v^c}, E^c_\Gamma)$$

and

$$\mathbf{s}_\Gamma^c = s^c \oplus \bigoplus_{\substack{i\in I(v^c),\, 1\le j\le d_i \\ (i,j)\neq(1,1)}} s_{ij}^{v^c},$$

where $\mathbf{s}_\Gamma^c$ is coherent and $\bar{\mathbf{s}}_\Gamma^c = \bar{s}^c \oplus \bigoplus s_{ij}^{v^c}$ is transverse to zero. We can write

$$\mathbf{s}_1|_{\overline{\mathcal{PM}}_\Gamma^{1/r}} = \text{✠}_{\Gamma,e}(s_\Gamma^o, \mathbf{s}_\Gamma^c).$$

Moreover, since $\mathbf{s}_\Gamma^o \pitchfork 0$, it does not vanish unless $\operatorname{rank}(E_\Gamma^o) \le \dim(\overline{\mathcal{PM}}_{v^o}^{1/r})$. Write

$$Z^o = \operatorname{Detach}_e^{-1}(\operatorname{Proj}_o^{-1}(Z(\mathbf{s}_\Gamma^o))) \subseteq \overline{\mathcal{PM}}_\Gamma^{1/r},$$

the preimage in $\overline{\mathcal{PM}}_\Gamma^{1/r}$ of the zero locus of $\mathbf{s}_\Gamma^o$. In case e is Neveu–Schwarz, $\mathbf{s}_\Gamma^c$ also does not vanish unless $\operatorname{rank}(E_\Gamma^c) \le \dim(\overline{\mathcal{PM}}_{v^c}^{1/r})$. In case e is Ramond, let $\bar{\mathbf{s}}_\Gamma^c$ be the restriction $\mathbf{s}_\Gamma^c|_{\overline{\mathcal{M}}_{v_\Gamma^c}}$, which we identify with a multisection of

$$\bar{E}_\Gamma^c = \bar{\mathcal{W}}^c \oplus \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i\in I(v_\Gamma^c)\setminus\{1\}} \mathbb{L}_i^{\oplus d_i} \longrightarrow \overline{\mathcal{M}}_{v_\Gamma^c}.$$

(We use the notation $\bar{\mathcal{W}}^c$ to denote the Witten bundle over $\overline{\mathcal{M}}_{v_\Gamma^c}$ in order to avoid confusion with the restriction to $\overline{\mathcal{M}}_{v_\Gamma^c}$ of $\mathcal{W}^c \to \overline{\mathcal{R}}_{v_\Gamma^c}$.) Then,

$$\mathbf{s}_1|_{Z^o} = \text{✠}_{\Gamma,e}(\mathbf{s}_\Gamma^o, \mathbf{s}_\Gamma^c)|_{Z^o} = (\operatorname{Proj}_c \circ \operatorname{Detach}_e)^* \bar{\mathbf{s}}_\Gamma^c, \tag{4.11}$$

where this equation should be interpreted using the injection of (2.10). By the transversality requirement of Lemma 4.10, $\bar{\mathbf{s}}_\Gamma^c$ is nowhere-vanishing unless $\operatorname{rank}(\bar{E}_\Gamma^c) \le \dim(\overline{\mathcal{M}}_{v_\Gamma^c})$.

Thus, whether e is Ramond or Neveu–Schwarz, $e(E_1|_{\overline{\mathcal{PM}}_\Gamma^{1/r}}; \mathbf{s}_1|_{\overline{\mathcal{PM}}_\Gamma^{1/r}}) = 0$ unless

$$\operatorname{rank}(E_\Gamma^o) = \dim(\overline{\mathcal{M}}_{v_\Gamma^o}),\ \operatorname{rank}(\bar{E}_\Gamma^c) = \dim \overline{\mathcal{M}}_{v_\Gamma^c}. \tag{4.12}$$

Assuming these equalities, we have

$$\#Z(\mathbf{s}_\Gamma^o) = \sum_{p\in Z(\mathbf{s}_\Gamma^o)} \varepsilon_p, \tag{4.13}$$

where ε_p is the weight of the zero p; see Notation A.4.

In case e is Ramond, using (4.11), we can write,

$$\#Z(\mathbf{s}_1|_{\overline{\mathcal{PM}}_\Gamma^{1/r}}) = \sum_{p\in Z(\mathbf{s}_\Gamma^o)} \varepsilon_p \#Z\left((\operatorname{Proj}_c \circ \operatorname{Detach}_e)^* \bar{\mathbf{s}}_\Gamma^c|_{\operatorname{Detach}_e^{-1}((\operatorname{Proj}_o)^{-1}(p))}\right). \tag{4.14}$$

Now, the map

$$(\operatorname{Proj}_c \circ \operatorname{Detach}_e)\big|_{\operatorname{Detach}_e^{-1}((\operatorname{Proj}_o)^{-1}(p))} : \operatorname{Detach}_e^{-1}((\operatorname{Proj}_o)^{-1}(p)) \to \overline{\mathcal{M}}_{v_\Gamma^c}^{1/r}$$

is a degree-1 map between smooth orbifolds (which even induces a diffeomorphism of the coarse spaces). Thus,

$$\#Z((\operatorname{Proj}_c \circ \operatorname{Detach}_e)^* \bar{\mathbf{s}}_\Gamma^c|_{\operatorname{Detach}_e^{-1}((\operatorname{Proj}_o)^{-1}(p))}) = \#Z(\bar{\mathbf{s}}_\Gamma^c), \tag{4.15}$$

where the right-hand side is the cardinality of the zero locus of $\bar{\mathbf{s}}^c_\Gamma$ as a transverse multisection over $\overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$. By transversality of $\bar{\mathbf{s}}^c_\Gamma$ and the interpretation of the Euler class of a bundle as the weighted zero count of a transversal multisection (Theorem A.5), we have

$$\#Z(\bar{\mathbf{s}}^c_\Gamma) = \int_{\overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}} e(\bar{E}^c_\Gamma) = \frac{1}{r}\left\langle \tau_0^{-1}\tau_{n-1}^{a_1} \prod_{\substack{i\in I(v^c_\Gamma)\\ i\neq 1}} \tau_{d_i}^{a_i} \right\rangle_0^{1/r,\mathrm{ext}}. \tag{4.16}$$

Furthermore, since $\mathbf{s}^o_\Gamma$ is transverse and canonical, we have

$$\#Z(\mathbf{s}^o_\Gamma) = \left\langle \tau_0^{r-1} \prod_{i\in I(v^o_\Gamma)} \tau_{d_i}^{a_i}\sigma^k \right\rangle_0^{1/r,o}. \tag{4.17}$$

Putting together equations (4.13), (4.14), (4.15), (4.16), and (4.17), and using (4.10), the Ramond case of the lemma is proven.

In case e is Neveu–Schwarz, we have the following analogue of equation (4.11):

$$\mathbf{s}_1|_{Z^o} = (\mathrm{Proj}_c \circ \mathrm{Detach}_e)^*\mathbf{s}^c_\Gamma$$

(more precisely, we use $(\mathrm{Proj}_c \circ \mathrm{Detach}_e)^*\mathbf{s}^c_\Gamma$ to denote $\mathrm{Detach}_e^*(\mathrm{Proj}_c^*\mathbf{s}^c_\Gamma \boxplus 0)$). From this it follows that

$$\#Z(\mathbf{s}_1|_{\overline{\mathcal{PM}}^{1/r}_\Gamma}) = \sum_{p\in Z(\mathbf{s}^o_\Gamma)} \varepsilon_p \#Z((\mathrm{Proj}_c \circ \mathrm{Detach}_e)^*\mathbf{s}^c_\Gamma|_{\mathrm{Detach}_e^{-1}((\mathrm{Proj}_o)^{-1}(p))}). \tag{4.18}$$

Again the restricted map

$$(\mathrm{Proj}_c \circ \mathrm{Detach}_e)\,\big|_{\mathrm{Detach}_e^{-1}((\mathrm{Proj}_o)^{-1}(p))} : \mathrm{Detach}_e^{-1}((\mathrm{Proj}_o)^{-1}(p)) \to \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$$

is a degree-1 map between smooth orbifolds, and thus

$$\#Z((\mathrm{Proj}_c \circ \mathrm{Detach}_e)^*\mathbf{s}^c_\Gamma|_{\mathrm{Detach}_e^{-1}((\mathrm{Proj}_o)^{-1}(p))}) = \#Z(\mathbf{s}^c_\Gamma), \tag{4.19}$$

where the right-hand side is the cardinality of the zero locus of $\mathbf{s}^c_\Gamma$ as a transverse multisection of $\mathcal{W} \to \overline{\mathcal{M}}^{1/r}_{v^c_\Gamma}$. By transversality of $\mathbf{s}^c_\Gamma$ and $\mathbf{s}^o_\Gamma$, and the fact that $\mathbf{s}^o_\Gamma$ is canonical, we obtain

$$\int_{\overline{\mathcal{PM}}^{1/r}_{v^o_\Gamma}} e(E^o_\Gamma, \mathbf{s}^o_\Gamma) = \left\langle \tau_0^{a} \prod_{i\in I(v^o_\Gamma)} \tau_{d_i}^{a_i}\sigma^k \right\rangle_0^{1/r,o}, \tag{4.20}$$

$$\int_{\overline{\mathcal{M}}_{v^c_\Gamma}} e(E^c_\Gamma) = \frac{1}{r}\left\langle \tau_0^{b}\tau_{n-1}^{a_1} \prod_{\substack{i\in I(v^c_\Gamma)\\ i\neq 1}} \tau_{d_i}^{a_i} \right\rangle_0^{1/r,c}, \tag{4.21}$$

where a is the twist of the half-edge of e that lies in the open side and b is the twist of the other half-edge. These equations, together with (4.13), (4.18), (4.19), and (4.10), prove the Neveu–Schwarz case, so Lemma 4.14 is proved. □

The next step is to analyze the difference between t and a canonical multisection.

Lemma 4.15. *We have*

$$\#Z(\mathbf{s}) \quad - \quad \#Z(\mathbf{t}) \qquad = \qquad \sum_{\substack{A \coprod B = \{2,\dots,l\} \\ k_1+k_2=k-1}} \binom{k}{k_1} \left\langle \tau_{n-1}^{a_1} \prod_{i\in A} \tau_{d_i}^{a_i} \sigma^{k_1} \right\rangle_0^{\frac{1}{r},o} \left\langle \prod_{i\in B} \tau_{d_i}^{a_i} \sigma^{k_2+2} \right\rangle_0^{\frac{1}{r},o}.$$

Proof. The proof of this lemma is very similar to the proof of Lemma 4.13 in [23]. Thus, for some of the technical details we shall refer the reader to the relevant places in [23].

Let $U_{oo} \subseteq \partial^0 \Gamma_{0,k,\vec{a}}$ be the collection of graphs Γ with exactly two vertices $v_\Gamma^\pm$, both open, and a single edge e_Γ, such that

$$1 \in I(v_\Gamma^-),\ 1 \in B(v_\Gamma^+),$$

and such that if $h^\pm \in H^B(\Gamma)$ is the half-edge belonging to $v_\Gamma^\pm$, then $\mathrm{alt}(h^-) = \mathrm{tw}(h^-) = 0$. In particular, this implies that $\mathrm{alt}(h^+) = 1$ and $\mathrm{tw}(h^+) = r-2$. Let U_{can} be the collection of graphs in $\partial^0 \Gamma_{0,k,\vec{a}} \setminus U_{oo}$ which either have a contracted boundary half edge or a single edge which is a boundary edge.

The restriction $t|_{\mathcal{M}_\Gamma}$ is canonical for every $\Gamma \in U_{\mathrm{can}}$. Indeed, this follows from the exact same proof as Lemma 4.8 in [23], which relies mainly on the observation that, in this case, t is pulled back from $\mathcal{M}_{\mathcal{B}\Gamma}$.

Let us now describe the difference in behavior between a canonical multisection of $\mathbb{L}_1$ and t on $\overline{\mathcal{PM}}_\Gamma^{1/r}$ for $\Gamma \in U_{oo}$, which is responsible for the extra contribution to the topological recursion. Let $p \in \overline{\mathcal{PM}}_{\mathcal{B}\Gamma}^{1/r}$ and let F_p be the fiber over p of the map $F_\Gamma : \overline{\mathcal{PM}}_\Gamma^{1/r} \to \overline{\mathcal{PM}}_{\mathcal{B}\Gamma}^{1/r}$, equipped with its natural orientation. Generically, F_p is the union of $a = k(v_\Gamma^-)$ closed intervals, each corresponding to a segment between marked points on which the illegal half-node can lie. By [23, Observation 4.9], we have:

Observation 4.16. The restriction of the tautological line $\mathbb{L}_i|_{F_p}$ is canonically trivialized.

We shall thus think of sections of $\mathbb{L}_i|_{F_p}$ as complex-valued functions, well-defined up to multiplication by a constant in $\mathbb{C}^*$. By definition, we have:

Observation 4.17 (c.f. Observation 4.10 of [23]). A canonical section of $\mathbb{L}_i|_{F_p}$ is constant.

The section t, on the other hand, twists non-trivially around F_p. Indeed, for $i \in B(v_\Gamma^-)$, let Γ_i be the unique graded graph in $\partial^B \Gamma$ with three open vertices v_i^0 and $v_i^\pm$ and two boundary edges $e^\pm$, such that $B(v_i^0) = \{i\}$. The boundary ∂F_p corresponds to two stable disks modelled on the graphs Γ_i for $i \in B(v_\Gamma^-)$, one for each cyclic order of $B(v_i^0)$. Denote by $\hat{F}_p$ the quotient space of F_p obtained by identifying, for each $i \in B(v_\Gamma^-)$, the two boundary points corresponding to Γ_i. Thus, $\hat{F}_p$ is homeomorphic to S^1. We have ([23, Observations 4.11 and 4.12]):

Observation 4.18. The section $t|_{F_p}$ descends to a continuous function $\theta_p : \hat{F}_p \to \mathbb{C}^*$. Furthermore, the winding number of θ_p is -1.

Returning to the proof of Lemma 4.15, let

$$E_2 = \mathbb{L}_1 \to \overline{\mathcal{PM}}_{0,k,\vec{a}}^{\frac{1}{r}},$$

so that $E = E_1 \oplus E_2$. Since $t|_{\mathcal{M}_\Gamma}$ is canonical for every $\Gamma \in U_{\mathrm{can}}$, we may apply Lemma 4.11 to the multisections $\mathbf{s}$ and $\mathbf{t}$, with the preceding choice of E_1 and E_2, $C = \coprod_{\Gamma \in U_{\mathrm{can}}} \mathcal{M}_\Gamma^{1/r}$ and $V_+ = U_+$. Thus, we may find a homotopy H between $\mathbf{s}$ and $\mathbf{t}$ of the form (4.7) such that

(i) the restriction of H to each stratum $\mathcal{M}_\Lambda^{1/r} \times [0,1]$ is transverse to zero;
(ii) H is nowhere-vanishing on $\mathcal{M}_\Gamma^{1/r} \times [0,1]$ for $\Gamma \in U_{\mathrm{can}}$;
(iii) H is nowhere-vanishing on $V'_+ \times [0,1]$, where V'_+ is as in the statement of Lemma 4.11;
(iv) the projection of H to E_1 equals $\mathbf{s}_1$ at all times

In particular, by transversality, H does not vanish on $\mathcal{M}_\Gamma \times [0,1]$ if

$$\dim \mathcal{M}_\Gamma \leq \dim \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} - 2.$$

Hence, by Lemma 4.12, we can write

$$\#Z(\mathbf{s}) - \#Z(\mathbf{t}) = -\#Z(H) = -\sum_{\Gamma \in U_{oo}} \#Z\left(H|_{\overline{\mathcal{PM}}_\Gamma^{1/r} \times [0,1]}\right). \tag{4.22}$$

Write $\pi : \partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \times [0,1] \to \partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ for the projection to the first factor, and decompose $H = H_1 \oplus H_2$, where $H_i \in C^\infty_m(\pi^* E_i)$. Then $H_1 = \pi^* \mathbf{s}_1$.

Since $\mathbf{s}_1$ is canonical, we can write $\mathbf{s}_1|_{\overline{\mathcal{PM}}_\Gamma^{1/r}} = F_\Gamma^* \mathbf{s}_1^{\mathcal{B}\Gamma}$ for each $\Gamma \in \partial^0 \Gamma_{0,k,\vec{a}}$. Transversality implies that $\mathbf{s}_1^{\mathcal{B}\Gamma}$ has isolated zeroes in $\mathcal{M}_{\mathcal{B}\Gamma}^{1/r}$ and $Z\left(\mathbf{s}_1|_{\overline{\mathcal{PM}}_\Gamma^{1/r}}\right) \subseteq F_\Gamma^{-1}\left(\mathcal{M}_{\mathcal{B}\Gamma}^{1/r}\right)$.

Write

$$\#Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right) = \sum_{p \in Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)} \varepsilon_p,$$

where again, ε_p is the weight of the zero p defined in Notation A.4. It follows from [4, Theorem 5.2, item (ii)] that for $\Gamma \in U_{oo}$,

$$\begin{aligned} \#Z\left(H|_{\overline{\mathcal{PM}}_\Gamma^{1/r} \times [0,1]}\right) &= \#Z\left(\pi^* F_\Gamma^* \mathbf{s}_1^{\mathcal{B}\Gamma}\right) \cap Z(H_2) \\ &= \sum_{p \in Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)} \varepsilon_p \cdot \#Z\left(H_2|_{F_p \times [0,1]}\right). \end{aligned} \tag{4.23}$$

Since H is of the form (4.7), we also have

$$H_2(q,s) = t(q)s + s_{11}(q)(1-s) + s(1-s)w(q), \tag{4.24}$$

where w is a canonical multisection.

