OPEN $\mathbb{CP}^1$ DESCENDENT THEORY I: THE STATIONARY SECTOR

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ABSTRACT. We define stationary descendent integrals on the moduli space of stable maps from disks to $(\mathbb{CP}^1, \mathbb{RP}^1)$. We prove a localization formula for the stationary theory involving contributions from the fixed points and from all the corner-strata. We use the localization formula to prove a recursion relation and a closed formula for all genus 0 disk cover invariants in the stationary case. For all higher genus invariants, we propose a conjectural formula.

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CONTENTS

1. Introduction

1.1. **Background.** Closed Gromov-Witten theory concerns integrals over the moduli spaces of stable maps to nonsingular projective target varieties X. The Gromov-Witten theory of a point, the simplest target variety, consists of all descendent integrals over the moduli spaces of curves,

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \, .$$

The associated generating series satisfies the KdV equation as conjectured by Witten [34] and proven by Kontsevich [19] (see [27, 22, 17] for other proofs). In case the dimension of the target X is 1, the closed Gromov-Witten theory has been solved completely in a series of papers [28, 29, 30] more than 10 years ago. Fundamental to the solution is a special study of the target $\mathbb{CP}^1$ via virtual localization [15], Hodge integrals [12], and the ELSV formula [8]. The complete solution for the closed Gromov-Witten theory of all 1-dimensional targets has two steps:

 (i) Virasoro constraints are used to reduce all descendent integrals to the *stationary sector* consisting of descendents of points,
 (ii) the stationary sector is solved via the Gromov-Witten/Hurwitz correspondence.

More recently, there have been parallel developments in *open* Gromov-Witten theory. The open case concerns integrals over the moduli spaces of maps of Riemann surfaces with boundary $(\Sigma, \partial\Sigma)$ to a symplectic target (X, L), where the boundary $\partial\Sigma$ is required to map to the Lagrangian

$$L \subset X.$$

The open Gromov-Witten theory of a point has been defined and solved in a series of papers [6, 31, 32, 33] resulting in open analogues of the KdV equations and Virasoro constraints.

Our goal here is to begin a systematic study of the open Gromov-Witten theory of 1-dimensional targets in the basic case

$$(X, L) = (\mathbb{CP}^1, \mathbb{RP}^1) \, .$$

The open theory is technically much more difficult to define than the closed theory because of nontrivial boundary, corner, and orientation issues for the moduli spaces of maps. Our main results are the definition and calculation of the stationary descendent theory of disk maps to $(\mathbb{CP}^1, \mathbb{RP}^1)$.

One of the main driving forces for our work was the localization formula of the second author [23]. We develop here an analogous localization formula from first principles. Even though in our geometry the fixed-point components never intersect the boundary, our formula involves contributions from the corners of the moduli spaces (which contribute also to the formula of [23]).

The construction of the stationary descendent theory of $(\mathbb{CP}^1, \mathbb{RP}^1)$ for open Riemann surfaces of higher genus is not yet complete. However, using our solution of the disk theory, we conjecture a full solution for the open stationary descendent invariants $(\mathbb{CP}^1, \mathbb{RP}^1)$ in all genera.

1.2. **General notations.** Throughout the paper we consider closed Riemann surfaces as a special case of Riemann surfaces with boundary. For a smooth or stable Riemann surface Σ with non-empty boundary, let the *double* $\Sigma_{\mathbb{C}}$ be the Riemann surface obtained by gluing the surface and the conjugate surface (same surface but with opposite complex structure) along the common boundary, using the Schwartz reflection principle. Whenever Σ has non-empty boundary, its *doubled genus*, which we will simply call the *genus*, is the genus of $\Sigma_{\mathbb{C}}$, which we denote by

$$g = g(\Sigma).$$

The Riemann surface $\Sigma_{\mathbb{C}}$ is endowed a canonical involution b called *conjugation*. We denote by $h(\Sigma)$ the number of boundary components. The *small genus* $g_s(\Sigma)$ is the genus of the closed surface obtained by capping each boundary component of Σ with a disk. When Σ is connected with $\partial \Sigma \neq \varnothing$, the numbers g, h and g_s are related by

$$(1) \qquad\qquad g_s = \frac{g - h + 1}{2},$$

and hence

$$(2) \qquad\qquad 1 \leq h \leq g + 1, \qquad h = g + 1 \mod 2.$$

When $\partial \Sigma = \varnothing$, we write $g_s = g$, the usual genus and then $h = 0$. Two elements from the set $\{g, h, g_s\}$ determine, in the connected case, the third. When working with fixed point graphs, to be defined below, g_s, h are more convenient parameters than g, h, but when considering dimensions of moduli, working with the parameter g simplifies the equations. We shall therefore work interchangeably with different elements from $\{g, h, g_s\}$, according to the context.

We view S^1 as given by the unit circle in the complex plane, $e^{\sqrt{-1}\theta} \in S^1$. The circle S^1 acts on $(\mathbb{CP}^1, \mathbb{RP}^1)$ by the rule

$$e^{\sqrt{-1}\theta} \cdot [z_0 : z_1] = [\cos\theta \cdot z_0 - \sin\theta \cdot z_1 : \sin\theta \cdot z_0 + \cos\theta \cdot z_1],$$

which preserves the Lagrangian $\mathbb{RP}^1 \subset \mathbb{CP}^1$. The S^1-action has two fixed points

$$\{p_+, p_-\} \subset \mathbb{CP}^1$$

labeled in such a way that $e^{\sqrt{-1}\theta} \in S^1$ acts on $T_{p_\pm}\mathbb{CP}^1$ by $z \mapsto e^{\pm 2\sqrt{-1}\theta} z$. The form $d\theta$ will denote the canonical normalized angular form on $\mathbb{RP}^1$, which satisfies $\int_{\mathbb{RP}^1} d\theta = 1$, where $\mathbb{RP}^1$ is oriented as the boundary of the upper hemisphere.

We identify the relative homology group

$$H_2\left(\mathbb{CP}^1, \mathbb{RP}^1\right) = \mathbb{Z}_+ \oplus \mathbb{Z}_-$$

so that the two oriented hemispheres containing p_+ and p_- represent the generators $(1, 0)$ and $(0, 1)$, respectively. The *degree* of a map $u : (\Sigma, \partial\Sigma) \to (\mathbb{CP}^1, \mathbb{RP}^1)$ is the element

$$\vec{d} = (d_+, d_-) = u_*[\Sigma] \in H_2\left(\mathbb{CP}^1, \mathbb{RP}^1\right).$$

We write ∂^H for the connecting map $H_2\left(\mathbb{CP}^1, \mathbb{RP}^1\right) \to H_1\left(\mathbb{RP}^1\right)$,

$$(3) \qquad\qquad \partial^H(a, b) = a - b.$$

We also set

$$|(a, b)| = \max\{a, b\}.$$

A map $u : (\Sigma, \partial\Sigma) \to (\mathbb{CP}^1, \mathbb{RP}^1)$ induces a map $u_{\mathbb{C}} : \Sigma_{\mathbb{C}} \to \mathbb{CP}^1$, which satisfies

$$u_{\mathbb{C}}(b(z)) = \overline{u(z)},$$

where, written in the standard projective coordinates, $\overline{[w:v]} = [\bar{w}:\bar{v}]$, and $z \to \bar{z}$ is the standard conjugation.

For a positive integer n, let $[n] = \{1, \ldots, n\}$. We will use natural numbers to label usual marked points and symbols such as $\{\star, \divideontimes\}$ to label marked points which are created from normalizing nodes in nodal surfaces. We will consider stable disk maps

$$u : (\Sigma, \partial\Sigma, z_1, \ldots, z_n) \to (\mathbb{CP}^1, \mathbb{RP}^1).$$

We will only consider the stationary sector, meaning that marked points are constrained to fixed points in the target.

In a sequel [5], we will relate the stationary theory developed here to the enumeration of open Hurwitz covers. We will only consider internal markings. Boundary markings play no interesting role in the stationary theory, since the divisor relation removes boundary markings. In a future work, where we will consider the full equivariant theory, which includes also non-stationary insertions, boundary markings will play a non-trivial role.

Notation 1.1. For a set I of labels, we use the notation $\vec{a} = (a_i)_{i \in \mathsf{I}} \in \mathbb{Z}_{\geq 0}^{\mathsf{I}}$ to denote a *vector of descendents*, which indicates the number of descendents the ith marking carries, and

$$\vec{\epsilon} = (\epsilon_i)_{i \in \mathsf{I}} \in \{\pm\}^{\mathsf{I}}$$

to denote a *vector of point constraints*, which indicates to which component of $\mathbb{CP}^1 \backslash \mathbb{RP}^1$ the point constraining the ith marking belongs.

1.3. **Stationary open Gromov-Witten theory of $(\mathbb{CP}^1, \mathbb{RP}^1)$ in genus** 0. The goal of the paper is to construct and study the stationary open GW theory of maps to $(\mathbb{CP}^1, \mathbb{RP}^1)$ with descendents. In Section 2, we will present the main geometric construction of this paper: a geometric definition of the equivariant stationary descendent invariants for the moduli of maps from disks to $(\mathbb{CP}^1, \mathbb{RP}^1)$,

(4)
$$\left\langle \prod_{i \in \mathsf{I}} \tau_{a_i}^{\epsilon_i} \right\rangle_{0, \vec{d}} = \int_{\overline{\mathcal{M}}_{0, \varnothing, \mathsf{I}}(\vec{d})} \prod_{i \in \mathsf{I}} \psi_i^{a_i} \mathrm{ev}_i^{\star} \rho_{\epsilon_i}.$$

Here, $\overline{\mathcal{M}}_{0, \varnothing, \mathsf{I}}(\vec{d})$ is the moduli space of stable disk maps with internal markings labeled by I and no boundary markings (indicated by $\varnothing$). The moduli space $\overline{\mathcal{M}}_{0, \varnothing, \mathsf{I}}(\vec{d})$ is an orbifold with corners which is oriented with the orientation defined in Section 6, the integrand ψ_i is the equivariant Chern form associated with a connection on the ith cotangent line bundle $\mathbb{L}_i$, and $\rho_{\pm}$ is the equivariant Poincaré dual of the corresponding point (see Section 2.5).

Because $\partial \overline{\mathcal{M}}_{0, \varnothing, \mathsf{I}} \neq \varnothing$, the integral (4) depends on a choice of boundary conditions. There is an infinite dimensional space of such choices which gives rise to different integrals. It is natural to ask whether there exist distinguished boundary conditions which are better than others in some sense. Criteria for "good" boundary conditions might be that they can be described in ways which interact with the stratification of the moduli, so that the resulting intersection numbers are related to the closed Gromov-Witten theory of $\mathbb{CP}^1$, or that a connection with an interesting integrable hierarchy exists.

The construction we provide satisfies most of these criteria. The main exception is that we do not know, at the moment, the precise relation to the Toda hierarchy which governs the closed theory of $\mathbb{CP}^1$. We define the integrals (4) in terms of equivariant forms. These forms are defined in a recursive way with respect to the stratification of the moduli. Although the space of these recursive integrands will also be infinite dimensional, we will show (in the first part of Theorem 2.18) that

different choices yield the same integrals. We will compute these integrals and find closed formulas in special cases. In the sequel [5], we will show (in genus 0) a correspondence with a Hurwitz theory with completed cycles which is the open analog of the Gromov-Witten/Hurwitz correspondence of [28].

1.4. **Boundary conditions on sections.** We postpone the full definition via recursive equivariant forms to Section 2. Instead, we briefly describe here an alternative definition in the language of sections of bundles. The approach using sections is simpler to describe and is useful for building intuition, but describes only the non-equivariant limit of our theory. The equivalence of the two definitions can be proven along the lines of [23, Corollary 15]. To prove our main results, we will work exclusively with the full definition in terms of equivariant forms.

We would like to describe the integral (4) as a weighted count of zeroes of a multisection

$$s = \bigoplus_{i \in \mathsf{I}, \, j \in [a_i]} s_{ij}$$

of $\bigoplus \mathbb{L}_i^{\oplus a_i}$ restricted to the locus

$$\bigcap \mathrm{ev}_i^{-1}(p_i) \subset \overline{\mathcal{M}}_{0,\varnothing,\mathsf{I}}\big(\vec{d}\big) \,,$$

where $p_i \in \mathbb{CP}^1 \backslash \mathbb{RP}^1$ are arbitrary different points and

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,\varnothing,\mathsf{I}}\big(\vec{d}\big) \to \mathbb{CP}^1$$

are the evaluation maps. Since (4) depends on the boundary behaviour of s, we must describe the boundary conditions for a multisection of the line bundle $\mathbb{L}_i$.

We will concentrate only on real codimension 1 boundary components. There are two types of real codimension 1 boundary strata. First, there are strata which parameterize stable disks with two disk components that share a common boundary node. The second type concerns strata which parameterize stable disks whose boundary is contracted to a point. These are equivalent to spheres with one special point which lies in $\mathbb{RP}^1$.

In the first case, let $\mathcal{B}$ be such a boundary stratum described by

$$\mathcal{B} = \overline{\mathcal{M}}_{0,\star_1,\mathsf{l}_1}\big(\vec{d}_1\big) \times_L \overline{\mathcal{M}}_{0,\star_2,\mathsf{l}_2}\big(\vec{d}_2\big) \xrightarrow{i_{\mathcal{B}}} \partial\overline{\mathcal{M}}_{0,\varnothing,\mathsf{I}}\big(\vec{d}\big) \,,$$

where the fibered product $\times_L$ is taken with respect to the evaluation maps at $\star_1, \star_2$, see the discussion which precedes Proposition 2.7 for details and notations. There is a map

$$f_{\mathcal{B}} : \mathcal{B} \to \overline{\mathcal{M}}_1^{\diamond} \times \overline{\mathcal{M}}_2^{\diamond} \,,$$

defined by *forgetting the node*, where

$$\overline{\mathcal{M}}_i^{\diamond} = \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_i}\big(\vec{d}_i\big) \,.$$

A section $s \in C^\infty(\mathbb{L}_i \to \mathcal{B})$ is said to be *coherent* if there exists a section

$$s' \in C^\infty(\mathbb{L}_i \to \overline{\mathcal{M}}_1^{\diamond} \times \overline{\mathcal{M}}_2^{\diamond})$$

with $s = f_{\mathcal{B}}^* s'$. If $\mathsf{l}_j = \{k\}$ and $\vec{d}_j = \vec{0}$, then $\overline{\mathcal{M}}_j^{\diamond}$ is not stable and the line bundle $\mathbb{L}_k$ is not pulled back from it. However, this boundary is not relevant for the *stationary* sector. Overcoming this issue in a theory which also involves non-stationary descendent insertions requires more work.

The second type of codimension 1 boundaries is the exceptional boundary

$$\mathcal{E} = \mathrm{ev}_\star^{-1}(L) \subset \overline{\mathcal{M}}_{0,\mathsf{l}\sqcup\star}(d_+) \,,$$

which may appear only when $d_+ = d_-$. We say that a section $s \in C^\infty(\mathbb{L}_i \to \mathcal{E})$ is coherent if there exists a section $s' \in C^\infty(\mathbb{L}_i \to \overline{\mathcal{M}}_{0,\mathsf{l}}(d_+))$ with $s = f^* s'$, where

$$f : \mathcal{E} \to \overline{\mathcal{M}}_{0,\mathsf{l}}(d_+)$$

is the forgetful map.

A (multi)section is coherent if its restrictions to all codimension 1 boundaries are coherent. An example of a coherent section in the stationary theory is the following. Consider the ith marked point q_i which is mapped to p_i. In a generic point of the moduli space given by the stable map $u : \Sigma \to \mathbb{CP}^1$ and the markings data, the map u induces an isomorphism near p_i of the cotangent lines $T^*_{q_i}\Sigma$ and $T^*_{p_i}\mathbb{CP}^1 \simeq \mathbb{C}$. Choose some fixed $v \in \mathbb{C}$. Then the section $s \in \Gamma(\mathbb{L}_i)$ given by $s_u = u^*v$, whenever u^* maps $T^*_{p_i}$ to $T^*_{q_i}$ isomorphically, and extended by 0 otherwise is coherent. The zero locus of this section is exactly the subspace where the map u is ramified at q_i. This simple observation is the first hint of the open Gromov-Witten/Hurwitz correspondence to be discussed in the sequel [5], and may also serve as a different path to proving it in genus 0 (see also Proposition 1.1 in [28]).

Consider now the closed setting and assume $d - 1 = \sum_{i \in I} a_i$. We may calculate the closed Gromov-Witten invariant

$$\left\langle \prod_{i \in I} \tau_i^{\mathrm{pt}} \right\rangle_{0,d}$$

as

$$\# \left(Z(\bigoplus_{i \in I, j \in [a_i]} s_{ij}) \cap \bigcap \mathrm{ev}_i^{-1}(p_i) \right),$$

where $p_i \in \mathbb{CP}^1$ are generic points, s_{ij} are generic smooth multisections of $\mathbb{L}_i$, and $\#Z(s)$ is the properly defined zero count of s.

Consider next the open setting and assume $\sum_{i \in I} a_i = d_+ + d_- - 1$. Suppose we have coherent multisections s_{ij} of $\mathbb{L}_i$ without a common zero on the boundary. Let $p_i \in \mathbb{CP}^1 \backslash \mathbb{RP}^1$, $i \in I$ be distinct points which p_i in the ϵ_i hemisphere (+ and − stand for the upper and lower hemispheres, respectively). We may then consider the number

(5)
$$\# \left(Z(\bigoplus_{i \in I, j \in [a_i]} s_{ij}) \cap \bigcap \mathrm{ev}_i^{-1}(p_i) \right).$$

It turns out that such coherent multisections exist. The remarkable fact, which is one of our main results here, is that the count (5) is independent of the generic choices. The number (5) will be denoted by

$$\left\langle \prod \tau_{a_i}^{\epsilon_i} \right\rangle_{0,\vec{d}}.$$

The main idea behind coherent sections is that over the boundary they are pulled back from moduli spaces of dimensions which are strictly smaller than the dimension of $\partial \overline{\mathcal{M}}_{0,\varnothing,I}(\vec{d})$. This fact, and transversality arguments, show that if such multisections can be found, the intersection numbers that they define will be independent of the choice of the specific coherent multisections. A similar idea has already appeared in [3, 4, 31] and related papers in their definitions of open intersection numbers. In those papers, and here as well, the scheme of boundary conditions gives rise to a beautiful enumerative theory.

In order to compute the invariants (5) using torus localization, we are lead to impose more refined conditions involving the recursive structure of the corners of the space, see Section 2.4. Instead of coherent sections, we will work with *coherent equivariant forms*. Although there are no fixed points on the boundary, we will see that the localization formula expresses the invariants as polynomials involving contributions from all the moduli spaces appearing at the corners. A similar situation occurs when computing Welschinger's invariants in terms of fixed-point contributions, see [23].

1.5. **Equivariant localization formula for stationary descendents.** In order to write the equivariant localization formula for the open stationary descendents in $g = 0$, we begin with some definitions which are relevant also for the $g > 0$ case. In Section 5, we will use these definitions to write the all-genera formula.

1.5.1. *Moduli specifications.* The discrete data which classifies the connected components of moduli spaces of maps from marked surfaces with boundary, but without boundary markings, to $(\mathbb{CP}^1, \mathbb{RP}^1)$ can be encapsulated in the definition of a *moduli specification*. Denote by Υ the set of possible labels of marked points (containing $\mathbb{N}$ and other symbols such as $\star, \star$).

Definition 1.2. A moduli specification is a decorated graph $S = (V_b \sqcup V_w, E, g_s, \mathsf{l}, \vec{d}, d)$, where
(i) V_b is the set of black vertices, V_w is the set of white vertices, and E is the set of edges,
(ii) g_s, l, and $\vec{d}$ are functions, $g_s : V_b \to \mathbb{Z}_{\geq 0}$, $\mathsf{l} : V_b \to 2^{\Upsilon}$,
$$\vec{d} = (d^+, d^-) : V_b \to H_2(\mathbb{CP}^1, \mathbb{RP}^1; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \,,$$
(iii) d is a function $d : V_w \to \mathbb{Z}$.
We require
(a) Each connected component of the graph contains exactly one black vertex. The graph is bipartite and each white vertex has a single (black) neighbor. We write $n(S)$ for the number of connected components. For a connected component $S' \subset S$, we denote the set of white vertices by $V_w(S')$ which we call the set of *boundaries*. Let
$$h(S') = |V_w(S')|$$
be the number of *boundaries*. For the unique black vertex v of S', we define
$$\mathsf{l}(S') = \mathsf{l}(v) \,, \quad \vec{d}(S') = \vec{d}(v) \,, \quad g_s(S') = g_s(v)$$
which we call the *boundary labels, total degree* and *small genus*, respectively, of the component. The *genus* of S' is defined via (1) to be $g(S') = 2g_s(S') + h(S') - 1$. The *boundary degrees* of S' is the multiset $d|_{V_w(S')}$.
(b) The sets $\mathsf{l}(S')$ for different connected components are pairwise disjoint.
(c) The stability condition for every connected component S' holds:
$$3\left(d^+(S') + d^-(S')\right) + 2g_s(S') + 2\left|\mathsf{l}(S')\right| + h(S') > 2.$$
(d) For every connected component S', we have
$$d^+(S') - \sum_{\{w \in V_w(S') | d(w) > 0\}} d(w) = d^-(S') + \sum_{\{w \in V_w(S') | d(w) < 0\}} d(w) \geq 0.$$
In particular, $d^+(S') - d^-(S') = \sum_{w \in V_w(S')} d(w)$.

An isomorphism between two moduli specifications S_1 and S_2 is a graph isomorphism which respects the decorations. An automorphism is similarly defined, and we write $\mathrm{Aut}(S)$ for the automorphism group of S. A moduli specification S is *connected* if $n(S) = 1$. It is *closed* if for every connected component $S' \subset S$, we have $h(S') = 0$.

There is an obvious way to read the data of a moduli specification off of a holomorphic map
$$(\Sigma, \partial\Sigma) \to \left(\mathbb{CP}^1, \mathbb{RP}^1\right)$$
from a Riemann surface with boundary Σ. We associate a black vertex for each connected component of Σ, a white vertex for each connected component of the boundary, and an edge from a white vertex to a black vertex if the corresponding boundary belongs to the corresponding connected component. The functions $\vec{d}, \mathsf{l}, g_s$ are the total degree, internal labels and small genus of the connected component which corresponds to the vertex, and d is the degree of the corresponding

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boundary. The degree d is with respect to the orientation on $\partial\Sigma$ induced from Σ, and the chosen orientation of $\mathbb{RP}^1$.

A moduli specification S is

- A *disk moduli specification* if $n(S) = 1$, $g_s(S) = 0$, and $h(S) = 1$. In case we consider such a moduli specification, we denote it simply by $(\mathsf{I}, \vec{d})$, omitting the rest of the notation.
- *sphere moduli specification* if $n(S) = 1$, $g_s(S) = 0$, and $h(S) = 0$. In case we consider such a moduli specification, we denote it simply by (I, d), where $d = |\vec{d}(S)|$ is the usual degree of a map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$.
- *disk-cover moduli specification* if for every black vertex v, $\vec{d}(v)$ is of the form $(d, 0)$, or for every black vertex v, $\vec{d}(v)$ is of the form $(0, d)$.

We denote the set of all disk moduli specifications with $\mathsf{I} \subseteq \mathbb{N}$ by $\mathcal{D}$ and the set of all sphere moduli specifications with $\mathsf{I} \subseteq \mathbb{N}$ by $\mathcal{S}$.

1.5.2. *Fixed-point graphs.*

Definition 1.3. A *fixed-point graph* Γ is a tuple $\Gamma = (V \sqcup V_\circ, F, \delta, \gamma, \mu, \lambda)$, where

   (i) $V \sqcup V_\circ$ is a set of *vertices*.
  (ii) F is the set of *hemisphere edges*.
 (iii) $\delta : V_\circ \to \mathbb{Z}_{\geq 0}^2 \setminus \{(0,0)\}$ is the *degree function*.
  (iv) $\gamma : V \to \mathbb{Z}_{\geq 0}$ is the *(vertex) genus function*.
   (v) $\mu : V \to \{\pm 1\}$ is the *(vertex) map function*.
  (vi) $\lambda : V \to 2^\Upsilon$ is the association of *vertex labels*.

The data is required to satisfy the following conditions:

(a) Every hemisphere edge is incident to one vertex of V and one vertex of $V_\circ$.
(b) Every vertex $v \in V_\circ$ is incident to either one hemisphere edge or two. In the former case, v is called a *boundary*, otherwise v is called an *equator*.
(c) $\delta(v) = \{(d,d) \,|\, d \in \mathbb{Z}_{>0}\}$ for an equator and $\delta(v) \in \{(d,0), (0,d) \,|\, d \in \mathbb{Z}_{>0}\}$ for a boundary.
(d) $\mu(v_1) \neq \mu(v_2)$ whenever v_1, v_2 are neighbors of the same equator.
(e) For $v \neq u \in V$, the set $\lambda(v)$ is finite and $\lambda(u) \cap \lambda(v) = \varnothing$.

There is an equivalence relation on F defined by declaring two hemisphere edges to be equivalent if they are incident to the same element of $V_\circ$. Equivalence classes of this relation which are of size one are called *disk edges*. We denote their set by H and consider H as a subset of F. The remaining equivalence classes are of size 2 and are called *sphere edges*. We denote their set by E. The set of sphere edges is in bijection with equators, and the set of disk edges is in bijection with boundaries.

The *contraction* of Γ is the moduli specification S obtained as follows. Associate a black vertex v' to every connected component Γ' of Γ. Define

$$g_s(v') = h_1(\Gamma') + \sum_{v \in V(\Gamma')} \gamma(v), \quad \mathsf{I}(v') = \bigsqcup_{v \in V(\Gamma')} \lambda(v), \quad \vec{d}(v') = \sum_{u \in V_\circ(\Gamma')} \delta(u).$$

Associate a white vertex v_h for any disk edge h' of Γ', connect it to v', and set $d(v_h) = \partial^H(\delta(h))$. A fixed-point graph Γ is said to be of type S, for a moduli specification S, if its contraction yields S.

Remark 1.4. Every $f \in F$ is incident to a unique $v \in V$, which we denote by $v(f)$. We extend the domain of μ to include F by setting $\mu(f) = \mu(v(f))$. We can also define degrees for elements of $F \sqcup E$. For $e \in E$ we define $\delta(e) = \delta(v_\circ)$, where $v_\circ$ is the equator of e. For $f \in F$ if $\mu(f) = +1$ we put $\delta(f) = (|\delta(v_\circ)|, 0)$, where $v_\circ$ is the unique vertex in $V_\circ$ incident to f, otherwise we set $\delta(f) = (0, |\delta(v_\circ)|)$.
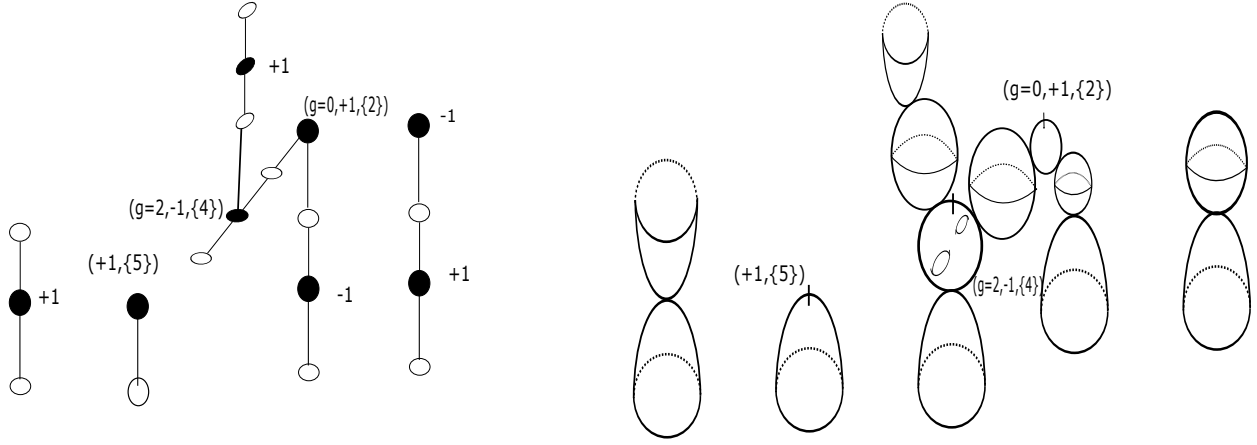
FIGURE 1. Fixed points graphs and the corresponding geometric strata. For the purpose of illustration, we have drawn degree 0 components as closed curves rather than points. Empty vertices stand for boundary and equator vertices. Next to the remaining vertices we write their genus, if they correspond to contracted component, μ and the markings.

Given an S^1-invariant stable open map $(\Sigma, \partial\Sigma) \xrightarrow{f} (\mathbb{CP}^1, \mathbb{RP}^1)$, we can read off its fixed-point graph Γ as follows.