Let $p \in Z(\mathbf{s}_1^{\mathcal{B}\Gamma})$. Observations 4.16, 4.17, and 4.18 and equation (4.24) imply that $H_2|_{F_p \times [0,1]}$ descends to a homotopy $\hat{H}_{2,p}$ on $\hat{F}_p \times [0,1]$ that can be viewed as taking values in $\mathbb{C}$. As a consequence,

$$\#Z\left(H_2|_{F_p \times [0,1]}\right) = \#Z\left(\hat{H}_{2,p}\right). \tag{4.25}$$

Now, $\hat{H}_{2,p}|_{\hat{F}_p \times \{0\}}$ is canonical and $\hat{H}_{2,p}|_{\hat{F}_p \times \{1\}} = \theta_p$, so by Observations 4.17 and 4.18, we obtain

$$\#Z\left(\hat{H}_{2,p}\right) = -1. \tag{4.26}$$

Putting together equations (4.22), (4.23), (4.25), and (4.26), we have

$$\#Z(\mathbf{s}) - \#Z(\mathbf{t}) = \sum_{\Gamma \in U_{oo}} \#Z(\mathbf{s}_1^{\mathcal{B}\Gamma}). \tag{4.27}$$

In order to finish, we need to calculate $\#Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)$ for $\Gamma \in U_{oo}$. Denote by $v^{\pm}_{\mathcal{B}\Gamma} \in V(\mathcal{B}\Gamma)$ the vertices corresponding to $v^{\pm}_{\mathcal{B}\Gamma} \in V(\Gamma)$. We consider them both as vertices and as graphs with single vertex. We have a canonical identification

$$\overline{\mathcal{PM}}^{1/r}_{\mathcal{B}\Gamma} \simeq \overline{\mathcal{PM}}^{1/r}_{v^+_{\mathcal{B}\Gamma}} \times \overline{\mathcal{PM}}^{1/r}_{v^-_{\mathcal{B}\Gamma}}.$$

From now on write $\mathcal{W}$ for $\mathcal{W} \to \overline{\mathcal{PM}}^{1/r}_{\mathcal{B}\Gamma}$ and $\mathcal{W}^{\pm}$ for $\mathcal{W} \to \overline{\mathcal{PM}}^{1/r}_{v^{\pm}_{\mathcal{B}\Gamma}}$. In addition, write

$$E^+_{\mathcal{B}\Gamma} = \mathcal{W}^+ \oplus \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i \in I(v^+_\Gamma)\setminus\{1\}} \mathbb{L}_i^{\oplus d_i} \longrightarrow \overline{\mathcal{PM}}^{1/r}_{v^+_{\mathcal{B}\Gamma}},$$

$$E^-_{\mathcal{B}\Gamma} = \mathcal{W}^- \oplus \bigoplus_{i \in I(v^-_\Gamma)} \mathbb{L}_i^{\oplus d_i} \longrightarrow \overline{\mathcal{PM}}^{1/r}_{v^-_{\mathcal{B}\Gamma}},$$

$$E_{\mathcal{B}\Gamma} = \mathcal{W} \oplus \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i=2}^{l} \mathbb{L}_i^{\oplus d_i} \longrightarrow \overline{\mathcal{PM}}^{1/r}_{\mathcal{B}\Gamma},$$

and let

$$\mathrm{Proj}_{\pm} : \overline{\mathcal{PM}}^{1/r}_{v^+_{\mathcal{B}\Gamma}} \times \overline{\mathcal{PM}}^{1/r}_{v^-_{\mathcal{B}\Gamma}} \longrightarrow \overline{\mathcal{PM}}^{1/r}_{v^{\pm}_{\mathcal{B}\Gamma}}$$

denote the projection maps. We clearly have

$$E_{\mathcal{B}\Gamma} = \mathrm{Proj}_+^* E^+_{\mathcal{B}\Gamma} \boxplus \mathrm{Proj}_-^* E^-_{\mathcal{B}\Gamma}.$$

Since $\mathbf{s}_1$ is special canonical, and using Observation 4.8, we can decompose

$$\mathbf{s}_1^{\mathcal{B}\Gamma} = \mathrm{Proj}_+^* \mathbf{s}^+_{\mathcal{B}\Gamma} \oplus \mathrm{Proj}_-^* \mathbf{s}^-_{\mathcal{B}\Gamma},$$

where $\mathbf{s}^{\pm}_{\mathcal{B}\Gamma} \in C^\infty_m(\overline{\mathcal{PM}}^{1/r}_{v^{\pm}_{\mathcal{B}\Gamma}}, E^{\pm}_{\mathcal{B}\Gamma})$ are canonical. Since $\mathbf{s}$ was chosen to satisfy the refined transversality of Lemma 4.10, the multisections $\mathbf{s}^{\pm}_{\mathcal{B}\Gamma}$ are transverse to zero. It follows that

$$\#Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right) = \#Z\left(\mathrm{Proj}_+^* \mathbf{s}^+_{\mathcal{B}\Gamma}\right) \cap Z\left(\mathrm{Proj}_-^* \mathbf{s}^-_{\mathcal{B}\Gamma}\right) = \#\mathrm{Proj}_+^{-1}(Z\left(\mathbf{s}^+_{\mathcal{B}\Gamma}\right)) \cap \mathrm{Proj}_-^{-1}(Z\left(\mathbf{s}^-_{\mathcal{B}\Gamma}\right)). \tag{4.28}$$

Now, $\#Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)$ vanishes unless $\operatorname{rank} E^{\pm}_{\mathcal{B}\Gamma} = \dim_{\mathbb{C}} \overline{\mathcal{M}}^{\pm}_{\mathcal{B}\Gamma}$, by transversality and dimension counting. In case the ranks of the bundles $E^{\pm}_{\mathcal{B}\Gamma}$ do agree with the dimensions of the corresponding moduli spaces, transversality implies that $\mathbf{s}^{\pm}_{\mathcal{B}\Gamma}|_{\partial^0 \overline{\mathcal{M}}^{\pm}_{\mathcal{B}\Gamma}}$ is nowhere-vanishing. Thus,

$$\#\mathrm{Proj}_+^{-1}(Z\left(\mathbf{s}^+_{\mathcal{B}\Gamma}\right)) \cap \mathrm{Proj}_-^{-1}(Z\left(\mathbf{s}^-_{\mathcal{B}\Gamma}\right)) = \#Z\left(\mathbf{s}^+_{\mathcal{B}\Gamma}\right) \#Z\left(\mathbf{s}^-_{\mathcal{B}\Gamma}\right). \tag{4.29}$$

Since $\mathbf{s}^{\pm}_{\mathcal{B}\Gamma}$ is canonical and transverse,

$$\#Z\left(\mathbf{s}^{\pm}_{\mathcal{B}\Gamma}\right) = \left\langle \prod_{i \in I(v^{\pm}_{\mathcal{B}\Gamma})} \tau_{a_i} \sigma^{k\left(v^{\pm}_{\mathcal{B}\Gamma}\right)} \right\rangle_0^{\frac{1}{r},o}. \tag{4.30}$$

For each $\Gamma \in U_{oo}$, we have $1 \in B(v^+_\Gamma)$, and e^+ is legal with twist $r-2$. It follows that

$$k\left(v^+_{\mathcal{B}\Gamma}\right) \geq 2$$

for $\Gamma \in U_{oo}$. Equations (4.27), (4.28), (4.29), and (4.30), and the facts

$$k\left(v^+_{\mathcal{B}\Gamma}\right) + k\left(v^-_{\mathcal{B}\Gamma}\right) = k+1, \quad I(v^+_{\mathcal{B}\Gamma}) \cup I(v^-_{\mathcal{B}\Gamma}) = \{2, \ldots, l\},$$

now imply Lemma 4.15. □

The first item of Theorem 4.1 now follows immediately from Lemmas 4.14 and 4.15. The proof of the second item of Theorem 4.1 is similar, except that φ_C is defined to be the unique meromorphic differential on the normalization of C with simple poles at $\bar{z}_1$ and z_2 and possibly at the nodes. The residues at $\bar{z}_1$ and z_2 are is -1 and 1, respectively, and at any node, the sum of residues at the two half-nodes is zero. The section $t(C)$ is defined as the evaluation of φ_C at z_1. The rest of the proof is exactly as for the first item. This completes the proof of Theorem 4.1. □

5. Open intersection numbers and the Gelfand-Dickey hierarchy

In this section, we prove Theorems 1.1, 1.2, and 1.3, and we derive open string and dilaton equations for the open r-spin intersection numbers.

5.1. **Primary extended closed r-spin intersection numbers.** As a preliminary step, let us derive an explicit formula for the primary closed extended r-spin intersection numbers in genus zero. We will often omit the index 0 in the symbol τ_0^α in the notation for correlators and write $\langle \prod_{i=1}^n \tau^{\alpha_i} \rangle_0^{\frac{1}{r},\mathrm{ext}}$ instead of $\langle \prod_{i=1}^n \tau_0^{\alpha_i} \rangle_0^{\frac{1}{r},\mathrm{ext}}$.

Proposition 5.1. *Let $n \geq 2$ and $0 \leq \alpha_1, \ldots, \alpha_n \leq r-1$. Then we have*

$$\left\langle \tau^{-1} \prod_{i=1}^n \tau^{\alpha_i} \right\rangle_0^{\frac{1}{r},ext} = \begin{cases} \frac{(n-2)!}{(-r)^{n-2}}, & \text{if } \frac{\sum \alpha_i - (r-1)}{r} = n-2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Proof. Equivalently, we have to prove that

$$\left\langle \tau^{-1} \prod_{i=1}^l \tau^{\alpha_i} (\tau^{r-1})^k \right\rangle_0^{\frac{1}{r},\mathrm{ext}} = \frac{(k+l-2)!}{(-r)^{k+l-2}}, \tag{5.2}$$

where $l + k \geq 2$, $0 \leq \alpha_i \leq r-2$, and the condition

$$\frac{\sum \alpha_i - (r-1)}{r} = n-2 \Leftrightarrow \sum_{i=1}^l (r-\alpha_i) + k = r+1$$

is satisfied. For $l \leq 1$, formula (5.2) says that

$$\left\langle \tau^{-1} \tau^\alpha (\tau^{r-1})^{\alpha+1} \right\rangle_0^{\frac{1}{r},\mathrm{ext}} = \frac{\alpha!}{(-r)^\alpha}, \quad 0 \leq \alpha \leq r-1, \tag{5.3}$$

and it was proved in [3, Section 4.4].

Suppose that $l \geq 2$. For a subset $I \subseteq [l]$, let

$$k_I := r + 1 - \sum_{i \in I} (r - \alpha_i) \quad \text{and} \quad A_I := \left\langle \tau^{-1} \prod_{i \in I} \tau^{\alpha_i} (\tau^{r-1})^{k_I} \right\rangle_0^{\frac{1}{r},\mathrm{ext}}.$$

In [3], we found the recursion

$$\frac{(r+k-1)!}{k!(-r)^{r-1}} A_{[l]} = \sum_{\substack{I \sqcup J = [l] \\ 1 \in I,\, l \in J}} \binom{r+k-1}{k_I - 1} A_I A_J - \sum_{\substack{I \sqcup J = [l] \\ 1, l \in I,\, J \neq \emptyset}} \binom{r+k-1}{k_I} A_I A_J, \tag{5.4}$$

that allows all primary closed extended intersection numbers to be computed starting from the intersection numbers (5.3). Therefore, we have to check that the right-hand side of (5.2)

satisfies recursion (5.4). Thus, for any $l \geq 2$, $k \geq 0$, and $0 \leq \alpha_1, \ldots, \alpha_l \leq r-2$ satisfying $\sum(r-\alpha_i)+k = r+1$, we have to prove the identity

$$\begin{aligned}\frac{(r+k-1)!}{k!(-r)^{r-1}}\frac{(k+l-2)!}{(-r)^{k+l-2}} &= \sum_{\substack{I\sqcup J=[l]\\ 1\in I,\, l\in J}} \binom{r+k-1}{k_I-1}\frac{(|I|+k_I-2)!}{(-r)^{|I|+k_I-2}}\frac{(|J|+k_J-2)!}{(-r)^{|J|+k_J-2}} \\ &\quad - \sum_{\substack{I\sqcup J=[l]\\ 1,l\in I,\, J\neq\emptyset}} \binom{r+k-1}{k_I}\frac{(|I|+k_I-2)!}{(-r)^{|I|+k_I-2}}\frac{(|J|+k_J-2)!}{(-r)^{|J|+k_J-2}}.\end{aligned} \tag{5.5}$$

Note that if $I \sqcup J = [l]$, then $k_I + k_J = r+1+k$. Therefore, identity (5.5) is equivalent to

$$\begin{aligned}\frac{(k+l-2)!}{k!} &= \sum_{\substack{I\sqcup J=[l]\\ 1\in I,\, l\in J}} \left(\prod_{i=1}^{|I|-1}(k_I-1+i)\right)\left(\prod_{j=1}^{|J|-1}(k_J-1+j)\right) \\ &\quad - \sum_{\substack{I\sqcup J=[l]\\ 1,l\in I,\, J\neq\emptyset}} \left(\prod_{i=1}^{|I|-2}(k_I+i)\right)\left(\prod_{j=1}^{|J|}(k_J-2+j)\right).\end{aligned} \tag{5.6}$$

Let $b_i := r-\alpha_i$ for $i = 1, \ldots, l$, and define

$$b_I := \sum_{i\in I} b_i, \quad \text{for } I \subseteq [l].$$

Then $k_I = r+1-b_I$, and we can rewrite identity (5.6) in the following way:

$$\begin{aligned}\frac{(k+l-2)!}{k!} &= \sum_{\substack{I\sqcup J=[l]\\ 1\in I,\, l\in J}} \left(\prod_{i=1}^{|I|-1}(r-b_I+i)\right)\left(\prod_{j=1}^{|J|-1}(k+b_I-1+j)\right) \\ &\quad - \sum_{\substack{I\sqcup J=[l]\\ 1,l\in I,\, J\neq\emptyset}} \left(\prod_{i=1}^{|I|-2}(k+b_J+i)\right)\left(\prod_{j=1}^{|J|}(r-b_J-1+j)\right).\end{aligned} \tag{5.7}$$

Consider now $b_1, \ldots, b_l$ as formal variables. Note that the right-hand side of (5.7) is a polynomial in the variables b_i, and, moreover, it does not depend on b_l. Let us prove that identity (5.7) holds as an identity between polynomials in the ring $\mathbb{C}[b_1, \ldots, b_{l-1}]$. We prove this by induction on l. The case $l = 2$ is trivial. Suppose $l \geq 3$. Denote the right-hand side of (5.7) by $R_l(b_1, \ldots, b_{l-1})$. Note that the degree of R_l is equal to $l-2$. So, in order to prove identity (5.7), it is sufficient to check that

$$R_l(b_1, \ldots, b_{l-1})|_{b_i=0} = \frac{(k+l-2)!}{k!}, \quad \text{for any } 2 \leq i \leq l-1, \tag{5.8}$$

$$\mathrm{Coef}_{b_2\cdots b_{l-1}} R_l(b_1, \ldots, b_{l-1}) = 0. \tag{5.9}$$

Let us check (5.8). Note that the polynomial R_l is symmetric with respect to permutations of the variables $b_2, \ldots, b_{l-1}$. So, without loss of generality, we can assume that $i = 2$. We

compute

$$\left[\sum_{\substack{I\sqcup J=[l]\\1\in I,\, l\in J}}\left(\prod_{i=1}^{|I|-1}(r-b_I+i)\right)\left(\prod_{j=1}^{|J|-1}(k+b_I-1+j)\right)\right]\Bigg|_{b_2=0}=$$
$$=(r+k+l-2)\sum_{\substack{I\sqcup J=[l]\setminus\{2\}\\1\in I,\, l\in J}}\left(\prod_{i=1}^{|I|-1}(r-b_I+i)\right)\left(\prod_{j=1}^{|J|-1}(k+b_I-1+j)\right).$$

In the same way, we compute

$$\left[\sum_{\substack{I\sqcup J=[l]\\1,l\in I,\, J\neq\emptyset}}\left(\prod_{i=1}^{|I|-2}(k+b_J+i)\right)\left(\prod_{j=1}^{|J|}(r-b_J-1+j)\right)\right]\Bigg|_{b_2=0}=$$
$$=(r+k+l-2)\sum_{\substack{I\sqcup J=[l]\setminus\{2\}\\1,l\in I,\, J\neq\emptyset}}\left(\prod_{i=1}^{|I|-2}(k+b_J+i)\right)\left(\prod_{j=1}^{|J|}(r-b_J-1+j)\right)+\frac{(k+l-3)!}{k!}r.$$

As a result, we obtain

$$R_l(b_1,\dots,b_{l-1})|_{b_2=0}=(r+k+l-2)R_{l-1}(b_1,b_3,\dots,b_{l-1})-\frac{(k+l-3)!}{k!}r\overset{\text{by the induction assumption}}{=}$$
$$=(r+k+l-2)\frac{(k+l-3)!}{k!}-\frac{(k+l-3)!}{k!}r=\frac{(k+l-2)!}{k!}.$$

Thus, equation (5.8) is true.

Let us check equation (5.9). It is easy to see that the coefficient of $b_2\cdots b_{l-1}$ in a term in the first sum on the right-hand side of (5.7) is equal to $(-1)^l(l-2)!$, if $I=[l]\setminus\{l\}$, and is equal to zero otherwise. Similarly, the coefficient of $b_2\cdots b_{l-1}$ in a term in the second sum on the right-hand side of (5.7) is equal to $(-1)^l(l-2)!$, if $J=[l]\setminus\{1,l\}$, and is equal to zero otherwise. So, equation (5.9) is also true. This completes the proof of the proposition. □

5.2. **Proof of Theorem 1.2.** We see that the open intersection numbers satisfy the following properties:

$$\left\langle \prod_{i=1}^{l} \tau_{d_i}^{\alpha_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = 0 \quad \text{unless} \quad \frac{(k-1)(r-2)+2\sum \alpha_i}{r} + 2\sum d_i = k+2l-3, \tag{5.10}$$

$$\left\langle \prod_{i=1}^{l} \tau^{\alpha_i} \right\rangle_0^{\frac{1}{r},o} = 0, \tag{5.11}$$

$$\frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_{p+1}^{\alpha} \partial t_q^{\beta}} = \sum_{\mu+\nu=r-2} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^{\alpha} \partial t_0^{\mu}} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_0^{\nu} \partial t_q^{\beta}} + \frac{\partial F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_p^{\alpha}} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_0^{r-1} \partial t_q^{\beta}} + \frac{\partial F_0^{\frac{1}{r},o}}{\partial t_p^{\alpha}} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial s \partial t_q^{\beta}}, \tag{5.12}$$

$$\frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_{p+1}^{\alpha} \partial s} = \sum_{\mu+\nu=r-2} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^{\alpha} \partial t_0^{\mu}} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_0^{\nu} \partial s} + \frac{\partial F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_p^{\alpha}} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_0^{r-1} \partial s} + \frac{\partial F_0^{\frac{1}{r},o}}{\partial t_p^{\alpha}} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial s^2}, \tag{5.13}$$

$$\left\langle \tau^1 \sigma^2 \right\rangle_0^{\frac{1}{r},o} = 1. \tag{5.14}$$

Property (5.10) follows from the formula for the rank of the open Witten bundle. Vanishing (5.11) is a consequence of Proposition 3.19. Equations (5.12) and (5.13) are the topological recursion relations from Theorem 4.1. Intersection number (5.14) was computed in Example 3.25.

We prove that these properties imply formula (1.10).

For $l = 1$, formula (1.10) says that

$$\left\langle \tau^{\alpha} \sigma^{\alpha+1} \right\rangle_0^{\frac{1}{r},o} = \alpha!, \quad 0 \leq \alpha \leq r-1. \tag{5.15}$$

We prove equation (5.15), using ideas borrowed from the proof of Theorem 4.6 in [3]. Namely, we consider the intersection number

$$\left\langle \tau_1^1 \tau^{\gamma} \sigma^{\gamma+2} \right\rangle_0^{\frac{1}{r},o}, \quad 0 \leq \gamma \leq r-2, \tag{5.16}$$

and apply two topological recursions to it. First, we can apply recursion (5.12) with $\alpha = 1$, $p = 0$, $\beta = \gamma$, $q = 0$, and we get that correlator (5.16) is equal to

$$\binom{\gamma+2}{2} \left\langle \tau^1 \sigma^2 \right\rangle_0^{\frac{1}{r},o} \left\langle \tau^{\gamma} \sigma^{\gamma+1} \right\rangle_0^{\frac{1}{r},o}. \tag{5.17}$$

On the other hand, by formula (5.13) with $\alpha = 1$ and $p = 0$, intersection number (5.16) is equal to

$$\begin{aligned} \binom{\gamma+1}{2} \left\langle \tau^1 \sigma^2 \right\rangle_0^{\frac{1}{r},o} \left\langle \tau^{\gamma} \sigma^{\gamma+1} \right\rangle_0^{\frac{1}{r},o} + \left\langle \tau^1 \tau^{\gamma} \tau^{r-3-\gamma} \right\rangle_0^{\frac{1}{r},\mathrm{ext}} \left\langle \tau^{\gamma+1} \sigma^{\gamma+2} \right\rangle_0^{\frac{1}{r},o} + \\ + \delta_{\gamma,r-2} \left\langle \tau^1 \tau^{r-2} \right\rangle_0^{\frac{1}{r},o} \left\langle \sigma^{r+1} \right\rangle_0^{\frac{1}{r},o}. \end{aligned} \tag{5.18}$$

By property (5.11), the last summand here vanishes. Note also that $\left\langle \tau^1 \tau^{\gamma} \tau^{r-3-\gamma} \right\rangle_0^{\frac{1}{r},\mathrm{ext}} = 1$ (see Proposition 5.1 and, for example, [21, Section 0.6]). Therefore, equating expressions (5.17) and (5.18), we obtain

$$(\gamma+1) \left\langle \tau^1 \sigma^2 \right\rangle_0^{\frac{1}{r},o} \left\langle \tau^{\gamma} \sigma^{\gamma+1} \right\rangle_0^{\frac{1}{r},o} = \left\langle \tau^{\gamma+1} \sigma^{\gamma+2} \right\rangle_0^{\frac{1}{r},o}, \quad 0 \leq \gamma \leq r-2.$$

Since $\langle \tau^1 \sigma^2 \rangle_0^{\frac{1}{r},o} = 1$, we get formula (5.15).

Denote $-\frac{1}{r!} \langle \sigma^{r+1} \rangle_0^{\frac{1}{r},o} = C$. Let us prove that

$$\left\langle \prod_{i=1}^{l} \tau^{\alpha_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = \begin{cases} C^{1-l} \frac{(k+l-2)!}{(-r)^{l-1}}, & \text{if } k \geq 1 \text{ and } \frac{(k-1)(r-2)+2\sum \alpha_i}{r} = 2l+k-3, \\ 0, & \text{otherwise.} \end{cases} \tag{5.19}$$

The idea is again the same as in the proof of Theorem 4.6 in [3]. Note that

$$\frac{(k-1)(r-2)+2\sum \alpha_i}{r} = 2l+k-3 \Leftrightarrow \sum (r-\alpha_i) + k = r+1. \tag{5.20}$$

Suppose $l \geq 2$ and numbers $0 \leq \alpha_1, \ldots, \alpha_l \leq r-1$ satisfy condition (5.20). For a subset $I \subseteq [l]$ denote

$$k_I := r+1-\sum_{i \in I} (r-\alpha_i) \quad \text{and} \quad A_I := \left\langle \prod_{i \in I} \tau^{\alpha_i} \sigma^{k_I} \right\rangle_0^{\frac{1}{r},o}.$$

Consider the intersection number

$$\left\langle \tau_1^{\alpha_1} \prod_{i=2}^{l} \tau_0^{\alpha_i} \sigma^{k+r} \right\rangle_0^{\frac{1}{r},o}, \tag{5.21}$$

and let us apply to it relation (5.12) with $\alpha = \alpha_1$, $\beta = \alpha_l$ and $p = q = 0$. We obtain

$$\sum_{\substack{I \sqcup J = [l] \\ 1 \in I,\, l \in J}} \sum_{\substack{\mu \geq -1,\, \nu \geq 0 \\ \mu+\nu = r-2}} \left\langle \prod_{i \in I} \tau^{\alpha_i} \tau^{\mu} \right\rangle_0^{\frac{1}{r},\mathrm{ext}} \underline{\left\langle \tau^{\nu} \prod_{j \in J} \tau^{\alpha_j} \sigma^{k+r} \right\rangle_0^{\frac{1}{r},o}} + \sum_{\substack{I \sqcup J = [l] \\ 1 \in I,\, l \in J}} \binom{r+k}{k_I} A_I A_J. \tag{5.22}$$

Note that the underlined term here vanishes, because otherwise we would have

$$r - \nu + \sum_{j \in J} (r-\alpha_j) + k + r = r+1 \Rightarrow \sum_{i \in I} (r-\alpha_i) = r + (r-\nu) \Rightarrow$$

$$\Rightarrow \sum_{i \in I} (r-\alpha_i) \geq r+1 \overset{I \neq [l]}{\Rightarrow} \sum_{i=1}^{l} (r-\alpha_i) \geq r+2,$$

which contradicts (5.20). On the other hand, by relation (5.13) with $\alpha = \alpha_1$ and $p = 0$, intersection number (5.21) is equal to

$$\delta_{k,0} \left\langle \prod_{i=1}^{l} \tau^{\alpha_i} \tau^{-1} \right\rangle_0^{\frac{1}{r},\mathrm{ext}} \langle \tau^{r-1} \sigma^r \rangle_0^{\frac{1}{r},o} + \sum_{\substack{I \sqcup J = [l] \\ 1 \in I}} \binom{r+k-1}{k_I} A_I A_J. \tag{5.23}$$

Equating expressions (5.22) and (5.23), we get

$$\binom{r+k-1}{k}\left\langle\sigma^{r+1}\right\rangle_0^{\frac{1}{r},o}A_{[l]}+\delta_{k,0}\left\langle\prod_{i=1}^{l}\tau^{\alpha_i}\tau^{-1}\right\rangle_0^{\frac{1}{r},\mathrm{ext}}\left\langle\tau^{r-1}\sigma^r\right\rangle_0^{\frac{1}{r},o}= \tag{5.24}$$
$$=\sum_{\substack{I\sqcup J=[l]\\1\in I,\,l\in J}}\binom{r+k-1}{k_I-1}A_IA_J-\sum_{\substack{I\sqcup J=[l]\\1,l\in I,\,J\neq\emptyset}}\binom{r+k-1}{k_I}A_IA_J.$$

This recursion allows one to compute all primary intersection numbers starting from the numbers (5.15) and the single number $\langle\sigma^{r+1}\rangle_0^{\frac{1}{r},o}$.

We see that in order to prove formula (5.19), it is sufficient to check that the right-hand of (5.19) satisfies equation (5.24). Therefore, we have to check the identity

(5.25)

$$\binom{r+k-1}{k}(-Cr!)\frac{(k+l-2)!}{(-r)^{l-1}}=$$
$$=\sum_{\substack{I\sqcup J=[l]\\1\in I,\,l\in J}}\binom{r+k-1}{k_I-1}C^{1-|I|}\frac{(|I|+k_I-2)!}{(-r)^{|I|-1}}C^{1-|J|}\frac{(|J|+k_J-2)!}{(-r)^{|J|-1}}$$
$$-\sum_{\substack{I\sqcup J=[l]\\1,l\in I,\,J\neq\emptyset}}\binom{r+k-1}{k_I}C^{1-|I|}\frac{(|I|+k_I-2)!}{(-r)^{|I|-1}}C^{1-|J|}\frac{(|J|+k_J-2)!}{(-r)^{|J|-1}},$$

where $k\geq 1$ and numbers $0\leq\alpha_1,\dots,\alpha_l\leq r-1$ satisfy the condition $\sum(r-\alpha_i)+k=r+1$. Identity (5.25) is equivalent to the equation

$$\frac{(k+l-2)!}{k!}=\sum_{\substack{I\sqcup J=[l]\\1\in I,\,l\in J}}\frac{(|I|+k_I-2)!}{(k_I-1)!}\frac{(|J|+k_J-2)!}{(k_J-1)!}-\sum_{\substack{I\sqcup J=[l]\\1,l\in I,\,J\neq\emptyset}}\frac{(|I|+k_I-2)!}{k_I!}\frac{(|J|+k_J-2)!}{(k_J-2)!},$$

which is exactly equation (5.6). (Strictly speaking, in equation (5.6) we required the condition $0\leq\alpha_i\leq r-2$, but we actually proved it for any integers $\alpha_1,\dots,\alpha_l$ satisfying $\sum(r-\alpha_i)+k=r+1$.)

It remains to prove that $C=1$. For this, we write relation (5.24) with $l=3$, $k=0$, $\alpha_1=\alpha_3=r-1$, and $\alpha_2=1$. Since, by property (5.11), $\langle\tau^{r-1}\tau^1\tau^{r-1}\rangle_0^{\frac{1}{r},o}=0$, we obtain

(5.26)

$$\left\langle\tau^{r-1}\tau^1\tau^{r-1}\tau^{-1}\right\rangle_0^{\frac{1}{r},\mathrm{ext}}\left\langle\tau^{r-1}\sigma^r\right\rangle_0^{\frac{1}{r},o}=2\left\langle\tau^{r-1}\tau^1\sigma\right\rangle_0^{\frac{1}{r},o}\left\langle\tau^{r-1}\sigma^r\right\rangle_0^{\frac{1}{r},o}-\left\langle\tau^{r-1}\tau^{r-1}\sigma^{r-1}\right\rangle_0^{\frac{1}{r},o}\left\langle\tau^1\sigma^2\right\rangle_0^{\frac{1}{r},o}.$$

We have

$$\left\langle\tau^{r-1}\tau^1\tau^{r-1}\tau^{-1}\right\rangle_0^{\frac{1}{r},\mathrm{ext}}\overset{\text{Prop. 5.1}}{=}-\frac{1}{r},\qquad\left\langle\tau^{r-1}\sigma^r\right\rangle_0^{\frac{1}{r},o}\overset{\text{eq. (5.15)}}{=}(r-1)!,$$
$$\left\langle\tau^{r-1}\tau^1\sigma\right\rangle_0^{\frac{1}{r},o}\overset{\text{eq. (5.19)}}{=}-\frac{1}{Cr},\qquad\left\langle\tau^{r-1}\tau^{r-1}\sigma^{r-1}\right\rangle_0^{\frac{1}{r},o}\overset{\text{eq. (5.19)}}{=}-\frac{(r-1)!}{Cr},$$
$$\left\langle\tau^1\sigma^2\right\rangle_0^{\frac{1}{r},o}\overset{\text{eq. (5.14)}}{=}1.$$

Substituting these values in (5.26), we get $C=1$. Theorem 1.2 is proved.

5.3. **Proof of Theorems 1.1 and 1.3.** In [3, Theorem 4.6], we proved that

$$F_0^{\frac{1}{r},\mathrm{ext}}(t_*^{\leq r-2}, t_*^{r-1}) = \sqrt{-r}\phi_0\left(t_*^{\leq r-2}, \frac{1}{\sqrt{-r}}t_*^{r-1}\right).$$

Therefore, Theorems 1.1 and 1.3 are equivalent.

Let us prove Theorem 1.3. Denote

$$\widetilde{F}(t_*^*, s) := -\frac{1}{r}\, F_0^{\frac{1}{r},\mathrm{ext}}\Big|_{t_d^{r-1}\mapsto t_d^{r-1}-r\delta_{d,0}s} + \frac{1}{r}F_0^{\frac{1}{r},\mathrm{ext}},$$

$$\left\langle \prod_{i=1}^{l}\tau_{d_i}^{\alpha_i}\sigma^k \right\rangle^{\widetilde{F}} := \left.\frac{\partial^{l+k}\widetilde{F}}{\partial t_{d_1}^{\alpha_1}\cdots\partial t_{d_l}^{\alpha_l}\partial s^k}\right|_{\substack{t_*^*=0\\ s=0}}. \tag{5.27}$$

We claim that the correlators (5.27) satisfy properties (5.10) – (5.14). Indeed, from the definition of the closed extended r-spin correlators [3, Section 3.1], it follows that a correlator $\left\langle \prod_{i=1}^{l}\tau_{d_i}^{\alpha_i}\sigma^k \right\rangle^{\widetilde{F}}$ is zero unless

$$\frac{-(r-1)+\sum\alpha_i + k(r-1)}{r} + \sum d_i = k+l-2,$$

or equivalently,

$$\frac{(k-1)(r-2)+2\sum\alpha_i}{r} + 2\sum d_i = k+2l-3.$$

The property $\left\langle \prod_{i=1}^{l}\tau^{\alpha_i} \right\rangle^{\widetilde{F}} = 0$ is obvious. In [3, Lemma 3.6], we proved that the function $F_0^{\frac{1}{r},\mathrm{ext}}$ satisfies the equations

$$\frac{\partial^2 F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_{p+1}^{\alpha}\partial t_q^{\beta}} = \sum_{\mu+\nu=r-2}\frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^{\alpha}\partial t_0^{\mu}}\frac{\partial^2 F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_0^{\nu}\partial t_q^{\beta}} + \frac{\partial F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_p^{\alpha}}\frac{\partial^2 F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_0^{r-1}\partial t_q^{\beta}},$$

which imply that

$$\frac{\partial^2\widetilde{F}}{\partial t_{p+1}^{\alpha}\partial t_q^{\beta}} = \sum_{\mu+\nu=r-2}\frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^{\alpha}\partial t_0^{\mu}}\frac{\partial^2\widetilde{F}}{\partial t_0^{\nu}\partial t_q^{\beta}} + \frac{\partial F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_p^{\alpha}}\frac{\partial^2\widetilde{F}}{\partial t_0^{r-1}\partial t_q^{\beta}} + \frac{\partial\widetilde{F}}{\partial t_p^{\alpha}}\frac{\partial^2\widetilde{F}}{\partial s\partial t_q^{\beta}},$$

$$\frac{\partial^2\widetilde{F}}{\partial t_{p+1}^{\alpha}\partial s} = \sum_{\mu+\nu=r-2}\frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^{\alpha}\partial t_0^{\mu}}\frac{\partial^2\widetilde{F}}{\partial t_0^{\nu}\partial s} + \frac{\partial F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_p^{\alpha}}\frac{\partial^2\widetilde{F}}{\partial t_0^{r-1}\partial s} + \frac{\partial\widetilde{F}}{\partial t_p^{\alpha}}\frac{\partial^2\widetilde{F}}{\partial s^2}.$$

Finally, in [3, Lemma 3.8], we proved that $\langle\tau^{-1}\tau^1(\tau^{r-1})^2\rangle_0^{\frac{1}{r},\mathrm{ext}} = -\frac{1}{r}$ and, therefore, $\langle\tau^1\sigma^2\rangle^{\widetilde{F}} = 1$.

In Section 5.2, we proved that properties (5.10) – (5.14) are sufficient to reconstruct all primary open intersection numbers $\left\langle \prod_{i=1}^{l}\tau^{\alpha_i}\sigma^k \right\rangle_0^{\frac{1}{r},o}$. Since the correlators $\left\langle \prod_{i=1}^{l}\tau_{d_i}^{\alpha_i}\sigma^k \right\rangle^{\widetilde{F}}$ also satisfy these properties, we can conclude that $\left\langle \prod_{i=1}^{l}\tau^{\alpha_i}\sigma^k \right\rangle_0^{\frac{1}{r},o} = \left\langle \prod_{i=1}^{l}\tau^{\alpha_i}\sigma^k \right\rangle^{\widetilde{F}}$. Note that the dimension constraint in property (5.10) implies that $k+l \geq 2$. Therefore, using topological recursion relations (5.12) and (5.13), one can reconstruct all open intersection

numbers starting from primary numbers. The same is true for the correlators (5.27) and, thus, $F_0^{\frac{1}{r},o} = \widetilde{F}$. Theorem 1.3 is proved.

5.4. **Open string and dilaton equations.**

Proposition 5.2. *We have*

$$\left\langle \tau_0^0 \prod_{i=1}^{l} \tau_{d_i}^{\alpha_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = \begin{cases} \displaystyle\sum_{\substack{1\le i\le l \\ d_i>0}} \left\langle \tau_{d_i-1}^{\alpha_i} \prod_{j\neq i} \tau_{d_j}^{\alpha_j} \sigma^k \right\rangle_0^{\frac{1}{r},o}, & \text{if } l \ge 1, \\ \delta_{k,1}, & \text{if } l = 0. \end{cases} \tag{5.28}$$

Proof. In [3, Lemma 3.7], we proved that

$$\left(\frac{\partial}{\partial t_0^0} - \sum_{n\ge 0} t_{n+1}^{\alpha} \frac{\partial}{\partial t_n^{\alpha}} \right) F_0^{\frac{1}{r},\mathrm{ext}} = t_0^{r-1}.$$

By Theorem 1.3, this implies that

$$\left(\frac{\partial}{\partial t_0^0} - \sum_{n\ge 0} t_{n+1}^{\alpha} \frac{\partial}{\partial t_n^{\alpha}} \right) F_0^{\frac{1}{r},o} = s,$$

which is equivalent to equation (5.28). □

Proposition 5.3. *We have*

$$\left\langle \tau_1^0 \prod_{i=1}^{l} \tau_{d_i}^{\alpha_i} \sigma^k \right\rangle_0^{\frac{1}{r},o} = (k+l-1) \left\langle \prod_{i=1}^{l} \tau_{d_i}^{\alpha_i} \sigma^k \right\rangle_0^{\frac{1}{r},o}.$$

Proof. Let us check that

$$\left(\frac{\partial}{\partial t_1^0} - \varepsilon \frac{\partial}{\partial \varepsilon} - \sum_{n\ge 0} t_n^{\alpha} \frac{\partial}{\partial t_n^{\alpha}} \right) \phi = \frac{1}{2}. \tag{5.29}$$

By Theorem 1.1, this is sufficient for the proof of the proposition. In the variables T_i, equation (5.29) looks as follows:

$$\left(\frac{1}{r+1} \frac{\partial}{\partial T_{r+1}} - \varepsilon \frac{\partial}{\partial \varepsilon} - \sum_{i\ge 1} T_i \frac{\partial}{\partial T_i} \right) \phi = \frac{1}{2}. \tag{5.30}$$

Before we proceed with the proof of equation (5.30), let us recall some more facts from the theory of the Gelfand–Dickey hierarchy.

Consider a pseudo-differential operator $A = \sum_{n=-\infty}^{m} a_n(T_*, \varepsilon)\partial_x^n$. The negative-degree part of A is defined by $A_- := A - A_+$. The Laurent series

$$\widehat{A}(T_*, \varepsilon, z) := \sum_{n=-\infty}^{m} a_n(T_*, \varepsilon) z^n,$$

in which z is a formal variable, is called the *symbol* of the operator A. Suppose an operator L is a solution of the Gelfand–Dickey hierarchy (1.4). Then there exists a pseudo-differential

operator P of the form

$$P = 1 + \sum_{n\geq 1} p_n(T_*, \varepsilon)\partial_x^{-n},$$

satisfying $L = P \circ \partial_x^r \circ P^{-1}$ and

$$\frac{\partial P}{\partial T_n} = -\varepsilon^{n-1}\left(L^{n/r}\right)_- \circ P, \quad n \geq 1.$$

The operator P is called the *dressing operator* of the operator L.

Denote by G_z the shift operator, which acts on a power series $f \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_1, T_2, \ldots]]$ as follows:

$$G_z(f)(T_1, T_2, T_3, \ldots) := f\left(T_1 - \frac{1}{z}, T_2 - \frac{1}{2\varepsilon z^2}, T_3 - \frac{1}{3\varepsilon^2 z^3}, \ldots\right).$$

Let $P = 1 + \sum_{n\geq 1} p_n(T_*, \varepsilon)\partial_x^{-n}$ be the dressing operator of some operator L satisfying the Gelfand–Dickey hierarchy (1.4). Then there exists a series $\tau \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_1, T_2, T_3, \ldots]]$ with constant term $\tau|_{T_i=0} = 1$ for which

$$\widehat{P} = \frac{G_z(\tau)}{\tau}.$$

The series τ is called a *tau-function* of the KP hierarchy. The operator L can be reconstructed from the tau-function τ by the following formula:

$$\operatorname{res} L^{n/r} = \varepsilon^{1-n}\frac{\partial^2 \log \tau}{\partial T_1 \partial T_n}, \quad n \geq 1.$$

This and further material about the Gelfand–Dickey hierarchy can be found in the book [9].

Denote the linear differential operator in the brackets in equation (5.30) by O. Let L be the solution of the Gelfand–Dickey hierarchy specified by the initial condition (1.5). Let us show that

$$\left(z\frac{\partial}{\partial z} + O\right)\widehat{L} = r\widehat{L}. \tag{5.31}$$

Witten's conjecture [28], proved by Faber, Shadrin, and Zvonkine [11], says that the function $\tau^{\frac{1}{r},c} := \exp(F^{\frac{1}{r},c})$ is a tau-function of the Gelfand–Dickey hierarchy, corresponding to the operator L. Therefore, the dressing operator P of the operator L is given by $\widehat{P} = \frac{G_z(\tau^{1/r,c})}{\tau^{1/r,c}}$. The function $\tau^{\frac{1}{r},c}$ satisfies the dilaton equation

$$O\tau^{\frac{1}{r},c} = \frac{(r-1)^2}{24}\tau^{\frac{1}{r},c}.$$

We compute

$$\left(z\frac{\partial}{\partial z} + O\right)G_z(\tau^{\frac{1}{r},c}) = G_z(O\tau^{\frac{1}{r},c}) = \frac{(r-1)^2}{24}G_z(\tau^{\frac{1}{r},c}),$$

and, thus,

$$\left(z\frac{\partial}{\partial z} + O\right)\widehat{P} = \left(z\frac{\partial}{\partial z} + O\right)\frac{G_z(\tau^{\frac{1}{r},c})}{\tau^{\frac{1}{r},c}} = 0.$$

Since $L = P \circ \partial_x^r \circ P^{-1}$, we conclude that equation (5.31) is true.