- A sphere edge $e \in E$ corresponds to a rational irreducible component $\mathbb{CP}^1 \simeq C \subset \Sigma$ with $f|_C$ given by the degree d cover of $\mathbb{CP}^1$ branched at $p_\pm$ and $\delta(e) = (d, d)$.
- A disk edge $h \in H$ corresponds to a disk component $D^2 \simeq C \subset \Sigma$ with $f|_C$ given by the standard degree d cover of either the northern or southern hemisphere (branched at the pole). And $\delta(h)$ is $(d, 0)$ or $(0, d)$, accordingly.
- Vertices correspond either to a contracted component of the domain or to a single point of the domain. Either way, f maps such components to $p_\pm$ according to $\mu(v) = \pm 1$.

Remark 1.5. E can be naturally thought of as the edge set of a graph with vertices V, and H can be thought of as half-edges, or tails of such a graph. From the data of $V, E, H, \delta|_{E \sqcup H}$, we can reconstruct $F, V_\circ, \delta$.

We include $F, V_\circ$ in the definition because later, in Section 5, where we will have to work with more complicated graphs, the addition of $F, V_\circ$ will simplify definitions. However, we will sometimes denote a fixed-point graph by the more standard notation $(V, E, H, \delta|_{E \sqcup H}, \gamma, \mu, \lambda)$.

An *isomorphism* of fixed-point graphs

$$(V, E, H, \delta, \gamma, \mu, \lambda) \to (V', E', H', \delta', \gamma', \mu', \lambda')$$

is a triple of bijections $V \to V', E \to E', H \to H'$ that preserve the rest of the structure. Let $\mathrm{Aut}(\Gamma)$ be the finite group automorphisms of the fixed-point graph Γ.

We define the finite group of *covering automorphisms*,

$$A_\Gamma^0 = \prod_{e \in E} \mathbb{Z}/|\delta(e)| \times \prod_{h \in H} \mathbb{Z}/|\delta(h)|.$$

We define the finite group A_Γ of *geometric automorphisms*, as the semidirect product

$$1 \to A_\Gamma^0 \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1,$$

where $\mathrm{Aut}(\Gamma)$ acts on A_Γ^0 by permuting the hemisphere edges in the obvious way.

For $v \in V(\Gamma)$, we let

$$\mathrm{val}(v) = \{f \in F(\Gamma) \,|\, v(f) = v\}$$

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denote the *set* of incident hemisphere edges (so $|\mathrm{val}\,(v)|$ is the usual "valency" of v in the graph). We write
$$\mathrm{val}_+ (v) = \mathrm{val}\,(v) \sqcup \lambda\,(v)\,.$$
We set
$$V_-\,(\Gamma) = \{v \in V\,(\Gamma) \mid (\gamma\,(v), |\mathrm{val}_+\,(v)|) \in \{(0,0)\,,(0,1)\,,(0,2)\,,(1,0)\}\}$$
and $V_+\,(\Gamma) = V\,(\Gamma)\backslash V_-\,(\Gamma)$. $v \in V_+\,(\Gamma)$ corresponds to a contracted component of the domain, while $v \in V_-\,(\Gamma)$ corresponds to a single point of the domain.

We let
$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V_+(\Gamma)} \overline{\mathcal{M}}_{\gamma(v),\mathrm{val}_+(v)}.$$

Observation 1.6. Given an S^1-invariant stable open map $(\Sigma, \partial\Sigma) \xrightarrow{f} (\mathbb{CP}^1, \mathbb{RP}^1)$, which corresponds to a fixed-point graph Γ as above, the isotropy group of f can be identified with the group of geometric automorphisms A_Γ. The space of all stable open maps corresponding to Γ is isomorphic to the stack quotient of $\overline{\mathcal{M}}_\Gamma$ by the group A_Γ.

1.5.3. *Contribution of a fixed-point graph.* Let Γ be a fixed-point graph. We construct vector bundles on $\overline{\mathcal{M}}_\Gamma$ by pulling back the Hodge and cotangent bundles along the projections $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\gamma(v),\mathrm{val}_+(v)}$. First, for $v \in V_+\,(\Gamma)$, we let $\mathbb{E}_v^\vee$ denote the pullback of the dual of the Hodge bundle on $\overline{\mathcal{M}}_{\gamma(v),\mathrm{val}_+(v)}$. The corresponding Chern polynomial is denoted
$$c\,(\mathbb{E}_v^\vee) \in H^\bullet\left(\overline{\mathcal{M}}_\Gamma\right)[t]\,.$$
For $f \in F\,(\Gamma)$ with $v\,(f) = v \in V_+\,(\Gamma)$, we can identify f with a marking on $\overline{\mathcal{M}}_{\gamma(v),\mathrm{val}_+(v)}$, and we then let $\mathbb{L}_f$ denote the pullback of the corresponding cotangent line bundle. We denote the first Chern class of $\mathbb{L}_f$ by
$$\psi_f = c_1\,(\mathbb{L}_f) \in H^\bullet\left(\overline{\mathcal{M}}_\Gamma\right).$$

Our invariants are (a priori) defined over the localized equivariant cohomology of a point $\mathbb{Q}\,[u, u^{-1}]$, where u is a generator of degree +2. Let
$$\omega_f = \mu\,(f)\,\frac{2u}{|\delta\,(f)|}.$$
We define a class $e_\Gamma^{-1} \in H^\bullet\left(\overline{\mathcal{M}}_\Gamma; \mathbb{Q}\,[u, u^{-1}]\right)$ by

$$(6) \qquad e_\Gamma^{-1} = \prod_{\{f \in F(\Gamma)\,|\,v(f) \in V_+(\Gamma)\}} \frac{1}{\omega_f - \psi_f}\mu\,(f)\,(2u) \prod_{v \in V_+(\Gamma)} c\,(\mathbb{E}_v^\vee)\left(t = \frac{\mu\,(v)}{2u}\right)(2u)^{\gamma(v)-1}$$

$$\times \prod_{\{v \in V_-(\Gamma)\,|\,\mathrm{val}_+(v)=\{f_1,f_2\}\}} \frac{1}{\omega_{f_1} + \omega_{f_2}} \prod_{\{v \in V_-(\Gamma)\,|\,\mathrm{val}_+(v)=\{f\}\}} \omega_f$$

$$\times \prod_{e \in E} \frac{(-1)^{|\delta(e)|}\,|\delta\,(e)|^{2|\delta(e)|}}{(|\delta\,(e)|!)^2\,(2u)^{2|\delta(e)|}} \prod_{h \in H} \mu(h)^{|\delta(h)|+1}\,\frac{(|\delta\,(h)|)^{|\delta(h)|}}{|\delta\,(h)|!\,(2u)^{|\delta(h)|}}.$$

We define a class $\alpha_\Gamma^{\vec{a},\vec{\epsilon}} \in H^\bullet\left(\overline{\mathcal{M}}_\Gamma; \mathbb{Q}\,[u, u^{-1}]\right)$ to be 0 if $\epsilon_i \neq \mu(v)$, for some $v \in V_+ \cup V_-$ and $i \in \lambda(v)$, and otherwise by

$$(7) \qquad \alpha_\Gamma^{\vec{a},\vec{\epsilon}} = \left(\prod_{v \in V_+(\Gamma)} \prod_{i \in \lambda(v)} \psi_i^{a_i}\mu\,(v)\,2u\right)\left(\prod_{\substack{v \in V_-(\Gamma), \\ \lambda(v)=\{i\}, \mathrm{val}(v)=\{f\}}} (-\omega_f)^{a_i}\,\mu\,(v)\,2u\right).$$

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Remark 1.7. In all genera, e_Γ^{-1} is conjectured to be the inverse to the equivariant Euler class of the virtual normal bundle to the fixed-point component $\mathcal{F}_\Gamma$ parameterized by Γ. In genus 0, which is what we consider in all sections except Section 5, e_Γ^{-1} is as claimed. Moreover, in genus 0, the second term in (6) is simplified to $\prod_{v \in V_+(\Gamma)} (2u)^{-1} = (2u)^{-|V_+(\Gamma)|}$. The class $\alpha_\Gamma^{\vec{a},\vec{\epsilon}}$ should be the pullback of the integrand, a product of stationary descendents as in (4), to $\mathcal{F}_\Gamma$.

Definition 1.8. Let S be a moduli specification, $\vec{a}, \vec{\epsilon}$ vectors as in Notation 1.1. The *fixed-point contribution* $I(S, \vec{a}, \vec{\epsilon})$ *associated to* $S, \vec{a}, \vec{\epsilon}$, is defined by the equation

$$(8) \qquad \frac{I(S, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(S)|} = \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|} I(\Gamma, \vec{a}, \vec{\epsilon}),$$

where the sum ranges over all isomorphism types of fixed-point graphs Γ corresponding to S, and

$$(9) \qquad I(\Gamma, \vec{a}, \vec{\epsilon}) = \frac{1}{|A_\Gamma^0|} \int_{\overline{\mathcal{M}}_\Gamma} e_\Gamma^{-1} \cdot \alpha_\Gamma^{\vec{a},\vec{\epsilon}}$$

is the *fixed-point contribution of* Γ. In case S is a disk moduli specification $(\mathsf{l}, \vec{d})$ or sphere moduli specification (l, d), we write $I(\mathsf{l}, \vec{d}, \vec{a}, \vec{\epsilon})$, $I(\mathsf{l}, d, \vec{a}, \vec{\epsilon})$ respectively for the associated fixed-point contributions.

For later convenience we set $I(S, \vec{a}, \vec{\epsilon})$ and $I(\Gamma, \vec{a}, \vec{\epsilon})$ to be 0 if some $a_i < 0$.

1.5.4. *Genus* 0 *equivariant localization formula.* We can now state our result in genus 0.

Definition 1.9. For a disk moduli specification $(\mathsf{l}, \vec{d})$ denote by $\mathcal{T}(\mathsf{l}, \vec{d})$ the set of trees T such that each vertex v is decorated by a disk moduli specification $(\mathsf{l}_v, \vec{d}_v)$ such that

$$\bigsqcup_{v \in V(T)} \mathsf{l}_v = \mathsf{l}, \qquad \sum_{v \in V(T)} \vec{d}_v = \vec{d}.$$

An *automorphism* of an element of $\mathcal{T}(\mathsf{l}, \vec{d})$ is an automorphism of the underlying tree which preserves the moduli specifications of the vertices. Let $\mathrm{Aut}(T)$ denote the automorphism group of T.

The *amplitude* of T is defined by

$$(10) \qquad \mathcal{A}(T, \vec{a}, \vec{\epsilon}) = \left(\frac{(-2u)^{-|E(T)|}}{|\mathrm{Aut}(T)|} \prod_{v \in V(T)} (\vec{d}^+(v) - \vec{d}^-(v))^{\mathrm{val}(v)} \right) I(S(T), \vec{a}, \vec{\epsilon}),$$

where $V(T)$, $E(T)$ are the vertex and edge sets of T, $\mathrm{val}(v)$ is the number of neighbors of v in T, and $S(T) = \{(\mathsf{l}_v, \vec{d}_v)\}_{v \in V(T)}$ is the moduli specification associated to T. [1]
Define[2]

$$(11) \qquad \mathrm{OGW}(\mathsf{l}, \vec{d}, \vec{a}, \vec{\epsilon}) = \sum_{T \in \mathcal{T}(\mathsf{l}, \vec{d})} \mathcal{A}(T, \vec{a}, \vec{\epsilon}) + \delta_{d^+ - d^-} \frac{|\vec{d}|}{2u} I(\mathsf{l}, |\vec{d}|, \vec{a}, \vec{\epsilon}).$$

The main result of the paper is the following theorem, see Theorem 2.18 for the precise statement of the boundary conditions.

Theorem. *The open equivariant stationary descendent* $\langle \prod \tau_{a_i}^{\epsilon_i} \rangle_{0,\vec{d}}$ *defined via coherent boundary conditions equals* $\mathrm{OGW}(\mathsf{l}, \vec{d}, \vec{a}, \vec{\epsilon})$. *In particular,* (11) *vanishes if* $1 + \sum a_i < d^+ + d^-$.

---

[1]If we denote by $\vec{a}|_{\mathsf{l}_v}, \vec{\epsilon}|_{\mathsf{l}_v}$ the l_v–components of $\vec{a}, \vec{\epsilon}$ respectively, then $I(S(T), \vec{a}, \vec{\epsilon}) = \prod_{v \in V(T)} I(\mathsf{l}_v, \vec{d}_v, \vec{a}|_{\mathsf{l}_v}, \vec{\epsilon}|_{\mathsf{l}_v})$.

[2]The dependence on l is only notational. (11) depends only on the degree $\vec{d}$, vector of descendents $\vec{a}$ and the points constraints $\vec{\epsilon}$, and is by definition permutation invariant.
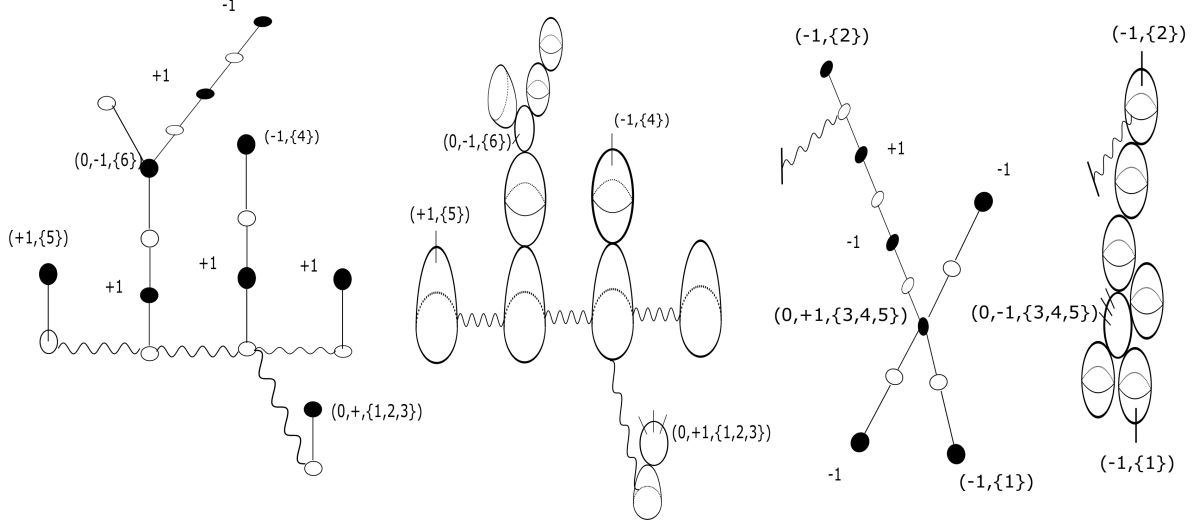
FIGURE 2. Localization graphs which appear in the formula for genus 0 invariants. We draw a graph and on its right the corresponding geometric picture. Again for illustration reasons we do not contract degree 0 components, and again next to a contracted component vertex we write the genus, μ and the markings, while next to other vertices we write the labels and μ.

1.6. **Explicit formulas and recursion.** In the case of disk covers, we have the following explicit formula for the open stationary descendents.

Theorem 1.10. *Suppose $a_i \geq 0$ and $1 + \sum_{i=1}^n a_i = d$, then*

$$(12) \qquad \left\langle \prod_{i=1}^n \tau_{a_i}^+ \right\rangle_{0,(d,0)} = \frac{(1 + \sum a_i)^{n-2}}{\prod a_i!}.$$

An additional nice formula is the vanishing of intersection numbers for $\vec{d} = (d, d)$, see Lemma 4.4 below. These formulas will play a significant role in the sequel.

By analogy to the closed GW theory of $\mathbb{CP}^1$, we have divisor equations and topological recursions (compare to the TRRs of [31, 3, 4]) also in the open setting.

Theorem 1.11. *The equivariant divisor equation is*

$$\left\langle \tau_0^{\epsilon_1} \prod_{i=2}^l \tau_{a_i}^{\epsilon_i} \right\rangle_{0,\vec{d}} = d^{\epsilon_1} \left\langle \prod_{i=2}^l \tau_{a_i}^{\epsilon_i} \right\rangle_{0,\vec{d}} + (2u) \sum_{\substack{j>1, \\ \epsilon_j = \epsilon_1, \ a_j > 0}} \left\langle \tau_{a_j-1}^{\epsilon_j} \prod_{i=2, i\neq j}^l \tau_{a_i}^{\epsilon_i} \right\rangle_{0,\vec{d}}.$$

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In particular, if $1 + \sum_{i=2}^{l} a_i = d_+ + d_- > 0$, *then*

$$\left\langle \tau_0^{\epsilon_1} \prod_{i=2}^{l} \tau_{a_i}^{\epsilon_i} \right\rangle_{0,\vec{d}} = d^{\epsilon_1} \left\langle \prod_{i=2}^{l} \tau_{a_i}^{\epsilon_i} \right\rangle_{0,\vec{d}}.$$

Theorem 1.12. *Suppose* $a_i \geq 0$, *then*

(13) $$\left\langle \tau_{a_i+1}^{+} \prod_{i=2}^{l} \tau_{a_i}^{+} \right\rangle_{0,(d,0)} = \sum_{R \sqcup S = \{3,\ldots,l\}} (2u)^{|R|} \left\langle \tau_{a_1} \left(\prod_{i \in R} \tau_{a_i} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^{+} \tau_{a_2}^{+} \prod_{i \in S} \tau_{a_i}^{+} \right\rangle_{0,(d,0)} +$$

$$+ \sum_{\substack{R \sqcup S = \{3,\ldots,l\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1}^{+} \prod_{i \in R} \tau_{a_i}^{+} \right\rangle_{0,(d_1,0)} \left\langle \tau_{a_2}^{+} \prod_{i \in S} \tau_{a_i}^{+} \right\rangle_{0,(d_2,0)},$$

where $\langle \cdots \rangle_0^c$ *stands for the closed descendent integrals over the Deligne-Mumford moduli space* $\overline{\mathcal{M}}_{0,n}$. *In particular, in the non-equivariant limit,* $2 + \sum_{i=1}^{l} a_i = d$, *we have*

(14) $$\left\langle \tau_{a_i+1}^{+} \prod_{i=2}^{l} \tau_{a_i}^{+} \right\rangle_{0,(d,0)} = \sum_{\substack{R \sqcup S = \{3,\ldots,l\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1}^{+} \prod_{i \in R} \tau_{a_i}^{+} \right\rangle_{0,(d_1,0)} \left\langle \tau_{a_2}^{+} \prod_{i \in S} \tau_{a_i}^{+} \right\rangle_{0,(d_2,0)},$$

where in the sum only non-equivariant intersection numbers appear.

Remark 1.13. There is also a recursion for sphere covers which involves non-stationary insertions and will be handled in a future paper.

1.7. **Main conjecture.** In Section 5, we propose a formula, motivated by equivariant localization, for all higher genus open stationary descendents which generalizes in a non-trivial way the proven formula for disks. Our main conjecture, Conjecture 1 of Section 5.5, is that our formula is geometric: *the open invariants in all higher genus can be defined geometrically via coherent boundary condition, and the resulting numbers agree with our high genus proposal.*

1.8. **Structure of the paper.** In Section 2, we define the relevant moduli spaces, tautological line bundles, and the key objects for our calculations and definitions – coherent integrands. We also explain how to obtain coherent integrands from the tautological line bundles. In Section 3, we prove the fixed-point localization formula and prove an algorithm for integrating coherent integrands. In Section 4, we prove the main results: Theorems 2.18, 1.12, and 1.10. In Section 5, we define the higher genus intersection numbers via localization. In Section 6, we construct and analyze the canonical orientation for the moduli spaces.

1.9. **Sequel.** The sequel to this paper will be devoted to the higher genus theory and to the open Gromov-Witten/Hurwitz correspondence:

- We will sketch a proof, modulo missing foundations, of the higher genus formula and provide more evidence for its correctness, including a genus 1 recursion. In addition, we will prove a *map decomposition* theorem which will allow us, in particular, to write all closed Gromov-Witten invariants of $\mathbb{CP}^1$ in terms of the (conjectural higher genus) *disk cover* invariants defined in Section 5.
- We will define *open Hurwitz theory*, which is a natural generalization of classical Hurwitz theory to surfaces with boundary, and prove that the genus 0 Gromov-Witten theory of disk covers satisfies an open GW/H correspondence, using the same completed cycles as in [28]. In fact, assuming a weak form of Conjecture 1, we will prove the open GW/H result in all genera.

2. Moduli of stable disk-maps and stationary descendent integrals

The goal of this section is to define the stationary descendent integrals over the moduli space of stable disk-maps. We will first review the basics of equivariant localization, and continue to define stable maps to $(\mathbb{CP}^1, \mathbb{RP}^1)$, the moduli of stable genus 0 maps to $(\mathbb{CP}^1, \mathbb{RP}^1)$, the tautological line bundles and the natural S^1 action. We will then define the notion of *coherent integrand* and show how to obtain a coherent integrand out of the tautological line bundles. We shall end by defining stationary descendent integrals over the moduli space of stable disk-maps, and stating the main theorem, Theorem 2.18.

2.1. S^1**–orbifolds and equivariant forms.** All our orbifolds will have corners unless explicitly mentioned otherwise. We refer the reader to [25] Section 3 for a precise definition of this and related differential-geometric notions (differential forms, vector bundles, various types of maps, etc.) which we will use without further discussion.

An S^1-orbifold $\mathcal{Y}$ is an orbifold with corners which is equipped with an S^1 action. Let ξ denote the vector field generating the action, and consider the differential graded algebra of *equivariant forms*

$$\mathcal{A}_{S^1}(\mathcal{Y}) = \left(\Omega(\mathcal{Y}; \mathbb{R}[u])^{S^1}, D = d - u\iota_\xi, \cdot \right)$$

consisting of S^1-invariant differential forms on $\mathcal{Y}$ with values in the graded polynomial algebra $\mathbb{R}[u], \deg u = 2$. The differential is a deformation of the exterior derivative by contraction with ξ, and the product is the usual exterior product of forms (taking values in an evenly-graded ring). The grading is the total grading, coming from the De-Rham degree and the grading of $\mathbb{R}[u]$. The homology of this differential graded algebra is isomorphic to the cohomology of the homotopy quotient of $\mathcal{Y}$ by the S^1 action, also known as the *equivariant cohomology* of $\mathcal{Y}$, see [1].

The algebra of *Laurent equivariant forms* $\mathcal{A}_{S^1}(\mathcal{Y})[u^{-1}]$ is the localization of $\mathcal{A}_{S^1}(\mathcal{Y})$ by the multiplicative subset $\{1, u, u^2, ...\}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is an equivariant map, the usual pullback of forms restricts to a map $f^* : \mathcal{A}_{S^1}(\mathcal{Y}) \to \mathcal{A}_{S^1}(\mathcal{X})$. If f is an equivariant proper and oriented submersion then we can consider the usual pushforward of forms (defined by integration over the fiber) as a map

$$f_* : \mathcal{A}_{S^1}(\mathcal{X}) \to \mathcal{A}_{S^1}(\mathcal{Y}).$$

Our convention for the pushforward is such that

(15) $$f_*\left(f^*\alpha \cdot \beta \right) = \alpha \cdot f_*\beta$$

and
$$g_* \circ f_* = (-1)^{\deg g \cdot \deg f} (g \circ f)_* .$$
The induced orientation on $\partial \mathcal{X}$ is defined so that the orientation of $\mathcal{X}$ is recovered by appending an outward normal vector to the beginning of an oriented base for $\partial \mathcal{X}$, so that
$$(f_* D - D f_*) \alpha = (f_* d - d f_*) \alpha = (-1)^{\deg \alpha + \dim \mathcal{Y} - \dim \mathcal{X}} (f|_{\partial \mathcal{X}})_* \alpha.$$

In particular, if $\mathcal{X}$ is a compact and oriented orbifold with corners, then the pushforward to a point gives an integration map
$$\int : \mathcal{A}_{S^1}(\mathcal{X}) \to \mathbb{R}[u],$$
satisfying Stokes' theorem
$$\int_{\mathcal{X}} D\alpha = \int_{\partial \mathcal{X}} \alpha.$$
Since all of these operations are $\mathbb{R}[u]$-linear they extend to Laurent equivariant forms as well.

Definition 2.1. A Laurent equivariant form $Q \in \mathcal{A}_{S^1}(\mathcal{Y})[u^{-1}]$ is called *an equivariant primitive* (or just *a primitive*) if $DQ = 1$, the unit of $\mathcal{A}_{S^1}(\mathcal{Y})[u^{-1}]$.

If $\mathcal{Y}$ has no fixed points, $\mathcal{Y}^{S^1} = \varnothing$, then $Q = \frac{\eta}{D\eta}$ is a primitive where $\eta = g(\xi, -)$ is the dual to ξ with respect to any S^1-invariant Riemannian metric g on $\mathcal{Y}$.

Definition 2.2. An S^1-orbifold $\mathcal{X}$ is said to have *simple fixed points* if $\partial \mathcal{X} \cap \mathcal{X}^{S^1} = \varnothing$.

If $\mathcal{X}$ has simple fixed points, then $\mathcal{X}^{S^1}$ is represented by a disjoint union of orbifolds without boundary (not necessarily of the same dimension) together with a closed embedding $\mathcal{X}^{S^1} \to \mathcal{X}$ (see [23, Definition 36]) and for every $c \geq 1$ the orbifold with corners $\mathcal{Y} = \partial^c \mathcal{X}$ is fixed point free. All the orbifolds we will consider here have simple fixed points.

If $F \subset \mathcal{X}^{S^1}$ is a connected component, we denote by N_F the normal bundle to $F \to \mathcal{X}$ and let
$$e_{S^1}(N_F) \in \mathcal{A}_{S^1}(F)^{\times} \subset \mathcal{A}_{S^1}(F)$$
denote an associated equivariant Euler form (see the proof of Lemma 3.5 below).

2.2. **Stable maps to** $(\mathbb{CP}^1, \mathbb{RP}^1)$. In this section we briefly recall the definition of stable maps to $(\mathbb{CP}^1, \mathbb{RP}^1)$, for a definition in the more general case of a target pair (X, L), see [21].

Definition 2.3. A *Riemann surface with boundary* Σ is the result of removing finitely many disjoint extendably embedded open disks from a compact Riemann surface. Given disjoint finite sets $\mathsf{k}, \mathsf{l} \subset \Upsilon$, a (k, l)–*smooth marked Riemann surface (with boundary)* is a triple $(\Sigma, \{x_i\}_{i \in \mathsf{k}}, \{z_i\}_{i \in \mathsf{l}})$ where

(a) Σ is a Riemann surface (with boundary).
(b) $\{x_i\}_{i \in \mathsf{k}}$ are distinct boundary points.
(c) $\{z_i\}_{i \in \mathsf{l}}$ are distinct internal points.

The points x_i are called boundary markings, the points z_i are the internal markings. We will sometimes omit the marked points and (k, l) from our notations; we will sometimes identify a marking with its label from $\mathsf{k} \cup \mathsf{l}$.

The *genus* $g(\Sigma)$ of a connected marked smooth Riemann surface with boundary Σ is defined as the genus of the doubled surface $\Sigma_{\mathbb{C}}$, if $\partial \Sigma \neq \varnothing$, and as the usual genus otherwise. Σ is *stable* if its automorphism group is finite.

When Σ is connected and $\partial \Sigma = \varnothing$ stability is equivalent to $2g(\Sigma) + |\mathsf{l}| \geq 3$, while when $\partial \Sigma \neq \varnothing$ stability is equivalent to $2g(\Sigma) + 2|\mathsf{l}| + |\mathsf{k}| \geq 3$. When Σ is not connected, then stability is equivalent to the stability of all the connected components.

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Definition 2.4. A (k,l)–*pre-stable marked surface with boundary* is a tuple $\Sigma = ((\Sigma_\alpha)_{\alpha\in C\sqcup O}, \sim_B$ $,\sim_I, \mathcal{CB})$, where

(a) C, O are finite sets. For $\alpha \in C$ Σ_α is a smooth marked Riemann surface without boundary, for $\alpha \in O$ Σ_α is a smooth marked Riemann surface with boundary.
(b) An equivalence relation $\sim_B$ on the set of all boundary marked points, with equivalence classes of size at most 2. An equivalence relation $\sim_I$ on the set of all internal marked points, with equivalence classes of size at most 2.
(c) A subset $\mathcal{CB}$ of the equivalence classes of size 1 of $\sim_I$.

We require that k is the set of labels of points belonging to $\sim_B$ –equivalence classes of size 1. I is the set of labels of points belonging to $\sim_I$ equivalence classes of size 1 which do not lie in $\mathcal{CB}$.

The two equivalence relations $\sim_B, \sim_I$ taken together are denoted by $\sim$. Equivalence classes of $\sim$ $(\sim_B, \sim_I)$ of size 2 are called *nodes* (*boundary nodes, internal nodes*). Elements of $\mathcal{CB}$ are called *contracted boundaries*.