Let us finally prove equation (5.30). Equivalently, we have to prove that $O\Phi = \frac{1}{2}\Phi$. We compute

$$L^{\frac{1}{r}}\Big|_{T_{\geq 2}=0} = \partial_x + \varepsilon^{-r} x \partial_x^{-r+1} - \frac{r-1}{2}\varepsilon^{-r}\partial_x^{-r} + \dots,$$
$$(L^{\frac{r+1}{r}})_+\Big|_{T_{\geq 2}=0} = \partial_x^{r+1} + (r+1)\varepsilon^{-r} x \partial_x + \frac{r+1}{2}\varepsilon^{-r},$$

and, therefore,

$$O\Phi|_{T_{\geq 2}=0} = \frac{1}{r+1}\frac{\partial \Phi}{\partial T_{r+1}}\Big|_{T_{\geq 2}=0} = \frac{\varepsilon^r}{r+1}\widehat{(L^{\frac{r+1}{r}})_+}\Big|_{z=T_{\geq 2}=0} = \frac{1}{2}. \tag{5.32}$$

We also have

$$\frac{\partial}{\partial T_n} O\Phi = (O-1)(\varepsilon^{n-1}(L^{n/r})_+\Phi) \overset{\text{by eq. (5.31)}}{=} \varepsilon^{n-1}(L^{n/r})_+(O\Phi).$$

We see that the function $O\Phi$ satisfies the same system of linear differential equations (1.7) as the function Φ, with the initial condition (5.32). Thus, $O\Phi = \frac{1}{2}\Phi$ and the proposition is proved. □

6. Prediction for higher genus

Here we give a prediction for the behavior of open r-spin intersection numbers in higher genus, and provide geometric and algebraic evidence for its correctness.

6.1. **Main conjecture.** We conjecture that, for any genus $g \geq 1$, there is a geometric construction of open r-spin intersection numbers

$$\left\langle \tau_{d_1}^{\alpha_1} \cdots \tau_{d_l}^{\alpha_l} \sigma^k \right\rangle_g^{\frac{1}{r},o} \tag{6.1}$$

generalizing our construction in genus zero. Given such intersection numbers, define a generating series $F_g^{\frac{1}{r},o}(t_*^*, s)$ by

$$F_g^{\frac{1}{r},o}(t_*^*, s) := \sum_{l,k\geq 0} \frac{1}{l!k!} \sum_{\substack{0\leq \alpha_1,\dots,\alpha_l \leq r-1 \\ d_1,\dots,d_l \geq 0}} \left\langle \tau_{d_1}^{\alpha_1} \cdots \tau_{d_l}^{\alpha_l} \sigma^k \right\rangle_g^{\frac{1}{r},o} t_{d_1}^{\alpha_1} \cdots t_{d_l}^{\alpha_l} s^k.$$

Conjecture 1. *For any $g \geq 1$, we have*

$$F_g^{\frac{1}{r},o} = (-r)^{\frac{g-1}{2}} \phi_g\Big|_{t_d^{r-1} \mapsto \frac{1}{\sqrt{-r}}(t_d^{r-1} - \delta_{d,0} r s)}.$$

Remark 6.1. In open Gromov–Witten theory, unlike the closed theory, not much is known or even conjectured about higher-genus invariants. The conjecture presented here is one of the few conjectures that describes the full, all-genus open Gromov–Witten theory in particular cases, and also one of the few that relates the potential to an integrable hierarchy.[4]

6.2. **Evidence.**

[4]As mentioned, an analogous conjecture was made in [23] and [25] for the $r = 2$ case, and it was proven in [6]. In [5], the full stationary open Gromov–Witten theory for maps to $(\mathbb{CP}^1, \mathbb{RP}^1)$ is conjectured.

6.2.1. *Dimension constraint, open string and dilaton equations.* It is natural to expect that the open intersection number (6.1) is zero unless

$$\frac{(g+k-1)(r-2)+2\sum\alpha_i}{r}+2\sum d_i = 3g-3+2l+k.$$

Indeed, the right-hand side is the dimension of the moduli space of r-spin genus-g surfaces with boundary, while the left-hand side is the virtual dimension of the purported open analogue of the virtual fundamental cycle that should correspond to the intersection problem. This is equivalent to the constraint

$$\sum\Big(\frac{\alpha_i}{r}+d_i-1\Big)-\frac{k}{r}=\frac{(r+1)(g-1)}{r}. \tag{6.2}$$

On the other hand, in [3, Lemma 4.4], we prove that the derivative $\frac{\partial^n\phi_g}{\partial t_{d_1}^{\alpha_1}\cdots\partial t_{d_n}^{\alpha_n}}\Big|_{t_*^*=0}$ is zero unless $\sum\big(\frac{\alpha_i}{r}+d_i-1\big)=\frac{(r+1)(g-1)}{r}$. This property and constraint (6.2) agree with Conjecture 1.

Furthermore, one would expect the function $F_g^{\frac{1}{r},o}$ to satisfy the open string and dilaton equations

$$\begin{aligned}\frac{\partial F_g^{\frac{1}{r},o}}{\partial t_0^0}=&\sum_{n\geq 0}t_{n+1}^\alpha\frac{\partial F_g^{\frac{1}{r},o}}{\partial t_n^\alpha}+\delta_{g,0}s,\\ \frac{\partial F_g^{\frac{1}{r},o}}{\partial t_1^0}=&(g-1)F_g^{\frac{1}{r},o}+\sum_{n\geq 0}t_n^\alpha\frac{\partial F_g^{\frac{1}{r},o}}{\partial t_n^\alpha}+s\frac{\partial F_g^{\frac{1}{r},o}}{\partial s}+\delta_{g,1}\frac{1}{2}.\end{aligned}$$

The $r=2$, Neveu–Schwarz case of these equations is proven in [23] in genus zero, and in [25] for all genera, using geometric techniques. Although in this paper, we used algebraic techniques (Propositions 5.2 and 5.3) to prove the analogous equations in the genus-zero case for all r and with Ramond insertions, we could have also proven them geometrically, imitating the proofs in [23]. In particular, we expect the geometric proof to work in all genera, all sectors, and all $r\geq 2$, given a construction of an open virtual fundamental cycle for the higher-genus Witten bundle that satisfies certain expected properties (being pulled back from the moduli without markings of twist zero, for example, and boundary behavior similar to that of canonical sections in genus zero). By [3, Lemmas 4.5] and equation (5.29), we have

$$\begin{aligned}\frac{\partial\phi_g}{\partial t_0^0}=&\sum_{n\geq 0}t_{n+1}^\alpha\frac{\partial\phi_g}{\partial t_n^\alpha}+\delta_{g,0}t_0^{r-1},\\ \frac{\partial\phi_g}{\partial t_1^0}=&(g-1)\phi_g+\sum_{n\geq 0}t_n^\alpha\frac{\partial\phi_g}{\partial t_n^\alpha}+\delta_{g,1}\frac{1}{2}.\end{aligned}$$

This agrees with Conjecture 1.

6.2.2. *The case $r=2$.* Suppose $r=2$, and consider the Neveu–Schwarz sector $t_d^1=0$.

Proposition 6.2. *We have an equality*

$$\left\langle \prod_{i\in[l]} \tau_{d_i}^0 \sigma^k \right\rangle_0^{\frac{1}{2},o} = (-2)^{\frac{k-1}{2}} \left\langle \prod_{i\in[l]} \tau_{d_i} \sigma^k \right\rangle_0^{o}, \tag{6.3}$$

where the right-hand side refers to the intersection numbers defined in [23]*. In other words, the Neveu–Schwarz sector of our theory agrees with the genus-zero part of the theory defined in* [23]*.*

Proof. When $r = 2$ and there are no Ramond markings, the Witten bundle is of rank zero.

In [23], the intersection theory is defined over $\overline{\mathcal{M}}_{0,k,l}$, which is the $r = 2$ case. There is a map

$$f = \mathrm{For}_{\mathrm{spin}} : \overline{\mathcal{PM}}^{1/r}_{0,k,\{0\}_{i\in[l]}} \to \overline{\mathcal{M}}_{0,k,l},$$

which is generically one-to-one. Note that when $r = 2$ and there are no Ramond insertions $\overline{\mathcal{PM}}^{1/r}_{0,k,\{0\}_{i\in[l]}} = \overline{\mathcal{M}}_{0,k,\{0\}_{i\in[l]}}$. Now, $\left\langle \prod_{i\in[l]} \tau_{d_i}\sigma^k \right\rangle^o$ were defined using a notion of special canonical multisections, and a simple verification of definitions shows that, if s is special canonical in the sense of [23], then f^*s is canonical in the sense of this paper.[5] Furthermore, Lemmas 3.53 and 3.56 of [23] show that nowhere-vanishing special canonical boundary conditions s exist, and

$$\left\langle \prod_{i\in[l]} \tau_{d_i} \sigma^k \right\rangle_0^{o} = 2^{-\frac{k-1}{2}} \int_{\overline{\mathcal{M}}_{0,k,l}} e\left(\bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus d_i}; s \right).$$

On the other hand, Lemma 4.10 implies that

$$\left\langle \prod_{i\in[l]} \tau_{d_i}^0 \sigma^k \right\rangle_0^{\frac{1}{2},o} = \int_{\overline{\mathcal{PM}}^{1/r}_{0,k,\{0\}_{i\in[l]}}} e\left(\bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus d_i}; f^*s \right).$$

Finally, the $(-1)^{\frac{k-1}{2}}$ comes from the slight difference in our choice of orientation in the current work compared to the choice in [23]; see Remark 3.13 of [4]. □

In higher genus, open intersection numbers $\left\langle \tau_{d_1} \cdots \tau_{d_l} \sigma^k \right\rangle_g^o$ are constructed in [25]; the details also appear in [26], where these numbers are calculated. In [6], the authors proved that

$$\sum_{l,k\geq 0} \sum_{d_1,\ldots,d_l\geq 0} \left\langle \tau_{d_1} \cdots \tau_{d_l} \sigma^k \right\rangle_g^o \frac{t_{d_1} \cdots t_{d_l} s^k}{l!k!} = \phi_g \Big|_{\substack{t_0^1 = s \\ t_{\geq 1}^1 = 0}},$$

where we identify $t_d^0 = t_d$. Note that the factor $2^{-\frac{g+k-1}{2}}$ is forcibly included in the definition of the intersection numbers $\left\langle \tau_{d_1} \cdots \tau_{d_l} \sigma^k \right\rangle_g^o$. Similarly, if we change the orientation of $\overline{\mathcal{M}}_{g,k,l}$ by $(-1)^{\frac{g+k-1}{2}}$ when $g + k$ is odd, as we know is necessary when $g = 0$, then Conjecture 1 is confirmed in the case $r = 2$ after setting $t_d^1 = 0$.

[5] If s is canonical in the sense of [23], it is not automatically the case that f^*s is canonical in the sense of this paper, because of a difference in the choice of boundary labels for the base. In our case, a canonical multisection may have to be more symmetric with respect to renaming boundary markings.

6.2.3. *Topological recursion relations in genus one.* We expect the following topological recursion relations for the genus-one open r-spin intersection numbers:

$$\frac{\partial F_1^{\frac{1}{r},o}}{\partial t_{p+1}^\alpha} = \sum_{\mu+\nu=r-2} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^\alpha \partial t_0^\mu} \frac{\partial F_1^{\frac{1}{r},o}}{\partial t_0^\nu} + \frac{\partial F_0^{\frac{1}{r},\mathrm{ext}}}{\partial t_p^\alpha} \frac{\partial F_1^{\frac{1}{r},o}}{\partial t_0^{r-1}} + \frac{\partial F_0^{\frac{1}{r},o}}{\partial t_p^\alpha} \frac{\partial F_1^{\frac{1}{r},o}}{\partial s} + \frac{1}{2} \frac{\partial^2 F_0^{\frac{1}{r},o}}{\partial t_p^\alpha \partial s}. \tag{6.4}$$

These relations were conjectured by the authors of [23], and were proven geometrically in the $r = 2$ case (for Neveu–Schwarz insertions only) in [25]. The proof presented there cannot work here, due to the lack of a rigorous construction of the open virtual fundamental cycle. Still, as in the case of the open string and dilaton equations, the claim is expected to be true, and with a similar proof, under some mild assumptions on the open virtual fundamental cycle. Thus, the following proposition gives further evidence to support Conjecture 1:

Proposition 6.3. *The function* $\phi_1|_{t_d^{r-1} \mapsto \frac{1}{\sqrt{-r}}(t_d^{r-1} - \delta_{d,0} rs)}$ *satisfies equation* (6.4).

Proof. Let us prove that the function ϕ_1 satisfies the equation

$$\frac{\partial \phi_1}{\partial t_{p+1}^\alpha} = \sum_{\mu+\nu=r-2} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial t_p^\alpha \partial t_0^\mu} \frac{\partial \phi_1}{\partial t_0^\nu} + \frac{\partial \phi_0}{\partial t_p^\alpha} \frac{\partial \phi_1}{\partial t_0^{r-1}} + \frac{1}{2} \frac{\partial^2 \phi_0}{\partial t_p^\alpha \partial t_0^{r-1}}. \tag{6.5}$$

Together with Theorem 1.1, this immediately implies the proposition. Let us rewrite equation (6.5) in the variables T_i:

$$\frac{\partial \phi_1}{\partial T_{a+r}} = \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \frac{\partial \phi_1}{\partial T_{r-b}} + \frac{a+r}{r} \frac{\partial \phi_0}{\partial T_a} \frac{\partial \phi_1}{\partial T_r} + \frac{a+r}{2r} \frac{\partial^2 \phi_0}{\partial T_a \partial T_r}, \quad a \geq 1. \tag{6.6}$$

In [3, Lemma 4.2], we proved that the operator L has the form

$$L = \partial_x^r + \sum_{i=0}^{r-2} \sum_{j \geq 0} \varepsilon^{i-r+j} f_i^{[j]} \partial_x^i, \quad f_i^{[j]} \in \mathbb{C}[[T_*]].$$

Denote

$$L_0 := \partial_x^r + \sum_{i=0}^{r-2} f_i^{[0]} \partial_x^i, \qquad L_1 := \sum_{i=0}^{r-2} f_i^{[1]} \partial_x^i, \qquad (f_i^{[0]})^{(k)} := \partial_x^k f_i^{[0]}.$$

For any $a \geq 1$, the Laurent series $\widehat{L_0^{\frac{a}{r}}}$ has the form

$$\widehat{L_0^{\frac{a}{r}}} = \sum_{i=-\infty}^{a} P_i((f_*^{[0]})^{(*)}) z^i,$$

where P_i are polynomials in $(f_j^{[0]})^{(k)}$ for $0 \leq j \leq r-2$ and $k \geq 0$. Let us assign to $(f_j^{[0]})^{(k)}$ differential degree k. Then we can decompose the polynomials $P_i((f_*^{[0]})^{(*)})$ as $P_i((f_*^{[0]})^{(*)}) = \sum_{m \geq 0} P_{i,m}((f_*^{[0]})^{(*)})$, where a polynomial $P_{i,m}$ has differential degree m. Introduce the following notation:

$$\left(\widehat{L_0^{\frac{a}{r}}}\right)_m := \sum_{i=-\infty}^{a} P_{i,m}((f_*^{[0]})^{(*)}) z^i, \qquad \left(\widehat{L_0^{\frac{a}{r}}}\right)_{+,m} := \sum_{i=0}^{a} P_{i,m}((f_*^{[0]})^{(*)}) z^i.$$

Lemma 6.4. *For any $a \geq 1$ we have*

$$\frac{\partial \phi_1}{\partial T_a} = \left[(\phi_1)_x \partial_z \left(\widehat{L_0^{\frac{a}{r}}} \right)_+ + \frac{(\phi_0)_{xx}}{2} \partial_z^2 \left(\widehat{L_0^{\frac{a}{r}}} \right)_+ + \left(\widehat{L_0^{\frac{a}{r}}} \right)_{+,1} + \frac{a}{r} \left(\widehat{L_0^{\frac{a}{r}-1}} \widehat{L}_1 \right)_+ \right] \Bigg|_{z=(\phi_0)_x}, \tag{6.7}$$

where ∂_z denotes the partial derivative $\frac{\partial}{\partial z}$.

Proof. We have $\frac{\partial \phi}{\partial T_a} = \varepsilon^{a-1} \frac{(L^{a/r})_+ e^\phi}{e^\phi}$. The operator $L^{\frac{a}{r}}$ has the form $L^{\frac{a}{r}} = \sum_{i=-\infty}^{a} R_i \partial_x^i$, where $R_i = \sum_{j \geq 0} \varepsilon^{i-a+j} R_{i,j}$, $R_{i,j} \in \mathbb{C}[[T_*]]$. We have to check that

$$\begin{aligned} \operatorname{Coef}_{\varepsilon^0} \left[\varepsilon^{a-1} \sum_{i=0}^{a} R_i \frac{\partial_x^i e^\phi}{e^\phi} \right] = \\ = \left[(\phi_1)_x \partial_z \left(\widehat{L_0^{\frac{a}{r}}} \right)_+ + \frac{(\phi_0)_{xx}}{2} \partial_z^2 \left(\widehat{L_0^{\frac{a}{r}}} \right)_+ + \left(\widehat{L_0^{\frac{a}{r}}} \right)_{+,1} + \frac{a}{r} \left(\widehat{L_0^{\frac{a}{r}-1}} \widehat{L}_1 \right)_+ \right] \Bigg|_{z=(\phi_0)_x}, \end{aligned} \tag{6.8}$$

where $\operatorname{Coef}_{\varepsilon^0}$ means the coefficient of ε^0.

By induction on i, it is easy to prove that

$$\frac{\partial_x^i e^\phi}{e^\phi} = i! \sum_{\substack{m_1, m_2, \ldots \geq 0 \\ \sum j m_j = i}} \prod_{j \geq 1} \frac{(\partial_x^j \phi)^{m_j}}{(j!)^{m_j} m_j!}, \quad i \geq 0. \tag{6.9}$$

We have

$$\prod_{j \geq 1} (\partial_x^j \phi)^{m_j} = \begin{cases} \varepsilon^{-i} (\phi_0)_x^i + \varepsilon^{-i+1} i (\phi_0)_x^{i-1} (\phi_1)_x + O(\varepsilon^{-i+2}), & \text{if } m_1 = i \text{ and } m_{\geq 2} = 0, \\ \varepsilon^{-i+1} (\phi_0)_x^{i-2} (\phi_0)_{xx} + O(\varepsilon^{-i+2}), & \text{if } m_1 = i-2,\ m_2 = 1 \text{ and } m_{\geq 3} = 0, \\ O(\varepsilon^{-i+2}), & \text{otherwise.} \end{cases}$$

As a result,

$$\frac{\partial_x^i e^\phi}{e^\phi} = \varepsilon^{-i} (\phi_0)_x^i + \varepsilon^{-i+1} \left(i (\phi_0)_x^{i-1} (\phi_1)_x + \frac{i(i-1)}{2} (\phi_0)_x^{i-2} (\phi_0)_{xx} \right) + O(\varepsilon^{-i+2}),$$

and

$$\operatorname{Coef}_{\varepsilon^0} \left[\varepsilon^{a-1} \sum_{i=0}^{a} R_i \frac{\partial_x^i e^\phi}{e^\phi} \right] = \sum_{i=0}^{a} R_{i,0} \left(i (\phi_0)_x^{i-1} (\phi_1)_x + \frac{i(i-1)}{2} (\phi_0)_x^{i-2} (\phi_0)_{xx} \right) + \sum_{i=0}^{a} R_{i,1} (\phi_0)_x^i. \tag{6.10}$$

Note that

$$\sum_{i=0}^{a} R_{i,0} z^i = \left(\widehat{L_0^{\frac{a}{r}}} \right)_+ \quad \text{and} \quad \sum_{i=0}^{a} R_{i,1} z^i = \left(\widehat{L_0^{\frac{a}{r}}} \right)_{+,1} + \frac{a}{r} \left(\widehat{L_0^{\frac{a}{r}-1}} \widehat{L}_1 \right)_+.$$

We can see now that the first sum on the right-hand side of (6.10) gives the first and the second terms on the right-hand side of (6.8). The second sum on the right-hand side of (6.10) gives the third and the fourth terms on the right-hand side of (6.8). This completes the proof of the lemma. □

In [3, Lemma 4.7], we proved that

$$\frac{\partial \phi_0}{\partial T_a} = \left(\widehat{L}_0^{\frac{a}{r}}\right)_+\Big|_{z=(\phi_0)_x}, \tag{6.11}$$

which implies

$$\frac{\partial^2 \phi_0}{\partial T_a \partial T_r} = \frac{\partial}{\partial T_r}\left[\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\Big|_{z=(\phi_0)_x}\right] = \left(\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_z \widehat{L}_0 (\phi_0)_{xx} + \partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_x \widehat{L}_0\right)\Big|_{z=(\phi_0)_x}.$$

Therefore, equation (6.6) is equivalent to

$$\begin{aligned}\frac{\partial \phi_1}{\partial T_{a+r}} =& \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \frac{\partial \phi_1}{\partial T_{r-b}} + \frac{a+r}{r}\frac{\partial \phi_0}{\partial T_a}\frac{\partial \phi_1}{\partial T_r} + \\ &+ \frac{a+r}{2r}\left((\phi_0)_{xx}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_z \widehat{L}_0 + \partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_x \widehat{L}_0\right)\Big|_{z=(\phi_0)_x}.\end{aligned} \tag{6.12}$$

Using formulas (6.7) and (6.11), we see that equation (6.12) is a consequence of the following four equations:

$$\partial_z\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+ = \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \partial_z\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_+ + \frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_z \widehat{L}_0, \tag{6.13}$$

$$\begin{aligned}\partial_z^2\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+ =& \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \partial_z^2\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_+ + \frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_z^2 \widehat{L}_0 + \\ &+ \frac{a+r}{r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_z \widehat{L}_0,\end{aligned} \tag{6.14}$$

$$\left(\widehat{L_0^{\frac{a+r}{r}}}\right)_{+,1} = \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \left(\widehat{L_0^{\frac{r-b}{r}}}\right)_{+,1} + \frac{a+r}{2r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \partial_x \widehat{L}_0, \tag{6.15}$$

$$\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a+r}{r}-1}\widehat{L}_1\right)_+ = \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \frac{r-b}{r}\left(\widehat{L}_0^{\frac{r-b}{r}-1}\widehat{L}_1\right)_+ + \frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ \widehat{L}_1. \tag{6.16}$$

In [3, Proof of Proposition 4.8], we proved that

$$d\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+ = \sum_{b=1}^{r-1} \frac{a+r}{b(r-b)} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} d\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_+ + \frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+ d\widehat{L}_0,$$

from which equation (6.13) clearly follows. Applying the derivative ∂_z to the both sides of (6.13), we get equation (6.14). Equation (6.16) follows from the property

$$\left(\widehat{L}_0^{\frac{a}{r}}\right)_- - \sum_{b=1}^{r-1} \frac{1}{b} \frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a \partial T_b} \widehat{L}_0^{-\frac{b}{r}} \in z^{-r-1}\mathbb{C}[[f_*^{[0]}]][[z^{-1}]], \tag{6.17}$$

which was proved in [3, Proof of Proposition 4.8].

Let us prove equation (6.15). Let us first prove that for any $a \geq 1$ we have

$$\left(\widehat{L_0^{\frac{a}{r}}}\right)_1 = \frac{a(a-r)}{2r^2} \widehat{L}_0^{\frac{a}{r}-2} \partial_z \widehat{L}_0 \partial_x \widehat{L}_0. \tag{6.18}$$

It is easy to see that

$$\begin{aligned}\left(\widehat{\left(L_0^{\frac{1}{r}}\right)^r}\right)_1 =& r\widehat{L}_0^{\frac{r-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1 + \sum_{j=0}^{r-1}\widehat{L}_0^{\frac{r-j-1}{r}}\partial_z\widehat{L}_0^{\frac{1}{r}}\partial_x\widehat{L}_0^{\frac{j}{r}} = \\ =& r\widehat{L}_0^{\frac{r-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1 + \sum_{j=0}^{r-1}\frac{j}{r^2}\widehat{L}_0^{-1}\partial_z\widehat{L}_0\partial_x\widehat{L}_0 = \\ =& r\widehat{L}_0^{\frac{r-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1 + \frac{r-1}{2r}\widehat{L}_0^{-1}\partial_z\widehat{L}_0\partial_x\widehat{L}_0.\end{aligned}$$

Since we obviously have $\left(\widehat{\left(L^{\frac{1}{r}}\right)^r}\right)_1 = 0$, equation (6.18) is proved for $a = 1$. For an arbitrary $a \geq 1$ we compute

$$\begin{aligned}\left(\widehat{\left(L_0^{\frac{1}{r}}\right)^a}\right)_1 =& a\widehat{L}_0^{\frac{a-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1 + \sum_{j=0}^{a-1}\widehat{L}_0^{\frac{a-1-j}{r}}\partial_z\widehat{L}_0^{\frac{1}{r}}\partial_x\widehat{L}_0^{\frac{j}{r}} = \\ =& \frac{a(1-r)}{2r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0 + \sum_{j=0}^{a-1}\frac{j}{r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0 = \\ =& \frac{a(a-r)}{2r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0.\end{aligned}$$

Thus, equation (6.18) is proved.

Let us rewrite formula (6.18) in the following way:

$$\left(\widehat{L_0^{\frac{a}{r}}}\right)_1 = \frac{a}{2r}\partial_z\widehat{L}_0^{\frac{a}{r}-1}\partial_x\widehat{L}_0.$$

Then we see that equation (6.15) is equivalent to

$$\frac{a+r}{2r}\left(\partial_z\widehat{L}_0^{\frac{a}{r}}\partial_x\widehat{L}_0\right)_+ = \sum_{b=1}^{r-1}\frac{a+r}{2rb}\frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\left(\partial_z\widehat{L}_0^{-\frac{b}{r}}\partial_x\widehat{L}_0\right)_+ + \underline{\frac{a+r}{2r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0}. \tag{6.19}$$

We obviously have

$$\frac{a+r}{2r}\left(\partial_z\widehat{L}_0^{\frac{a}{r}}\partial_x\widehat{L}_0\right)_+ = \underline{\frac{a+r}{2r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0} + \frac{a+r}{2r}\left(\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_-\partial_x\widehat{L}_0\right)_+.$$

Note that the underlined term here cancels the underlined term on the right-hand side of (6.19). Therefore, equation (6.19) is equivalent to the identity

$$\left(\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_-\partial_x\widehat{L}_0\right)_+ = \sum_{b=1}^{r-1}\frac{1}{b}\frac{\partial^2 F_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\left(\partial_z\widehat{L}_0^{-\frac{b}{r}}\partial_x\widehat{L}_0\right)_+, \tag{6.20}$$

which follows from equation (6.17). The proposition is proved. □

7. Constructions of canonical multisections, independence of choices

This section is devoted to constructing the different types of multisections we use throughout the article, and to prove Theorem 3.17. First, we discuss the construction of canonical boundary conditions.

7.1. **Constructions of a single global canonical multisection.** The following definition is required:

Definition 7.1. Let Γ be a graded graph. A canonical multisection s is *strongly positive* if, for any $\Lambda \in \partial^{\mathrm{I}}\Gamma \setminus \partial^{+}\Gamma$ such that $\mathcal{B}\Lambda$ has some connected components $\Xi_1, \dots, \Xi_a$ without boundary tails, and every $C \in \mathcal{M}_{\Lambda}^{1/r}$, each local branch of the projection of s_C to a Ξ_i-component evaluates positively on some nonempty subset of the boundary $\partial(\mathcal{B}\Sigma^{\Xi_i})$ of the Ξ_i-component of $\mathcal{B}\Sigma$ (defined in Remark 3.9).

The first step towards constructing canonical boundary conditions was the local Proposition 3.20. The second step is the following global proposition.

Proposition 7.2. *(1) For any smooth, graded, connected Γ, there exists a transverse special canonical multisection of Witten's bundle.*

(2) For any smooth, graded, connected Γ, there exists a transverse special canonical strongly positive multisection of the Witten bundle. Moreover, for any strongly positive multisection s one can find a strongly positive multisection $\hat{s}$ such that, for any $C \in \overline{\mathcal{PM}}_{\Gamma}^{1/r}$ and for any local branch $\hat{s}_C^i$ of $\hat{s}_C$, the subset of $\partial\Sigma \subseteq C$ on which $\hat{s}_C^i$ is positive intersects, for every local branch s_C^j of s_C, the subset of $\partial\Sigma$ on which s_C^j is positive.

(3) For any smooth, graded, connected Γ, and any $p \in \overline{\mathcal{PM}}_{\Gamma}^{1/r}$, there exist multisections $s_1, \dots, s_N \in C_m^{\infty}(\overline{\mathcal{PM}}_{\Gamma}^{1/r}, \mathcal{W})$ with compact support such that for any $\Lambda \in \partial^0\Gamma$, the section $s_i|_{\overline{\mathcal{PM}}_{\Lambda}^{1/r}}$ is pulled back from $\overline{\mathcal{PM}}_{\mathcal{B}\Lambda}^{1/r}$, and for any choice of local branches $i_1, \dots, i_N$ of $s_1(p), \dots, s_N(p)$ respectively, the set of vectors $\{s_j^{i_j}(p)\}_{j=1,\dots,N}$ spans $\mathcal{W}_p$.

(4) For any $C \in \partial\overline{\mathcal{M}}_{0,k,\vec{a}}^{\frac{1}{r}}$ such that z_i does not belong to a partially stable component of $\mathcal{B}C$, one can find $s \in \mathcal{S}_i$ that does not vanish at C.

Proof. We begin with a transversality claim.

Observation 7.3. Let Λ be a graded graph (open or closed) without boundary edges, and suppose that we are given multisections s^v for each $v \in \mathrm{Conn}(\mathrm{detach}(\Lambda))$ such that

(1) if v is closed and has an anchor of twist -1, then $s^v \in C_m^{\infty}(\mathcal{W}, \overline{\mathcal{R}}_v)$ is a coherent multisection;
(2) if v is closed without an anchor of twist -1, then $s^v \in C_m^{\infty}(\mathcal{W}, \overline{\mathcal{M}}_v)^{1/r}$;
(3) if v is open, then $s^v \in C_m^{\infty}(\mathcal{W}, \overline{\mathcal{PM}}_v^{1/r})$;
(4) $\bar{s}^v \pitchfork 0$ (where, we recall, if the anchor of v does not have twist -1, then $\bar{s}^v = s^v$).

Then also $\bar{s} \pitchfork 0$ for

$$s = \maltese_{\Lambda, E(\Lambda)}\big((s^v)_{v \in \mathrm{Conn}(\mathrm{detach}(\Lambda))}\big).$$

The proof of Observation 7.3 is as follows. By induction, we may restrict to the case where Λ has a single edge e and $\mathrm{Conn}(\mathrm{detach}(\Lambda)) = \{v_1, v_2\}$. If v_1 and v_2 are not connected by a Ramond internal node, then the Witten bundle on $\overline{\mathcal{M}}_{\Lambda}$ decomposes as a direct sum, so if each s^{v_i} is transverse to zero, then so is s.

Suppose, then, that Λ is obtained by gluing the anchor of v_2 to a Ramond internal tail of v_1. Let $u_i \in \overline{\mathcal{M}}_{v_i}^{1/r}$ be a zero of $\bar{s}^{v_i}$. Then $\mathrm{Detach}_e^{-1}(u_1, u_2)$ is a zero of $\bar{s} = \overline{\maltese_{\Lambda,e}(s^{v_1}, s^{v_2})}$. By the transversality assumption, the derivatives of (each local branch of) $\bar{s}^{v_i}(u_i)$ span $\mathcal{W}_{u_i}$.

Thus the derivatives of the branches of $\overline{\mathbb{F}_{\Lambda,e}(s^{v_1}, s^{v_2})}(\mathrm{Detach}_e^{-1}(u_1,u_2))$ in the $\mathcal{M}_{v_i}^{1/r}$ directions span subspaces of W_i of $\mathcal{W}_{\mathrm{Detach}_e^{-1}(u_1,u_2)}$. By (2.10) of Proposition 2.7, the sequence

$$0 \to \mathcal{W}_{u_2} \to \mathcal{W}_{\mathrm{Detach}_e^{-1}(u_1,u_2)} \to \mathcal{W}_{u_1} \to 0 \tag{7.1}$$

is exact. The space W_2 is just the image, under the left map of (7.1), of $\mathcal{W}_{u_2}$ in $\mathcal{W}_{u_1,u_2}$, while W_1 surjects to $\mathcal{W}_{u_1}$ by the right map of (7.1). Thus, at $\mathrm{Detach}_e^{-1}(u_1,u_2)$, the derivatives of the branches of $\overline{\mathbb{F}_{\Lambda,e}(s^{v_1}, s^{v_2})}$ span $\mathcal{W}_{\mathrm{Detach}_e^{-1}(u_1,u_2)}$, and the vanishing at $\mathrm{Detach}_e^{-1}(u_1,u_2)$ is transverse. This proves Observation 7.3.

The constructions of multisections for the Witten bundle having the properties specified in items (1), (2), and (3) of Proposition 7.2 is inductive. For the first two items, we shall require the following extension lemmas.

Lemma 7.4. *Let Γ be a smooth, connected, closed, graded r-spin graph. Suppose*

$$s^v \in C_m^\infty(\overline{\mathcal{R}}_v, \mathcal{W})$$

are coherent multisections for each $v \in \mathcal{V}(\Gamma) \setminus \Gamma$ such that

(1) $\bar{s}^v \pitchfork 0$;
(2) for any $\Lambda \in \partial v$,

$$s^v|_{\overline{\mathcal{R}}_\Lambda} = \mathbb{F}_{\Lambda,E(\Lambda)}((s^\Xi)_{\Xi \in Conn(detach(\Lambda))});$$

(3) if the anchor of v has twist $r-1$, then s^v evaluates positively (with respect to the grading) at the anchor.

Then one may construct a multisection $s^\Gamma \in C_m^\infty(\overline{\mathcal{R}}_\Gamma, \mathcal{W})$ that satisfies the above properties with v replaced by Γ.

Lemma 7.5. *Let Γ be a smooth, connected, open, graded r-spin dual graph. Let ζ be a multisection of $\mathcal{W}$ defined over*

$$\bigcup_{\Lambda \in \partial\Gamma \setminus \partial^+\Gamma} \mathcal{M}_\Lambda^{1/r}.$$

(1) Suppose that for any $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$, the multisection

$$\zeta|_{\overline{\mathcal{PM}}_\Lambda^{1/r}} = F_\Lambda^* \zeta^{\mathcal{B}\Lambda},$$

where $\zeta^{\mathcal{B}\Lambda} \in C_m^\infty(\overline{\mathcal{PM}}_{\mathcal{B}\Lambda}^{1/r}, \mathcal{W})$, is positive with respect to $\mathcal{B}\Lambda$, $Aut(\mathcal{B}\Gamma)$-invariant, and transverse. Then one can extend ζ to a positive, transverse, $Aut(\Gamma)$-invariant multisection $\sigma \in C_m^\infty(\overline{\mathcal{PM}}_\Gamma^{1/r}, \mathcal{W})$.

(2) Suppose we add the additional assumption on ζ that for any $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$ such that $\mathcal{B}\Lambda$ has at least one vertex without boundary tails, $\zeta|_{\mathcal{M}_\Lambda^{1/r}}$ is strongly positive in the sense of Definition 7.1. Then the multisection σ of item (1) can be chosen to be strongly positive on $\overline{\mathcal{PM}}_\Gamma^{1/r}$.

(3) Let s be a strongly positive multisection on $\overline{\mathcal{PM}}_\Gamma^{1/r}$, and assume, in addition to the assumptions of item 1, that for any $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$ such that $\mathcal{B}\Lambda$ has at least one vertex without boundary tails, and for any $C \in \mathcal{M}_\Lambda^{1/r}$ and any branch of ζ_C, the subset of $\partial\Sigma \subseteq C$ on which ζ_C evaluates positively intersects all of the subsets of $\partial\Sigma$ on which the branches of s_C are positive. Then the multisection σ of item (1) satisfies the following further condition: if Γ has no boundary tails, then for any $C \in \overline{\mathcal{PM}}_\Gamma^{1/r}$ and

any branch of σ_C, the subset of $\partial\Sigma$ on which σ_C evaluates positively intersects all of $\partial\Sigma$ on which the branches of s_C are positive.

We return to the proofs of Lemmas 7.4 and 7.5 below. First, though, we show how to deduce Proposition 7.2 from these lemmas.

To prove item (1) of the proposition, we first note that, by Lemma 7.4 and induction on $\dim(\mathcal{M}_v^{1/r})$, there exists a family of multisections s^v for each closed $v \in \mathcal{V}(\Gamma)$ satisfying the conditions of the proposition. What remains, then, is to consider open vertices.

We shall construct, for any abstract open vertex $v \in \mathcal{V}(\Gamma)$, a transverse positive multisection $s^v \in C_m^\infty(\overline{\mathcal{PM}}_v^{1/r}, \mathcal{W})$ such that for any $\Lambda \in \partial v \setminus \partial^+ v$,

$$s^v|_{\overline{\mathcal{PM}}_\Lambda^{1/r}} = F_\Lambda^* \left(\maltese_{\mathcal{B}\Lambda, E(\mathcal{B}\Lambda)}((s^f)_{f \in \mathrm{Conn}(\mathrm{detach}(\mathcal{B}\Lambda))}) \right). \tag{7.2}$$

The resulting s^Γ will satisfy the requirements of the proposition by its definition.

The construction of s^v is by induction on $\dim(\overline{\mathcal{M}}_v)$. The base case is when v is partially stable, and this case is settled by the first item of Proposition 3.20. Suppose, then, that we have constructed multisections s^v for any $v \in \mathcal{V}(\Gamma)$ with $\dim \overline{\mathcal{M}}_v < n \leq \dim \overline{\mathcal{M}}_\Gamma$, and consider $v \in \mathcal{V}(\Gamma)$ with $\dim \overline{\mathcal{M}}_v = n$. We define a multisection ζ on

$$\coprod_{\Lambda \in \partial v \setminus \partial^+ v} \overline{\mathcal{PM}}_\Lambda^{1/r}$$

by (7.2). Thanks to Observation 3.10, the multisection ζ descends to

$$\bigcup_{\Lambda \in \partial v \setminus \partial^+ v} \overline{\mathcal{PM}}_\Lambda^{1/r}$$

as a continuous multisection that is smooth on each $\overline{\mathcal{PM}}_\Lambda^{1/r}$. This section is automatically $\mathrm{Aut}(v)$-invariant, and it is transverse on each $\overline{\mathcal{PM}}_\Lambda^{1/r}$, by Observation 7.3. Moreover, ζ satisfies the assumptions of item (1) of Lemma 7.5 by construction, so we can use the lemma to extend ζ to a smooth, $\mathrm{Aut}(v)$-invariant and transverse multisection s^v over all $\overline{\mathcal{PM}}_v^{1/r}$. The inductive step for the construction of s^v follows.

We now turn to item 2 of the proposition. The proof of the first part of item (2) follows by induction, similarly to the one performed in the previous case, by using the fact that the boundary conditions in the partially stable case (which is the base case of the induction) trivially satisfy the strong positivity requirement, as they evaluate positively on the whole boundary of the partially stable components. We then apply item (2) of Lemma 7.5 for the inductive step, and the first part of item 2 is proved.

The "moreover" part of Item (2) is proven similarly. This time, we start with a strongly positive multisection s and again use induction. As in the previous case, the basis of induction is the partially stable case, and this satisfies the requirements by the same reasoning as above. The induction step follows by applying item (3) of Lemma 7.5.

For item (3) of the proposition, we first note that the claim cannot be achieved using only multisections that satisfy the second property of special canonical multisections in Definition 4.7. The reason is that such sections have too much symmetry: if C is a graded surface such that $\mathrm{Aut}(\mathcal{B}C)$ has an element that permutes two components D_1 and D_2 of $\mathcal{B}C$, then for any canonical multisection there is some branch with the same projections onto $\mathcal{W}_{D_1}$ and $\mathcal{W}_{D_2}$ (with respect to the natural identification). The second difficulty in working with multisections that have the second property of special canonical multisections

is that, because of their strong decomposition properties, it is difficult to construct them with compact support in $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$, unless they vanish on the closed moduli space.