We identify Σ with the topological space $\left(\bigsqcup_\alpha \Sigma_\alpha\right)/\sim$.

The *doubling* $\Sigma_\mathbb{C}$ of Σ is the pre-stable marked surface without boundary obtained by gluing Σ and an isomorphic copy of it $\overline{\Sigma}$ with an opposite complex structure along the common boundary (using Schwartz reflection), and then taking the topological quotient which identifies a point $z \in \mathcal{CB} \subset \Sigma$ with its partner in $\overline{\Sigma}$. The *genus* of a connected pre-stable marked surface with non-empty boundary or with contracted boundaries is defined as the genus of the doubled (pre-stable) surface. If there are no boundaries or contracted boundaries, the genus is taken to be the usual genus of a pre-stable surface.

The *normalization of* Σ is the pair $(\widehat{\Sigma}, q)$, where $\widehat{\Sigma} = \bigsqcup_\alpha \Sigma_\alpha$, and $q : \widehat{\Sigma} \to \Sigma$ is the quotient map. Σ is stable if each Σ_α is stable.

Note that Σ is smooth precisely if all the $\sim$ –equivalence classes are of size 1 and $\mathcal{CB} = \varnothing$. We will usually omit $\sim_B, \sim_I$ and $\mathcal{CB}$ from the notations.

Definition 2.5. A *stable map* $u : \Sigma \to (\mathbb{CP}^1, \mathbb{RP}^1)$, where $\Sigma = (\{\Sigma_\alpha\}, \sim_B, \sim_I, \mathcal{CB})$ is a pre-stable marked surface with boundary, is a map u which satisfies

(a) The restrictions $u|_{\Sigma_\alpha}$ are smooth, holomorphic on the interior of Σ_α, and map $\partial\Sigma_\alpha$ to $\mathbb{RP}^1$.
(b) If $x \sim y$ then $u(x) = u(y)$; equivalently, u descends to a (continuous) map from Σ to $\mathbb{CP}^1$.
(c) $u(\mathcal{CB}) \subset \mathbb{RP}^1$.
(d) If some Σ_α is mapped to a point, then Σ_α is stable.

u is said to be *smooth* if Σ is smooth. It is said to be a *disk map* if $g(\Sigma) = 0$ and Σ is connected with either $\partial\Sigma \neq \varnothing$ or $\mathcal{CB} \neq \varnothing$.

The *degree* of u is defined as $u_*[\Sigma, \partial\Sigma] \in H_2(\mathbb{CP}^1, \mathbb{RP}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$. u is a *disk cover* if its degree is either of the form $(d, 0)$ or of the form $(0, d)$.

The notions of isomorphisms and automorphisms between stable maps are the expected ones.

Throughout most of this paper we will consider only disk maps. In addition, we will consider mainly internal markings; boundary markings will appear through normalization of boundary nodes.

2.3. **Moduli space, its corners and the tautological line bundles.** In this section we discuss the moduli spaces of stable disk maps, the structure of the corners, cotangent line bundles and maps between the moduli spaces. Many of the constructions appeared in [25] and [26]. For those, we will allow ourselves to be quite brief here, and the reader should consult these references for more detail.

Consider finite disjoint subsets $\mathsf{k}, \mathsf{l} \subseteq \Upsilon$ and $\vec{d} = (d^+, d^-) \in \mathbb{Z}_{\geq 0}^2 \subset H_2(\mathbb{CP}^1, \mathbb{RP}^1)$. Write $\sum \vec{d} = d^+ + d^-$ and $\mathsf{l}_{\mathbb{C}} = \mathsf{k} \sqcup (\mathsf{l} \times \{1, 2\})$. Suppose the stability condition

(16) $$|\mathsf{k}| + 2\,|\mathsf{l}| + 3 \sum \vec{d} > 2$$

holds.

Proposition 2.6. *The moduli space* $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$ *is an* S^1-*orbifold with corners admitting an* S^1-*equivariant map*

(17) $$\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big) \to \overline{\mathcal{M}}_{0,\mathsf{l}_{\mathbb{C}}}\big(\textstyle\sum \vec{d}\big).$$

Moreover, $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$ *has simple fixed points.*

Proof. The moduli map is induced from the doubling map $(\Sigma, u) \to (\Sigma_{\mathbb{C}}, u_{\mathbb{C}})$, and the first statement of the proposition is a special case of [25, Theorem 1] (see Example 3 *ibid.*). Since the induced action on $\mathbb{RP}^1 \subset \mathbb{CP}^1$ has no fixed points, the fixed points of $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$ have no special points on $\mathbb{RP}^1$. In particular, $\partial\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$ has no fixed points. $\qquad\square$

Recall the underlying groupoid of $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$ is equivalent to the groupoid whose objects are *fundamental configurations* $\sigma = ((\Sigma, \nu, \lambda, w), b, \Sigma^{1/2})$. Here $(\Sigma, \nu, \lambda, w)$ represents a point in $\overline{\mathcal{M}}_{0,\mathsf{l}_{\mathbb{C}}}\big(\sum \vec{d}\big)$, so

- Σ is a disjoint union of $\mathbb{CP}^1$'s,
- $\nu : \Sigma \to \Sigma$ is an involution whose size 2 orbits are *the nodes*,
- $\lambda : \mathsf{l}_{\mathbb{C}} \to \Sigma^\nu$ is an injective map which specifies the position of the marked points, and
- $w : \Sigma \to X$ is a holomorphic map with $w_*([\Sigma, \partial\Sigma]) = \sum \vec{d}$.

$b : \Sigma \to \Sigma$ is an anti-holomorphic involution, so that $((\Sigma, \nu, \lambda, w), b)$ represents a point in the fixed-point stack $\overline{\mathcal{M}}_{0,\mathsf{l}_{\mathbb{C}}}\big(\sum \vec{d}\big)^{\mathbb{Z}/2}$ for a suitable $\mathbb{Z}/2$ action. Finally, $\Sigma^{1/2} \subset \Sigma$ is a fundamental domain for the action of b, subject to some conditions, so $\Sigma^{1/2}/\nu$ is the domain of the stable disk-map.

Let us discuss the structure of the corners. Consider some fundamental configuration $\sigma = ((\Sigma, \nu, \lambda, w), b, \Sigma^{1/2})$ representing a point $[\sigma] \in \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}$. A *real node* of σ is an orbit $o = \{o_1, o_2\}$ of the involution ν such that $b(o) = o$. We call a node *standard* if $b(o_1) = o_1$ and *exceptional* otherwise (these are called E-type and H-type nodes in [25], following [21]). Let $\mathcal{R}(\sigma)$ denote the set of real nodes of σ. It is not hard to see that $\mathcal{R}(\sigma)$ either consists of a single exceptional node, or some number of standard nodes.

The set $\mathcal{R}(\sigma)$ is naturally identified with the set of local boundary components at $[\sigma] \in \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$, so a point $q \in \partial^c \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$ is represented by (σ, ρ) where σ is a fundamental configuration, and $\rho : \{1, ..., c\} \hookrightarrow \mathcal{R}(\sigma)$ is an injective map. Our numbering convention is such that $\big(\sigma, \rho|_{\{2,...,c\}}\big)$ represents the image of q in $\partial^{c-1}\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\big(\vec{d}\big)$.

Consider $\rho(c)$. If $\rho(c)$ is standard, we can write

$$\big((\Sigma, \nu, \lambda, w), b, \Sigma^{1/2}\big) = \Big((\Sigma_1 \sqcup \Sigma_2, \nu_1 \sqcup \nu_2, \lambda_1 \sqcup \lambda_2, w_1 \sqcup w_2), b_1 \sqcup b_2, \Sigma_1^{1/2} \cup \Sigma_2^{1/2}\Big)$$

so that

$$\sigma_i = \Big((\Sigma_i, \nu_i, \lambda_i, w_i), b_i, \Sigma_i^{1/2}\Big)$$

specifies a point $p_i \in \overline{\mathcal{M}}_{0,\mathsf{k}_i,\mathsf{l}_i}\big(\vec{d}_i\big)$ for $i = 1, 2$. This decomposition is unique up to a $\mathbb{Z}/2$ action which swaps the labels $i = 1$ and $i = 2$. We have $\mathcal{R}(\sigma) = \mathcal{R}(\sigma_1) \sqcup \mathcal{R}(\sigma_2) \sqcup \{\rho(c)\}$. Set $\kappa_i = \rho^{-1}(\mathcal{R}(\sigma_i))$, $c_i = |\kappa_i|$, and let $\rho_i = \rho|_{\kappa_i} \circ \gamma_i$ where $\gamma_i : \{1, ..., c_i\} \to \kappa_i$ is the unique order-preserving bijection. We can interpret (σ_i, ρ_i) as specifying a point of $\partial^{c_i}\overline{\mathcal{M}}_{0,\mathsf{k}_i,\mathsf{l}_i}\big(\vec{d}_i\big)$. Note that in order to reconstruct σ, we must keep track of the partition $\{1, ..., c-1\} = \kappa_1 \sqcup \kappa_2$; henceforth if $\kappa \subset \mathbb{N}$ is any finite subset,

we write $\partial^\kappa \mathcal{X}$ for a copy of $\partial^c \mathcal{X}$, $c = |\kappa|$, where the local boundary components are labeled by κ instead of $[c]$. This allows us to write

$$\partial^c (\mathcal{X} \times \mathcal{Y}) = \bigsqcup \partial^{\kappa_1} \mathcal{X} \times \partial^{\kappa_2} \mathcal{Y},$$

where the disjoint union ranges over partitions $\kappa_1 \sqcup \kappa_2 = [c]$, which specify how to merge the local boundary components of $\mathcal{X}$ and of $\mathcal{Y}$ to form a corner of the product. A similar Leibnitz rule holds for transverse fibered product of orbifolds over the manifold without boundary $L = \mathbb{RP}^1$ (cf. [16, Proposition 6.7]).

If $\rho(c)$ is an exceptional node, then $\Sigma^b = \varnothing$ and $\Sigma = \Sigma^{1/2} \sqcup b\left(\Sigma^{1/2}\right)$, $\mathsf{k} = \varnothing$ and $d_+ = d_-$.

Proposition 2.7. *We have* $\partial^c \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\left(\vec{d}\right) = \mathcal{C} \sqcup \mathcal{E}.$
The standard corners

$$\mathcal{C} = \mathcal{C}^c\left(\mathsf{k},\mathsf{l},\vec{d}\right) = \left(\bigsqcup \partial^{\kappa_1} \overline{\mathcal{M}}_{0,\mathsf{k}_1 \sqcup \star_1, \mathsf{l}_1}\left(\vec{d}_1\right) \times_L \partial^{\kappa_2} \overline{\mathcal{M}}_{0,\mathsf{k}_2 \sqcup \star_2, \mathsf{l}_2}\left(\vec{d}_2\right)\right)_{\mathbb{Z}/2}$$

are the $\mathbb{Z}/2$ quotient of the disjoint union over $\mathsf{k}_1 \sqcup \mathsf{k}_2 = \mathsf{k}$, $\mathsf{l}_1 \sqcup \mathsf{l}_2 = \mathsf{l}$, $\vec{d}_1 + \vec{d}_2 = \vec{d}$ *and* $\kappa_1 \sqcup \kappa_2 = \{1,...,c-1\}$ *such that* $(k_i + 1) + 2l_i + 3\sum \vec{d}_i > 2$ *for* $i = 1,2$, *where* $k_i = |\mathsf{k}_i|$ *and* $l_i = |\mathsf{l}_i|$. *The fibered product is over the evaluation maps* $\mathrm{ev}_{\star_1}, \mathrm{ev}_{\star_2}$ *to* $L = \mathbb{RP}^1$. *$\mathbb{Z}/2$ acts by swapping the two fibered factors.*
The exceptional boundary *is*

$$\mathcal{E} = \begin{cases} \mathrm{ev}_\star^{-1}(L) \subset \overline{\mathcal{M}}_{0,\mathsf{l}\sqcup\star}(d_+), & \text{if } d_+ = d_-, \mathsf{k} = \varnothing, \text{ and } c = 1, \\ \varnothing, & \text{otherwise}, \end{cases}$$

where $\overline{\mathcal{M}}_{0,\mathsf{l}\sqcup\star}(d_+)$ *is the moduli space of closed stable maps.*

Proof. The discussion preceding the proposition defines a bijection of groupoids of sets. Using the real gluing maps this is seen to be an equivalence of orbifolds. $\qquad \square$

The standard corners $\mathcal{C}^c\left(\mathsf{k},\mathsf{l},\vec{d}\right)$ correspond to the familiar disk bubbling in Floer theory. The exceptional boundary $\mathcal{E}$ represents points where the boundary of a disk with no special points shrinks to a point on $\mathbb{RP}^1$. See also the illustrations in 5.

The product of symmetric groups $\mathrm{Sym}(\mathsf{k}) \times \mathrm{Sym}(\mathsf{l})$ acts on $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\left(\vec{d}\right)$ by permuting the labels, and there is an $S^1 \times \mathrm{Sym}(\mathsf{k}) \times \mathrm{Sym}(\mathsf{l})$ equivariant evaluation map $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\left(\vec{d}\right) \to L^\mathsf{k} \times X^\mathsf{l}$.

If $\overline{\mathcal{M}}_{\mathsf{k}_i \sqcup \star_i, \mathsf{l}_i}\left(\vec{d}_i\right)$ are moduli spaces of disks, $i = 1,2$, we have a gluing map

$$\overline{\mathcal{M}}_{\mathsf{k}_1 \sqcup \star_1, \mathsf{l}_1}\left(\vec{d}_1\right) \,{}_{\mathrm{ev}_{\star_1}}\!\times_{\mathrm{ev}_{\star_2}} \overline{\mathcal{M}}_{\mathsf{k}_2 \sqcup \star_2, \mathsf{l}_2}\left(\vec{d}_2\right) \to \overline{\mathcal{M}}_{\mathsf{k}_1 \sqcup \mathsf{k}_2, \mathsf{l}_1 \sqcup \mathsf{l}_2}\left(\vec{d}_1 + \vec{d}_2\right)$$

obtained by taking the $\mathbb{Z}/2$-invariants of the corresponding closed gluing map. Gluing maps at a complex node can be defined similarly.

Suppose (16) holds. We have a forgetful map

$$\overline{\mathcal{M}}_{0,\mathsf{k}\sqcup\{y\},\mathsf{l}}\left(\vec{d}\right) \xrightarrow{f_y} \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\left(\vec{d}\right)$$

corresponding to forgetting a boundary marked point.

Remark 2.8. In case $\mathsf{k} = \varnothing$ and $\vec{d} = (d,d)$ this map is only *weakly smooth* over points of $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}\left(\vec{d}\right)$ with an exceptional node, see the proof of Lemma 8 in [25]. It is straightforward to extend the category of orbifolds with corners as defined there to allow such maps. Some fibered products (cf. Lemma 26 *ibid.*) may only satisfy the universal property with respect to the smaller subcategory of smooth maps, but this is sufficient for our needs. In particular, if $\star' \in \mathsf{k}'$ and $\star'' \in \mathsf{k}''$, we can use the universal property to define maps such as

$$\overline{\mathcal{M}}_{0,\mathsf{k}'\sqcup\{y\},\mathsf{l}'}\left(\vec{d}'\right) \,{}_{\mathrm{ev}_{\star'}}\!\times_{\mathrm{ev}_{\star''}} \overline{\mathcal{M}}_{0,\mathsf{k}'',\mathsf{l}''}\left(\vec{d}''\right) \xrightarrow{(f_y,\mathrm{Id})} \overline{\mathcal{M}}_{0,\mathsf{k}',\mathsf{l}'}\left(\vec{d}'\right) \,{}_{\mathrm{ev}_{\star'}}\!\times_{\mathrm{ev}_{\star''}} \overline{\mathcal{M}}_{0,\mathsf{k}'',\mathsf{l}''}\left(\vec{d}''\right),$$

since $\mathsf{k}' \neq \varnothing$ in this case.

The usual relations (e.g., associativity of gluing, and the various compatibilities of the evaluation maps, forgetful maps, gluing maps, and the group actions) are shown to hold using the corresponding results for the closed curves.

For $i \in \mathsf{l}$ we have a cotangent line bundle $\mathbb{L}_i = \mathbb{L}_i^{\mathsf{k},\mathsf{l},\vec{d}}$ on $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d})$ which is the pullback of the cotangent line bundle $\mathbb{L}_{i\times\{1\}}^{\mathsf{l}_{\mathbb{C}},\Sigma\vec{d}}$ on $\overline{\mathcal{M}}_{0,\mathsf{l}_{\mathbb{C}}}(\Sigma\vec{d})$ along (17).

Lemma 2.9. *Suppose the stability condition* (16) *holds. Then the map*

$$df_j : \left(\mathbb{L}_i^{\mathsf{k}\sqcup\{j\},\mathsf{l},\vec{d}}\right)^{\vee} \to \left(\mathbb{L}_i^{\mathsf{k},\mathsf{l},\vec{d}}\right)^{\vee}$$

is cartesian, *that is, it induces an isomorphism* $\left(\mathbb{L}_i^{\mathsf{k}\sqcup\{j\},\mathsf{l},\vec{d}}\right)^{\vee} \simeq f_j^*\left(\mathbb{L}_i^{\mathsf{k},\mathsf{l},\vec{d}}\right)^{\vee}$.

Proof. Consider some fundamental configuration $\sigma = ((\Sigma,\nu,\lambda,w),b,\Sigma^{1/2})$ representing a point of $\overline{\mathcal{M}}_{0,\mathsf{k}\sqcup\{j\},\mathsf{l}}(\vec{d})$. Forgetting the point j never makes an irreducible component of Σ which contains $i \times \{1\}$ unstable (if an irreducible component contains both j and $i \times \{1\}$, it must also contain $i \times \{2\}$, and either have positive degree or contain at least one other special point). The claim follows. $\square$

We shall sometimes omit the superscript $\mathsf{k},\mathsf{l},\vec{d}$ from the notation of $\mathbb{L}_i$.

2.4. **Coherent integrands.** Let $\alpha \in \mathcal{A}_{S^1}(\mathcal{X})$ satisfy $D\alpha = 0$. Because $\partial\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d}) \neq \varnothing$, in order to define

$$\int_{\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(d)} \alpha$$

we must introduce boundary conditions on α. In fact, we will use the recursive structure of the boundary of the moduli spaces to impose a more detailed condition which will place additional constraints on α at the corners. This is done for two reasons. First, it allows us to construct α recursively. Second, we need this detailed condition to compute the contributions of the corners $\partial^c\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(\vec{d})$ in Proposition 3.1 below. In this section we formulate these conditions precisely, and in the next two sections we will use this to define the open descendent integrals.

Consider disk specifications $(\mathsf{l}_1,\vec{d}_1)$, $(\mathsf{l}_2,\vec{d}_2)$ and $(\mathsf{l},\vec{d})$ in $\mathcal{D}$. The equation

$$(18) \qquad\qquad (\mathsf{l}_1,\vec{d}_1)\#(\mathsf{l}_2,\vec{d}_2) = (\mathsf{l},\vec{d})$$

will be taken to mean that $\mathsf{l}_1 \sqcup \mathsf{l}_2 = \mathsf{l}$ and $\vec{d}_1 + \vec{d}_2 = \vec{d}$. In this case we have a boundary component $\mathcal{B} \subset \partial\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(\vec{d})$ given by

$$(19) \qquad\qquad \mathcal{B} = \mathcal{B}\left(\mathsf{l}_1,\vec{d}_1,\mathsf{l}_2,\vec{d}_2\right) = \overline{\mathcal{M}}_{0,\star_1,\mathsf{l}_1}(\vec{d}_1) \times_L \overline{\mathcal{M}}_{0,\star_2,\mathsf{l}_2}(\vec{d}_2).$$

For $(\mathsf{l},(d,d)) \in \mathcal{D}$ we also have the exceptional boundary component

$$\mathcal{E} = \mathcal{E}(\mathsf{l},(d,d)) \subset \partial\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}((d,d)).$$

We write $\mathcal{B} \xrightarrow{i_{\mathcal{B}}} \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(\vec{d})$ and $\mathcal{E} \xrightarrow{i_{\mathcal{E}}} \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}((d,d))$ for the maps that forget the local boundary component. We may write $E|_{\mathcal{B}}, E|_{\mathcal{E}}$ instead of $i_{\mathcal{B}}^*E, i_{\mathcal{E}}^*E$.

Definition 2.10. A *coherent integrand* is a collection of equivariant forms

$$\left\{\alpha_{\mathsf{l},\vec{d}} \in \mathcal{A}_{S^1}\left(\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(\vec{d})\right)\right\}_{(\mathsf{l},\vec{d})\in\mathcal{D}} \sqcup \left\{\beta_{\mathsf{l},d} \in \mathcal{A}_{S^1}\left(\overline{\mathcal{M}}_{0,\mathsf{l}}(d)\right)\right\}_{(\mathsf{l},d)\in\mathcal{S}}.$$

We assume the degrees of all forms is even, and that the following conditions hold.

(a) $D\alpha_{\mathsf{l},\vec{d}} = 0, D\beta_{\mathsf{l},d} = 0$.
(b) $\alpha_{\mathsf{l},\vec{0}} = 0, \beta_{\mathsf{l},0} = 0$.

(c) For any $\left(\mathsf{l}_1, \vec{d}_1\right), \left(\mathsf{l}_2, \vec{d}_2\right)$ and $\left(\mathsf{l}, \vec{d}\right)$ satisfying (18) we have

(20)
$$i_{\mathcal{B}}^* \alpha_{\mathsf{l}, \vec{d}} = \left(\mathrm{pr}_1'\right)^* \alpha_{\mathsf{l}_1, \vec{d}_1} \wedge \left(\mathrm{pr}_2'\right)^* \alpha_{\mathsf{l}_2, \vec{d}_2},$$

for $\mathcal{B} = \mathcal{B}\left(\mathsf{l}_1, \vec{d}_1, \mathsf{l}_2, \vec{d}_2\right)$ and where pr_i' denotes the composition

$$\mathcal{B} \xrightarrow{\mathrm{pr}_i} \overline{\mathcal{M}}_{0, \star_i, \mathsf{l}_i}\left(\vec{d}_i\right) \to \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}_i}\left(\vec{d}_i\right)$$

of the projection with the forgetful map, if it is defined. In case the $\overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}_i}\left(\vec{d}_i\right)$ is unstable, we set the right-hand side of (20) to be zero.

(d) For any $(\mathsf{l}, (d, d)) \in \mathcal{D}$ and $\mathcal{E} \subset \partial \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}}((d, d))$ the exceptional boundary component, we have a natural identification

$$\mathcal{E} = \mathrm{ev}_\star^{-1}\left(L\right) \subset \overline{\mathcal{M}}_{0, \mathsf{l} \sqcup \star}\left(d\right)$$

and we require

(21)
$$\alpha_{\mathsf{l}, \vec{d}}|_{\mathcal{E}} = f^* \beta_{\mathsf{l}, d},$$

where f is the composition

$$\mathrm{ev}_\star^{-1}\left(L\right) \hookrightarrow \overline{\mathcal{M}}_{0, \mathsf{l} \sqcup \star}\left(d\right) \to \overline{\mathcal{M}}_{0, \mathsf{l}}\left(d\right)$$

of the inclusion followed by the forgetful map.

The map

$$\bigsqcup \overline{\mathcal{M}}_{0, \star_1, \mathsf{l}_1}\left(\vec{d}_1\right) \times_L \overline{\mathcal{M}}_{0, \star_2, \mathsf{l}_2}\left(\vec{d}_2\right) \to \mathcal{C}^1\left(\varnothing, \mathsf{l}, \vec{d}\right)$$

is $2:1$; the right hand side of (20) defines a form in the image of the pullback map since it is invariant under the $\mathbb{Z}/2$ action that swaps the two moduli factors. Here we use the assumption that $\deg \alpha_{\mathsf{l}, d}$ is even).

Remark 2.11. There is an obvious non-equivariant version of these conditions; the equivariant forms are replaced by ordinary differential forms; conditions (a)-(d) are formally the same (with d used in place of D in (a)).

2.5. Coherent integrands coming from the tautological line bundles.
The main step towards the definition of the stationary descendent integrals (4) is constructing a suitable coherent integrand. First, we recall the construction of the equivariant Chern form, following Atiyah-Bott [1, §8], focusing on the necessary changes needed to accommodate orbifolds.

Throughout this section we fix vectors $\vec{a}, \vec{\epsilon}$ as in Notation 1.1, so we may omit them from the notation.

Let $\mathcal{M}$ be an S^1-orbifold, and let $E \to \mathcal{M}$ be an equivariant complex line bundle. Let $\Gamma(E)$ denote the *sheaf* of sections of E. The S^1 action defines a map of sheaves

$$X_E : \Gamma(E) \to \Gamma(E).$$

A connection is specified by a map of sheaves

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^* \mathcal{M}),$$

and we say the connection is *equivariant* if $\nabla \circ X_E = X_{E \otimes T^* \mathcal{M}} \circ \nabla$. If ∇ is equivariant, the *equivariant Chern form* $c_1(E, \nabla) \in \mathcal{A}_{S^1}(\mathcal{M})$ can be defined locally using a generating section (but is independent of the choice of such a section), cf. [1, equation (8.8)]. We have $D c_1(E, \nabla) = 0$.

We let $\rho_\pm \in \mathcal{A}_{S^1}(\mathbb{CP}^1)$ be equivariant forms representing the Poincaré dual to $p_\pm$. We assume that the support of $\rho_\pm$ is contained in the connected component of $\mathbb{CP}^1 \backslash \mathbb{RP}^1$ containing $p_\pm$.

For every $(\mathsf{l}, d) \in \mathcal{S}$ and every $i \in \mathsf{l}$, we fix once and for all an S^1-equivariant connection $\nabla_{\mathrm{cl},i}^{\mathsf{l},d}$ for the cotangent line bundle $\mathbb{L}_{\mathrm{cl},i}^{\mathsf{l},d}$ on $\overline{\mathcal{M}}_{0,\mathsf{l}}(d)$. Set

$$\psi_{\mathrm{cl},i}^{\mathsf{l},d} = c_1\left(\mathbb{L}_{\mathrm{cl},i}^{\mathsf{l},d}, \nabla_{\mathrm{cl},i}^{\mathsf{l},d}\right),$$

(22)
$$\beta_{\mathsf{l},d} = \begin{cases} \beta_{\mathsf{l},d}^{\vec{a},\vec{\epsilon}} = \prod_{i\in\mathsf{l}}\left(\psi_{\mathrm{cl},i}^{\mathsf{l},d}\right)^{a_i} \mathrm{ev}_i^* \rho_{\epsilon_i}, & \text{if } d \neq 0, \\ \beta_{\mathsf{l},0} = 0, & \text{otherwise.} \end{cases}$$

Proposition 2.12. *There exist S^1-equivariant connections $\left\{\nabla_i^{\mathsf{l},\vec{d}}\right\}_{(\mathsf{l},\vec{d})\in\mathcal{D}, i\in\mathsf{l}}$ for the complex line bundles $\mathbb{L}_i^{\mathsf{l},\vec{d}}$ on $\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}\left(\vec{d}\right)$ such that, if we define*

$$\psi_i^{\mathsf{l},\vec{d}} = c_1\left(\mathbb{L}_i^{\mathsf{l},\vec{d}}, \nabla_i^{\mathsf{l},\vec{d}}\right),$$

(23)
$$\alpha_{\mathsf{l},\vec{d}} = \alpha_{\mathsf{l},\vec{d}}^{\vec{a},\vec{\epsilon}} = \prod_{i\in\mathsf{l}}\left(\psi_i^{\mathsf{l},\vec{d}}\right)^{a_i} \mathrm{ev}_i^* \rho_{\epsilon_i},$$

then the collection $\left\{\alpha_{\mathsf{l},\vec{d}}\right\}, \{\beta_{\mathsf{l},d}\}$ is a coherent integrand.

The proof of this proposition occupies the remainder of this section.

Let $\mathcal{X}$ be an orbifold with corners. We denote by $i : \partial\mathcal{X} \to \mathcal{X}$ and by $i_j : \partial^2\mathcal{X} \to \partial\mathcal{X}$, for $j = 1, 2$, the structure maps forgetting a local boundary component.