To overcome these issues, we construct multisections as follows. We assume that the internal labels are all different. Let $r(\Lambda)$ be the maximal possible number of vertices with $r+1$ boundary labels and no internal labels that may appear in any $\Xi' = \mathrm{detach}(\Xi)$ for any $\Xi \in \partial^! \Lambda$. Note that $r(\Lambda) = r(\mathrm{detach}(\Lambda))$ and $r(\Lambda) \geq r(\Xi)$ for any $\Xi \in \partial^! \Lambda$.

Fix $C \in \overline{\mathcal{PM}}^{1/r}_{\Gamma}$, and suppose that $C \in \mathcal{M}^{1/r}_{\Lambda}$. Observe that

$$\mathcal{W}_{\Sigma} \simeq \boxplus_{\Xi \in \mathrm{Conn}\mathcal{B}\Lambda} \mathcal{W}_{\Sigma^{\Xi}},$$

where C^{Ξ} is the Ξ-component of $\mathcal{B}C$. The construction of multisections now splits into two cases.

The first case is when $\mathrm{Aut}(\mathcal{B}C)$ preserves C^{Ξ}, which happens, in particular, if Ξ has internal tails. In this case, all elements of $\mathrm{Aut}(C^{\Xi})$ are induced by permuting the boundary markings[6], though there are additional automorphisms of the bundle if C^{Ξ} has internal nodes. Let $w \in \mathcal{W}_{C^{\Xi}}$ be an arbitrary vector, and let $G(w) = \{w_1 = w, \ldots, w_a\}$ be its orbit under the action of $\mathrm{Aut}(C^{\Xi})$, lifted to the Witten bundle. We shall show, in this case, that one can construct a multisection as in the statement of item (3) whose branches at C are $w_1, \ldots, w_a$ when projected to the Ξ component, and are zero when projected to other components.

The second case is when $\mathrm{Aut}(\mathcal{B}C)$ does not preserve C^{Ξ}. This means, in particular, that Ξ has no internal half-edges or tails. Let $\Xi_1, \ldots, \Xi_b$ be the components of the $\mathrm{Aut}(\mathcal{B}C)$-orbit of C^{Ξ}. For a vector $w \in \mathcal{W}_{C^{\Xi}}$, let $G(w)$ be the orbit of w under the action of $\mathrm{Aut}(C^{\Xi})$, and for $w_1, \ldots, w_b \in \mathcal{W}_{C^{\Xi}}$, let $G(w_1, \ldots, w_b)$ be the collection of vectors in $\widetilde{\mathcal{W}}_{C^{\Xi}} := \bigoplus_{i \in [b]} \mathcal{W}_{C^{\Xi_i}}$ defined by

$$G(w_1, \ldots, w_b) = \left\{ \bigoplus_{i=1}^{b} w_i' | w_1' \in G(w_1), \ldots, w_b' \in G(w_b) \right\},$$

where we use the canonical identification between the different $\mathcal{W}_{C^{\Xi_i}}$. Fix a set $F \subseteq \mathbb{N}$ of size at least $r(\Gamma)$, and let $W_F \subseteq \mathcal{W}_{C^{\Xi}}$ be a set of vectors w_Q for each $Q \subseteq F$ such that $|Q| = r(\Xi)$. Let

$$G(W_F) = \bigcup_{\substack{Q_1, \ldots, Q_b \subseteq F, \\ |Q_i| = r(\Xi),\ i \in [b],\ \sqcup Q_i \subseteq F}} G(w_{Q_1}, \ldots, w_{Q_b}).$$

We show how to construct a multisection s as in the statement of item (3) whose branches at C are precisely $G(W_F)$, under the natural injection $\widetilde{\mathcal{W}}_{C^{\Xi}} \to \mathcal{W}$ as a direct summand.

Write $m = \operatorname{rank} \mathcal{W}_{C^{\Xi}}$. There is no difficulty, in the first case, in finding $w^1, \ldots, w^m$ such that each element of $G(w^1) \times \cdots \times G(w^m)$ is a basis for $\mathcal{W}_{C^{\Xi}}$, and in the second case, in finding sets $W_F^1, \ldots, W_F^{mb}$ as above such that each element of

$$G(W_F^1) \times \cdots \times G(W_F^{mb})$$

is a basis for $\widetilde{\mathcal{W}}_{C^{\Xi}}$. Thus, constructing multisections as above for any w (in the first case) or any W_F (in the second case) will prove item (3).

Write $\mathcal{V}'(\Gamma)$ for the collection of graphs that appear as connected components of $\mathcal{B}\Upsilon$ for graphs $\Upsilon \in \partial^0 \Gamma$ with no internal edges or contracted boundary tails. We define, for any Ξ and w as in the first case and any $\Upsilon \in \mathcal{V}'(\Gamma)$, a multisection s^{Υ}. Similarly, for any Ξ and

[6]Such permutations induce automorphisms of C^{Ξ} only when the open vertex Ξ^o of $\mathrm{detach}(\Xi)$ has exactly one internal half edge, and for very specific configurations of markings.

W_F as in the second case, any $\Upsilon \in \mathcal{V}'(\Gamma)$, and any $Q \subseteq F$ of size at least $r(\Upsilon)$, we define a multisection s_Q. We require these multisections to have the following properties:

(1) For any $\Upsilon' \in \partial^0\Upsilon$ with no internal edges or contracted boundary tails, in the first case
$$s^\Upsilon|_{\overline{\mathcal{PM}}^{1/r}_{\Upsilon'}} = F^*_{\Upsilon'}\left(\boxplus_{\Omega\in\mathrm{Conn}(\mathcal{B}\Upsilon')} s^\Omega\right),$$
and in the second case,
$$s^\Upsilon_Q|_{\overline{\mathcal{PM}}^{1/r}_{\Upsilon'}} = \biguplus_{\{(Q_\Omega)_{\Omega\in\mathrm{Conn}(\mathcal{B}\Upsilon')} \,|\, |Q_\Omega|=r(\Omega), \bigsqcup_\Omega Q_\Omega\subseteq Q\}} F^*_{\Upsilon'}\left(\boxplus_{\Omega\in\mathrm{Conn}(\mathcal{B}\Upsilon')} s^\Omega_{Q_\Omega}\right),$$
where $\uplus$ is the operation defined in Notation A.2.

(2) In the first case, let Υ be the smoothing of Ξ (c.f. Definition 3.1). We take the branches of $s^\Upsilon(\Sigma^\Xi)$ to be $\{w_1,\dots,w_a\}$, with equal weights, in the sense of Definition A.1. For any other component $\Xi' \in \mathrm{Conn}(\mathcal{B}\Lambda)$ with a smoothing Υ', we put $s^{\Upsilon'}(C^{\Xi'}) = 0$. In the second case, each Ξ_i is smooth. We take the branches of $s^{\Xi_1}_Q(C^{\Xi_1})$, $|Q| = r(\Xi_1)$, to be $G(w_Q)$ with equal weights[7], and for any other component $\Xi' \in \mathrm{Conn}(\mathcal{B}\Lambda)$ with a smoothing Υ', we put $s^{\Upsilon'}_Q(C^{\Xi'}) = 0$.

(3) For every $\Upsilon \in \mathcal{V}'(\Gamma)$, both s^Υ and s^Υ_Q have compact support.

The construction of s^Υ and s^Υ_Q is by induction on $\dim \overline{\mathcal{M}}^{1/r}_\Upsilon$. In the second case, we first define s^Υ_Q for $Q \subseteq F$ with $|Q| = r(\Upsilon)$ and then define, for $|Q| > r(\Upsilon)$,
$$s^\Upsilon_Q = \biguplus_{Q'\subseteq Q,\ |Q'|=r(\Upsilon)} s^\Upsilon_{Q'}.$$
Using (A.2), we see that this definition is consistent with the required properties of the multisections.

The base case of the inductive construction of s^Υ and s^Υ_Q is the partially stable case, where we have to choose one vector. In fact, the construction from the proof of the previous items works here as well.

Suppose, then, that we have constructed multisections with the above three properties for all $\Omega \in \mathcal{V}'(\Gamma)$ with $\dim \overline{\mathcal{M}}^{1/r}_\Omega < n$. We define $s^\Upsilon|_{\partial^0\overline{\mathcal{PM}}^{1/r}_v}$ and $s^\Upsilon_Q|_{\partial^0\overline{\mathcal{PM}}^{1/r}_\Upsilon}$ by item (1). By Observation 3.10, and using (A.2) in the second case, we see that this multisection is indeed well-defined and compactly-supported on $\partial^0\overline{\mathcal{PM}}^{1/r}_v$. Let Υ' be a graph identical to Υ but with injective labelings. Represent the quotient $\tilde{q} : \overline{\mathcal{PM}}^{1/r}_{\Upsilon'}/G \simeq \overline{\mathcal{PM}}^{1/r}_\Upsilon$, where G is a finite group, as in the last paragraph of Proposition 3.20. Pull back $s^\Upsilon|_{\partial^0\overline{\mathcal{PM}}^{1/r}_\Upsilon}$ and $s^\Upsilon_Q|_{\partial^0\overline{\mathcal{PM}}^{1/r}_\Upsilon}$ to G-invariant multisections $s^{\Upsilon'}|_{\partial^0\overline{\mathcal{PM}}^{1/r}_{\Upsilon'}}$ and $s^{\Upsilon'}_Q|_{\partial^0\overline{\mathcal{PM}}^{1/r}_{\Upsilon'}}$. Extend these to multisections $\tilde{s}^{\Upsilon'}$ and $\tilde{s}^{\Upsilon'}_Q$ with compact support over $\overline{\mathcal{PM}}^{1/r}_{\Upsilon'}$. Define
$$s^{\Upsilon'} = \biguplus_{g\in G} g\cdot\tilde{s}^{\Upsilon'}, \qquad s^{\Upsilon'}_Q = \biguplus_{g\in G} g\cdot\tilde{s}^{\Upsilon'}_Q.$$
These multisections descend to the quotient $\overline{\mathcal{PM}}^{1/r}_{\Upsilon'}/G$, and we pull back using $\tilde{q}^{-1}$ to define $s^{\Upsilon'}$ and s^Υ_Q on all $\overline{\mathcal{PM}}^{1/r}_\Upsilon$, extending their previous definition over boundary. If Υ is the

[7]This implies that the branches of $s(C)$ are precisely $G(W_F)$ with equal weights.

smoothing of Ξ (respectively, Ξ_1) we perform the extensions and symmetrizations of s^Υ (respectively, s_Q^Υ) so that item 2 also holds. The induction follows.

The resulting multisections, defined on $\overline{\mathcal{PM}}_\Gamma^{1/r}$ by pulling back from $\overline{\mathcal{PM}}^{1/r}_{\mathrm{for}_{\mathrm{marking}}(\Gamma)}$, satisfy the required properties, so item (3) of the proposition is proven.

The last item of the proposition, item 4, is proven completely analogously to Proposition 3.49 of [23]. The only comment that should be made is that, during the recursive construction process, some initial conditions involve defining the sections on partially stable components, an issue that did not appear in [23]. On such components, we define the section to be zero. This does not affect the resulting intersection theory, as the section of the Witten bundle has a prescribed, nonvanishing behavior on such components.

This completes the proof of Proposition 7.2, modulo the two lemmas. □

We now return to the proofs of Lemmas 7.4 and 7.5.

Proof of Lemma 7.4. The claim is trivial when the dimension of $\mathcal{M}_\Gamma$ is zero or when the rank of the Witten bundle is zero, so we assume that both are positive. Write

$$N = \bigcup_{\Lambda \in \partial\Gamma} \overline{\mathcal{R}}_\Lambda.$$

Then N is a normal crossings divisor in $\overline{\mathcal{R}}_\Gamma$, and on N, a multisection s is defined by the second item. More precisely, the second item defines a multisection on $\coprod_{\Lambda\in\partial\Gamma}\overline{\mathcal{R}}_\Lambda$, but this multisection descends to N by Observation 4.3.

Suppose first that the anchor does not have twist $r-1$. By the first item and Observation 7.3, the multisection $\bar{s}|_{\overline{\mathcal{M}}_\Lambda^{1/r}}$ is transverse to zero for any $\Lambda \in \partial\Gamma$. We can thus extend it to a transverse multisection $\bar{s} \in C_m^\infty(\overline{\mathcal{M}}_\Gamma^{1/r}, \mathcal{W})$. This gives a multisection $s \in C_m^\infty(N', \mathcal{W})$ where $N' = N \cup \overline{\mathcal{M}}_\Gamma^{1/r} \hookrightarrow \overline{\mathcal{R}}_\Gamma$, is just $\overline{\mathcal{M}}_\Gamma^{1/r}$ if the anchor is not Ramond, and a normal crossing divisor in $\overline{\mathcal{R}}_\Gamma$ otherwise. If the anchor is not twisted -1 then s satisfies the requirements of the lemma, and we are done. Otherwise recall the coherent section u defined in Example 4.1.2. The multisection $s|_{N'} - u|_{N'}$ is a multisection of $\mathcal{W}' \to N'$. Extend it to a smooth multisection $s' \in C_m^\infty(\overline{\mathcal{R}}_\Gamma, \mathcal{W})$. Then $s' + u$ extends s to a coherent multisection of $\overline{\mathcal{R}}_\Gamma$ which satisfies all the requirements.

If the anchor has twist $r-1$, then the multisection s, currently defined only on N, evaluates positively at the anchor over any point of N. Extend s to a multisection s_0 in neighborhood U of N, where we assume that U is small enough that s still evaluates positively at the anchor over any point of U. For any point $p \in \overline{\mathcal{M}}_v^{1/r}$, the first part of Proposition 3.20 allows one to construct a multisection s_p that evaluates positively at the anchor over any point in a neighborhood U_p of p. Cover $\overline{\mathcal{M}}_v^{1/r} \setminus U$ by a finite number of such neighborhoods $U_{p_1}, \dots U_{p_m}$ that do not intersect N. Let s_i be the multisection that corresponds to p_i, and let $\rho_0, \dots, \rho_m$ be a partition of unity subordinate to the cover $U, U_{p_1}, \dots, U_{p_m}$. Then $\sum_{i=0}^m \rho_i s_i$ extends s and satisfies the requirements of the lemma. □

Proof of Lemma 7.5. Write

$$W = \bigcup_{\Lambda \in \partial\Gamma \setminus \partial^+\Gamma} \overline{\mathcal{PM}}_\Lambda^{1/r}.$$

Step 1**:** Let Ξ be a graph containing some internal edges or illegal boundary half-edges of twist zero, and let $u \in \mathcal{M}_\Xi^{1/r} \subseteq \partial^+\overline{\mathcal{M}}_\Gamma^{1/r}$ be a moduli point with corresponding r-spin surface

C_u. The first step is to construct, for each such Ξ and u, a Ξ-neighborhood U_u in $\overline{\mathcal{M}}_\Gamma^{1/r}$, a Ξ-family of intervals $\{I_{u,h}\}_{h\in H^+(\Xi)}$, and a $(U_u, \{I_{u,h}\}_h)$-positive multisection s_u that restricts to the given multisection ζ on $W \cap U_u$.

(The naïve approach to the lemma would be to try to use standard local arguments to show that one can extend the multisection to a neighborhood of W. However, due to noncompactness, this does not work directly; it only shows that one can extend the multisection to a set of the form $U \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$, where U is an open set containing u in its closure. This is why we first construct $(U_u, I_{u,h})$ as above, and then we explicitly construct a $(U_u, I_{u,h})$-positive extension of ζ to all U_u.)

To construct U_u, $I_{u,h}$, and s_u, let $\{e_1, \dots, e_L\}$ be the set of edges of Ξ that are either internal or such that their illegal half-edge has twist zero. In the restriction of the universal curve to a small contractible Ξ-neighborhood U'_u of u, one can number the nodes corresponding to $e_1, \dots, e_L$ by $n_1, \dots, n_L$. Write $\Gamma^{(Q)}$ for the graph obtained by smoothing all edges not numbered $1, \dots, L$, as well as all edges of Q, for $Q \subseteq [L]$. Using the assumptions of the lemma, one can write

$$\zeta^{(Q)} = \zeta|_{\overline{\mathcal{PM}}^{1/r}_{\Gamma^{(Q)}} \cap U'_u}$$

for any $Q \subsetneq [L]$. Recalling notation 3.8, define $\Xi^{(Q)} \in \mathcal{B}\Gamma^{(Q)}$ by $\Xi^{(Q)} = f_{\Gamma^{(Q)}}(\Xi)$.

By the assumptions on ζ, for each $Q \subsetneq [L]$ one has

$$\zeta^{(Q)} = F^*_{\Gamma^{(Q)}} \zeta^{\mathcal{B}(Q)},$$

where $\zeta^{\mathcal{B}(Q)}$ is a restriction of a positive multisection of $\overline{\mathcal{PM}}^{1/r}_{\mathcal{B}\Gamma^{(Q)}}$. Thus, for any $Q \subsetneq [L]$, there is a $\Xi^{(Q)}$-neighborhood $U^{\mathcal{B}(Q)} \subseteq \overline{\mathcal{M}}^{1/r}_{\mathcal{B}\Gamma^{(Q)}}$ of $F_{\Gamma^{(Q)}}(u)$ and a family of intervals $\{I_h^{\mathcal{B}(Q)}\}_{h\in H^+(\Xi^{(Q)})}$ for $U^{\mathcal{B}(Q)}$ on which $\zeta^{\mathcal{B}(Q)}$ evaluates positively. Pulling back to $\overline{\mathcal{M}}^{1/r}_{\Gamma^{(Q)}}$, we can find a Ξ-neighborhood $U^{(Q)}$ of u inside $F^{-1}_{\Gamma^Q}(U^{\mathcal{B}(Q)}) \subseteq \overline{\mathcal{M}}^{1/r}_{\Gamma^{(Q)}}$, and a family of intervals

$$\{I^{(Q)}_{h'}(u') = n(\phi^{-1}_{\Gamma^{(Q)}}(I^{\mathcal{B}(Q)}_{h'}(F_{\Gamma^{(Q)}}(u'))))\}_{h'\in H^+(\Xi^{(Q)})},$$

where $\phi_{\Gamma^{(Q)}}$ is the map from Remark 3.9 and n is the normalization map. For any $u' \in U^{(Q)}$ and any h, the set $I^{(Q)}_h(u')$ is either an interval or a union of intervals inside the surface $\Sigma_{u'}$; see Figure 5. Moreover, the endpoints of these intervals vary smoothly with respect to the smooth structure on the universal curve restricted to $U^{(Q)}$. By replacing with a smaller Ξ-neighborhood U'_u of u, we may assume that $U'_u \cap \overline{\mathcal{M}}_{\Gamma^{(Q)}} \subseteq U^{(Q)}$.

The simple yet crucial observation is that if $h \in H^+(\Xi)$ is mapped by $f_{\Gamma^{(Q)}}$ to $h^{(Q)} := f_{\Gamma^{(Q)}}(h) \in H^+(\Xi^{(Q)})$, then for any Σ in $U^{(Q)} \cap \overline{\mathcal{M}}^{1/r}_{\Xi^{(Q)}}$ the node $n_h = n_h(C)$ belongs to an interval contained in $I^{(Q)}_{h^{(Q)}}(C)$. In Figure 5, both the case where $f_{\Gamma^{(Q)}}$ is bijective and the case where $f_{\Gamma^{(Q)}}$ is not bijective are illustrated.

Thus, we can find a family of non-intersecting intervals $I_h(u) \subseteq \partial C_u$ containing $n_h(u)$ and no other special point, such that $I_h(u) \subseteq I^{(Q)}_{h^{(Q)}}(u)$ for all $h \in H^+(\Xi)$ and all $Q \subseteq L$. We extend smoothly the family $\{I_h(u)\}_{h\in H^+(\Xi)}$ to a Ξ-family of intervals $\{I_{u,h}\}_{h\in H^+(\Xi)}$ on U'_u. By continuity of the endpoints of the intervals $I^{(Q)}_h$ with respect to the topology of the universal curve, there exists a contractible Ξ-neighborhood $U_u \subseteq U'_u$ of u such that, for all $u' \in U_u \cap \overline{\mathcal{PM}}^{1/r}_{\Gamma^{(Q)}} \subseteq U_u \cap W$, we have $I_{u,h}(u') \subseteq I^{(Q)}_{h^{(Q)}}(u')$.