Lemma 2.13. *Let E be an S^1-equivariant complex line over an S^1-orbifold with corners $\mathcal{X}$. Let $K \subset \partial\mathcal{X}$ be a clopen (closed and open) component of the boundary, and let $K_2 = i_1^{-1}K \cap i_2^{-1}K$. Write $i^K : K \to \mathcal{X}$ and $i_j^K : K_2 \to K$ for the restrictions of i, i_j. Let ∇_K be an S^1-equivariant connection on $\left(i^K\right)^* E$, such that $\left(i_1^K\right)^* \nabla_K = \left(i_2^K\right)^* \nabla_K$. Then there exists an S^1-equivariant connection ∇ with $\left(i^K\right)^* \nabla = \nabla_K$.*

Proof. Fix a reference S^1-equivariant connection ∇_0 for E, so that $\nabla_K - \left(i^K\right)^* \nabla_0$ corresponds to a 1-form on K, and the problem becomes: given an S^1-invariant 1-form $\omega_K \in \Omega^1\left(K;\mathbb{C}\right)^{S^1}$ such that $\left(i_1^K\right)^* \omega_K = \left(i_2^K\right)^* \omega_K$, find an S^1-invariant 1-form $\omega \in \Omega^1\left(\mathcal{X};\mathbb{C}\right)^{S^1}$ with $\left(i^K\right)^* \omega = \omega_K$. This is an immediate generalization of [23, Lemma 82] (there, the claim is proved for the case $K = \partial\mathcal{X}$, but the proof extends *mutatis mutandis* to $K \subset \partial\mathcal{X}$ clopen). $\square$

Fix some $i \in \mathbb{N}$. We denote

$$\mathcal{D}(i) = \left\{\left(\mathsf{l}, \vec{d}\right) \in \mathcal{D} \,\middle|\, i \in \mathsf{l}, \vec{d} \neq (0,0)\right\}.$$

The equation

(24)
$$\left(\mathsf{l}, \vec{d}\right) = \left(\vec{d}_1, \mathsf{l}_1\right) \#_i \left(\vec{d}_2, \mathsf{l}_2\right)$$

will be taken to mean $\left(\mathsf{l}, \vec{d}\right) = \left(\vec{d}_1, \mathsf{l}_1\right) \# \left(\vec{d}_2, \mathsf{l}_2\right)$, $\vec{d}_1, \vec{d}_2 \neq (0,0)$ and $i \in \mathsf{l}_1$. Take a triple $\left(\mathsf{l}, \vec{d}\right), \left(\vec{d}_1, \mathsf{l}_1\right), \left(\vec{d}_2, \mathsf{l}_2\right)$ which satisfies (24), and consider

(25)
$$\mathcal{B} = \overline{\mathcal{M}}_{0,\star_1,\mathsf{l}_1}\left(\vec{d}_1\right) \times_L \overline{\mathcal{M}}_{0,\star_2,\mathsf{l}_2}\left(\vec{d}_2\right) \xrightarrow{\mathrm{pr}_1'} \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_1}\left(\vec{d}_1\right).$$

It follows from Lemma 2.9 that $d\mathrm{pr}_1'$ induces an isomorphism

(26)
$$\mathbb{L}_i^{\mathsf{l},\vec{d}}\big|_{\mathcal{B}} \simeq \left(\mathrm{pr}_1'\right)^* \mathbb{L}_i^{\mathsf{l}_1,\vec{d}_1}.$$

Next we consider the exceptional boundary $\mathcal{E} \subset \partial\overline{\mathcal{M}}_{0,\mathsf{l}}((d,d))$. We fix some precompact open subset

$$\mathrm{supp}\,\rho_\pm \subset U_\pm \Subset \mathbb{CP}^1 \backslash \mathbb{RP}^1$$

containing the support of $\rho_\pm$. Write $\mathcal{E}' = \mathcal{E} \cap \mathrm{ev}_i^{-1}(U_{\epsilon_i})$, and let $\mathcal{E}' \xrightarrow{f'} \overline{\mathcal{M}}_{0,\mathsf{l}}(d)$ denote the pullback of $\mathrm{ev}_\star^{-1}(L) \xrightarrow{f} \overline{\mathcal{M}}_{0,\mathsf{l}}(d)$ (see Definition 2.10).

Observation 2.14. df' induces an isomorphism

$$\tag{27} \mathbb{L}_i^{\mathsf{l},\vec{d}}|_{\mathcal{E}'} \simeq (f')^* \mathbb{L}_{\mathrm{cl},i}^{\mathsf{l},d}.$$

Proof. The image of $\mathcal{E}'$ in $\overline{\mathcal{M}}_{0,\mathsf{l}}(d)$ does not contain any configurations where $i, \star$ are on a degree zero component. $\qquad\square$

Definition 2.15. A collection of connections $\nabla_i^{\mathsf{l},\vec{d}}$ for $\mathbb{L}_i^{\mathsf{l},\vec{d}}$, defined for all $(\mathsf{l}, \vec{d})$ in some subset $\mathcal{D}' \subset \mathcal{D}(i)$, will be called *coherent* if:

(a) For all $(\mathsf{l}, \vec{d}), (\mathsf{l}_1, \vec{d}_1) \in \mathcal{D}'$ and $(\mathsf{l}_2, \vec{d}_2) \in \mathcal{D}$ which satisfy (24), we have

$$\tag{28} \left(i^* \nabla_i^{\mathsf{l},\vec{d}}\right)\big|_{\mathcal{B}} = (\mathrm{pr}_1')^* \nabla_i^{\mathsf{l}_1, \vec{d}_1}.$$

(b) For all $(\mathsf{l}, (d, d)) \in \mathcal{D}'$, we have

$$\tag{29} \nabla_i^{\mathsf{l},(d,d)}\big|_{\mathcal{E}'} = (f')^* \nabla_{\mathrm{cl},i}^{\mathsf{l},d}.$$

Lemma 2.16. *There exists a coherent collection of connections* $\left\{\nabla_i^{\mathsf{l},\vec{d}}\right\}$ *defined for all* $(\mathsf{l}, \vec{d}) \in \mathcal{D}(i)$.

Proof. Fix some linear order on the set $\mathcal{D}(i)$ with the property that (24) implies $(\mathsf{l}_1, \vec{d}_1) < (\mathsf{l}, \vec{d})$. For $(\mathsf{l}, \vec{d}) \in \mathcal{D}(i)$ we denote the corresponding prefix of $\mathcal{D}(i)$ by

$$\mathcal{D}_{\leq(\mathsf{l},\vec{d})} = \left\{(\mathsf{l}_1, \vec{d}_1) \in \mathcal{D}(i) \,\middle|\, (\mathsf{l}_1, \vec{d}_1) \leq (\mathsf{l}, \vec{d})\right\}.$$

We define $\mathcal{D}_{<(\mathsf{l},\vec{d})}$ similarly. We construct, by induction on $(\mathsf{l}, \vec{d}) \in \mathcal{D}(i)$, a collection of coherent connections on $\mathcal{D}_{\leq(\mathsf{l},\vec{d})}$. Suppose we have a coherent collection on $\mathcal{D}_{<(\mathsf{l},\vec{d})}$. Let

$$K \subset \partial \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(\vec{d})$$

be the clopen component of the boundary corresponding to $\mathcal{B}$ of the form (25) for $(\mathsf{l}_1, \vec{d}_1), (\mathsf{l}_2, \vec{d}_2)$ satisfying (24), as well as the exceptional boundary component (which is non-empty only if $\vec{d} = (d, d)$). Define ∇_K on $(i^K)^* \mathbb{L}_i^{\mathsf{l},\vec{d}}$ so that (28) and (29) hold (for the latter, this involves using a partition of unity to extend the pulled-back connection from $\mathcal{E}'$ to $\mathcal{E}$).

We check that $(i_1^K)^* \nabla_K = (i_2^K)^* \nabla_K$. Indeed, $\partial\mathcal{E} = \varnothing$ so there is nothing to check at the exceptional boundary. The corner components in $i_1^{-1}K \cap i_2^{-1}K$ can be written as fibered products

$$\mathcal{M}_1 \times_L \mathcal{M}_2 \times_L \mathcal{M}_3.$$

By symmetry we may assume the ith marking is on the disk parameterized by $\mathcal{M}_j$ for $j = 1$ or $j = 2$. Consider first the case $j = 1$. Let $\mathcal{M}_1^\diamond = \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_1}(\vec{d}_1)$ denote the moduli obtained from $\mathcal{M}_1$ by forgetting the boundary node, and let $\mathcal{M}_2^\diamond$ denote the moduli obtained from $\mathcal{M}_2$ by forgetting the boundary node connecting it to $\mathcal{M}_3$. The triangle

$$\mathcal{M}_1 \times_L \mathcal{M}_2 \times_L \mathcal{M}_3$$

$$\mathcal{M}_1 \times_L \mathcal{M}_2^\diamond \longrightarrow \mathcal{M}_1^\diamond$$

commutes, and so does its linearization, so the cartesian lifts of the maps to the cotangent line bundles $\mathbb{L}_i$ also form a commutative triangle. Thus we can use the inductive hypothesis to conclude

that $i_1^* \nabla_K = i_2^* \nabla_K$, since both are equal to the pullback of $\nabla_i^{l_1, \vec{d}_1}$ from $\mathcal{M}_1^\diamond$. The case $j = 2$ is similar (though the triangle needs to be replaced by a square). Now apply Lemma 2.13 to construct $\nabla_i^{l, \vec{d}}$ compatible with ∇_K, completing the inductive step and the proof of the lemma. $\qquad \square$

Proof of Proposition 2.12. We define $\alpha_{l, \vec{d}}$ by (23) applied to the connections constructed in Lemma 2.16. Let us check the conditions of Definition 2.10. Condition (a) holds since the Chern form and $\rho_\pm$ are D-closed. Condition (b) holds for $\beta_{l,0}$ by definition, and for $\alpha_{l,(0,0)}$ since $l \neq \varnothing$ and $\rho_\pm|_{\mathbb{RP}^1} = 0$. Condition (c) holds by (28) and the fact the evaluation maps commute with pr_i'. Let us check that condition (d) holds. If $l = \varnothing$, then $\alpha_{l, \vec{d}} = 1$ and $\beta_{l,d} = 1$, since (22),(23) involve empty products. Therefore (21) holds trivially. Assume $l \neq \varnothing$. By construction, the evaluation maps commute with (17). At

$$\mathcal{E}'' = \mathcal{E} \cap \bigcap_{i \in l} \left(\mathrm{ev}_i^{l,(d,d)} \right)^{-1} (U_{\epsilon_i})$$

(21) holds by (29), while at $\mathcal{E} \backslash \mathcal{E}''$ both sides of (21) vanish. $\qquad \square$

2.6. **Stationary descendent integrals.**

Definition 2.17. Fix $\vec{a}, \vec{\epsilon}$. A *coherent integrand* for the tautological line bundles is a coherent integrand $\left\{ \alpha_{l, \vec{d}} \right\}, \{ \beta_{l,d} \}$ defined via (22),(23).

For given $l, \vec{d}, \vec{a}, \vec{\epsilon}$, we define the *open stationary descendent integral* or *open stationary intersection number* by

$$\left\langle \prod_{i \in l} \tau_{a_i}^{\epsilon_i} \right\rangle_{0, \vec{d}} = \int_{\overline{\mathcal{M}}_{0, \varnothing, l}(\vec{d})} \alpha_{l, \vec{d}}^{\vec{a}, \vec{\epsilon}},$$

where $\alpha_{l, \vec{d}}^{\vec{a}, \vec{\epsilon}}$ comes from a family of coherent integrands for the tautological line bundles, and $\int_{\overline{\mathcal{M}}_{0, \varnothing, l}(\vec{d})}$ is oriented via the canonical orientation defined in Section 6.

Definition 2.17 gives a rigourous definition for the integral on the right-hand side of (4). The next theorem says that the open stationary intersection number is independent of the specific coherent integrand used in its definition, and gives a formula for this number.

Theorem 2.18. *For given $l, \vec{d}, \vec{a}, \vec{\epsilon}$, $\langle \prod_{i \in l} \tau_{a_i}^{\epsilon_i} \rangle_{0, \vec{d}}$ is independent of the coherent integrand used to define it. Moreover, it holds that*

$$\left\langle \prod_{i \in l} \tau_{a_i}^{\epsilon_i} \right\rangle_{0, \vec{d}} = \mathrm{OGW}(l, \vec{d}, \vec{a}, \vec{\epsilon}),$$

where

$$\mathrm{OGW}(l, \vec{d}, \vec{a}, \vec{\epsilon}) = \sum_{T \in \mathcal{T}(l, \vec{d})} \mathcal{A}(T, \vec{a}, \vec{\epsilon}) + \delta_{d^+ - d^-} \frac{d^+}{2u} I(l, d, \vec{a}, \vec{\epsilon})$$

was defined in (11), and $\mathcal{A}(-,-,-), \mathcal{T}(-,-)$ and $I(-,-,-,-)$ are defined in Definitions 1.9 and 1.8. In particular, (11) vanishes if $1 + \sum a_i < d^+ + d^-$.

Remark 2.19. Several comments are in place.

(a) First, in the introduction another definition for open intersection numbers was sketched, as the weighted cardinality of the intersection of the zero loci of multisections of $\mathbb{L}_i$, given some boundary conditions. That approach, which is similar to the approach taken in [31, 3, 4], can be made fully rigorous by a recursive construction analogous to those given in these papers. The equivalence of that definition with the definition in terms of coherent integrands (in the non-equivariant limit) can be proven along the lines of [23, Corollary 15].

(b) Second, the first sum in (11) can be represented in a less compact, but perhaps more suggestive way, by expanding the expressions (8) in the definition of the amplitude in terms of fixed-point graph contributions (9). Doing that, we get a sum over tuples $(T, (\Gamma_v)_{v \in V(T)})$, in which $T \in \mathcal{T}$ are trees which have no vertex v with $d^+(v) = d^-(v)$ (otherwise the amplitude clearly vanishes), and Γ_v are fixed point graphs for the specifications $(\mathsf{l}_v, \vec{d}_v)$ of the vertices of T. Each tuple can be graphically presented as follows:

(i) Replace each vertex v by Γ_v. Since $d^+(v) \neq d^-(v)$, the graph Γ_v has a single disk edge H_v with

$$\partial^H(\delta(H_v)) = (d^+(v) - d^-(v)),$$

where ∂^H is the connecting map of (3).

(ii) Draw the edge $e = \{u, v\}$ as a wavy edge which connects the boundary vertices of H_v, H_u and associate this wavy edge e the weight

$$\frac{\partial^H(\delta(H_v))\partial^H(\delta(H_u))}{-2u} = \frac{(d^+(v) - d^-(v))(d^+(u) - d^-(u))}{-2u}.$$

(iii) Associate the vertex v of T the weight

$$\frac{1}{|A_{\Gamma_v}|} \int_{\overline{\mathcal{M}}_{\Gamma_v}} e_{\Gamma_v}^{-1} \cdot \alpha_{\Gamma_v}^{\vec{a}, \vec{\epsilon}}.$$

(iv) Take the product of weights of vertices and wavy edges, and divide by $|\mathrm{Aut}(T)|$.

Similarly, the second term in (11) can be graphically presented as a sum over all isomorphism types of pairs (Γ, e), where Γ is a fixed point graph for (l, d), the sphere moduli specifications obtained from shrinking the boundary of a disk map of type $(\mathsf{l}, \vec{d})$, and e is a sphere edge of Γ. An isomorphism between (Γ, e) to (Γ', e') is an isomorphism between Γ and Γ' which takes e to e'. For such a pair one draws a wavy half-edge emanating from the equator of e. This wavy half-edge is associated the weight $\frac{-|\delta(e)|}{-2u}$. The contribution of (Γ, e) to (11) is the product

$$\frac{-|\delta(e)|}{-2u} \frac{1}{|A'_{(\Gamma, e)}|} \int_{\overline{\mathcal{M}}_\Gamma} e_\Gamma^{-1} \cdot \alpha_\Gamma^{\vec{a}, \vec{\epsilon}}$$

of the graph contribution and the wavy half-edge weight

$$|A'_{(\Gamma, e)}| = |A_\Gamma^0||\mathrm{Aut}_{(\Gamma, e)}|,$$

where $\mathrm{Aut}_{(\Gamma, e)}$ is the automorphism group of Γ preserving e. See Figure 2 for the graphical description described above (where we draw sphere edges of fixed point graphs as spheres and disk edges as disks).

The proof that the two representations are the same is straightforward (see also Lemma 5.11 for a similar statement in higher genus).

(c) One could have hoped that the simpler formula (8) will give rise to the stationary open descendent integrals. This expression does not correspond to the intersection theory we define, and moreover, there is no geometric intersection theory that would give rise to this expression. Indeed, a formula which calculates an intersection number must vanish in underdetermined cases (when the dimension of the moduli is bigger than the dimension of the constraints). This is not the case with $I(S, \vec{a}, \vec{\epsilon})$.

3. Fixed-point localization for coherent integrands

In this section we first prove a general localization formula for S^1−orbifolds with corners and simple fixed points, Proposition 3.1. We then prove a localization expression for the integration of a coherent integrand, Theorem 3.9.

3.1. **Fixed-point formula for orbifolds with corners.** The following fixed-point formula is the main computational tool that we will use.

Proposition 3.1. *Let $\mathcal{X}$ be a compact oriented S^1-orbifold with simple fixed points, and for $c \geq 1$ let $Q_c \in \mathcal{A}_{S^1}(\partial^c \mathcal{X})[u^{-1}]$ be an equivariant primitive which is invariant under the $\mathrm{Sym}(c)$ action on $\partial^c \mathcal{X}$, permuting local boundary components. Then for any $\alpha \in \mathcal{A}_{S^1}(\mathcal{X})$ such that $D\alpha = 0$ we have*

$$\int_{\mathcal{X}} \alpha = \sum_{F \subset \mathcal{X}^{S^1}} \int_F \frac{\alpha|_F}{e(N_F)} + \sum_{c \geq 1} (-1)^{c-1} \int_{\partial^c \mathcal{X}} Q_c \cdots Q_2 Q_1 \, \alpha|_{\partial^c \mathcal{X}},$$

where the first sum ranges over connected components $F \subset \mathcal{X}^{S^1}$.

Remark 3.2. Note that as a special case of Proposition 3.1 we obtain the well-known Atiyah-Bott fixed-point formula [1],

$$(30) \qquad \qquad \int_{\mathcal{X}} \alpha = \sum \int_F \frac{\alpha|_F}{e_{S^1}(N_F)}, \quad \text{if} \quad \partial \mathcal{X} = \varnothing.$$

There are easy counterexamples where (30) does not hold in case $\partial \mathcal{X} \neq \varnothing$.

Remark 3.3. It is possible to obtain a fixed-point formula even when the fixed points of $\mathcal{X}$ intersect the boundary, see [23].

The proof of Proposition 3.1 will occupy the remainder of this section. Let $\widetilde{\mathcal{X}}$ be an oriented orbifold with corners on which S^1 acts with *no* fixed points, and suppose $Q_c \in \mathcal{A}_{S^1}(\partial^c \widetilde{\mathcal{X}})[u^{-1}]$ are $\mathrm{Sym}(c)$-invariant primitives as above. We write $Q_{\leq c} = Q_c \cdots Q_1$.

Lemma 3.4. *For any equivariant form α with $D\alpha = 0$ we have $\int_{\widetilde{\mathcal{X}}} \alpha = \sum_{c \geq 1} (-1)^{c-1} \int_{\partial^c \widetilde{\mathcal{X}}} Q_{\leq c} \alpha$.*

Proof. Since $\widetilde{\mathcal{X}}$ has no fixed points, we may take Q to be an equivariant primitive on $\widetilde{\mathcal{X}}$. We will show by induction on m that

$$\int_{\widetilde{\mathcal{X}}} \alpha = \sum_{1 \leq c \leq m} (-1)^{c-1} \int_{\partial^c \widetilde{\mathcal{X}}} Q_{\leq c} \alpha + (-1)^{m-1} \int_{\partial^m \widetilde{\mathcal{X}}} (Q - Q_m) Q_{\leq (m-1)} \alpha,$$

from which the result follows by taking m so large that $\partial^m \widetilde{\mathcal{X}} = \varnothing$. Let us prove the base case $m = 1$. By Stokes' theorem, we have

$$\int_{\widetilde{\mathcal{X}}} \alpha = \int_{\widetilde{\mathcal{X}}} D(Q\alpha) = \int_{\partial \widetilde{\mathcal{X}}} Q\alpha = \int_{\partial \widetilde{\mathcal{X}}} Q_1 \alpha + \int_{\partial \widetilde{\mathcal{X}}} (Q - Q_1) \alpha.$$

Suppose the result holds for m, let us prove it for $m + 1$:

$$(-1)^{m-1} \int_{\partial^m \widetilde{\mathcal{X}}} (Q - Q_m) Q_{\leq (m-1)} \alpha = (-1)^m \int_{\partial^m \widetilde{\mathcal{X}}} D(QQ_m) Q_{\leq (m-1)} \alpha \overset{(\star)}{=} (-1)^m \int_{\partial^m \widetilde{\mathcal{X}}} D(QQ_{\leq m}\alpha) =$$

$$= (-1)^m \int_{\partial \partial^m \widetilde{\mathcal{X}}} QQ_{\leq m}\alpha = (-1)^m \int_{\partial^{m+1} \widetilde{\mathcal{X}}} Q_{m+1} Q_{\leq m} \alpha + (-1)^m \int_{\partial^{m+1} \widetilde{\mathcal{X}}} (Q - Q_{m+1}) Q_{\leq m}\alpha.$$

To establish the equality marked with $(\star)$ it suffices to show that

$$\int_{\partial^m \widetilde{\mathcal{X}}} Q Q_m \cdots \widehat{Q_j} \cdots Q_1 \alpha = 0$$

for any $1 \leq j \leq m - 1$. Consider the orientation-reversing involution $\sigma : \partial^m \widetilde{\mathcal{X}} \to \partial^m \widetilde{\mathcal{X}}$ which swaps the j and $j+1$ boundary components. We have $\sigma^* Q = Q$, $\sigma^* \alpha = \alpha$ and $\sigma^* Q_i = Q_i$ for all $i < j$ since σ commutes with the structure map $\partial^j \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$. We have $\sigma^* Q_i = Q_i$ for $i \geq j+1$ by the invariance assumption. In short, we have

$$\int_{\partial^m \widetilde{\mathcal{X}}} Q_m \cdots \widehat{Q_j} \cdots Q_1 Q \alpha = -\int_{\partial^m \widetilde{\mathcal{X}}} \sigma^* \left(Q_m \cdots \widehat{Q_j} \cdots Q_1 Q \alpha \right) = -\int_{\partial^m \widetilde{\mathcal{X}}} Q_m \cdots \widehat{Q_j} \cdots Q_1 Q \alpha,$$

from which it follows that $\int_{\partial^m \widetilde{\mathcal{X}}} Q_m \cdots \widehat{Q_j} \cdots Q_1 Q \alpha = 0$. This completes the proof. $\qquad \square$

Now suppose that $\mathcal{X}$ has simple fixed points, and fix some S^1-invariant Riemannian metric on $\mathcal{X}$. For each connected component $F \subset \mathcal{X}^{S^1}$, let $S(N_F) \xrightarrow{\pi} F$ denote the sphere bundle associated with the normal bundle N_F. We take $\epsilon > 0$ so small so that the geodesic flow

$$\bigsqcup_{F \subset \mathcal{X}^{S^1}} (S(N_F) \times (0, \epsilon)) \to \mathcal{X}$$

is well-defined and injective, and use it to construct the blow up

$$\widetilde{\mathcal{X}} = (\mathcal{X} \backslash \mathcal{X}^{S^1}) \cup \left(\bigsqcup_{F \subset \mathcal{X}^{S^1}} (S(N_F) \times [0, \epsilon)) \right)$$

of the fixed points of $\mathcal{X}$. It is not hard to see $\widetilde{\mathcal{X}}$ is a compact oriented orbifold with corners with no fixed points and that

$$\partial^c \widetilde{\mathcal{X}} = \begin{cases} \partial \mathcal{X} \sqcup \bigsqcup_{F \subset \mathcal{X}} S(N_F), & \text{if } c = 1, \\ \partial^c \mathcal{X}, & \text{if } c \neq 1. \end{cases}$$

Note that the induced outward normal orientation on $S(N_F)$ is opposite to the standard orientation induced from that of the total space N_F. Thus, Proposition 3.1 follows from Lemma 3.4 and the following.

Lemma 3.5. *If Q is any equivariant primitive on $S(N_F)$, then we have*

(31)
$$-\int_{S(N_F)} Q\pi^* (\alpha|_F) = \int_F \frac{\alpha|_F}{e_{S^1}(N_F)}.$$

Proof. We need to compute

$$-\int_{S(N_F)} Q\pi^* (\alpha|_F) = \int_F (-\pi_* Q) \alpha|_F.$$

Since π is an S^1-equivariant map from an orbifold without boundary to an orbifold with trivial S^1 action, we have

$$\pi_* D = D\pi_* = d\pi_*.$$

Now if Q' is any other equivariant primitive, then

$$0 = (\pi_* D - d\pi_*) (QQ') = \pi_* (Q') - \pi_* (Q) + d\epsilon,$$

so the cohomology class $[-\pi_* Q]$, and the left hand side of (31), is independent of Q. To complete the proof, we identify $[-\pi_* Q]$ with the inverse of the equivariant Euler form. In [24] the second author proves the existence of an *equivariant angular form*

$$\phi \in \Omega (S(N); \mathrm{Or}(N) \otimes \mathbb{R}[u])^{S^1}$$

for $S(N) \xrightarrow{\pi} F$. This is a form such that (a) $D\phi \in \mathrm{Im}(\pi^*)$ and (b) $\pi_* \phi = 1$. By definition the equivariant Euler form associated with ϕ is specified by the equation

$$D\phi = -\pi^* e_{S^1}(N_F).$$

On the other hand,

$$0 = (\pi_* D - d\pi_*) (Q\phi) = 1 + (\pi_* Q) \cdot e_{S^1}(N_F) + d\epsilon,$$

which shows that $e_{S^1}(N_F)$ is invertible (since $d\epsilon$ is nilpotent), and $[-\pi_* Q] \cdot [e_{S^1}(N_F)] = 1$. The proof of the lemma, hence also of Proposition 3.1, is now complete. $\qquad\square$

Remark 3.6. In the proof of Lemma 3.5, we can take $Q = -(\pi^* e_{S^1}(N_F))^{-1} \phi$ as an equivariant primitive, so $\pi_* Q = -e_{S^1}(N_F)^{-1}$ and $\epsilon = 0$.

3.2. Statement of fixed-point formula for coherent integrands.

Definition 3.7. Fix a disk moduli specification $(\mathsf{l}, \vec{d}) \in \mathcal{D}$. Write
$$\mathcal{D}_{\neq 0} = \left\{ (\mathsf{l}, \vec{d}) \in \mathcal{D} \,|\, \vec{d} \neq (0,0) \right\}.$$
For $r \geq 0$ a non-negative integer, an $(r, \mathsf{l}, \vec{d})$-*labeled tree* T consists of
(a) A tree with r oriented edges labeled $1, \ldots, r$. We denote by $V(T)$ the set of vertices of T.
(b) A function
$$\delta : V(T) \to \mathcal{D}_{\neq 0}, \qquad \delta(v) = \left(\mathsf{l}(v), \vec{d}(v) \right) = \left(\mathsf{l}(v), (d^+(v), d^-(v)) \right),$$
such that $\bigsqcup \mathsf{l}(v) = \mathsf{l}$ and $\sum \vec{d}(v) = \vec{d}$.

There is an obvious notion of isomorphism of such trees, and we fix a set of representatives $\hat{\mathcal{T}}(r, \mathsf{l}, \vec{d})$ for the isomorphism classes of $(r, \mathsf{l}, \vec{d})$-labeled trees.

Notation 3.8. Consider some coherent integrand $\{\alpha_{\mathsf{l}, \vec{d}}\}_{(\mathsf{l}, \vec{d}) \in \mathcal{D}}, \{\beta_{\mathsf{l}, d}\}_{(\mathsf{l}, d) \in \mathcal{S}}$ as in Definition 2.10. For $(\mathsf{l}, \vec{d}) \in \mathcal{D}_{\neq 0}$ we set
$$F(\mathsf{l}, \vec{d}) = F^{\{\alpha_\delta\}_{\delta \in \mathcal{D}}, \{\beta_\delta\}_{\delta \in \mathcal{S}}}(\mathsf{l}, \vec{d}) = \sum_{F \subset \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}}(\vec{d})} \int_F \frac{\alpha_{\mathsf{l}, \vec{d}}|_F}{e^{S^1}(N_F)},$$
where F ranges over connected components of the fixed-point stack $\overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}}(\vec{d})^{S^1}$, and
$$E(\mathsf{l}, \vec{d}) = E^{\{\alpha_\delta\}_{\delta \in \mathcal{D}}, \{\beta_\delta\}_{\delta \in \mathcal{S}}}(\mathsf{l}, \vec{d}) = \begin{cases} \frac{-d}{(-2u)} \int_{\overline{\mathcal{M}}_{0, \mathsf{l}}(d)} \beta_{\mathsf{l}, d}, & \text{if } \vec{d} = (d, d), \\ 0, & \text{otherwise}, \end{cases}$$
$$G(\mathsf{l}, \vec{d}) = G^{\{\alpha_\delta\}_{\delta \in \mathcal{D}}, \{\beta_\delta\}_{\delta \in \mathcal{S}}}(\mathsf{l}, \vec{d}) = F(\mathsf{l}, \vec{d}) + E(\mathsf{l}, \vec{d}).$$
We define the *amplitude* $\hat{\mathcal{A}}(T) = A^{\{\alpha_\delta\}_{\delta \in \mathcal{D}}, \{\beta_\delta\}_{\delta \in \mathcal{S}}}(T) \in \mathbb{R}[u, u^{-1}]$ of a labeled tree $T \in \hat{\mathcal{T}}(r, \mathsf{l}, \vec{d})$ by
$$\hat{\mathcal{A}}(T) = \frac{1}{2^r r!} \left(\frac{-1}{2u} \right)^r \prod_{v \in V} \left(d^+(v) - d^-(v) \right)^{\mathrm{val}(v)} \cdot G(\delta(v)).$$
Observe that if T has more than one vertex, and if $E(\delta(v)) \neq 0$, then $(d^+(v) - d^-(v))^{\mathrm{val}(v)} = 0$, hence also $\hat{\mathcal{A}}(T) = 0$. Thus, we can also write
$$\hat{\mathcal{A}}(T) = \begin{cases} \frac{1}{2^r r!} \left(\frac{-1}{2u} \right)^r \prod_{v \in V} \left(d^+(v) - d^-(v) \right)^{\mathrm{val}(v)} \cdot F(\delta(v)), & \text{if } r > 1, \\ G(\delta(v)), & \text{if } V(T) = \{v\}. \end{cases}$$

Theorem 3.9. *For all* $(\mathsf{l}, \vec{d}) \in \mathcal{D}$ *we have*

(32)
$$\int_{\overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}}(\vec{d})} \alpha_{\mathsf{l}, \vec{d}} = \sum_{r \geq 0} \sum_{T \in \hat{\mathcal{T}}(r, \mathsf{l}, \vec{d})} \hat{\mathcal{A}}(T).$$

The proof of this theorem appears at the end of §3.6 below.

3.3. Equivariant primitives.
To apply Proposition 3.1 we need to construct equivariant primitives $Q_c^{\mathsf{l}, \vec{d}} \in \mathcal{A}_{S^1}\left(\partial^c \overline{\mathcal{M}}_{0, \mathsf{k}, \mathsf{l}}(\vec{d}) \right)[u^{-1}]$.

If $\vec{d} = (d, d)$, $\mathsf{k} = \varnothing$, and $\mathcal{E} \subset \overline{\mathcal{M}}_{0, \mathsf{l} \sqcup \star}(d)$ denotes the exceptional boundary component, we take
$$Q_1^{\mathsf{l}, \vec{d}}|_{\mathcal{E}} = \left(-\frac{1}{2u} \right) \cdot (\mathrm{ev}_\star)^* \, d\theta,$$
where $d\theta$ is the canonical angular form on $\mathbb{RP}^1$, which satisfies $\iota_\xi d\theta = 2$.

Recall the standard codimension c corners, $C^c(\varnothing, \mathsf{l}, \vec{d})$, parameterize nodal configurations of $c+1$ disks connected by c real nodes, which are numbered $i = 1, \ldots, c$. We define

$$Q_c^{\mathsf{l},\vec{d}} = \frac{-1}{2u} \frac{1}{c} \sum_{i=1}^c d\theta_i,$$

where $d\theta_i$ is the pullback of $d\theta$ along the evaluation map corresponding to the ith node.

It follows that for $s \le c$ we have

$$Q_s|_{C^c(\varnothing, \mathsf{l}, \vec{d})} = \frac{-1}{2u} \frac{1}{s} \left(d\theta_s + \cdots + d\theta_c\right)$$

so

$$Q_{\le c}^{\mathsf{l},\vec{d}} = Q_c^{\mathsf{l},\vec{d}} \cdots Q_1^{\mathsf{l},\vec{d}} = \frac{1}{c!} \frac{1}{(-2u)^c} d\theta_1 \cdots d\theta_c.$$

We say that a connected component of standard corners $\mathcal{B} \subset \mathcal{C}^c(\varnothing, \mathsf{l}, \vec{d})$ is *degenerate* if an interior point of $\mathcal{B}$ has a degree zero disk with two or more nodes attached, and *non-degenerate* otherwise. Note that if $\mathcal{B}$ is degenerate, then $Q_{\le c}^{\mathsf{l},\vec{d}}|_{\mathcal{B}} = 0$, since $d\theta_i = d\theta_j$ for some $i \ne j$.

3.4. **Recursion for corner contributions.** The following recursion is the basis for the proof of the fixed-point formula in Theorem 3.9.

Proposition 3.10. *We have*

$$\int_{\mathcal{C}^c(\varnothing, \mathsf{l}, \vec{d})} Q_{\le c}^{\mathsf{l},\vec{d}} \alpha_{\mathsf{l},\vec{d}} = \frac{1}{2} \frac{(d_1^+ - d_1^-)(d_2^+ - d_2^-)}{(-2u)} \frac{1}{c} \sum \int_{\mathcal{C}^{c_1}(\varnothing, \mathsf{l}_1, \vec{d}_1)} Q_{\le c_1}^{\mathsf{l}_1, \vec{d}_1} \alpha_{\mathsf{l}_1, \vec{d}_1} \times \int_{\mathcal{C}^{c_2}(\varnothing, \mathsf{l}_2, \vec{d}_2)} Q_{\le c_2}^{\mathsf{l}_2, \vec{d}_2} \alpha_{\mathsf{l}_2, \vec{d}_2},$$

where the sum ranges over all $(\mathsf{l}_1, \vec{d}_1) \# (\mathsf{l}_2, \vec{d}_2) = (\mathsf{l}, \vec{d})$ *and* $c_1 + c_2 = c - 1$.

The proof of this proposition occupies the remainder of this section. Since $Q_{\le c}$ vanishes on degenerate corners, and $\alpha_i^{\mathsf{l}_i, \vec{d}_i}$ vanishes if $\vec{d}_i = (0,0)$, we have

$$\int_{\mathcal{C}^c(\varnothing, \mathsf{l}, \vec{d})} Q_{\le c}^{\mathsf{l},\vec{d}} \alpha_{\mathsf{l},\vec{d}} = \frac{1}{2} \int_{\widetilde{\mathcal{C}}^c(\mathsf{l}, \vec{d})} Q_{\le c}^{\mathsf{l},\vec{d}} \alpha_{\mathsf{l},\vec{d}},$$

where

$$\widetilde{\mathcal{C}}^c(\mathsf{l}, \vec{d}) = \bigsqcup \partial_{nd}^{\sigma_1} \overline{\mathcal{M}}_{0,\star_1,\mathsf{l}_1}(\vec{d}_1) \times_L \partial_{nd}^{\sigma_2} \overline{\mathcal{M}}_{0,\star_2,\mathsf{l}_2}(\vec{d}_2),$$

is the disjoint union ranging over all $\sigma_j, \mathsf{l}_j, \vec{d}_j$, $j = 1, 2$, as in Proposition 2.7, where $\vec{d}_1, \vec{d}_2$ are both not equal to $(0,0)$, and

$$\partial_{nd}^{\sigma_j} \overline{\mathcal{M}}_{0,\star_j,\mathsf{l}_j}(\vec{d}_j) \subset \partial^{\sigma_j} \overline{\mathcal{M}}_{0,\star_j,\mathsf{l}_j}(\vec{d}_j)$$

are the non-degenerate corners.

Now fix some σ_1, σ_2, $\sigma_1 \sqcup \sigma_2 = \{1, \ldots, c-1\}$ and write $C = \partial_{nd}^{\sigma_1} \overline{\mathcal{M}}_{0,\star_1,\mathsf{l}_1}(\vec{d}_1) \times_L \partial_{nd}^{\sigma_2} \overline{\mathcal{M}}_{0,\star_2,\mathsf{l}_2}(\vec{d}_2)$. Non-degenerate corners are vertical in the sense that the forgetful map $\overline{\mathcal{M}}_{0,\star_j,\mathsf{l}_j}(\vec{d}_j) \to \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_j}(\vec{d}_j)$ induces a map

$$(33) \qquad \mathcal{M}_j = \partial_{nd}^{\sigma_j} \overline{\mathcal{M}}_{0,\star_j,\mathsf{l}_j}(\vec{d}_j) \xrightarrow{f_j} \check{\mathcal{M}}_j = \partial_{nd}^{\sigma_j} \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_j}.$$

Indeed, when we forget $\star_j$ the *depth* of a point (i.e., the number of local boundary components incident to that point) decreases only if a disk component becomes unstable; for non-degenerate boundaries this is never the case and so the depth is preserved. See [25, §2.7] for more details. We will also need to consider the vertical/horizontal boundary decomposition of f_j itself:

$$\partial \mathcal{M}_j = \partial^- \mathcal{M}_j \sqcup \partial^+ \mathcal{M}_j.$$

The *vertical* clopen component $\partial^- \mathcal{M}_j$ admits a map $\partial^- \mathcal{M}_j \to \partial \check{\mathcal{M}}_j$. The *horizontal* clopen component $\partial^+ \mathcal{M}_j \subset \partial \mathcal{M}_j$ parameterizes configurations with a *ghost disk*: an irreducible component which is a degree $(0,0)$ disk with no special points on the interior and three special points on the

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boundary: two boundary nodes labeled by $\{a_1, a_2\} \subset \sigma_j$, and the boundary marking $\star_j$ (this is the only type of disk that becomes unstable when we forget $\star_j$). We can define an involution

$$\mathrm{inv}_j : \partial^+ \mathcal{M}_j \to \partial^+ \mathcal{M}_j$$

that switches the labels $\{a_1, a_2\}$ of the nodes. This is an orientation-*reversing* involution and we have

(34) $$f_j \circ i \circ \mathrm{inv}_j = f_j \circ i \text{ and } \mathrm{ev}_j \circ i \circ \mathrm{inv}_j = \mathrm{ev}_j \circ i,$$

where $i : \partial^+ \mathcal{M}_j \to \mathcal{M}_j$.

Set $f = f_1 \times f_2$ and write $C \overset{g}{\hookrightarrow} \mathcal{M}_1 \times \mathcal{M}_2$ for the structure map. Set $\Gamma \in \mathcal{A}_{S^1}\left(\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2\right)$ to be

$$\Gamma = \mathrm{pr}_1^* Q_{\leq c_1} \alpha_{l_1, \vec{d}_1} \wedge \mathrm{pr}_2^* Q_{\leq c_2} \alpha_{l_2, \vec{d}_2} =: \mathrm{pr}_1^* \Gamma_1 \wedge \mathrm{pr}_2^* \Gamma_2.$$

We have the following version of the projection formula.

Lemma 3.11. *For* $\vec{d}^j = \left(d_+^j, d_-^j\right) \neq (0,0)$, *if we orient* C *as a corner of* $\overline{\mathcal{M}}_{0,\varnothing,l_1 \sqcup l_2}\left(\vec{d}^1 + \vec{d}^2\right)$ *and* $\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2$ *as a product of corners, then*

(35) $$\int_C d\theta_c \, g^* f^* \Gamma = (-1)^{c-1} (-1)^\sigma \left(d_+^1 - d_-^1\right)\left(d_+^2 - d_-^2\right) \int_{\check{\mathcal{M}}_1} \Gamma_1 \times \int_{\check{\mathcal{M}}_2} \Gamma_2,$$

where, writing the elements of σ_j *in order* $\sigma_j^1 < \sigma_j^2 < \cdots < \sigma_j^{c_j}$, *we define*

$$(-1)^\sigma = \mathrm{sign}\begin{pmatrix} 1 & \cdots & & & \cdots & c-1 \\ \sigma_1^1 & \cdots & \sigma_1^{c_1} & \sigma_2^1 & \cdots & \sigma_2^{c_2} \end{pmatrix}$$

to be the shuffle sign.

Proof. Let ev be the evaluation at the cth node, so that $d\theta_c = \mathrm{ev}^* d\theta$. Consider the diagram

$$\begin{array}{ccccc} C & \overset{g}{\longrightarrow} & \mathcal{M}_1 \times \mathcal{M}_2 & \overset{f}{\longrightarrow} & \check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2 \\ {\scriptstyle \mathrm{ev}} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_{12}} & & \\ L & \underset{\delta}{\longrightarrow} & L^2 & & \end{array}$$

Here $L = \mathbb{RP}^1$ and δ denotes the diagonal map. Using the S^1-action one checks that the map $\mathrm{ev}_{12} = \mathrm{ev}_1 \times \mathrm{ev}_2$ is a b-submersion (this means the differential is onto, as well as all of its restrictions to the corners of the domain, see [26]) and so the square is a b-transverse cartesian square.

We let $\gamma \in \Omega^{\dim(\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2)}\left(\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2\right)$ be the *top de Rham degree* part of Γ:

$$\Gamma = \gamma \, u^m \quad \mathrm{mod} \, u^{m+1},$$

where $m = \frac{\deg \Gamma - \dim(\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2)}{2}$. We define γ_1, γ_2 similarly, so $\gamma = \gamma_1 \wedge \gamma_2$.

The lemma follows immediately from the following computation:

(36) $$(-1)^\sigma \int_C \mathrm{ev}^* d\theta \, g^* f^* \gamma \overset{(\mathrm{I})}{=} (-1)^{c_1} \int_{\mathcal{M}_1 \times \mathcal{M}_2} \left(g_* \mathrm{ev}^* d\theta\right) f^* \gamma =$$

$$\overset{(\mathrm{II})}{=} (-1)^{c_1} \int_{\mathcal{M}_1 \times \mathcal{M}_2} \left(\mathrm{ev}_1 \times \mathrm{ev}_2\right)^* \left(d\theta_1 \wedge d\theta_2\right) f^* \gamma =$$

$$\overset{(\mathrm{III})}{=} \left(\int_{\mathcal{M}_1} \mathrm{ev}_1^* d\theta_1 \, f_1^* \gamma_1\right) \times \left(\int_{\mathcal{M}_2} \mathrm{ev}_2^* d\theta_2 \, f_2^* \gamma_2\right) =$$

$$\overset{(\mathrm{IV})}{=} (-1)^{c-1} \left(\left(d_1^+ - d_1^-\right) \int_{\check{\mathcal{M}}_1} \gamma_1\right) \times \left(\left(d_2^+ - d_2^-\right) \int_{\check{\mathcal{M}}_2} \gamma_2\right).$$

Let us justify this computation step by step. The discussion of orientations and signs appears at the end of the proof.

First, we define g_*. Let $\partial_1 - \partial_2$ denote the vector field on $L \times L$ which is normal to the diagonal $L \xrightarrow{\delta} L \times L$. By [23, Lemma 55] there exists a vector field η on an open neighbourhood $U \subset \mathcal{M}_1 \times \mathcal{M}_2$ of C, such that (i) η is *b-tangent*, that is, for any $p \in U$ it is tangent to all the boundary faces incident to p, and (ii) $(\mathrm{ev}_1 \times \mathrm{ev}_2)_* \eta = \partial_1 - \partial_2$. Flowing along η and $\partial_1 - \partial_2$ we construct a pair of compatible tubular neighbourhoods $C \subset V \subset \mathcal{M}_1 \times \mathcal{M}_2$ and $L \subset V_0 \subset L \times L$. We denote by $V \xrightarrow{\pi} C$ and $V_0 \xrightarrow{\pi_0} L$ the associated projections, which satisfy $\mathrm{ev} \circ \pi = \pi_0 \circ \mathrm{ev}_{12}$.

Let $r_0 : V_0 \to [0, \epsilon)$ be the distance function from the diagonal associated with the flow. Let $\widetilde{V_0} \xrightarrow{B_0} V_0$ denote the hyperplane blowup of the diagonal (see [25, §3.3]); it is isomorphic to $(-1, 0] \sqcup [0, 1) \times L \to (-1, 1) \times L$, and extends to a blow up $\widetilde{L \times L} \to L \times L$. We construct a Thom form τ_0 for δ supported in V_0, following Bott and Tu [2]. We define τ_0 by

$$B_0^* \tau_0 = d \left(\sigma \left(r_0 \right) \phi_0 \right),$$

where $\sigma : [0, \epsilon) \to [-1, 0]$ is a smooth compactly supported monotone function with $\frac{\partial^a \sigma}{\partial r^a} \big|_{r=0} = 0$ for all $a \geq 1$, and $\phi_0 = \pm \frac{1}{2}$ is a locally constant function obtaining opposite values on the two connected components of $\widetilde{V_0}$ (ϕ_0 is the pullback of the angular form for $S^0 = \mathrm{pt} \sqcup \mathrm{pt}$ to $S(N_\delta) = S^0 \times L$; see below for the orientation convention which fixes the trivialization). Let $\tau = \mathrm{ev}_{12}^* \tau_0$ be the corresponding Thom form for $C \subset \mathcal{M}_1 \times \mathcal{M}_2$. We define

$$\delta_* \omega = \tau_0 \wedge \pi_0^* \omega, \qquad g_* \omega = \tau \wedge \pi^* \omega,$$

the expressions on the right-hand side should be interpreted as the extension by zero of the indicated forms, which have compact support in tubular neighbourhoods of the diagonals, to $L \times L$ and to $\mathcal{M}_1 \times \mathcal{M}_2$, respectively.

Let us justify equation (I). Let $\widetilde{V} \xrightarrow{B_V} V$ denote the hyperplane blowup of V along C (more directly, it can be defined as the pullback of B_0 along $\mathrm{ev}_1 \times \mathrm{ev}_2$). We have $\partial \widetilde{V} = (S^0 \times C) \sqcup (\partial V)^\sim$, where $(\partial V)^\sim$ is an open substack of $\partial (\mathcal{M}_1 \times \mathcal{M}_2)^\sim$, the hyperplane blowup of $\partial (\mathcal{M}_1 \times \mathcal{M}_2)$ along ∂C. We denote the map associated with this blowup by

$$\partial (\mathcal{M}_1 \times \mathcal{M}_2)^\sim \xrightarrow{B_\partial} \partial (\mathcal{M}_1 \times \mathcal{M}_2).$$

There is a lift $\partial (\mathcal{M}_1 \times \mathcal{M}_2)^\sim \xrightarrow{\widetilde{\mathrm{ev}}_{12}} \widetilde{L \times L}$ of ev_{12},

$$B_\partial \circ \widetilde{\mathrm{ev}}_{12} = \mathrm{ev}_{12} \circ B_\partial.$$

We have

$$(37) \quad \int_{\mathcal{M}_1 \times \mathcal{M}_2} (g_* \lambda) \, \rho = \int_{\widetilde{V}} d \left(\sigma \left(r_0 \right) \phi_0 \left(\pi^* \lambda \right) \rho \right) = \int_{\partial \widetilde{V}} \sigma \left(r_0 \right) \phi_0 \left(\pi^* \lambda \right) \rho =$$

$$= (-1)^\sigma (-1)^{c_1} \int_{S^0 \times C} \phi_0 \left(g \circ \mathrm{pr}_C \right)^* \left(\left(\pi^* \lambda \right) \rho \right) + R = (-1)^\sigma (-1)^{c_1} \int_C \lambda \, g^* \rho + R,$$

where

$$(38) \quad R = \int_{\partial (\mathcal{M}_1 \times \mathcal{M}_2)^\sim} \widetilde{\mathrm{ev}}_{12}^* \left(\sigma \left(r_0 \right) \phi_0 \right) B^* \left(\left(\pi^* \lambda \right) \rho \right).$$

In general, R doesn't have to vanish, but for $\lambda = \mathrm{ev}^* d\theta$ and $\rho = f^* \gamma$ we claim that it does, completing the justification of step (I). Define

$$\partial_1^+ = \left(\partial^+ \mathcal{M}_1 \right) \times \mathcal{M}_2 \subset \partial (\mathcal{M}_1 \times \mathcal{M}_2), \qquad \tilde{\partial}_1^+ = \left(\partial^+ \mathcal{M}_1 \times \mathcal{M}_2 \right)^\sim \subset \left(\partial (\mathcal{M}_1 \times \mathcal{M}_2) \right)^\sim,$$

and define $\partial_j^\pm, \tilde{\partial}_j^\pm$ similarly, so

$$\partial (\mathcal{M}_1 \times \mathcal{M}_2)^\sim = \tilde{\partial}_1^+ \sqcup \tilde{\partial}_2^+ \sqcup \tilde{\partial}_1^- \sqcup \tilde{\partial}_2^-.$$

The involution inv_j lifts to an orientation-reversing involution $\widetilde{\text{inv}}_j : \tilde{\partial}_j^+ \to \tilde{\partial}_j^+$ with $B_\partial \circ \widetilde{\text{inv}}_j = \text{inv}_j \circ B_\partial$. We have $\pi^* \text{ev}^* d\theta = \text{ev}_{12}^* \pi_0^* d\theta$. Using this and (34) it follows that

$$\left[\tilde{\text{ev}}_{12}^* \left(\sigma\left(r_0\right)\phi_0\right) \wedge B^* \left(\pi^* \text{ev}^* d\theta \wedge f^* \gamma\right)\right]\big|_{\tilde{\partial}_j^+}$$

is invariant under $\widetilde{\text{inv}}_j^*$. Since $\widetilde{\text{inv}}_j$ is orientation-reversing, the contribution of $\int_{\tilde{\partial}_j^+} \cdots$ to R vanishes. On $\tilde{\partial}_1^- \sqcup \tilde{\partial}_2^-$, $f^* \gamma$ vanishes identically. Indeed, there exists a map f^- making the following square commute

$$
\begin{array}{ccc}
\partial_1^- \sqcup \partial_2^- & \longrightarrow & \mathcal{M}_1 \times \mathcal{M}_2 \\
\scriptstyle{f^-}\big\downarrow & & \big\downarrow\scriptstyle{f} \\
\partial\left(\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2\right) & \longrightarrow & \check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2
\end{array}
$$

so $\left(f^* \gamma\right)\big|_{\partial_1^- \sqcup \partial_2^-} = \left(f^-\right)^* \left(\gamma\big|_{\partial\left(\check{\mathcal{M}}_1 \times \check{\mathcal{M}}_2\right)}\right) = 0$, since γ is a top form.

Let us justify equation (II). We have

$$g_* \text{ev}^* d\theta = \tau \wedge \pi^* \text{ev}^* d\theta = \text{ev}_{12}^* \left(\tau_0 \wedge \pi_0^* d\theta\right) = \text{ev}_{12}^* \delta_* d\theta.$$

Since $d\theta$ is Poincaré dual to a point in L, $\delta_* d\theta$ is Poincaré dual to a point in $L \times L$, so there exists some ϵ with

$$\text{ev}_{12}^* d\epsilon = g_* \text{ev}^* d\theta - \text{ev}_{12}^* \left(d\theta_1 \wedge d\theta_2\right).$$

It follows that

$$\int_{\mathcal{M}_1 \times \mathcal{M}_2} \left(g_* \text{ev}^* d\theta\right) f^* \gamma = \int_{\mathcal{M}_1 \times \mathcal{M}_2} \text{ev}_{12}^* \left(d\theta_1 \wedge d\theta_2\right) f^* \gamma + \int_{\partial\left(\mathcal{M}_1 \times \mathcal{M}_2\right)} \text{ev}_{12}^* \epsilon \, f^* \gamma.$$

Again, the boundary term on the right-hand side vanishes, the argument is essentially the same as for (38).

Step (III) follows from $\deg \gamma_1 = c_1$ and Fubini's theorem. To prove step (IV) we need to show

$$\int_{\mathcal{M}_1} \text{ev}_1^* d\theta_1 \wedge f_1^* \gamma_1 = (-1)^{c_1} \left(d_1^+ - d_1^-\right) \int_{\check{\mathcal{M}}_1} \gamma_1.$$

Consider some point $p \in \check{\mathcal{M}}_1$, represented by a fundamental configuration $\sigma = \left(\left(\Sigma, \nu, \lambda, w\right), b, \Sigma^{1/2}\right)$. Let $y_p \in L$ be a regular value for $w|_{\partial \Sigma^{1/2}}$. It is not hard to see that for a small open interval $y_p \in V_p \subset L$ and a small open neighbourhood $p \in U_p \subset \check{\mathcal{M}}_1$ $f_1^{-1}\left(U_p\right) \cap \text{ev}_1^{-1}\left(V_p\right) \xrightarrow{f_1 \times \text{ev}_1} U_p \times V_p$ is a (possibly empty) finite covering map, whose sheets are in bijection with $w|_\partial^{-1}\left(y_p\right)$. Choose a finite open subcover $\{U_\alpha = U_{p_\alpha}\}$ with a subordinate partition of unity ρ_α. Let $\lambda_\alpha \in \Omega^1\left(L\right)$ be Poincaré dual to a point with compact support in $V_\alpha = V_{p_\alpha}$, and let $\epsilon_\alpha \in \Omega^0\left(L\right)$ satisfy $d\epsilon_\alpha = d\theta - \lambda_\alpha$. We compute

$$\int_{\mathcal{M}_1} \text{ev}_1^* d\theta_1 \wedge f_1^* \gamma_1 = \sum_\alpha \int_{\mathcal{M}_1} \text{ev}_1^* d\theta_1 \wedge f_1^* \left(\rho_\alpha \gamma_1\right) =$$

$$= \sum_\alpha \int_{\mathcal{M}_1} \text{ev}_1^* \lambda_\alpha \wedge f_1^* \left(\rho_\alpha \gamma_1\right) + \sum_\alpha \int_{\partial \mathcal{M}_1} \text{ev}_1^* \epsilon_\alpha \wedge f_1^* \left(\rho_\alpha \gamma_1\right).$$

We have used the fact $d\left(\rho_\alpha \gamma_1\right) = 0$, since $\rho_\alpha \gamma_1$ is a top form. The boundary terms vanish as before (by using inv_1 to show that the contribution of $\partial^+ \mathcal{M}_1$ vanishes, and arguing that $f^* \rho_\alpha \gamma_1|_{\partial^- \mathcal{M}_1} \equiv 0$). To evaluate $\int_{\mathcal{M}_1} \text{ev}_1^* \lambda_\alpha \wedge f_1^* \left(\rho_\alpha \gamma_1\right)$, use $f_1 \times \text{ev}_1$ as local coordinates. The contribution of each sheet is $\pm \int_{U_\alpha} \rho_\alpha \gamma_1$, where the sign is ± 1 according to (46). This signed count satisfies

(39) $$\# w|_\partial^{-1}\left(y_p\right) = (-1)^{c_1} \left(d_1^+ - d_1^-\right),$$

where only the sign needs some explanation, see below. So we get

$$\sum_\alpha \int_{\mathcal{M}_1} \mathrm{ev}_1^* \lambda_\alpha \wedge f_1^* (\rho_\alpha \gamma_1) = (-1)^{c_1} (d_1^+ - d_1^-) \int_{\check{\mathcal{M}}_1} \left(\sum \rho_\alpha \right) \gamma = (-1)^{c_1} (d_1^+ - d_1^-) \int_{\check{\mathcal{M}}_1} \gamma,$$

completing the justification of step (IV).

It remains to explain our orientation conventions and discuss signs. A generic point $p \in C$ satisfies $g(p) \notin i(\partial_1^+ \sqcup \partial_2^+)$, that $dev|_{g(p)}, df|_{g(p)}$ are submersions, and that the isotropy at p is trivial. Fix such a point. For the remainder of the proof we will pull back all of the vector bundles to p, and work exclusively with the vector spaces we obtain in this way and with their oriented bases. Henceforth, if the pullback to p is unambiguously defined we omit it from the notation. Let $[X]$ denote an oriented base for $T_q X$, for an orbifold X and a point q that will be clear from the context. We write $\mathrm{ev}^* TL, \mathrm{ev}_1^* TL, \mathrm{ev}_2^* TL$ for the positive vectors (dual to $d\theta$) on the corresponding vector spaces.