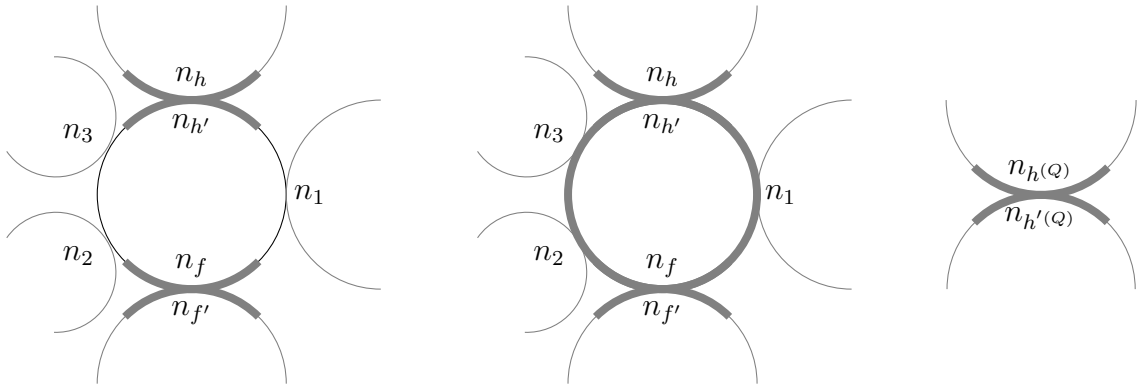


FIGURE 5. The left and middle part of the figure show part of a nodal disk C_u, in which the half-nodes n_1, n_2, and n_3 are legal with twist $r-2$ (so the other half-nodes of these nodes are illegal of twist zero, hence are forgotten when one passes to $\mathcal{B}C_u$), $h, h', f, f' \in H^+$, and the disk component of $n_{h'}$ becomes unstable after forgetting the half-nodes. In the left part of the picture, we consider the case $1,2,3 \in Q$, so that the intervals $I_h^{(Q)}, I_f^{(Q)}, I_{h'}^{(Q)}$, and $I_{f'}^{(Q)}$ (marked in bold) are pullbacks of intervals $I_h^{\mathcal{B}(Q)}, I_f^{\mathcal{B}(Q)}, I_{h'}^{\mathcal{B}(Q)}$, and $I_{f'}^{\mathcal{B}(Q)}$. In the middle and right side of the figure, we consider the case $1,2,3 \notin Q$; the right side of the picture shows part of the nodal disk obtained by the base operation. We see that n_h and n_f are mapped to the same $n_{h(Q)}$ on the right, and $n_{h'}$ and $n_{f'}$ are mapped to the same $n_{h'(Q)}$. In addition, the intervals $I_{h(Q)}^{\mathcal{B}(Q)}$ and $I_{h'(Q)}^{\mathcal{B}(Q)}$ (marked in bold on the right) are pulled back to $I_h^{(Q)} = I_f^{(Q)}$ and $I_{h'}^{(Q)} = I_{f'}^{(Q)}$, respectively, in the middle, which are unions of boundary intervals containing intervals around the nodes $n_h, n_f, n_{h'}$, and $n_{f'}$.

We now show that one can extend the given multisection $\zeta|_{W\cap U_u}$ to all $U_u \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$. By the definition of $I_{h,u}$ and U_u, the multisection $\zeta|_{W\cap U_u}$ evaluates positively at $\{I_{h,u}\}$. We can extend $\zeta|_{W\cap U_u}$ to a multisection s' on a neighborhood $V_u \subseteq U_u$ of $W \cap U_u$ such that s' evaluates positively at each $I_{u,h}(u')$ for $u' \in V_u$ and $h \in H^+(\Xi)$. Let V_u' be another neighborhood of $W \cap U_u$ whose closure is contained in V_u. Let ρ_u' be a smooth nonnegative function that equals 1 on V_u' and 0 on $U_u \setminus V_u$.

Denote by $\hat{\Xi}$ the graph obtained by smoothing the internal edges of Ξ and the edges that have an illegal half-edge of twist zero; then $H^+(\Xi) = H^+(\hat{\Xi})$ canonically. Let $\overline{W}$ be the closure of W in $\overline{\mathcal{M}}_\Gamma^{1/r}$, and let $U_{u,1}, U_{u,2}, \ldots \subseteq U_u \setminus \overline{W}$ be a locally finite cover of $U_u \setminus \overline{W}$ by contractible $\hat{\Xi}$-sets.

By Proposition 3.20, we can construct sections $s'_{u,i} \in C^\infty(\mathcal{W} \to U_{u,i})$ that are positive with respect to $(U_{u,i}, \{I_{u,h}\}_{h\in H^+(\hat{\Xi})})$. We now take subsets $U'_{u,i}$ with $\overline{U'}_{u,i} \subseteq U_{u,i}$ that also cover $U_u \setminus \overline{W}$, and let $\rho_{u,i}$ be smooth nonnegative functions that equal 1 on $U'_{u,i}$ and 0 outside of $U_{u,i}$. Then $s_u = \rho_u' s' + \sum_{i\geq 1} \rho_{u,i} s'_{u,i}$ is defined on all U_u, is smooth and $(U_u, \{I_{u,h}\}_h)$-positive, and restricts to ζ on $W \cap U_u$. This completes the first step of the proof.

Step 2: Since $\bigcup_{\Lambda\in\partial\Gamma\setminus\partial^+\Gamma}\partial^+\overline{\mathcal{M}}_\Lambda^{1/r}$ is compact, the previous step allows one to construct a finite number of tuples $(\Xi_j, V_j, \{I_{j,h}\}_{h\in H^+(\Xi_j)}, s'_j)$, with $j=1,\dots,M$, where V_j is a Ξ_j-set,

$$\bigcup_{\Lambda\in\partial\Gamma\setminus\partial^+\Gamma}\partial^+\overline{\mathcal{M}}_\Lambda^{1/r}\subseteq\bigcup_{j=1}^{M}V_j,$$

and the smooth multisection s'_j is $(V_j,\{I_{j,h}\}_h)$-positive and agrees with ζ on $W\cap V_j$. Using a partition of unity $\{\hat\rho_j\}_{j\in[M]}$ subordinate to $\{V_j\}_{j\in[M]}$, we construct a smooth multisection

$$s_0=\sum\hat\rho_j s'_j$$

defined on $\hat U=\bigcup V_j$ that agrees with ζ on $\hat U\cap W$.

We claim that s_0 is positive. Indeed, consider any $u\in\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}\cap\hat U$ with $u\in\mathcal{M}_\Lambda^{1/r}$, where Λ has at least one internal edge or illegal boundary half-edge of twist zero. Let S_u be the set of indices i with $u\in V_i$. Then for any $i\in S_u$, the definition of a Ξ_i-set implies that $\Xi_i\in\partial^!\Lambda$. Let $Z_u=\bigcap_{i\in S_u}V_i$, and for $u'\in Z_u$, write $I_h(u')=\bigcap_{i\in S_u}I_{i,h}(u')$. Note that for $h\in H^+(\Lambda)$, the intervals $I_h(u)$ are nonempty, since each is the intersection of intervals that contain a specific node n_h. Thus, these intervals are also nonempty in a neighborhood $Z'_u\subseteq Z_u$ of u. By the definition of S_u and of Λ-sets, there is a neighborhood Z_u^0 of u that does not meet $\mathrm{Supp}(\rho_i)$ for any $i\notin S_u$. Thus, on $Z'_u\cap Z_u^0$, each s'_i that is defined evaluates positively at the intervals I_h, and hence so does $s_0=\sum\hat\rho_j s'_j$.

Step 3: We now construct, for any Λ without internal edges or illegal boundary half-edges of twist zero and any point $u\in\mathcal{M}_\Lambda^{1/r}\subseteq\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}$, a contractible Λ-neighborhood U_u, a Λ-family of intervals $\{I_{u,h}\}_{h\in H^+(\Lambda)}$ for U, and a $(U_u,\{I_{u,h}\}_h)$-positive section s_u. This is straightforward: on the surface Σ_u, we draw intervals $I_{u,h}(u)$ that satisfy the requirements of a Λ-family of intervals at u. We then extend these in a smooth way to Λ-family $I_{u,h}$ in a small enough contractible Λ-neighborhood U_u of u and use Proposition 3.20 to construct sections $s_u\in C^\infty(U_u,I_u)$ that are (U_u,I_u)-positive.

By compactness of $\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}\setminus\hat U$,we can therefore find tuples (Λ_i,U_i,I_i,s_i), for $i=1,\dots,N$, such that each U_i is a contractible Λ_i-set for Λ_i without internal edges or illegal half-edges of twist zero, each $I_i=\{I_{i,h}\}_{h\in H^+(\Lambda_i)}$ is a Λ_i-family of intervals for U_i, each s_i is (U_i,I_i)-positive, and

$$\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}\setminus\hat U\subseteq\bigcup U_i.$$

For $i=1,\dots,N$, let U'_i be an open set whose closure is contained in U_i, such that $\bigcup U'_i$ also covers $\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}\setminus\hat U$. For each i, let ρ_i be a smooth nonnegative function defined on $\hat U\cup\bigcup_{i=1}^n U_i$ that equals 1 on U'_i and equals 0 outside U_i. Let ρ_0 be a smooth nonnegative function such that $\hat U\setminus(\bigcup U'_i)\subseteq\mathrm{Supp}(\rho)\subseteq\hat U$ and $\rho_0|_W=1$. Then

$$\sigma'=\sum_{i=0}^{N}\rho_i s_i$$

is a multisection, defined on W and on a neighborhood of $\overline{\mathcal{PM}}_\Gamma^{1/r}$, that extends ζ. In addition, σ' is positive, as one can see by applying the same argument used for the positivity of s_0.

Step 4: The multisection σ' is transverse on W and positive near $\partial^+\overline{\mathcal{M}}_\Gamma^{1/r}$. Let $\widetilde{\Gamma}$ be a graph identical to Γ but with injective labels, and let $\tilde{q} : \overline{\mathcal{M}}_{\widetilde{\Gamma}}^{1/r} \to \overline{\mathcal{M}}_\Gamma^{1/r}$ be the quotient map. Extend $\tilde{q}^*\sigma'$ transversally to all of $\overline{\mathcal{PM}}_{\widetilde{\Gamma}}^{1/r}$. Symmetrize the extension of $\tilde{q}^*\sigma'$ with respect to the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{W} \to \overline{\mathcal{PM}}_{\widetilde{\Gamma}}^{1/r}$, where symmetrization of a multisection is defined in (A.1) in Appendix A. The result is a pullback $\tilde{q}^*\sigma$ of a multisection σ of $\mathcal{W} \to \overline{\mathcal{PM}}_\Gamma^{1/r}$. Then σ is the required global multisection, and item (1) of the lemma is proved.

We now turn to items (2) and (3). First, an observation:

Observation 7.6. If Γ is smooth and graded without boundary tails, then there exists a smooth section s_+ of $\mathcal{W} \to \overline{\mathcal{PM}}_\Gamma^{1/r}$ such that s_+ vanishes on $\overline{\mathcal{PM}}_\Gamma^{1/r} \setminus \mathcal{M}_\Gamma^{1/r}$ and, for any smooth $C \in \mathcal{M}_\Gamma^{1/r}$, the section $(s_+)_C$ evaluates positively at $\partial\Sigma \subseteq C$.

Indeed, for any $C \in \mathcal{M}_\Gamma^{1/r}$, one can find a vector $v_C \in \mathcal{W}_C$ (that is, a global section of $|J|$ over C) that evaluates positively on $\partial\Sigma \subseteq C$ by choosing all of the zeroes of this global section to be internal. This is done using Lemma 3.21 and is an especially simple instance of the proof method of Proposition 3.20; in particular, since there are no boundary markings, the fact that all zeroes are internal implies that the section has the same sign at each boundary point. Thus, multiplying by -1 if required yields the desired v_C.

We can extend v_C to a section s_C defined in a neighborhood U_C of C with the same properties. From here, a standard argument with partitions of unity allows us to construct a global section $s_+ \in \Gamma(\mathcal{M}_\Gamma^{1/r}, \mathcal{W})$ with the required positivity property. We then obtain the desired section by replacing s_+ with ρs_+, where ρ is a function on $\overline{\mathcal{PM}}_\Gamma^{1/r}$ that is positive on $\mathcal{M}_\Gamma^{1/r}$ and vanishes quickly enough when approaching $\overline{\mathcal{PM}}_\Gamma^{1/r} \setminus \mathcal{M}_\Gamma^{1/r}$, and then extending ρs_+ to $\overline{\mathcal{PM}}_\Gamma^{1/r} \setminus \mathcal{M}_\Gamma^{1/r}$ by zero. This verifies Observation 7.6.

A second observation is:

Observation 7.7. If $\mathcal{B}\Gamma$ has a component with no boundary tails, then for any $\Xi \in \partial^0\Gamma$, there is a component of $\mathcal{B}\Xi$ with no boundary tails.

From here, in order to prove item (2) of the lemma, we proceed as in the proof of the previous item, only that after the third step in the proof, if Γ has no boundary tails, we extend ζ further to a neighborhood of $\overline{\mathcal{PM}}_\Gamma^{1/r} \setminus \mathcal{M}_\Gamma^{1/r}$ in $\overline{\mathcal{PM}}_\Gamma^{1/r}$. By Observation 7.7 and the strong positivity assumption, if we restrict to a small enough neighborhood U_0, then for any $C \in U_0$ the extended section (which we also denote by σ' as in the proof of Lemma 7.5) evaluates positively on a nonempty subset of $\partial\Sigma$. Let s_+ be the section constructed in Observation 7.6. Multiply σ' by a smooth function $\rho : \overline{\mathcal{PM}}_\Gamma^{1/r} \to [0,1]$ that equals 0 outside of U_0 and equals 1 on $\partial\overline{\mathcal{M}}_\Gamma^{1/r} \cap \overline{\mathcal{PM}}_\Gamma^{1/r}$, and extend the result to a global section σ'' that equals zero outside of U_0. Then $\sigma = s_+ + \sigma''$ is the required extension.

The proof of prove item (3) is similar to the previous case, and we use the same notation. In particular, we extend ζ to U_0 as above, continuing to use ζ to denote the result. Again, by the assumptions and Observation 7.7, for U_0 small enough, the requirement regarding the intersections between the positivity loci of the branches of s_C and ζ_C holds, since the locus in $\partial\Sigma$ on which a given branch of a section is positive is a union of open intervals. Now, defining ρ, σ'', s_+, and σ as above, we see that the resulting σ satisfies the modified requirements.

The proof of Lemma 7.5 is thus complete. □

7.2. **Proofs of Lemma 4.10 and Proposition 3.19.** We now turn to the proofs of previously-stated results concerning the existence of certain canonical multisections. We require the following theorem:

Theorem 7.8. *Let $E \to M$ be an orbifold vector bundle over a smooth orbifold with corners, and let $V = \mathbb{R}^n_+$. Fix smooth multisections $s_0, \ldots, s_n \in C^\infty_m(M, E)$, and let*

$$F : V \to C^\infty_m(M, E)$$

be the map

$$(\lambda_i)_{i\in[n]} \to F_\Lambda = s_0 + \sum \lambda_i s_i.$$

If the multisection

$$F^{ev} \in C^\infty_m(V \times M, p_M^* E), \qquad F^{ev}(\lambda, x) = F_\lambda(x)$$

is transverse to zero, where $p_M : V \times M \to M$ is the projection, then the set

$$\{v \in V | F_v \pitchfork 0\}$$

is residual.

This theorem is the orbifold analogue of [23, Theorem 3.52], and the proof is identical.

Proof of Lemma 4.10. Let s be a special canonical multisection of $\mathcal{W}$ with the properties guaranteed by item (1) of Proposition 7.2. Denote its components by s^v, for $v \in \mathcal{V}(\Gamma_{0,k,\vec{a}})$. Then s is positive on some set U_+ of the form given in (3.2), hence is nonvanishing in some neighborhood U'_+ of U_+. In addition, let

$$w_{ijk} \in \mathcal{S}_i, \;\; 1 \le i \le l, \;\; 1 \le j \le d_i, \;\; 1 \le k \le m_{ij}, \tag{7.3}$$

be a finite set of special canonical multisections of the jth copy of $\mathbb{L}_i$, which together span the fibers of $\mathbb{L}_i$ at any point of the compact $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \setminus U'_+$. The existence of these multisections is guaranteed by Proposition 7.2, Item (4). Set

$$J = \{ijk\}_{i,j,k}, \;\; V = (\mathbb{R}_+)^J.$$

where i, j, and k range over the values in (7.3).

For any $v \in \mathcal{V}(\Gamma_{0,k,\vec{a}})$ and any set K as in the statement of the lemma, write

$$J_{v,K} = \{abc \mid ab \in K, \; c \in [m_{ab}]\} \subseteq J.$$

Apply Theorem 7.8 with

$$M = \mathcal{M}_v^{1/r}, \;\; V_{v,K} = \mathbb{R}_+^{J_{\Lambda,K}}, \;\; E = \mathcal{W}_v \oplus \bigoplus_{ab\in K} \mathbb{L}_a \to \mathcal{M}_v^{1/r}$$

and

$$F = (F_{v,K})_\lambda = \bar{s^v} + \sum_{\alpha \in J_{\Lambda,K}} \lambda_\alpha w^v_\alpha, \;\; \lambda = \{\lambda_\alpha\}_{\alpha\in J_{\Lambda,K}} \in V_{\Lambda,K},$$

Observe that, in the notation of Theorem 7.8, $F^{\mathrm{ev}}_{v,K}$ is transverse to 0. Indeed, at any zero $(\lambda, u) \in V_{v,K} \times \mathcal{M}_v^{1/r}$, the derivatives of $F^{\mathrm{ev}}_{v,K}$ in the $\mathcal{M}_v^{1/r}$-directions span $\mathcal{W}_u$ by the assumption on s, while those in the $V_{v,K}$-direction span $\bigoplus_{ab\in K} \mathbb{L}_a$.

Set

$$X_{v,K} = \{\lambda \in V_{\Lambda,K} | (F_{\Lambda,K})_\lambda \pitchfork 0\}.$$

Then $X_{v,K}$ is residual, by Theorem 7.8. Moreover, if $p_{v,K} : V \to V_{v,K}$ is the projection, then

$$X = \bigcap_{v,K} p_{v,K}^{-1}(X_{v,K})$$

is residual. Write

$$s_{ij} = s_{ij;\lambda} = \sum_{k\in[m_{ij}]} \lambda_{ijk} w_{ijk}, \quad \mathbf{s} = F_\lambda$$

for some $\lambda \in X$. Then for any abstract vertex $v \in \mathcal{V}(\Gamma_{0,k,\vec{a}})$ and an set K, we have

$$\bar{s}^v \oplus \bigoplus_{ab\in K} s^v_{ab;\lambda} = (F_{v,K})_\lambda \pitchfork 0.$$

If (4.5) holds, then for any $\Lambda \in \partial^0\Gamma_{0,k,\vec{a}}$ there is at least one abstract vertex v with

$$\operatorname{rank}_{\mathbb{R}}(\mathcal{W}_v) + 2\sum_{i\in I(v)} d_i > \dim_{\mathbb{R}}(\mathcal{M}_v^{1/r}).$$

By choosing $K = \bigcup_{i\in I(v)} \{i\} \times [d_i]$, transversality shows that

$$\bar{s}^v \oplus \bigoplus_{ab\in K} s^v_{ab;\lambda}$$

does not vanish in $\mathcal{M}_v^{1/r}$, so $\mathbf{s}$ does not vanish in $\mathcal{M}_\Lambda^{1/r}$. □

We now turn to Proposition 3.19.

Proof of Proposition 3.19. This proposition is a trivial consequence of a much stronger statement:

Lemma 7.9. *For any smooth graded open graph Γ, there exist global strongly positive multisections. Moreover, for any two such multisections s_0 and s_1, there is a homotopy $H \in C^\infty_m(\overline{\mathcal{PM}}_\Gamma^{1/r} \times [0,1], \mathcal{W})$ with $H(-,i) = s_i$ for $i = 0,1$ such that every s_t is strongly positive.*

The lemma implies the proposition, since when $k = 0$ for every $\Lambda \in \partial^0\Gamma \cup \{\Gamma\}$, there is at least one open component of $\mathcal{B}\Lambda$ with no boundary tails. The strong positivity requirement (Definition 7.1) assures, in particular, that the multisection does not vanish on $\mathcal{M}_\Lambda^{1/r}$.