Recall

$$\check{\mathcal{M}}_j = \partial^{\sigma_j} \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_j} \left(\vec{d}_j \right),$$

and $\sigma_1 \sqcup \sigma_2 = \{1, ..., c - 1\}$. Let

$$O_j = o\left(\sigma_j^{c_j} \right) o\left(\sigma_j^{c_{j-1}} \right) \cdots o\left(\sigma_j^1 \right)$$

denote the sequence of outward normal vectors in $\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_j} \left(\vec{d}_j \right)$, with $o(i)$ corresponding to the ith local boundary component. Note the elements of σ_j appear in reverse order; this is because, iterating the definition of the boundary orientation, we find that there is an equality of orientations

$$O_j \left[\check{\mathcal{M}}_j \right] = \left[\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_j} \left(\vec{d}_j \right) \right].$$

Let $[\overline{\mathcal{M}}]$ denote an oriented basis for $\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_1 \sqcup \mathsf{l}_2} \left(\vec{d}_1 + \vec{d}_2 \right)$, and let $O = o(c)$ denote an outward normal vector corresponding to smoothing the $(\star_1, \star_2)$-node. Since C is oriented as a corner of $\overline{\mathcal{M}}$, we find that

$$(-1)^{\sigma + c_1 c_2} O O_1 O_2 [C] = [\overline{\mathcal{M}}],$$

where

$$(-1)^{\sigma + c_1 c_2} = \mathrm{sign} \begin{pmatrix} c - 1 & & & & \cdots & & & & 2 & 1 \\ \sigma_1^{c_1} & \sigma_1^{c_1 - 1} & \cdots & \sigma_1^2 & \sigma_1^1 & \sigma_2^{c_2} & \sigma_2^{c_2 - 1} & \cdots & \sigma_2^1 \end{pmatrix}.$$

Let $F_{f \circ g}$ denote a vector spanning $\ker d(f \circ g)$, oriented so that the following equality holds:

(40)
$$(-1)^{\sigma + c_1 c_2} O_1 O_2 [C] = F_{f \circ g} O_1 [\check{\mathcal{M}}_1] O_2 [\check{\mathcal{M}}_2].$$

To understand this equation, note that the standard convention for orienting the fiber of $f \circ g$ would lead to an equation of the form

$$C \overset{!}{=} F_{f \circ g}^! \check{\mathcal{M}}_1 \check{\mathcal{M}}_2.$$

Instead, (40) comes from identifying $F_{f \circ g}$ with the tangent space to the fiber of the map

$$\overline{\mathcal{M}}_{0,\star_1,\mathsf{l}_1} \left(\vec{d}_1 \right) \times_L \overline{\mathcal{M}}_{0,\star_2,\mathsf{l}_2} \left(\vec{d}_2 \right) \to \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_1} \left(\vec{d}_1 \right) \times \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}_2} \left(\vec{d}_2 \right),$$

where the domain is oriented as a boundary of $\overline{\mathcal{M}}$. We use this orientation for $F_{f \circ g}$ because in Lemma 6.16 it is computed to be

$$F_{f \circ g} = (-1)^{s_1 + s_2} \mathrm{ev}^* TL,$$

where s_j is the sign of the region incident to $\star_j$.

Plugging this in (40) and using that $\dim \check{\mathcal{M}}_1 = c_1 \mod 2$ we get

(41)
$$[C] = (-1)^\sigma (-1)^{s_1 + s_2} (-1)^{c-1} (\mathrm{ev}^* TL) [\check{\mathcal{M}}_1] [\check{\mathcal{M}}_2].$$

We turn to the construction of the Thom forms. We orient the normal lines N_g, N_δ associated with g, δ by

(42) $\qquad [\overline{\mathcal{M}}_{0,\star_1,l_1}(\vec{d}_1)][\overline{\mathcal{M}}_{0,\star_2,l_2}(\vec{d}_2)] = N_g[C] = (-1)^\sigma (-1)^{s_1+s_2} (-1)^{c-1} N_g (\mathrm{ev}^\star TL) [\check{\mathcal{M}}_1][\check{\mathcal{M}}_2],$

(43) $\qquad (\mathrm{ev}_1^* TL)(\mathrm{ev}_2^* TL) = N_\delta \mathrm{ev}^* TL.$

Since g is the pullback of δ, N_g, N_δ are naturally identified as vector spaces. We claim that we have an equality of oriented vector spaces

(44) $$N_g = (-1)^\sigma (-1)^{c_1} N_\delta.$$

To see this, let F_{f_j} be a vector spanning $\ker df_j$ oriented by the fiber convention

(45) $$[\overline{\mathcal{M}}_{0,\star_j,l_j}(\vec{d}_j)] = F_{f_j}[\check{\mathcal{M}}_j].$$

We have

$$O_j F_{f_j}[\check{\mathcal{M}}_j] = [\overline{\mathcal{M}}_{0,\star_j,l_j}(\vec{d}_j)] = (-1)^{s_j}(\mathrm{ev}_j^* TL)[\overline{\mathcal{M}}_{0,\varnothing,l_j}(\vec{d}_j)] = (-1)^{s_j}(\mathrm{ev}_j^* TL) O_j[\check{\mathcal{M}}_j],$$

where the second equality follows from the orientation formula (84), so

(46) $$F_{f_j} = (-1)^{s_j+c_j} \mathrm{ev}_j^* TL$$

Plugging (46) and (45) in (42) we find

$$(\mathrm{ev}_1^* TL)(\mathrm{ev}_2^* TL) = (-1)^\sigma (-1)^{c_1} N_g \mathrm{ev}^* TL.$$

Comparing this to (43) gives (44).

We use this to explain the sign appearing between the first and second line in (37). Indeed it is not hard to see that, had we used a Thom form $\tau^!$ constructed using the orientation of N_g, there would be no sign. Instead, we defined $\tau = \mathrm{ev}_{12}^* \tau_0$, so there is a sign from (44).

Step (II) is an equality of forms. The sign of step (III) requires no additional explanation. The sign of step (IV) comes from the calculation (39), which follows from (46). This concludes the proof of the lemma. $\qquad\square$

Proof of Proposition 3.10. By definition,

$$\frac{1}{2} \int_{\widetilde{\mathcal{C}}^c(\varnothing,l,\vec{d})} Q_{\leq c}^{l,\vec{d}} \alpha_{l,\vec{d}} = \frac{1}{2 \cdot c! \cdot (-2u)^c} \sum_{\sigma_1,\sigma_2,l_1,l_2,\vec{d}_1,\vec{d}_2} \int_C d\theta_1 \cdots d\theta_c \, g^* f^* \left(\alpha_{l_1,\vec{d}_1} \alpha_{l_2,\vec{d}_2} \right),$$

where

$$C = C\left(\sigma_1,\sigma_2,l_1,l_2,\vec{d}_1,\vec{d}_2\right) = \partial_{nd}^{\sigma_1}\overline{\mathcal{M}}_{0,\star_1,l_1}(\vec{d}_1) \times_L \partial_{nd}^{\sigma_2}\overline{\mathcal{M}}_{0,\star_2,l_2}(\vec{d}_2).$$

We have

$$\frac{1}{2 \cdot c! \cdot (-2u)^c} \int_C d\theta_1 \cdots d\theta_c g^* f^* \left(\alpha_{l_1,\vec{d}_1} \alpha_{l_2,\vec{d}_2} \right) =$$

$$= \frac{(-1)^{c-1}(-1)^\sigma}{2 \cdot c! \cdot (-2u)^c} \int_C d\theta_c \, g^* f^* \left(d\theta_{\sigma_1^1} \cdots d\theta_{\sigma_1^{c_1}} \alpha_{l_1,\vec{d}_1} \, d\theta_{\sigma_2^1} \cdots d\theta_{\sigma_2^{c_2}} \alpha_{l_2,\vec{d}_2} \right) =$$

$$= \frac{(-1)^{c-1}(-1)^\sigma}{2 \cdot c \cdot (-2u)} \cdot \frac{c_1! \, c_2!}{(c-1)!} \int_C \mathrm{ev}^* d\theta \, g^* f^* \Gamma =$$

$$= \frac{(d_+^1 - d_-^1)(d_+^2 - d_-^2)}{2 \cdot (-2u)} \cdot \frac{1}{c} \cdot \frac{c_1! \, c_2!}{(c-1)!} \int_{\partial_{nd}^{\sigma_1}\overline{\mathcal{M}}_{0,\varnothing,l_1}(\vec{d}^1)} \Gamma_1 \cdot \int_{\partial_{nd}^{\sigma_2}\overline{\mathcal{M}}_{0,\varnothing,l_2}(\vec{d}^2)} \Gamma_2,$$

where we used Lemma 3.11 in the last step.

Summing over all $\sigma_1 \sqcup \sigma_2 = \{1, ..., c-1\}$ with $|\sigma_j| = c_j$, then over all $c_1 + c_2 = c - 1$, $\mathsf{l}_1 \sqcup \mathsf{l}_2 = \mathsf{l}$ and $\vec{d}_1 + \vec{d}_2 = \vec{d}$, we find that

$$\int_{\mathcal{C}^c(\varnothing, \mathsf{l}, \vec{d})} Q_{\leq c}^{\mathsf{l}, \vec{d}} \alpha_{\mathsf{l}, \vec{d}} = \frac{1}{2} \cdot \frac{(d_1^+ - d_1^-)(d_2^+ - d_2^-)}{(-2u)} \frac{1}{c} \sum \int_{\partial_{nd}^{c_1} \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}_1}(\vec{d}_1)} Q_{\leq c_1}^{\mathsf{l}_1, \vec{d}_1} \alpha_{\mathsf{l}_1, \vec{d}_1} \times \int_{\partial_{nd}^{c_2} \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}_2}(\vec{d}_2)} Q_{\leq c_2}^{\mathsf{l}_2, \vec{d}_2} \alpha_{\mathsf{l}_2, \vec{d}_2}.$$

The statement of the proposition follows, since we can enlarge the domains of integration to include the degenerate corners

$$\mathcal{C}^{c_j}\left(\varnothing, \mathsf{l}_j, \vec{d}_j\right) \supset \partial_{nd}^{c_j} \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}_j}\left(\vec{d}_j\right)$$

without changing the value of the integral. $\qquad \square$

3.5. Contribution of the exceptional boundary.

Lemma 3.12. *The contribution of the exceptional boundary is*

$$\int_{\mathcal{E}} Q_1^{\mathsf{l}, \vec{d}}\Big|_{\mathcal{E}} \alpha_{\mathsf{l}, (d,d)} = \int_{\mathcal{E}} \left(-\frac{1}{2u}\right) \cdot (\mathrm{ev}_\star)^* d\theta f^* \beta_{\mathsf{l}, d} = \frac{-d}{(-2u)} \int_{\overline{\mathcal{M}}_{0, \mathsf{l}}(d)} \beta_{\mathsf{l}, d},$$

where f is the map which forgets $\star$.

Proof. By part (d) of Definition 2.10,

$$\int_{\mathcal{E}} Q_1^{\mathsf{l}, \vec{d}}\Big|_{\mathcal{E}} \alpha_{\mathsf{l}, (d,d)} = \int_{\mathcal{E}} \left(-\frac{1}{2u}\right) \cdot (\mathrm{ev}_\star)^* d\theta f^* \beta_{\mathsf{l}, d}.$$

Let $(\mathcal{M}_\varepsilon)_{\varepsilon \in (0,1]}$ be a decreasing family of compact, S^1–invariant suborbifolds with boundary of $\overline{\mathcal{M}}_{0, \mathsf{l}}(d)$ whose union is $\mathcal{M}_{0, \mathsf{l}}(d)$. Let $\mathcal{E}_\varepsilon = f^{-1}(\mathcal{M}_\varepsilon)$. We have

$$\int_{\mathcal{E}} \left(-\frac{1}{2u}\right) \cdot (\mathrm{ev}_\star)^* d\theta f^* \beta_{\mathsf{l}, d} = \lim_{\varepsilon \to 0} \int_{\mathcal{E}_\varepsilon} \left(-\frac{1}{2u}\right) \cdot (\mathrm{ev}_\star)^* d\theta f^* \beta_{\mathsf{l}, d}.$$

Indeed, $\overline{\mathcal{M}}_{0, \mathsf{l}}(d) \backslash \mathcal{M}_{0, \mathsf{l}}(d)$, and hence also $\overline{\mathcal{E}}_{0, \mathsf{l}}(d) \backslash \mathcal{E}_{0, \mathsf{l}}(d)$, are unions of strata with codimension at least 1 to which the forms extend. Similarly,

$$\int_{\overline{\mathcal{M}}_{0, \mathsf{l}}(d)} \beta_{\mathsf{l}, d} = \lim_{\varepsilon \to 0} \int_{\mathcal{M}_\varepsilon} \beta_{\mathsf{l}, d}.$$

It is therefore enough to show for all $\varepsilon > 0$

$$\int_{\mathcal{E}_\varepsilon} \left(-\frac{1}{2u}\right) \cdot (\mathrm{ev}_\star)^* d\theta f^* \beta_{\mathsf{l}, d} = (d/2u) \int_{\mathcal{M}_\varepsilon} \beta_{\mathsf{l}, d}.$$

Now, $f|_{\mathcal{E}_\varepsilon} : \mathcal{E}_\varepsilon \to \mathcal{M}_\varepsilon$ is a submersion. We can therefore calculate the left-hand side of the last equation by integration of $(\mathrm{ev}_\star)^* d\theta$ along the fibers of f. The integral along any fiber equals $-d$, by Lemma 6.19, which completes the proof of the claim. $\qquad \square$

3.6. Solution to recursion.
For $\delta = \left(\mathsf{l}, \vec{d}\right) \in \mathcal{S}\left(\tilde{\mathsf{l}}, \vec{\tilde{d}}\right)$ we write $\mathcal{M}_\delta = \overline{\mathcal{M}}_{0, \varnothing, \mathsf{l}}\left(\vec{d}\right)$, and define $P_c^\delta \in \mathbb{R}[u]$ for $c \geq 0$ by

$$P_c^\delta = \begin{cases} \int_{\partial^c \mathcal{M}_\delta} Q_{\leq c}^\delta \alpha_\delta, & \text{if } c \neq 1, \\ \int_{\partial \mathcal{M}_\delta \backslash \mathcal{E}} Q_1^\delta \alpha_\delta, & \text{if } c = 1. \end{cases}$$

Recall that $\mathcal{E}$ denotes the exceptional boundary and that we use the definitions of $G(\delta), \hat{\mathcal{A}}(T)$ from Notation 3.8.

We can combine Propositions 3.1, 3.10, Lemma 3.12 and the description of the boundary and corners, Proposition 2.7, to obtain the following pair of equations:

$$(47) \qquad \sum_{c \geq 0} (-1)^c P_c^\delta = G(\delta) \quad \text{for} \quad \delta \in \mathcal{S}\left(\tilde{\mathsf{l}}, \tilde{\vec{d}}\right),$$

$$(48) \qquad P_c^\delta = \sum_{\delta_1 \# \delta_2 = \delta} \frac{1}{2} \frac{(d_1^+ - d_1^-)(d_2^+ - d_2^-)}{-2u} \frac{1}{c} \sum_{c_1 + c_2 = c-1} P_{c_1}^{\delta_1} P_{c_2}^{\delta_2} \quad \text{for} \quad c \geq 1 \quad \text{and} \quad \delta \in \mathcal{S}\left(\tilde{\mathsf{l}}, \tilde{\vec{d}}\right).$$

Lemma 3.13. *Treating $\{P_c^\delta\}$ as unknowns, there exists a unique solution to (47) and (48).*

Proof. Define a partial order

$$\left(c_1, \mathsf{l}_1, \vec{d}_1\right) \leq \left(c_2, \mathsf{l}_2, \vec{d}_2\right)$$

iff $c_1 \leq c_2$ and $\mathsf{l}_1 \subseteq \mathsf{l}_2$ and $d_1^+ \leq d_2^+$ and $d_1^- \leq d_2^-$. We see that (47) and (48) are upper-triangular; more precisely, if $c = 0$ we can use (47) to express P_c^δ in terms of $\left\{ P_{c'}^{\delta'} \,|\, (c', \delta') < (c, \delta) \right\}$ and if $c \geq 1$ we can use (48) to do so. This proves existence and uniqueness of a solution $\{P_c^\delta\}$. $\qquad\square$

We now describe an anzats for this.

Lemma 3.14. *The unique solution to (47)-(48) is given by*

$$P_c^{\mathsf{l}, \vec{d}} = \sum_{n \geq 0} \binom{n}{c} \sum_{T \in \hat{\mathcal{T}}(n, \mathsf{l}, \vec{d})} \hat{\mathcal{A}}(T).$$

Proof. Let us check that this satisfies equation (47). Fix a tree $T \in \hat{\mathcal{T}}\left(n, \mathsf{l}, \vec{d}\right)$. If $n > 0$, the contribution of $\hat{\mathcal{A}}(T)$ to $\sum_{c \geq 0} (-1)^c P_c^{\mathsf{l}, \vec{d}}$ is given by

$$\left(\sum_{c \geq 0} (-1)^c \binom{n}{c} \right) \hat{\mathcal{A}}(T) = 0.$$

If $n = 0$, T has a single vertex labeled $\left(\mathsf{l}, \vec{d}\right)$ and the contribution of $\hat{\mathcal{A}}(T)$ to $\sum_{c \geq 0} (-1)^c P_c^{\mathsf{l}, \vec{d}}$ is $G\left(\mathsf{l}, \vec{d}\right)$.

Let us check that this satisfies equation (48). Rewrite the right-hand side of (48) as a sum over tuples $(\sigma, T_1, v_1, T_2, v_2)$, where $\sigma = (\sigma_1, \sigma_2)$ is a partition $\sigma_1 \sqcup \sigma_2 = [n-1]$ with $|\sigma_i| = n_i$ and for $i = 1, 2$ we have $T_i \in \hat{\mathcal{T}}\left(n_i, \mathsf{l}_i, \vec{d}_i\right)$ and $v_i \in V(T_i)$, and where the contribution of each such tuple is

$$(49) \qquad \binom{n-1}{n_1, n_2}^{-1} \frac{1}{2^n n!} \left(\frac{-1}{2u} \right)^n \sum_{c_1 + c_2 = c-1} \frac{1}{c} \frac{2^{n-1} n!}{2^{n_1} n_1! 2^{n_2} n_2!} \times$$

$$\times (d^+(v_1) - d^-(v_1)) \binom{n_1}{c_1} \prod_{v \in V(T_1)} (d^+(v) - d^-(v))^{\mathrm{val}(v)} \cdot G(\delta(v)) \times$$

$$\times (d^+(v_2) - d^-(v_2)) \binom{n_2}{c_2} \prod_{v \in V(T_2)} (d^+(v) - d^-(v))^{\mathrm{val}(v)} \cdot G(\delta(v)).$$

Now glue v_1 and v_2 by an oriented edge to obtain a tree T, so that

$$\prod_{j=1,2} (d^+(v_j) - d^-(v_j)) \prod_{v \in V(T_j)} (d^+(v) - d^-(v))^{\mathrm{val}(v)} = \prod_{v \in V(T)} (d^+(v) - d^-(v))^{\mathrm{val}(v)}.$$

We use the partition σ and the orders on the edges of T_1 and of T_2 to number the edges of T, labeling the new edge (v_1, v_2) by $n = n_1 + n_2 + 1$. This defines a bijection

$$\{(\sigma, v_1, T_1, v_2, T_2)\} \simeq \left\{ T \in \hat{\mathcal{T}}\left(n, \mathsf{l}, \vec{d}\right) \right\}.$$

Plugging in the identity

$$\sum_{c_1+c_2=c-1} \frac{1}{c}\binom{n_1}{c_1}\binom{n_2}{c_2}\frac{n!}{n_1!n_2!} = \binom{n}{c}\binom{n-1}{n_1,n_2}$$

completes the proof of (48). □

Proof of Theorem 3.9. Theorem 3.9 is the case $c = 0$ of Lemma 3.14. □

4. Proofs of the main theorems

4.1. **Fixed-point contributions for stationary descendent integrals.** Recall Notation 3.8. Let $\{\alpha_{l,\vec{d}}\}, \{\beta_{l,d}\}$ be a coherent integrand for the tautological line bundles, we set

(50) $$F\left(l,\vec{d},\vec{a},\vec{\epsilon}\right) = F^{\{\alpha_\delta\}_{\delta\in\mathcal{D}},\{\beta_\delta\}_{\delta\in\mathcal{S}}}\left(l,\vec{d}\right), \qquad E\left(l,\vec{d},\vec{a},\vec{\epsilon}\right) = E^{\{\alpha_\delta\}_{\delta\in\mathcal{D}},\{\beta_\delta\}_{\delta\in\mathcal{S}}}\left(l,\vec{d}\right).$$

The following lemma shows that for given $\vec{a},\vec{\epsilon}$, E, F of (50) are independent of the choices made in the definition of the coherent integrand, and calculates them.

Lemma 4.1. *(a) For a sphere moduli specification* (l, d)*, we have*

$$\frac{2u}{d}E(l,d,\vec{a},\vec{\epsilon}) = \int_{\overline{\mathcal{M}}_{0,l}(d)} \beta_{l,d} = I\left(l,d,\vec{a},\vec{\epsilon}\right),$$

where $I(S,\vec{a},\vec{\epsilon})$ *is given in Definition 1.8. In particular, for fixed* $l,d,\vec{a},\vec{\epsilon}$*,* $E(l,d,\vec{a},\vec{\epsilon})$ *is independent of choices.*
(b) For a disk moduli specification $\left(l,\vec{d}\right)$*, we have*

$$F\left(l,\vec{d},\vec{a},\vec{\epsilon}\right) = I\left(l,\vec{d},\vec{a},\vec{\epsilon}\right).$$

In particular, for fixed $l,\vec{d},\vec{a},\vec{\epsilon}$ *it is independent of choices.*
(c) Let S be a moduli specification with several connected components. Then we have

$$I\left(S,\vec{a},\vec{\epsilon}\right) = \prod I\left(S',\vec{a}|_{S'},\vec{\epsilon}|_{S'}\right),$$

where the product is taken over the connected components and $\vec{a}|_{S'},\vec{\epsilon}|_{S'}$ *are the restrictions of* $\vec{a},\vec{\epsilon}$ *to the labels of* S'*.*

Proof. Part (a) is standard [11, 20, 15], and part (c) is immediate, so we prove the formula of part (b). The independence of choices is clear from the formula.

The map (17) induces a map of fixed-point stacks

$$\overline{\mathcal{M}}_{0,\varnothing,l}\left(\vec{d}\right)^{S^1} \to \left(\overline{\mathcal{M}}_{0,l\times\{1,2\}}\left(\textstyle\sum\vec{d}\right)^{S^1}\right).$$

We find that this map factors through

$$\overline{\mathcal{M}}_{0,\varnothing,l}\left(\vec{d}\right)^{S^1} \to \left(\overline{\mathcal{M}}_{0,l\times\{1,2\}}\left(\textstyle\sum\vec{d}\right)^{S^1}\right)^{\mathbb{Z}/2},$$

which, assuming $l \ne \varnothing$, is the inclusion of a clopen component (see [23, §3.2] for a similar argument).

Now use the description of the fixed points of the moduli of stable *closed* maps to deduce that

$$\overline{\mathcal{M}}_{0,\varnothing,l}\left(\vec{d}\right) \simeq \bigsqcup_{\Gamma}\left(\overline{\mathcal{M}}_\Gamma\right)_{A_\Gamma},$$

where Γ ranges over isomorphism classes of fixed-point graphs Γ for $\left(l,\vec{d}\right)_{\mathcal{D}}$.

To compute the normal bundle of $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{0,\varnothing,l}\left(\vec{d}\right)$, we proceed as in [23, §3.2], and use the orientation results of Section 6. Using the short exact sequence (90), the short exact sequence in Lemma 6.18, and the well-known formula for the normal bundle to the fixed points in the closed

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case [15], we see that we only need to compute the normal bundle to an S^1-invariant smooth disk-map, justifying the contribution of the disk edges in the last factor of (6).

More precisely, write $\mathcal{M}_+ = \overline{\mathcal{M}}_{0,\varnothing,\varnothing}((d,0))$ and $\mathcal{M}_- = \overline{\mathcal{M}}_{0,\varnothing,\varnothing}((0,d))$. We denote $\mu = \mu(h) \in \{\pm\}$. There is a unique S^1 fixed point $q_\mu \in \mathcal{M}_\mu$, represented by a disk mapping to a hemisphere with one branch point at p_μ with ramification profile a single cycle of length d. We need to show that

$$(51) \qquad e_\mu = e^{S^1}\left(T_{q_\pm}\mathcal{M}_\pm\right) = \frac{\mu}{d!}\left(\mu\frac{2u}{d}\right)^{d-1}.$$

Let $q \in \overline{\mathcal{M}}_{0,\varnothing}(d)$ denote the double of q_μ, which is the unique fixed point of the moduli of stable disk-maps with smooth domain. The involution that conjugates the map acts on $T_q\overline{\mathcal{M}}_{0,\varnothing}(d)$ as an anti-holomorphic involution and we have

$$T_{q_\mu}\mathcal{M}_\pm = \left(T_q\overline{\mathcal{M}}_{0,\varnothing}(d)\right)^{\mathbb{Z}/2}$$

as S^1-representations, so

$$e_\mu^2 = e^{S^1}\left(T_q\overline{\mathcal{M}}_{0,\varnothing}(d)\right) = (-1)^d\frac{(d!)^2(2u)^{2d-2}}{d^{2d-2}},$$

which gives the value of e up to a sign:

$$e = s \cdot \frac{d^{d-1}}{d!\,(2u)^{d-1}}, \quad s = s(\mu) \in \{\pm 1\}.$$

Let us compute s. We have a diagram

$$\mathcal{M}_\pm \to \left(D^2\right)^{d-1}/\operatorname{Sym}(d-1) \leftarrow \left(D^2\right)^{d-1},$$

where both maps are equivariant, holomorphic, non-constant maps between spaces of the same dimension: the left map is induced from the branch divisor, the right map is the quotient. This implies that if we orient $T_{q_\pm}\mathcal{M}_\pm$ using the complex orientation, then

$$e^{S^1}\left(T_{q_\pm}\mathcal{M}_\pm\right) = c \cdot e^{S^1}\left(T_{(0,\dots,0)}\left(D^2\right)^{d-1}\right),$$

where $c \in \mathbb{Q}$ is *positive*. The sign of $e^{S^1}\left(T_{(0,\dots,0)}\left(D^2\right)^{d-1}\right)$ is $(\mu)^{d-1}$; since, by (84), the orientation of $T_{q_\pm}\mathcal{M}_\pm$ is twisted by $\operatorname{sgn}(d_- \le d_+)$ relative to the complex orientation, we conclude that $s = \mu^d$. Equation (51) is proved. $\qquad\square$

Proof of Theorem 2.18. The 'Moreover' part follows immediately from applying Lemma 4.1 to the calculation of the amplitudes $\hat{\mathcal{A}}(T)$ in (32). The passage from the trees with labeled oriented edges of $\bigsqcup_r \hat{\mathcal{T}}(r,\mathsf{I},\vec{d})$ to trees of $\mathcal{T}(\mathsf{I},\vec{d})$ without this additional data is responsible to the difference in combinatorial factors, by standard Orbit-Stabilizer argument. The independence of choices is a consequence of the 'Moreover' part.

The vanishing statement follows immediately from the fact the equivariant integral vanishes on forms whose de Rham degree is less than the dimension of the domain, and this is the case for $\langle\prod_{i\in\mathsf{I}}\tau_{a_i}^{\epsilon_i}\rangle_{0,\vec{d}}$ whenever $1 + \sum_{i\in\mathsf{I}}a_i < d^+ + d^-$.

Note that, because fixed points components can be of any dimension, the individual contributions to $\mathrm{OGW}(\mathsf{I},\vec{d},\vec{a},\vec{\epsilon})$ don't have to vanish, and so the vanishing provides a non-trivial relation. $\qquad\square$

We end this section with a simplification of the formula (11) for OGW. Recall the famous theorem of Cayley [7], in its weighted version.

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Theorem 4.2. *Associate a number x_v to every vertex $v \in [n]$, and associate the monomial $\prod_{v \in [n]} x_v^{\mathrm{val}_T(v)}$ to every vertex-labeled tree T on the set of vertices $[n]$. Then we have*

$$\sum_T \prod_{i \in [n]} x_i^{\mathrm{val}_T(i)} = \prod x_i \left(\sum_{i \in [n]} x_i \right)^{n-2},$$

where the summation is taken over all trees with vertices labeled by $[n]$.

Let $\mathcal{P}(\mathsf{l}, \vec{d})$ be the set of *ordered partitions* of $(\mathsf{l}, \vec{d})$ meaning sets of disk specifications

$$((\mathsf{l}_1, \vec{d}_1), \ldots, (\mathsf{l}_r, \vec{d}_r)) \text{ with } \bigsqcup \mathsf{l}_i = \mathsf{l}, \ \sum \vec{d}_i = \vec{d}.$$

Given $\vec{a}$ and $\vec{\epsilon}$ we define the *amplitude* of $P = ((\mathsf{l}_1, \vec{d}_1), \ldots, (\mathsf{l}_r, \vec{d}_r))$ to be

$$\mathcal{A}^{\mathcal{P}}(P, \vec{a}, \vec{\epsilon}) = \frac{1}{(-2u)^{r-1} r!} (d^+ - d^-)^{r-2} \prod_{i \in r} (d_i^+ - d_i^-) . \prod_{i=1}^r I(\mathsf{l}_i, \vec{d}_i, \vec{a}|_{\mathsf{l}_i}, \vec{\epsilon}|_{\mathsf{l}_i}).$$

Corollary 4.3. *We have*

$$\mathrm{OGW}(\mathsf{l}, \vec{d}, \vec{a}, \vec{\epsilon}) = \sum_{P \in \mathcal{P}(\mathsf{l}, \vec{d})} \mathcal{A}^{\mathcal{P}}(P, \vec{a}, \vec{\epsilon}) + \delta_{d^+ - d^-} \frac{d^+}{2u} I(\mathsf{l}, |\vec{d}|, \vec{a}, \vec{\epsilon}).$$

The proof is immediate from applying Cayley's theorem to formula (11). The difference in automorphism factors follows easily from the fact that trees in $\mathcal{T}(\mathsf{l}, \vec{d})$ are not vertex-labeled.

4.2. **Divisor and TRR for genus 0 disk covers.**
In this section we prove the divisor relation, Theorem 1.11, and the topological recursion for disk covers, Theorem 1.12.

Proof of Theorem 1.11. Let S be either a sphere moduli specification (l, d) or a disk moduli specification $(\mathsf{l}, \vec{d})$, S' be the moduli specification obtained by extending l to $\mathsf{l}' = \mathsf{l} \sqcup \{1\}$, without changing the degree. Let $\vec{a}, \vec{\epsilon}$ be vectors of descendents and constraints, respectively, for l, and $\vec{a}', \vec{\epsilon}'$ be their extensions to l' obtained by defining $a_i' = a_i, \epsilon_i' = \epsilon_i$ for $i \in \mathsf{l}$ and $a_1' = 0, \epsilon_1' = \pm$. We assume $\epsilon_1 = +$ and denote in the disk case d^+ by d. The case $\epsilon_1 = -$ is treated in an analogous way. We will first show, and then use to derive the theorem, that

$$(52) \qquad \frac{I(S', \vec{a}', \vec{\epsilon}')}{|\mathrm{Aut}(S')|} = d \frac{I(S, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(S)|} + \frac{2u}{|\mathrm{Aut}(S)|} \sum_{i>1 | \epsilon_i = +} I(S, \vec{a_{(i)}}, \vec{\epsilon}),$$

where the fixed point contribution I is given by (8) and $\vec{a_{(i)}}$ is the vector which is obtained from $\vec{a}$ by decreasing a_i by 1. There is an obvious forgetful map for_1 from fixed point graphs for S' to those of S. Equation (52) will follow if we could show that for any fixed point graph Γ for S we have

$$(53) \qquad \sum_{\Gamma' \in \mathrm{for}^{-1}(\Gamma)} \frac{I(\Gamma', \vec{a}', \vec{\epsilon}')}{|\mathrm{Aut}(\Gamma')|} = d \frac{I(\Gamma, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma)|} + 2u \sum_{i>1 | \epsilon_i = +} \frac{I(\Gamma, \vec{a_{(i)}}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma)|}.$$

Observe that $\mathrm{for}_1^{-1}(\Gamma)$ is in bijection with the orbits of the action of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$, where Γ' corresponds to $[v] \in V(\Gamma)/\mathrm{Aut}(\Gamma)$ if it is obtained from Γ by adding the tail marked 1 to some $v \in [v]$ with $\mu(v) = +$. Moreover, $\mathrm{Aut}(\Gamma')$ in this case is isomorphic to the subgroup of $\mathrm{Aut}(\Gamma)$ which fixes some $v \in [v]$, hence

$$(54) \qquad \frac{|\mathrm{Aut}(\Gamma)|}{|\mathrm{Aut}(\Gamma')|} = |\mathrm{Orb}(v)|,$$

the size of the orbit of v under $\mathrm{Aut}(\Gamma)$.

Recall (6) and (7). The analysis for Γ' which corresponds to $[v]$, depends on $[v]$ as follows.

(a) If $\lambda(v) = \varnothing$, $\operatorname{val} v = \{f\}$ then after adding the marking 1 to v, we see that

$$e_{\Gamma'}^{-1} = \frac{|\delta(f)|}{2u} e_{\Gamma}^{-1}, \quad \alpha_{\Gamma'}^{\vec{a}',\vec{\epsilon}'} = 2u\alpha_{\Gamma}^{\vec{a},\vec{\epsilon}} \Rightarrow I(\Gamma', \vec{a}', \vec{\epsilon}') = |\delta(f)|I(\Gamma, \vec{a}, \vec{\epsilon}).$$

The first equality follows from the change in the fourth term of (6) which corresponds to v and the second equality follows from the change in the second term of (7) which corresponds to v.

(b) If $\lambda(v) = \varnothing$, $\operatorname{val} v = \{f_1, f_2\}$ then after adding the marking 1 to v, v becomes a vertex in V_+ which represents the moduli $\overline{\mathcal{M}}_{0,3}$, with the three markings 1 and the two nodes which correspond to f_1, f_2. We have

$$e_{\Gamma'}^{-1} = \frac{(2u)(\omega_{f_1} + \omega_{f_2})}{\omega_{f_1}\omega_{f_2}} e_{\Gamma}^{-1}, \quad \alpha_{\Gamma'}^{\vec{a}',\vec{\epsilon}'} = (2u)\alpha_{\Gamma}^{\vec{a},\vec{\epsilon}} \Rightarrow I(\Gamma', \vec{a}', \vec{\epsilon}') = (|\delta(f_1)| + |\delta(f_2)|)I(\Gamma, \vec{a}, \vec{\epsilon}),$$

where the change in the inverse Euler term comes from the first three terms in (6).

(c) If $\lambda(v) = \{i\}$, with $a_i = 0$ then after adding the marking 1 to v, v becomes a vertex in V_+ which represents the moduli $\overline{\mathcal{M}}_{0,3}$, with the three markings $1, i$ and the node. The α term (7) changes by $(2u)$ again, and if we write $\operatorname{val} v = \{f\}$ we see that

$$e_{\Gamma'}^{-1} = \frac{|\delta(f)|}{2u} e_{\Gamma}^{-1} \Rightarrow I(\Gamma', \vec{a}', \vec{\epsilon}') = |\delta(f)|I(\Gamma, \vec{a}, \vec{\epsilon}),$$

where the change comes from the first term of (6) which corresponds to f, expanded to zeroth order in ψ_f, and the $(2u)^{-1}$ is contributed by the second term of (6). Note that in this case, before adding 1, $|\operatorname{Orb}(v)| = 1$ since it carries the ith label.

(d) Suppose now $\lambda(v) = \{i\}$, with $a_i > 0$. Write $\operatorname{val} v = \{f\}$. Again $|\operatorname{Orb}(v)| = 1$. After adding the marking 1 to v, v becomes a vertex in V_+ which represents the moduli $\overline{\mathcal{M}}_{0,3}$, with the three markings $1, i$ and the node. Since $a_i > 0$ but $\overline{\mathcal{M}}_{0,3}$ is zero dimensional, the α term, and hence also $I(\Gamma', \vec{a}', \vec{\epsilon}')$ vanish. However, comparing the second expressions in $\alpha_{\Gamma}^{\vec{a},\vec{\epsilon}}$ and in $\alpha_{\Gamma}^{\vec{a}_{(i)},\vec{\epsilon}}$

$$|\delta(f)|I(\Gamma, \vec{a}, \vec{\epsilon}) = -(2u)I(\Gamma, \vec{a}_{(i)}, \vec{\epsilon}) \Rightarrow I(\Gamma', \vec{a}', \vec{\epsilon}') = |\delta(f)|I(\Gamma, \vec{a}, \vec{\epsilon}) + (2u)I(\Gamma, \vec{a}_{(i)}, \vec{\epsilon}).$$

(e) The last case is that v represents a contracted component with $\lambda(v) = \mathsf{l}_1$ and $\operatorname{val} v = \{f_1, \ldots, f_s\}$. From combining the first terms in (6),(7) we see that the multiplicative contribution of v before adding 1 is

$$\sum_{\substack{b_1,\ldots,b_s \geq 0, \\ \sum b_j = |\mathsf{l}_1| + s - 3 - \sum_{i \in \mathsf{l}_1} a_i}} \prod_{j=1}^{s} \frac{|\delta(f_j)|^{b_j+1}}{(2u)^{b_j}} \left\langle \prod_{i \in \mathsf{l}_1} \tau_{a_i} \prod_{j=1}^{s} \tau_{b_j} \right\rangle^c,$$

where $\langle\cdots\rangle^c$ stands for closed intersection numbers over the moduli of stable marked spheres. Adding the marking 1 changes the contribution of v to

(55)

$$(2u) \sum_{\substack{b_1,\ldots,b_s \geq 0, \\ \sum b_j = |\mathsf{l}_1| + s - 2 - \sum_{i \in \mathsf{l}_1} a_i}} \prod_{j=1}^{s} \frac{|\delta(f_j)|^{b_j+1}}{(2u)^{b_j}} \left\langle \tau_0 \prod_{i \in \mathsf{l}_1} \tau_{a_i} \prod_{j=1}^{s} \tau_{b_j} \right\rangle^c =$$

$$= (2u) \sum_{\substack{b_1,\ldots,b_s \geq 0, \\ \sum b_j = |\mathsf{l}_1| + s - 2 - \sum_{i \in \mathsf{l}_1} a_i}} \prod_{j=1}^{s} \frac{|\delta(f_j)|^{b_j+1}}{(2u)^{b_j}} \left(\sum_{k \in \mathsf{l}_1} \left\langle \tau_{a_k-1} \prod_{i \in \mathsf{l}_1, i \neq k} \tau_{a_i} \prod_{j=1}^{s} \tau_{b_j} \right\rangle^c + \sum_{k \in [s]} \left\langle \tau_{b_k-1} \prod_{i \in \mathsf{l}_1, i \neq k} \tau_{a_i} \prod_{j \in [s] \setminus \{k\}} \tau_{b_j} \right\rangle^c \right),$$

where we used the string equation

(56)

$$\left\langle \tau_0 \prod_{i=1}^{n} \tau_{a_i} \right\rangle^c = \sum_{j=1}^{n} \left\langle \tau_{a_j-1} \prod_{i \in [n] \setminus \{j\}} \right\rangle^c.$$

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Substituting (55) in the expression for $I(\Gamma', \vec{a'}, \vec{\epsilon'})$ we obtain

$$(57) \qquad I(\Gamma', \vec{a'}, \vec{\epsilon'}) = (2u) \sum_{i \in I_1} I(\Gamma, \vec{a_{(i)}}, \vec{\epsilon}) + \left(\sum_{i=1}^{s} |\delta(f_i)| \right) I(\Gamma, \vec{a}, \vec{\epsilon}).$$

Summing the above items over all $[v] \in V(\Gamma)/\mathrm{Aut}(\Gamma)$, where the term for $[v]$ is taken with multiplicity $|\mathrm{Orb}(v)|$, using (54) and the definition of the fixed-point contributions (9) for $(\Gamma', \vec{a'}, \vec{\epsilon'}), (\Gamma, \vec{a}, \vec{\epsilon}), (\Gamma, \vec{a_{(i)}}, \vec{\epsilon})$ for the different i, we obtain (53) and hence (52).

We will now use (52) to derive the divisor relation. Write $I' = [l], I = \{2, \ldots, l\}$. We want to prove that

$$\mathrm{OGW}(I', \vec{d}, \vec{a'}, \vec{\epsilon'}) = d^{\pm} \mathrm{OGW}(I, \vec{d}, \vec{a}, \vec{\epsilon}) + (2u) \sum_{i \in I | \epsilon_i = \epsilon_1} \mathrm{OGW}(I, \vec{d}, \vec{a_{(i)}}, \vec{\epsilon}),$$

where again $\vec{a}, \vec{\epsilon}$ are vectors of descendents and constraints respectively for I, and $\vec{a'}, \vec{\epsilon'}, \vec{a_{(i)}}$ are defined as in the previous part of the proof.

We begin with considering the first term of (11) for OGW. We would like to show that

$$(58) \qquad \sum_{T \in \mathcal{T}(I', \vec{d'})} \mathcal{A}(T, \vec{a'}, \vec{\epsilon'}) = \sum_{T \in \mathcal{T}(I, \vec{d})} \left(d^{\pm} \mathcal{A}(T, \vec{a}, \vec{\epsilon}) + 2u \sum_{i \in I | \epsilon_i = \epsilon_1} \mathcal{A}(T, \vec{a_{(i)}}, \vec{\epsilon}) \right).$$

Denote by $\widetilde{\mathrm{for}}_1 : \mathcal{T}(I', \vec{d}) \to \mathcal{T}(I', \vec{d})$ the obvious forgetful map. Equation (58) will follow if we could show that for any $T \in \mathcal{T}(I, \vec{d})$ we have

$$\sum_{T' \in \widetilde{\mathrm{for}}_1^{-1}(T)} \mathcal{A}(T', \vec{a'}, \vec{\epsilon'}) = d^{\pm} \mathcal{A}(T, \vec{a}, \vec{\epsilon}) + 2u \sum_{i \in I | \epsilon_i = \epsilon_1} \mathcal{A}(T, \vec{a_{(i)}}, \vec{\epsilon}).$$

However, elements $T' \in \widetilde{\mathrm{for}}_1^{-1}(T)$ are in bijection with the equivalence classes of $[v] \in V(T)/\mathrm{Aut}(T)$, which specify to which vertex the marking 1 will be sent. For T' which corresponds to $[v]$, by (52) and the definition of the amplitude, equation (10),

$$\mathcal{A}(T', \vec{a'}, \vec{\epsilon'}) = \frac{|\mathrm{Aut}(T)|}{|\mathrm{Aut}(T')|} \left(d_v^{\pm} \mathcal{A}(T, \vec{a'}, \vec{\epsilon'}) + 2u \sum_{i \in I_v | \epsilon_i = \epsilon_1} \mathcal{A}(T, \vec{a_{(i)}}, \vec{\epsilon}) \right).$$

The group $\mathrm{Aut}(T')$ is canonically identified with the subgroup $\mathrm{Aut}(T, v)$ of $\mathrm{Aut}(T)$ preserving some $v \in [v]$. Therefore $\frac{|\mathrm{Aut}(T)|}{|\mathrm{Aut}(T')|} = |\mathrm{Orb}(v)|$, the size of the orbit of v under $\mathrm{Aut}(T)$, and in particular $\mathrm{Aut}(T') \simeq \mathrm{Aut}(T)$ when $I_v \neq \varnothing$. Thus,

$$\sum_{[v] \in V(T)/\mathrm{Aut}(T)} |\mathrm{Orb}(v)| d_v^{\pm} = \sum_{v \in V(T)} d_v^{\pm} = d^{\pm},$$

and equation (58) follows.

If $d^+ = d^-$, then the expressions of (11) for $\mathrm{OGW}(I', \vec{d}, \vec{a'}, \vec{\epsilon'}), \mathrm{OGW}(I, \vec{d}, \vec{a}, \vec{\epsilon})$ include the second (closed term). Applying (52) to $S = (I', d), \vec{a}, \vec{\epsilon}$ we see immediately that the second term satisfies the expected relation, and the divisor equation follows.

The 'In particular' part follows from taking the non-equivariant limit (order u^0 invariants) and using the fact that terms with negative powers of u vanish, by the 'In particular' part of Theorem 2.18. $\qquad \square$

Proof of Theorem 1.12. We work with $I = [l]$. Write $l_0 = |\{i > 2 | a_i = 0\}|$. We first consider the case $l_0 = 0$. In this case the first term on the RHS of (13) doesn't contribute, for degree reasons. By Theorem 2.18, we have

$$\left\langle \tau_{a_1+1}^+ \prod_{i=2}^{l} \tau_{a_i}^+ \right\rangle_{0,(d,0)} = \sum_{n \geq 1} \sum_{T \in \mathcal{T}([l],(d,0))} \hat{\mathcal{A}} \left(T, \vec{a'}, \vec{\epsilon} \right),$$

where $\vec{a}' = \vec{a} + (1, 0, \ldots, 0)$.

For any tree T which appears in the expression above and has a non-zero contribution, and any vertex v of T, it must hold that $\mathsf{l}(v)$ is either empty or a singleton ($\mathsf{l}(v)$ and the other tree notations are defined in Definition 3.7). Write $\mathcal{T}'([l], (d, 0))$ for the subset of $\mathcal{T}([l], (d, 0))$ whose elements are trees satisfying this constraint. Consider an arbitrary $T \in \mathcal{T}'([l], (d, 0))$. Let v be the vertex with $\mathsf{l}(v) = \{1\}$. There is a unique path in the tree between v and the vertex u with $\mathsf{l}(u) = \{2\}$. Let e be the first edge of this path, starting from v. Suppose it connects v to some other vertex w. Erasing e divides T into two trees, T_1 which contains v and T_2 which contains u. Denote by $R \cup \{1\}$ the label set of T_1 and by $S \cup \{2\}$ the label set of T_2. From the definition of the amplitude, equation (10), it follows that

$$(59) \qquad \mathcal{A}(T, \vec{a}', \vec{\epsilon}) = \frac{d^+(v)}{-2u} d^+(w) \mathcal{A}(T_1, \vec{a}'|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}) \mathcal{A}(T_2, \vec{a}|_{S \cup \{2\}}, \vec{\epsilon}|_{S \cup \{2\}}).$$

Note that this simple procedure gives a bijection

$$T \to J(T) = (T_1, T_2, w)$$

between $\mathcal{T}'([l], (d, 0))$ and

$$\bigsqcup_{\substack{d_1, d_2 > 0, \ d_1 + d_2 = d, \\ R, S \subseteq \{3, \ldots, l\}, \ R \sqcup S = \{3, \ldots, l\}}} \mathcal{T}'(R \cup \{1\}, (d_1, 0)) \times \bigsqcup_{T_2 \in \mathcal{T}'(S \cup \{2\}, (d_2, 0))} V(T_2).$$

For any $T_2 \in \mathcal{T}'(\mathsf{l}_{S \cup \{2\}}, (d_2, 0))$ we have

$$(60) \qquad \sum_{w \in V(T_2)} d^+(w) = d_2.$$

In addition, by Theorem 2.18, we have

$$(61) \qquad \sum_{T_2 \in \mathcal{T}'(\mathsf{l}_{S \cup \{2\}}, (d_2, 0))} \mathcal{A}(T_2, \vec{a}|_{S \cup \{2\}}, \vec{\epsilon}|_{S \cup \{2\}}) = \left\langle \tau^+_{a_2} \prod_{i \in R} \tau^+_{a_i} \right\rangle_{0, (d_2, 0)}.$$

Putting together, using (59), we see that for any d_1, R and T_1, it holds that

(62)

$$\sum_{T_2 \in \mathcal{T}'(\mathsf{l}_{S \cup \{2\}}, (d_2, 0))} \sum_{w \in V(T_2)} \mathcal{A}(J^{-1}(T_1, T_2, w), \vec{a}', \vec{\epsilon}) =$$

$$= \sum_{T_2 \in \mathcal{T}'(\mathsf{l}_{S \cup \{2\}}, (d_2, 0))} \sum_{w \in V(T_2)} \left(\frac{d^+(v)}{-2u} \mathcal{A}(T_1, \vec{a}'|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}) \right) \left(d^+(w) \mathcal{A}(T_2, \vec{a}|_{S \cup \{2\}}, \vec{\epsilon}|_{S \cup \{2\}}) \right) =$$

$$= d_2 \left(\frac{d^+(v)}{-2u} \mathcal{A}(T_1, \vec{a}'|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}, \vec{a}'|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}) \right) \left\langle \tau^+_{a_2} \prod_{i \in S} \tau^+_{a_i} \right\rangle_{0, (d_2, 0)}.$$

Observe that, by the definition of the amplitude, we have

$$\frac{d^+(v)}{-2u} \mathcal{A}(T_1, \vec{a}'|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}) = \mathcal{A}(T_1, \vec{a}|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}).$$

Thus, using Theorem 2.18 again, we get, for fixed d_1, R,

(63)

$$\sum_{T_1 \in \mathcal{T}'(\mathsf{l}_{R \cup \{1\}}, (d_1, 0))} \frac{d^+(v)}{-2u} \mathcal{A}(T_1, \vec{a}'|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}) = \sum_{T_1 \in \mathcal{T}'(\mathsf{l}_{R \cup \{1\}}, (d_1, 0))} \mathcal{A}(T_1, \vec{a}|_{R \cup \{1\}}, \vec{\epsilon}|_{R \cup \{1\}}) = \left\langle \tau^+_{a_1} \prod_{i \in R} \tau^+_{a_i} \right\rangle_{0, (d_1, 0)}.$$

Combining equations (62),(63), summing over all d_1, R and using the bijection J we obtain (13).

For the general TRR, we order the pairs (l_0, l) lexicographically, so that $(l_0', l') < (l_0, l)$ if either $l_0' < l_0$ or $l_0' = l_0$ but $l' < l$. We use induction on (l_0, l). We have proved (13) for $l_0 = 0$. Suppose we have proved it for all $(l_0', l') < (l_0, l)$. We now prove for (l_0, l). Since $l_0 \geq 1$, there is some $i > 2$ with $a_i = 0$. Without loss of generality take it to be l. Theorem 1.11 and the string equation (56) allow us to write the closed-open term of the RHS of (13),

$$(64) \qquad \sum_{R \sqcup S = \{3,\ldots,l\}} (2u)^{|R|} \left\langle \tau_{a_1} \left(\prod_{i \in R} \tau_{a_i} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

as the sum of four terms $Q_1 + \ldots + Q_4$, where

$$Q_1 = (2u) \delta_{a_1 = 0} \left\langle \tau_0^+ \prod_{i \neq 1, l} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

$$Q_2 = \sum_{R \sqcup S = \{3,\ldots,l-1\}} (2u)^{|R|+1} \sum_{j \in R \cup \{1\}} \left\langle \left(\prod_{i \in R \cup \{1\}} \tau_{a_i - \delta_{j=i}} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

$$Q_3 = d \sum_{R \sqcup S = \{3,\ldots,l-1\}} (2u)^{|R|} \left\langle \tau_{a_1} \left(\prod_{i \in R} \tau_{a_i} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

$$Q_4 = (2u) \sum_{R \sqcup S = \{3,\ldots,l-1\}} \sum_{j \in S \cup \{2\}} (2u)^{|R|} \left\langle \tau_{a_1} \left(\prod_{i \in R} \tau_{a_i} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \prod_{i \in S \cup \{2\}} \tau_{a_i - \delta_{j=i}}^+ \right\rangle_{0,(d,0)}.$$

Indeed, Q_1 is obtained from the case $R = \{l\}$ in (64), Q_2 from applying the string equation to sets $R \neq \{l\}$ which contain l, Q_3, Q_4 are obtained from applying the divisor equation to the case $l \in S$. Similarly, the open-open term of the RHS of (13),

$$(65) \qquad \sum_{\substack{R \sqcup S = \{3,\ldots,l\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1}^+ \prod_{i \in R} \tau_{a_i}^+ \right\rangle_{0,(d_1,0)} \left\langle \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d_2,0)},$$

can be written as the sum $Q_5 + Q_6$, where

$$Q_5 = (d_1 + d_2) \sum_{\substack{R \sqcup S = \{3,\ldots,l-1\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1}^+ \prod_{i \in R} \tau_{a_i}^+ \right\rangle_{0,(d_1,0)} \left\langle \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d_2,0)},$$

$$Q_6 = (2u) \sum_{\substack{R \sqcup S = \{3,\ldots,l-1\} \\ d_1 + d_2 = d}} \sum_{j \in [l-1]} d_2 \left\langle \prod_{i \in R \cup \{1\}} \tau_{a_i - \delta_{j=i}}^+ \right\rangle_{0,(d_1,0)} \left\langle \prod_{i \in S \cup \{2\}} \tau_{a_i - \delta_{j=i}}^+ \right\rangle_{0,(d_2,0)},$$

and both are obtained by applying the divisor equation to (65) in the two cases $l \in R$ and $l \in S$.

By the divisor equation, the LHS of (13) is equal to

$$(66) \qquad d \left\langle \tau_{a_1+1}^+ \prod_{i=2}^{l-1} \tau_{a_i}^+ \right\rangle_{0,(d,0)} + (2u) \left(\left\langle \tau_{a_1}^+ \prod_{i=2}^{l-1} \tau_{a_i}^+ \right\rangle_{0,(d,0)} + \sum_{j=2}^{l-1} \left\langle \tau_{a_1+1}^+ \prod_{i=2}^{l-1} \tau_{a_i - \delta_{j=i}}^+ \right\rangle_{0,(d,0)} \right).$$

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All summands in (66) are smaller in lexicographic order than (l_0, l), hence we can rely on the induction and apply TRR to (66). We will obtain the sum of seven terms $P_1 + \ldots + P_7$, where

$$P_1 = d \sum_{R \sqcup S = \{3,\ldots,l-1\}} (2u)^{|R|} \left\langle \tau_{a_1} \left(\prod_{i \in R} \tau_{a_i} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

$$P_2 = d \sum_{\substack{R \sqcup S = \{3,\ldots,l-1\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1}^+ \prod_{i \in R} \tau_{a_i}^+ \right\rangle_{0,(d_1,0)} \left\langle \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d_2,0)},$$

$$P_3 = \delta_{a_1=0}(2u) \left\langle \tau_0^+ \prod_{i=2}^{l-1} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

$$P_4 = (2u) \sum_{R \sqcup S = \{3,\ldots,l-1\}} (2u)^{|R|} \left\langle \tau_{a_1-1} \left(\prod_{i \in R} \tau_{a_i} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d,0)},$$

$$P_5 = (2u) \sum_{\substack{R \sqcup S = \{3,\ldots,l-1\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1-1}^+ \prod_{i \in R} \tau_{a_i}^+ \right\rangle_{0,(d_1,0)} \left\langle \tau_{a_2}^+ \prod_{i \in S} \tau_{a_i}^+ \right\rangle_{0,(d_2,0)},$$

$$P_6 = (2u) \sum_{j=2}^{l-1} \sum_{R \sqcup S = \{3,\ldots,l-1\}} (2u)^{|R|} \left\langle \tau_{a_1} \left(\prod_{i \in R} \tau_{a_i - \delta_{i=j}} \right) \tau_0 \right\rangle_0^c \left\langle \tau_0^+ \prod_{i \in S \cup \{2\}} \tau_{a_i - \delta_{i=j}}^+ \right\rangle_{0,(d,0)},$$

$$P_7 = (2u) \sum_{j=2}^{l-1} \sum_{\substack{R \sqcup S = \{3,\ldots,l-1\} \\ d_1 + d_2 = d}} d_2 \left\langle \tau_{a_1}^+ \prod_{i \in R} \tau_{a_i - \delta_{i=j}}^+ \right\rangle_{0,(d_1,0)} \left\langle \prod_{i \in S \cup \{2\}} \tau_{a_i - \delta_{i=j}}^+ \right\rangle_{0,(d_2,0)}.$$

Here P_1, P_2 are obtained from applying TRR to the first term in (66), P_4, P_5 are obtained from applying TRR to the second term, assuming $a_1 \neq 0$, P_6, P_7 are obtained from applying TRR to the last term. When $a_1 = 0$ we cannot apply TRR, and P_3 is the contribution in this case. The induction will follow if we could show that $Q_1 + \ldots + Q_6 = P_1 + \ldots + P_7$. Indeed,

$$Q_1 = P_3, \quad Q_3 = P_1, \quad Q_5 = P_2, \quad Q_6 = P_5 + P_7, \quad Q_2 + Q_4 = P_4 + P_6,$$

and the theorem follows.

The 'In particular' part follows from taking the non-equivariant limit (order u^0 invariants) and using the fact that terms with negative powers of u vanish, by the 'In particular' part of Theorem 2.18. In this case also the closed term $\langle \rangle_c$ vanishes. $\square$

4.3. **Genus** 0 **disk cover formula and genus** 0 (d, d)**–vanishing.**

Proof of Theorem 1.10. For shortness, throughout the proof we will omit the superscripts $+$ in the symbols $\tau_{a_i}^+$, as well as the genus and other superscripts and subscripts which are not the degree, since we consider disks which are mapped to upper hemisphere. Stationary invariants satisfy the following three properties:

$$(67) \qquad \langle \rangle_{(1,0)} = 1,$$

$$(68) \qquad \left\langle \tau_0 \prod_{i=1}^n \tau_{a_i} \right\rangle_{(1+a_{[n]},0)} = (1 + a_{[n]}) \left\langle \prod_{i=1}^n \tau_{a_i} \right\rangle_{(1+a_{[n]},0)}, \qquad\qquad n \geq 0,$$

$$(69) \qquad \left\langle \tau_{a_1+1} \prod_{i=2}^n \tau_{a_i} \right\rangle_{(2+a_{[n]},0)} = \sum_{\substack{I \sqcup J = [n] \\ 1 \in I, \, n \in J}} (1 + a_J) \left\langle \prod_{i \in I} \tau_{a_i} \right\rangle_{(1+a_I,0)} \left\langle \prod_{j \in J} \tau_{a_j} \right\rangle_{(1+a_J,0)}, \qquad n \geq 2.$$

Here for an n-tuple $a_1, \ldots, a_n$ and a subset $I \subset [n]$ we use the notation $a_I = \sum_{i \in I} a_i$. Note that (69) is a special case of Theorem 1.12, and (68) is a special case of Theorem 1.11.