The existence part of the lemma is an immediate consequence of Proposition 7.2, Item (2). The "moreover" part follows from the "moreover" part of that item, applied to $s = s_0 \uplus s_1$, where $\uplus$ is as defined in Definition A.2. If the multisection defined in Item (2) is $\hat{s}$, then

$$H(-,t) = \rho_0(t)s_0 + \rho_1(t)s_1 + t(1-t)\hat{s}$$

is a homotopy that satisfies our requirements, assuming, for example, that $\rho_\epsilon : [0,1] \to [0,1]$ satisfy $\operatorname{Supp}(\rho_0) = [0, \frac{1}{3}], \operatorname{Supp}(\rho_1) = [\frac{2}{3}, 1]$, and $\rho_\epsilon(\epsilon) = 1$. This concludes the proof of Lemma 7.9 and hence of Proposition 3.19. □

7.3. **Construction of canonical homotopies and proof of Theorem 3.17.** We start with the proof of Lemma 4.11.

Proof of Lemma 4.11. Let $\mathbf{s}_1$ be the projection of $\mathbf{s}$ to E_1, and extend $\mathbf{s}$ and $\mathbf{r}$ to a neighborhood Y of V_+.

By the proof of Observation 3.16 and the definition of canonical multisections, there is an open subset Y_+ of U that also contains V_+, such that $Y_+ = Y_0 \cup Y_1$ for Y_0 and Y_1 satisfying:

(1) All branches of the $\mathcal{W}$-components of both $\mathbf{s}$ and $\mathbf{r}$ are positive with respect to the same families of intervals in an open set containing Y_0.
(2) For any $C \in Y_1$, there is a point $p \in \partial\Sigma \subseteq C$ such that all branches of $\mathbf{s}$ and $\mathbf{r}$ evaluate positively at p.

(One should think of Y_1 as a neighborhood of the strongly positive boundary strata.) By making Y_+ smaller if necessary, we assume that $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \setminus Y_+$ is a compact orbifold with corners. Let Y'_+ be another open set satisfying the same properties, whose closure is contained in Y_+.

Take a finite set of multisections w_j of $E_2 \to \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$, where $1 \leq j \leq m$ and

$$\mathrm{Supp}(w_j) \subseteq \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \setminus Y'_+,$$

such that the multisections w_j span every fiber of $E_2 \to \partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \setminus Y_+$ in the sense that, for any choice of a local branch for each $w_1, \ldots, w_m$, the branches span the fiber. Assume, furthermore, that the multisection w_j projects to zero in all but one direct summand of E_2 (with respect to the decomposition in the statement of the lemma) and that w_j is canonical if this direct summand is one of the bundles $\mathbb{L}_i$, whereas w_j is pulled back from the base (in the sense of Definition 3.14) if this direct summand is $\mathcal{W}$. Such multisections exist by items (3) and (4) of Proposition 7.2 and the compactness of $\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \setminus Y_+$.

Write $\pi : \partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \times [0,1] \to \partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ for the projection onto the first factor. Let $h \in C^\infty_m(\partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}, \pi^*E|_{\partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}})$ be the linear homotopy defined by

$$h(p,t) = (1-t)\mathbf{s}(p) + t\mathbf{r}(p)$$

for $p \in \partial\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ and $t \in [0,1]$, and set

$$H_\lambda(p,t) = h(p,t) + t(1-t)\sum \lambda_i w_i, \ \lambda \in \mathbb{R}^m_+.$$

By positivity, H_λ is nowhere-vanishing on $Y'_+ \times [0,1]$, and in particular transverse, for each λ. For λ in a small enough neighborhood N of $0 \in \mathbb{R}^m$, the multisection $H_\lambda(p,t)$ is also nonzero on $Y_+ \times [0,1]$. Now, apply Theorem 7.8 to

$$M = \partial^0\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \times (0,1)\,, \ E = \pi^*E|_{\partial^0\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}}\,, \ V = N,$$

and $F = H$. For $p \notin Y_+$, the derivatives of H^{ev} in directions tangent to $\partial\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}$ span the fiber $(E_1)_p$ whenever $\mathbf{s}_1(p)$ vanishes, by assumption 2 of our current lemma. Since the multisections w_j span $(E_2)_p$, the derivatives of H^{ev} in the $\mathbb{R}^m$-directions span the fiber $(E_2)_p$

for all $p \in \partial^0 \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \setminus Y_+$. Thus, $H^{\mathrm{ev}}\big|_{\mathbb{R}^m_+ \times \partial \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}}$ is transverse to zero, and by Theorem 7.8, the set

$$X = \{\lambda \in V |\ H_\lambda \pitchfork 0\}$$

is residual.

For any $\Gamma \in A$, let $E_\Gamma \to \mathcal{M}_{\mathcal{B}\Gamma}$ be the bundle induced by E on $\mathcal{M}_{\mathcal{B}\Gamma}$, so that $F_\Gamma^* E_\Gamma = E$. Write

$$M_\Gamma = \mathcal{M}_{\mathcal{B}\Gamma} \times (0,1), \text{ and } E_\Gamma = \pi_\Gamma^* E_\Gamma,$$

where $\pi_\Gamma : M_\Gamma \to \mathcal{M}_{\mathcal{B}\Gamma}$ is the projection to the first factor. By the definitions of the multisections $\{w_j\}$ and of A, there exists $H^{\mathcal{B}\Gamma} : \mathbb{R}^m_+ \to C^\infty_m(E_\Gamma \to M_\Gamma)$ for any $\Gamma in A$ such that

$$H_\lambda|_{\mathcal{M}_\Gamma \times (0,1)} = (F_\Gamma \times id_{(0,1)})^* H^{\mathcal{B}\Gamma}_\lambda,\ \lambda \in \mathbb{R}^m_+.$$

Now, apply Theorem 7.8 with the same V, and with

$$M = M_\Gamma,\ E = E_\Gamma,\ F = H^{\mathcal{B}\Gamma}.$$

Since $\mathbf{s}_1 \pitchfork 0$, it follows that $\mathbf{s}_1^{\mathcal{B}\Gamma} \pitchfork 0$. Thus, the same argument that showed $H^{\mathrm{ev}} \pitchfork 0$ also shows $(H^{\mathcal{B}\Gamma})^{\mathrm{ev}} \pitchfork 0$. It follows that the set

$$X_\Gamma = \{\lambda \in \mathbb{R}^m \mid H^{\mathcal{B}\Gamma}_\lambda \pitchfork 0\}$$

is residual.

The intersection $X \cap \bigcap_{\Gamma \in A} X_\Gamma$ is nonempty, since X and X_Γ are residual, and for any $\lambda \in X \cap \bigcap_{\Gamma \in A} X_\Gamma$ and any $\Gamma \in A$ such that $\mathcal{B}\Gamma$ has no partially stable components, the homotopy $H^{\mathcal{B}\Gamma}_\lambda$ does not vanish at $\mathcal{M}_\Gamma \times [0,1]$ due to transversality and the dimension count of Observation 3.11. The homotopy H_λ does not vanish on $Y \times [0,1]$ by of the choice of N. Thus, H_λ satisfies our requirements for any $V'_+ \subseteq V_+ \cap Y_+$ of the form (4.6), and the proof is complete. □

We can now prove Theorem 3.17.

Proof of Theorem 3.17. By Lemma 4.10, one can find a global canonical multisection $\mathbf{s} \in C^\infty_m(\overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}, E)$ that does not vanish in $\partial \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$ or on U_+ of the form (3.2) with $\Gamma = \Gamma_{0,k,\vec{a}}$. Hence, $e\left(E; \mathbf{s}|_{\partial \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \cup U_+}\right)$ can be defined. If $\mathbf{r}$ is another canonical multisection whose domain contains $\partial \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \cup U'_+$, for U'_+ of the same form, then by Lemma 4.11 applied to $C = \partial^0 \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}}$, the trivial bundle E_1, and

$$V_+ = U_+ \cap U'_+,$$

one can find a nowhere-vanishing homotopy between $\mathbf{s}$ and $\mathbf{r}$. Thus, by Lemma 4.12, we have

$$\int_{\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}} e\left(E; \mathbf{s}|_{\partial \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \cup U_+}\right) = \int_{\overline{\mathcal{M}}^{\frac{1}{r}}_{0,k,\vec{a}}} e\left(E; \mathbf{r}|_{\partial \overline{\mathcal{PM}}^{\frac{1}{r}}_{0,k,\vec{a}} \cup U'_+}\right),$$

which completes the proof. □

Appendix A. Multisections of orbibundles and the relative Euler class

The aim of this section is to give references and, in some cases, briefly describe notions we use throughout the paper.

Our definitions for orbifolds with corners, operations between them, and orientations are those of [20, Section 3]. Analogous definitions for manifolds with corners can be found in [14]. In short, an orbifold is given by a proper étale groupoid $M = (M_0, M_1, s, t, e, i, m)$ (Definition 22 in [20]), which is a category with objects M_0 and morphisms M_1, where both M_0 and M_1 are manifolds with corners.[8] The source and target maps $s, t : G_1 \to G_0$ take a morphism to its source and target, respectively. The map $e : M_0 \to M_1$ takes an object x to its identity morphism 1_x, the map $i : M_1 \to M_1$ takes an element to its inverse, and m is the composition of morphisms, denoted by $m(g, h) = gh$ whenever defined. The maps s, t, e, i, m are required to be étale, and the map $s \times t : M_1 \to M_0 \times M_0$ is required to be proper. The *coarse space* or *orbit space* is the quotient $|M| = M_0/M_1$, with the quotient topology. The isotropy group of $x \in M_0$ is the group of elements $\gamma \in M_1$ with $s(\gamma) = t(\gamma) = x$. It is well known that an effective orbifold, meaning an orbifold such that the generic isotropy group is trivial, is completely determined from the local orbifold structure of a cover of its coarse space; see [19, Proposition 5.29]. The same holds for orbifolds with corners.

An orbibundle E over an orbifold with corners M is given by a vector bundle $\pi : E \to M_0$ and a fiberwise linear map

$$\mu : E \times_{M_0} M_1 := \{(e, \gamma) | \pi(e) = t(\gamma)\} \to E,$$

where we write $e \cdot \gamma$ for $\mu(e, \gamma)$. It is required that

$$\pi(e \cdot \gamma) = s(\gamma),\ e \cdot 1_{\pi(e)} = e,\ \text{and}\ (e \cdot \gamma) \cdot \delta = e \cdot (\gamma\delta)$$

whenever the last equality makes sense; see [18, Section 5]. This data defines, in a natural way, an orbifold whose objects are E, and the morphisms and other maps are defined using π and μ. Note that the isotropy group of x acts on the fiber E_x.

Our discussion of multisections, Euler classes, and their relations follows [16]. Although there the underlying category is manifolds without boundary, while for us it is the category of manifolds with corners, the constructions and proofs are easily adapted to our case. We also refer the reader to the Appendix of [23] for a short description of the analogous definitions and for manifolds with corners. Inspired by [16, Definition 4.13], we define:

Definition A.1. A *multisection* of an orbibundle $E \to M$ over an orbifold with corners is a function

$$\mathfrak{L} : E \to \mathbb{Q}_{\geq 0},$$

that satisfies:

(1) for every $e = (x, v) \in E,\ \gamma \in X_1$ with $x = t(\gamma)$, one has $\mathfrak{L}(x, v) = \mathfrak{L}((x, v) \cdot \gamma)$;
(2) for each $x \in M_0$ there is an open neighborhood U, a nonempty finite set of smooth local sections $s_i : U \to E,\ i = 1, \ldots, N$ called *local branches*, and a collection of positive rational numbers $\mu_1, \ldots, \mu_N$ called *weights*, such that for all $y \in U$,

$$\sum_{v \in E_y} \mathfrak{L}(y, v) = 1$$

[8] More precisely, the category of orbifolds is a 2-localization of the category of proper étale groupoids by a natural notion of refinement. A similar comment holds for orbibundles, as considered below.

and for all $v \in E_y$,

$$\mathfrak{L}(y,v) = \sum_{i \mid s_i(y)=v} \mu_i.$$

The triple $(U, \{s_i\}, \mu_i)$ is called a *local structure* for the multisection. We denote by $C_m^\infty(E)$ the collection of multisections of E.

When $N = 1$ for all x, the multisection is just an ordinary section in the category of orbibundles over orbifolds with corners. Any section descends to a map from the coarse space of M to that of E. The reason we have to work with multisections is that in the orbifold category, sections may be constrained to vanish at some points. Being canonical puts more vanishing constraints on sections, as noted in Remark 3.5 in [23].

We write $Z(s) = \{p \mid \mathfrak{L}(p,0) \neq 0\}$ for the zero locus of a multisection s. A multisection s is *transverse to zero* if every local branch is transverse to the zero section; in this case, we write $s \pitchfork 0$.

Addition and scalar multiplication of multisections are defined in [16, Page 38] via

$$(\mathfrak{L}_1 + \mathfrak{L}_2)(x,v) = \sum_{v_1+v_2=v} \mathfrak{L}_1(x,v_1)\mathfrak{L}_2(x,v_2), \ \lambda\mathfrak{L}(x,v) = \mathfrak{L}(x,\lambda v).$$

These operations define a vector space structure on the set of multisections. When a finite group G acts on $E \to M$, an induced action on multisections is defined via $(g \cdot \mathfrak{L})(x,v) = \mathfrak{L}(g^{-1}(x,v))$. The *symmetrization* of $\mathfrak{L}$ with respect to G is defined in [23, Definition A.10] as the G-invariant multisection

$$\mathfrak{L}^G(x,v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \mathfrak{L}(x,v). \tag{A.1}$$

Notation A.2. If $\mathfrak{L}_1, \ldots, \mathfrak{L}_m$ are multisections of an orbifold vector bundle E on an orbifold M, we write

$$\uplus_{i=1}^m \mathfrak{L}_i(x,v) := \frac{1}{m} \sum_{i=1}^m \mathfrak{L}_i(x,v).$$

The *support* of a multisection is the support of $\mathfrak{L}$, which is locally the union of the images of the local branches s_i. We often denote a multisection $\mathfrak{L}$ by its support s. A multisection descends to a function on the coarse space of E, considering E as an orbifold with corners, so we sometimes use the term "support" to refer, by slight abuse of notation, to the support on the coarse bundle.

If s_i is the support of $\mathfrak{L}_i$, then $\uplus_{i=1}^m s_i(x)$ denotes $\uplus_{i=1}^m \mathfrak{L}_i(x,v)$. In particular, the support of $\uplus_{i=1}^m s_i(x)$, as a set, is the union of the supports of the s_i. Under this convention, we have, for any multisection s,

$$\uplus_{i=1}^m s = s. \tag{A.2}$$

Remark A.3. The boundary ∂X of a manifold with corners X is itself defined as a manifold with corners, and is equipped with a (not necessarily injective) map $i_X : \partial X \to X$. We extend this map to $i_X : X \sqcup \partial X \to X$ by the identity. If $U \subseteq X$ and $V \subseteq \partial X$, then we say that a section s of a bundle over $U \sqcup V$ is *consistent* if $s(x) = s(y)$ whenever $i_X(x) = i_X(y)$,

where the equality is obtained using the natural identifications of the fibers E_x and E_y. Consistent multisections are similarly defined.[9]

Throughout this paper, we omit the maps i_X from the notation and identify x with $i_X(x)$. By a section or a multisection, we always mean a consistent section or multisection. In particular, canonical multisections are automatically consistent, as well as all other multisections constructed in this paper.

Let $E \to M$ be an orbibundle over an orbifold with corners such that $\operatorname{rank} E = \dim M$. Assume that E and M are oriented, and, while $|M|$ need not be compact, assume that it has finitely many connected components.

Suppose $U \subseteq |M|$ is an open subspace such that $|M| \setminus U$ is a compact orbifold with corners. Let s be (the support of) a multisection of $|E| \to U \sqcup \partial|M|$, which is, by definition, induced from a multisection of E on the preimage of $U \sqcup \partial|M|$ in M_0. Suppose that s vanishes nowhere on $U \sqcup \partial|M|$. Let $\tilde{s}$ be a global extension of s to a multisection of $E \to M$ with isolated zeroes. For any $|p| \in |M|$ in the zero locus of s, let $p \in M_0$ be a choice of preimage of $|p|$, and let $(U, \{s_i\}_{i=1}^N, \{\mu_i\}_{i=1}^N)$ be a local structure for s at p. By shrinking U if necessary, we can assume that the only possible zero of each local branch s_i in U is at p, and that the restriction of E to U is trivialized as $U \times \mathbb{R}^n$; then, give the fiber $\mathbb{R}^n$ the orientation induced from E.

Let $\deg_p(s_i)$ be the degree of vanishing of s_i at p. More specifically, this degree is zero if $s_i(p) \neq 0$, and otherwise is defined as follows. Let S be the boundary of a small ball in U containing p, with the orientation induced from the ball, which in turn is induced from M. Let S' be the unit sphere in $\mathbb{R}^n$ with the orientation induced from the unit ball, which in turn is induced from $\mathbb{R}^n$. We define $\deg_p(s_i)$ as the degree of the map $f : S \to S'$ given by $x \to \frac{s_i(x)}{|s_i(x)|}$, where the norm is the L^2 norm of $\mathbb{R}^n$.

Notation A.4. Under the above assumptions, we define the *weight* of $|p|$ as

$$\varepsilon_{|p|} = \frac{1}{|G|} \sum_{i=1}^{N} \mu_i \deg_p(s_i),$$

where G is the isotropy group of p. We define, for a global multisection $\tilde{s}$, the *weighted cardinality* of $Z(\tilde{s})$ as

$$\#Z(\tilde{s}) = \sum_{|p| \in |M|} \varepsilon_{|p|}.$$

The weight $\varepsilon_{|p|}$ is independent of choices. In fact, one need not choose orientations of both E and M in order to define $\deg_p(s_i)$; it is enough to choose a relative orientation for $E \to M$.

We use the following theorem.

Theorem A.5. *Let M, E, and s be as above, and let $\tilde{s}$ be a global extension with isolated zeroes. Then $\#Z(\tilde{s})$ and the homology class $\sum \varepsilon_{|p|}[|p|] \in H_0(|M|; \mathbb{Q})$ depend only on s and not on $\tilde{s}$. Changing the orientations of E and M but preserving the relative orientation does not change the class or the count. This homology class is therefore Poincaré dual to a relative*

[9]To be more precise, part of the information of a vector bundle $E \to X$ in the category of vector bundles over manifolds with corners is a vector bundle $\partial E \to \partial X$ together with an identification $\widetilde{i}_X : (\partial E)_x \to i_X^* E_x$. The map $\widetilde{i}_X$ also extends by the identity map to $\widetilde{i}_X : E \sqcup \partial E \to i_X^* E$. We use these natural identifications to define the equality of $s(x)$ and $s(y)$.

cohomology class with compact support $H^n_c(|M|, \partial|M|; \mathbb{Q})$*, which is called the* relative Euler class*, and by definition satisfies,*

$$\int_{|M|} e(E, s) = \#Z(\tilde{s}).$$

In case $\partial|M| = \emptyset$*, the relative Euler class is the usual Euler class of the orbibundle.*

The independence of choices follows from a standard cobordism argument, which is easily modified to the orbifold case, so we omit the proof. The fact that the weighted zero count of a transverse multisection is dual to the Euler class in the case without boundary is a special case of the results of [16], and it extends without difficulty to our setting.

A final useful observation is the following:

Observation A.6. Let E, M, U, and s be as above. Let $U' \subseteq U$ be another open set such that $|M| \setminus U'$ is a compact orbifold with corners, and let $s' = s|_{\partial|M| \cup U'}$. Then a transverse extension of s is also a transverse extension of s'. Therefore,

$$\int_{|M|} e(E; s') = \int_{|M|} e(E; s) \in \mathbb{Q}.$$

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