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We prove (12) by induction on n and on $\sum a_i$. The case $n = 0$ follows from (67). Suppose $n = 1$. By (68) and (67), $\langle \tau_0 \rangle_{(1,0)} = 1$. Suppose that $a \geq 1$. Then we have

$$\langle \tau_a \rangle_{(a+1,0)} \overset{\text{by (68)}}{=} \frac{1}{a+1} \langle \tau_a \tau_0 \rangle_{(a+1,0)} \overset{\text{by (69)}}{=} \frac{1}{a+1} \langle \tau_{a-1} \rangle_{(a,0)} \langle \tau_0 \rangle_{(1,0)} = \frac{1}{a+1} \langle \tau_{a-1} \rangle_{(a,0)}.$$

Therefore, $\langle \tau_a \rangle_{(a+1,0)} = \frac{1}{(a+1)!}$.

Suppose $n \geq 2$. If all a_i's are zero, then $\langle \tau_0^n \rangle_{(1,0)} = 1$. Suppose some of a_i's are not zero. Without loss of generality we can assume that $a_1 \geq 1$. Then, by (69), we have

$$\left\langle \prod_{i=1}^n \tau_{a_i} \right\rangle_{(1+a_{[n]},0)} = \sum_{\substack{I \sqcup J = \{2,\ldots,n\} \\ n \in J}} (1 + a_J) \left\langle \tau_{a_1-1} \prod_{i \in I} \tau_{a_i} \right\rangle_{(a_1+a_I,0)} \left\langle \prod_{j \in J} \tau_{a_j} \right\rangle_{(1+a_J,0)} \overset{\substack{\text{by the induction} \\ \text{assumption}}}{=}$$

$$= \frac{1}{\prod a_i!} \sum_{\substack{I \sqcup J = [n] \\ 1 \in I,\, n \in J}} a_1 a_I^{|I|-2} (1 + a_J)^{|J|-1}.$$

It remains to prove that for any $n \geq 2$ the following identity is true:

$$(70) \qquad \sum_{\substack{I \sqcup J = [n] \\ 1 \in I,\, n \in J}} a_1 a_I^{|I|-2} (1 + a_J)^{|J|-1} = (1 + a_{[n]})^{n-2},$$

where we consider $a_1, \ldots, a_n$ as formal variables.

We prove identity (70) by induction on n. Both sides of it are polynomials in $a_1, \ldots, a_n$. Denote the left-hand side by $L_n(a_1, \ldots, a_n)$. For $n = 2$ identity (70) is trivial. Suppose $n \geq 3$. Since both sides of (70) have degree $n - 2$, it is sufficient to check the following two properties:

- $L_n|_{a_i=0} = (1 + \sum_{j \neq i} a_j)^{n-2}$, for any $2 \leq i \leq n - 1$.
- Coefficient of $a_2 a_3 \cdots a_{n-1}$ in L_n is equal to $(n-2)!$.

For this we compute

$$L_n|_{a_i=0} = \sum_{\substack{I \sqcup J = [n] \setminus \{i\} \\ 1 \in I,\, n \in J}} \left(a_1 a_I^{|I|-2} (1 + a_J)^{|J|-1} \right)(a_I + 1 + a_J) =$$

$$= L_{n-1}(a_1, \ldots, \widehat{a_i}, \ldots, a_n) \left(1 + \sum_{j \neq i} a_j \right) \overset{\substack{\text{by the induction} \\ \text{assumption}}}{=} \left(1 + \sum_{j \neq i} a_j \right)^{n-2}.$$

Clearly, the coefficient of $a_2 \cdots a_{n-1}$ in $a_1 a_I^{|I|-2} (1 + a_J)^{|J|-1}$ is equal to zero, unless $|I| = 1$. If $|I| = 1$, then this coefficient is equal to $(n-2)!$. This completes the proof of identity (70) and, hence, the theorem is proved. $\qquad \square$

Lemma 4.4. *The genus* 0 *stationary intersection numbers for degree* (d, d) *vanish.*

Proof. Consider any intersection number $\langle \prod_{i \in I} \tau_{a_i}^{\epsilon_i} \rangle_{0,(d,d)}$. By Corollary 4.3, partitions P with $r > 2$ pieces which appear in the expression for $\mathrm{OGW}(I, (d,d), \vec{a}, \vec{\epsilon})$ vanish, since their amplitude contains the term $(d - d)^{r-2}$.

When $r = 2$ this argument does not guarantee vanishing. We would like to show that the $r = 2$ terms of (11) precisely cancel the second summand of that equation. In other words, we would like to show that

$$(71) \qquad \sum_{((I_+, \vec{d}_+), (I_-, \vec{d}_-))} \frac{-(\partial^H(\vec{d}_+))^2}{-2u} I(I_+, \vec{d}_+) I(I_-, \vec{d}_-) = \frac{d}{-2u} I(I, d),$$

where the sum is over pairs of moduli specifications $((l_+, \vec{d}_+), (l_-, \vec{d}_-) = (l \smallsetminus l_+, \vec{d} - \vec{d}_+))$, where $\vec{d}_+ = (b + a, b)$ with $a > 0$ and ∂^H is the connecting map in homology, given by (3), so that $\partial^H(\vec{d}_+) = a$.

By the definition of fixed-points contribution, Definition 1.8, the left-hand side equals

$$\sum_{(\Gamma_+, \Gamma_-) \in B(l, \vec{d})} \frac{a(\Gamma_+)^2}{2h} \prod_{s \in \{+,-\}} \frac{I(\Gamma_s, \vec{a}|_{l_s}, \vec{\epsilon}|_{l_s})}{|\mathrm{Aut}(\Gamma_s)|},$$

where $B(l, \vec{d})$ is the collection of pairs of fixed-point graphs $\Gamma_\pm$ for moduli specifications $(l_\pm, \vec{d}_\pm)$ as above with $a(\Gamma_+) = \partial^H(\vec{d}_+) > 0$, e_+ is the unique disk edge of Γ_+ and $\delta(e^+) = (a(\Gamma_+), 0)$. We have

$$I(\Gamma_s, \vec{a}|_{l_s}, \vec{\epsilon}|_{l_s}) = \frac{1}{|A^0_{\Gamma_s}|} \int_{\overline{\mathcal{M}}_{\Gamma_s}} e_{\Gamma_s}^{-1} \cdot \alpha_{\Gamma_s}^{\vec{a}|_{l_s}, \vec{\epsilon}|_{l_s}},$$

as in Definition 1.8.

Recall item (b) of Remark 2.19, concerning the graphical representation of (11). It allows us to write the right-hand side of (71) as the sum

$$\sum \frac{-|\delta(e)|}{-2u} \frac{I(\Gamma, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma, e)|}$$

over pairs (Γ, e), where Γ is a fixed-point graph for the specification (l, d), e a sphere edge of Γ and $\mathrm{Aut}(\Gamma, e)$ is the group of automorphisms of Γ which fix e. Let $A(l, \vec{d})$ be the collection of these pairs.

There is an obvious bijection $q : B(l, \vec{d}) \to A(l, \vec{d})$ obtained by sending (Γ_+, Γ_-) to (Γ, e), where Γ is the result of gluing $\Gamma_\pm$ along their boundaries and e is the new sphere edge. The inverse $q^{-1}(\Gamma, e)$ is obtained cutting Γ along the equator of e.

Under this bijection, if $(\Gamma, e) = q(\Gamma_+, \Gamma_-)$, then

$$|\delta(e)| = \partial^H(\delta(e_+)), \qquad |\mathrm{Aut}(\Gamma_+)||\mathrm{Aut}(\Gamma_-)| = |\mathrm{Aut}(\Gamma, e)|,$$

and, by Observation 4.5 below, we have

$$I(\Gamma_+, \vec{a}, \vec{\epsilon})I(\Gamma_-, \vec{a}, \vec{\epsilon})(-(\partial^H \delta(e_+))^2) + (-|\delta(e)|)I(\Gamma, \vec{a}, \vec{\epsilon}) = 0.$$

Thus,

$$\frac{I(\Gamma_+, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma_+)|} \frac{I(\Gamma_-, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma_-)|}(-(\partial^H \delta(e_+))^2) + (-|\delta(e)|)\frac{I(\Gamma, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma, e)|} = 0.$$

Summing over $B(l, \vec{d})$ we obtain (71). $\qquad\square$

Observation 4.5. If

$$\mathrm{he}\,(\mu, d) = \frac{1}{d}\frac{\mu}{d!}\left(\mu \frac{2u}{d}\right)^{-d}$$

denotes the factor in $I(\Gamma, \vec{a}, \vec{\epsilon})$, coming from (6), corresponding to a disk edge (taking into account also the size d group of geometric automorphisms) and

$$\mathrm{ed}\,(d) = \frac{1}{d}(-1)^d \frac{d^{2d}}{(2u)^{2d}(d!)^2}$$

represents the contribution of a sphere edge, then

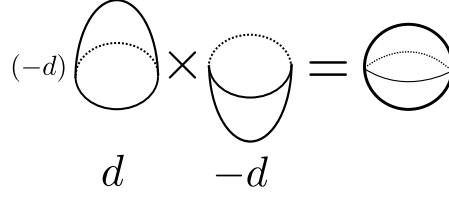(72) $$\mathrm{ed}\,(d) = (-d) \cdot \mathrm{he}\,(+1, d) \cdot \mathrm{he}\,(-1, d).$$

FIGURE 3. A graphical representation of Observation 4.5.

5. ALL GENERA DEFINITION BY LOCALIZATION

In this section we *define* via localization the stationary descendent integrals for all moduli specifications, using what should be the fixed-point formula expressing them. We will see that this definition agrees with the geometric definition for closed moduli specifications and for disk moduli specifications. We use the terminology of Sections 1.5.1, 1.5.2 and 1.5.3.

We begin with preliminary definitions of auxiliary objects.

5.1. BC (Boundary contribution) graphs.

Definition 5.1. A *bare boundary contribution graph* $G = (V, E, L)$ is the following data:

(a) A set of vertices V.
(b) A collection of edges $E = \vec{E} \cup \widetilde{E}$ between them.
(c) A collection L called the *loops* which should be thought of as oriented loops.

This data is required to satisfy

(a) All edges in $\vec{E}$ are oriented, and any vertex has exactly one such incoming edge, and one such outgoing edge.
(b) Every vertex touches a single edge of $\widetilde{E}$. These edges are called the *wavy edges*, they are not oriented and each touches two different vertices.

From this data one may construct the set F^{new} of oriented loops defined as L together with the oriented closed paths of the graph $(V, \vec{E})$. Another set of oriented loops which we define is F^{old}, made of L and of all oriented loops which can be written as a sequence $(v_0, e_1, v_1, e_2, \ldots, e_{2m})$ (up to a cyclic change of order), under the following constraints. First, all edges e_j which are directed are different. For an even i, $e_i \in \widetilde{E}$, and it connects v_{i-1} to $v_{(i \pmod{2m})}$. For an odd i, $e_i \in \vec{E}$, and it connects v_{i-1} to v_i, in agreement with its direction. Elements of F^{new} (F^{old}) are called new (old) faces. We write $e \in f$ whenever the edge e is a part of the closed path f.

Definition 5.2. A *boundary contribution graph* $G = (V, E, L, d)$, or a *BC graph* for shortness, is a bare boundary contribution graph $G = (V, E, L)$ together with a *perimeter function*

$$d : F^{new} \cup F^{old} \to \mathbb{Z},$$

which associates to a face its perimeter. It is required that the perimeters of new faces will be non-zero, and that the sum of perimeters of old faces equals the sum of perimeter of new faces, in any connected component H of G. We write $\vec{d}$ for the collection of perimeters of faces, and we sometimes use this notation instead of writing the function d.

An isomorphism of BC graphs $G = (V, E, L, d), G' = (V', E', L', d')$ is a collection of bijections

$$f^V : V \to V', \quad f^E : E \to E', \quad f^L : L \to L',$$

which preserve incidence relations between edges and vertices, directions of directed edges and perimeters of faces. In particular, there are induced bijections between new and old faces of G, G'.

An *enumeration* of a BC graph is an enumeration of its new faces by $1, \ldots, |F^{new}|$, of its old faces by $1, \ldots, |F^{old}|$, of its wavy edges by $1, \ldots, |\widetilde{E}|$, and a choice of orientation to each wavy edge.

If we define isomorphisms and automorphisms in the expected way, then the automorphism group of an *enumerated BC graph* is easily seen to be trivial.

Observe that any element of $\vec{E}$ belongs to one new and one old face, while every element of $\widetilde{E}$ belongs to two old faces (or one old, but appears twice in that face).

Definition 5.3. A *metric* on a BC graph G is an association of a (signed) length $x_e \in \mathbb{R}$ for any $e \in \vec{E}$ such that

(a) The sum of lengths of the oriented edges of any circuit of the graph is an integer.
(b) The sum of lengths of the oriented edges which belong to a new or old face is the perimeter of that face.
(c) If $e \in \vec{E}$ is an edge of a new face f, then $x_e d(f) > 0$.

We denote by $W_G \subseteq \mathbb{R}^{\vec{E}}$ the space of metrics on G and by $\overline{W}_G$ its closure in $\mathbb{R}^{\vec{E}}$.

We consider the lengths as functions on W_G, $x_e : W_G \to \mathbb{R}$.

Definition 5.4. Let $\mathcal{E} \subseteq \vec{E}$ be a set of edges that together with $\widetilde{E}$ forms a spanning forest of G. This means that $\mathcal{E} \sqcup \vec{E}$ do not contain any circuit, and any two vertices which belong to the same connected component of G are connected by a path in $\mathcal{E} \sqcup \vec{E}$. Write

$$\Omega_G = \bigwedge_{e \in \mathcal{E}} dx_e.$$

Observation 5.5. Ω_G is well-defined up to a sign.

Proof. Changing the order in which the wedge is taken results in a change of sign to Ω_G. Replacing $\mathcal{E}$ by another set of edges with the same properties $\mathcal{E}'$ can be achieved in a sequence of steps

$$\mathcal{E} = \mathcal{E}_0 \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_r = \mathcal{E}',$$

where for each i we have $\mathcal{E}_{i+1} = \mathcal{E}_i \backslash \{e_i\} \cup \{e_{i+1}\}$. Adding e_{i+1} to $\mathcal{E} \sqcup \widetilde{E}$ creates a circuit in the graph which contains e_i, e_{i+1}, additional oriented edges $\{e_\alpha\}_{\alpha \in A} \subseteq \mathcal{E} \backslash \{e_i\}$, and some elements of $\widetilde{E}$. Since any circuit has an integer perimeter, this perimeter is locally constant on W_G. Thus, on W_G we have

$$dx_{e_{i+1}} = -dx_{e_i} - \sum_{\alpha \in A} dx_\alpha,$$

hence,

$$\bigwedge_{e \in \mathcal{E}_i} dx_e = \pm \left(\bigwedge_{e \in \mathcal{E}_i \backslash e_i} dx_e \right) \wedge \left(-dx_{e_i} - \sum_{\alpha \in A} dx_\alpha \right) = \pm \left(\bigwedge_{e \in \mathcal{E}_i \backslash e_i} dx_e \right) \wedge dx_{e_{i+1}} = \pm \bigwedge_{e \in \mathcal{E}_{i+1}} dx_e,$$

and the proof follows. $\square$

Definition 5.6. Define the *volume* of a BC graph $G = (V, E, L, d)$ to be

$$\mathrm{Vol}_G = (-1)^{alt} \left| \left(\prod_{f \in F^{new} \backslash L} d(f) \right) \int_{\overline{W}_G} \Omega_G \right|,$$

where *alt* is the number of edges of $\widetilde{E}$ which touch two new faces f_1, f_2 with $d(f_1)d(f_2) < 0$.

In the sequel we will study those volumes and we will see, in particular, that they are piecewise polynomial functions in the perimeters of the new and old faces.

5.2. Morphisms between specifications and the compact form of the localization definition.
A moduli specification is *pure* if there is no boundary of degree 0 (0 is not in the image of d).

Definition 5.7. A *morphism* from a moduli specification S^{new} to a moduli specification S^{old} is a decorated graph $M = (V_b \sqcup V_w, E, H^{CB}, \widetilde{E}, g_s, \mathsf{l}, \vec{d}, d, \sigma, d^{old})$ such that

(a) $(V_b \sqcup V_w, E, \vec{d}, g_s, \mathsf{l}, \vec{d}, d)$ is the moduli specification S'.
(b) The *wavy edges* $\widetilde{E}$ connect white vertices.
(c) The *contracted boundary half-edges* H are half-edges which emanate from black vertices.
(d) A *wavy flag* is a pair of a wavy edge and an orientation of it. For any white vertex v, σ_v is a cyclic order of the wavy flags which are incident to v and oriented outward of v.
(e) There is an involution τ on wavy flags defined by associating a given wavy flag the other wavy flag which corresponds to the same wavy edge with opposite orientation. The function d^{old} is a mapping from the set of cycles of $\sigma^{-1} \circ \tau$ to $\mathbb{Z}$. Here and afterwards we consider a white vertex which is not incident to a wavy edge as a cycle of $\sigma^{-1} \circ \tau$ of its own.
(f) S^{old} is the *desingularization* of M, where the desingularization is the moduli specification defined as follows.
  - Associate a black vertex to any connected component of the graph $G = (V_b \sqcup V_w, E \sqcup \widetilde{E})$ (we consider G as a graph in the usual sense, with vertices $V_b \sqcup V_w$ and edges $E \sqcup \widetilde{E}$).
  - For such a vertex v' define

$$\vec{d}(v') = \sum_v \vec{d}(v), \quad \mathsf{l}(v') = \bigsqcup_v \mathsf{l}(v), \quad g_s(v') = h_1(G) + \sum_v g_s(v),$$

    where the summation or union are over all black vertices of G which belong to the connected component represented by v'.
  - We associate a white vertex for any cycle of $\sigma^{-1} \circ \tau$, connect it to the black vertex v which corresponds to the connected component which contains this cycle. Define, for a white vertex w, $d(w) = d^{old}(c)$, where c is the cycle which corresponds to w.
  - We associate a white vertex for any contracted boundary half-edge, define d for such a vertex to be 0, and connect this vertex to the black vertex which represents the connected component containing this contracted boundary half-edge.

As usual, isomorphisms or automorphisms are graph isomorphisms or automorphisms which respect the additional data, and $\mathrm{Aut}(M)$ is the automorphism group of M.

Write $\mathrm{Hom}(S', S)$ for the set of isomorphism types of morphisms from S' to S.

Definition 5.8. One can associate a BC graph G for any morphism

$$M = (V_b \sqcup V_w, E, H^{CB}, \widetilde{E}, \vec{d}, g_s, \mathsf{l}, \vec{d}, d, \sigma, d^{old}) \in \mathrm{Hom}(S^{new}, S^{old})$$

as follows:

- Associate a vertex v for any wavy flag e_v.
- Connect two vertices which correspond to flags which are paired via τ by a wavy edge of G.
- Draw a directed edge (v, u) if $\sigma(e_v) = e_u$.
- Associate a loop to any white vertex without wavy edges.
- So far we obtain a bare boundary contribution graph, whose new faces correspond white vertices of M or equivalently of S^{new}, and the old faces correspond to the $\sigma^{-1} \circ \tau$–cycles of M, or equivalently the white vertices of S. Define d for an old face to be d^{old} of the corresponding cycle, and d of a new face to be d of the corresponding vertex.

We denote G by $BC(M)$.

In addition, one can associate a degree, which we also denote by d to a contracted boundary half-edge h of M. Suppose h is connected to a black vertex v. Define

$$\delta(h) = \delta^{morph}(h) = d^+(v) - \sum_{w \in V_w(v) | d(w) > 0} d(w),$$

where $V_w(v)$ is the set of white neighbors of v.

Definition 5.9. Recall Definition 1.8. Let S be a moduli specification, $\vec{a}, \vec{\epsilon}$ be vectors of descendents and constraints. Define the *open Gromov-Witten invariant* of $(S, \vec{a}, \vec{\epsilon})$ as follows:

$$(73) \qquad \mathrm{OGW}(S) = \sum_{S' \text{ is pure}} \sum_{M \in \mathrm{Hom}(S',S)} \frac{1}{|\mathrm{Aut}(M)|} \frac{\mathrm{Vol}_{BC(M)}}{(-2u)^{|\widetilde{E}|}} \left(\prod_{h \in H^{CB}(M)} \frac{\delta^{morph}(h)}{2u} \right) I(S', \vec{a}, \vec{\epsilon}).$$

Note that, although the summation is over infinitely many specifications, only for finitely many of them $\mathrm{Hom}(S', S)$ is non-empty. Indeed, for a given S the only specifications S' for which $\mathrm{Hom}(S', S)$ is non-empty must have the same set of underlying labels, same total degree and total genus bounded by that of S. There are only finitely many such specifications. Even when $\mathrm{Hom}(S', S)$ is non-empty, it could still happen that the summand will vanish because of the volume term.

5.3. Equivalent definition in terms of graphs.

Definition 5.10. A *localization graph* is a tuple

$$\mathcal{G} = \left(V \sqcup V_\circ, F, \delta, \gamma, \mu, \lambda, H^{CB}, \widetilde{E}, \sigma, d^{old} \right)$$

such that
(a) $\Gamma = (V \sqcup V_\circ, F, \delta, \gamma, \mu, \lambda)$ is a fixed point graph. We denote Γ by $FP(\mathcal{G})$.
(b) H^{CB} are half-edges which emanate from equators of Γ and are called *contracted boundary edges*.
(c) $\widetilde{E}$ are the *wavy edges* which connect boundaries of Γ. Wavy flags are wavy edges together with orientations.
(d) σ_v, for a boundary vertex $v \in V_\circ$, is a cyclic order of wavy flags which emanate from v. We define τ as in Definition 5.7.
(e) d^{old} is a function from cycles of $\sigma^{-1} \circ \tau$ to $\mathbb{Z}$.

Given moduli specifications S^{new}, S^{old}, we say that $\mathcal{G}$ is a fixed point graph of type (S^{new}, S^{old}), if $FP(\mathcal{G})$ is of type S^{new} and the *contraction* of $\mathcal{G}$ belongs to $\mathrm{Hom}(S^{new}, S^{old})$, where the contraction $\mathrm{Contract}(\mathcal{G})$ is the tuple $(V_b \sqcup V_w, E, H^{CB}, \widetilde{E}, g_s, \mathsf{l}, \vec{d}, d, \sigma, d^{old})$ defined as follows.

$(V_b \sqcup V_w, E, g_s, \mathsf{l}, \vec{d}, d)$ is the contraction of $FP(\mathcal{G})$ as defined in the end of Definition 1.3. In particular there is an identification between boundaries in $V_\circ$ and V_w. With this identification we draw wavy edges, also denoted $\widetilde{E}$, between white vertices, and denote the induced cyclic orders also by $\sigma = \sigma^{\mathrm{Contract}(\mathcal{G})}$. This also allows us to define $\tau = \tau^{\mathrm{Contract}(\mathcal{G})}$, and hence induce the function, which is still denoted d^{old}, on cycles of $\sigma^{-1} \circ \tau$ of $\mathrm{Contract}(\mathcal{G})$. Finally, by the identification between V_b and connected components of $FP(\mathcal{G})$, we associate, for any contracted boundary half-edge $h \in H^{CB}(\mathcal{G})$ a contracted boundary half-edge, which emanates from the black vertex which corresponds to the connected component of $FP(\mathcal{G})$ which contains h.

Isomorphisms, automorphisms and the group $\mathrm{Aut}(\mathcal{G})$ are defined in the expected way.

We denote by $\mathrm{Graphs}(S', S)$ the set of isomorphism types of localization graphs of type (S', S).

We denote $BC(\mathrm{Contract}(\mathcal{G}))$ by $BC(\mathcal{G})$. Finally, for a contracted boundary half-edge h which is connected to an equator v we set $\delta(h) = \delta^{graph}(h) = |\delta(v)|$.

The following lemma provides an equivalent interpretation of OGW in terms of graph summation. We will consider it as an alternative definition.

Lemma 5.11. *Let S be a moduli specification, $\vec{a}, \vec{\epsilon}$ be vectors of descendents and constraints. We have*

$$(74) \qquad \mathrm{OGW}(S) = \sum_{S' \ is \ pure} \sum_{\mathcal{G} \in \mathrm{Graphs}(S',S)} \frac{1}{|\mathrm{Aut}(\mathcal{G})|} \frac{\mathrm{Vol}_{BC(\mathcal{G})}}{(-2u)^{|\widetilde{E}|}} \left(\prod_{h \in H^{CB}(\mathcal{G})} \frac{\delta^{graph}(h)}{2u} \right) I(FP(\mathcal{G}), \vec{a}, \vec{\epsilon}).$$

Definition 5.9 is a compact, renormalized packing of (74). The advantage of (74) is that it can be described graphically in a way which generalizes nicely the localization graphs of closed GW theory, see Figure 4 for an illustration.

Proof. Write $\Pi(\mathcal{G}) = (\mathrm{Contract}(\mathcal{G}), FP(\mathcal{G}))$. Then the RHS of (74) can be written as

$$\sum_{\substack{S' \ is \ pure}} \sum_{\substack{M \in \mathrm{Hom}(S',S) \\ \Gamma \ is \ a \ fixed-point \ graph \ for \ S'}} \sum_{\mathcal{G} \in \Pi^{-1}(M, \Gamma)} \frac{1}{|\mathrm{Aut}(\mathcal{G})|} \frac{\mathrm{Vol}_{BC(M)}}{(-2u)^{|\widetilde{E}|}} \left(\prod_{h \in H^{CB}(\mathcal{G})} \frac{\delta^{graph}(h)}{2u} \right) I(\Gamma, \vec{a}, \vec{\epsilon}).$$

On the other hand, using Definition 1.8, the RHS of (73) can be written as

$$\mathrm{OGW}(S) = \sum_{S' \ is \ pure} \sum_{M \in \mathrm{Hom}(S',S)} \frac{|\mathrm{Aut}(S')|}{|\mathrm{Aut}(M)|} \frac{\mathrm{Vol}_{BC(M)}}{(-2u)^{|\widetilde{E}|}} \left(\prod_{h \in H^{CB}(M)} \frac{\delta^{morph}(h)}{2u} \right) \sum_{\substack{\Gamma \ is \ a \ fixed-point \\ graph \ for \ S'}} \frac{I(\Gamma, \vec{a}, \vec{\epsilon})}{|\mathrm{Aut}(\Gamma)|}.$$

The lemma will follow if we could prove that for any $M \in \mathrm{Hom}(S', S)$ and a fixed-point graph Γ for S' we have

$$(75) \qquad \frac{|\mathrm{Aut}(S')|}{|\mathrm{Aut}(M)||\mathrm{Aut}(\Gamma)|} \left(\prod_{h \in H^{CB}(M)} \frac{\delta^{morph}(h)}{2u} \right) = \sum_{\mathcal{G} \in \Pi^{-1}(M, \Gamma)} \frac{1}{|\mathrm{Aut}(\mathcal{G})|} \left(\prod_{h \in H^{CB}(\mathcal{G})} \frac{\delta^{graph}(h)}{2u} \right).$$

Define an *enumeration* of a finite set A by a set B to be a bijection from $A \to B$. A moduli specification S is *enumerated* if for each connected component S' and $a \in \mathbb{Z}$ the white vertices of degree a are enumerated by $[m_a(S')]$, where $m_a(S')$ is the number of these white vertices, and, in addition, for each isomorphism type of connected moduli specification S', the connected components of S of this type are enumerated by $[m_{S'}(S)]$, where $m_{S'}(S)$ is the number of these components.

An *enumeration of a morphism* $M \in \mathrm{Hom}(S', S)$ is an enumeration of S' and S (where we identify white vertices of S with cycles of $\sigma^{-1} \circ \tau$) together with an enumeration of $\widetilde{E}$ by $[|\widetilde{E}|]$, an enumeration of $H^{CB}(M)$ by $[|H^{CB}(M)|]$, and a choice of orientation for each element of $\widetilde{E}$.

An *enumeration of a fixed point graph* Γ is an enumeration of its contraction S (where we identify boundaries of Γ with white vertices of S) together with enumerations $V(\Gamma)$, $E(\Gamma)$ by $[|V(\Gamma)|], [|E(\Gamma)|]$ respectively. An enumeration of a localization graph $\mathcal{G} \in \mathrm{Hom}(S', S)$ is a pair of enumerations, one for $FP(\mathcal{G})$ and the other for $\mathrm{Contract}(\mathcal{G})$ which are compatible in the sense that they induce the same enumeration on S'.

We define isomorphisms and automorphisms and the group Aut of each of the above enumerated objects in the expected way, and write $\mathrm{Enum}(X)$ for the set of isomorphism types of enumerations of the object X. We see that for each of the above enumerated objects the automorphism group is trivial, therefore, by the Orbit-Stabilizer theorem we have

Observation 5.12. (a) For a fixed point graph Γ of type S' we have

$$|\mathrm{Aut}(\Gamma)||\mathrm{Enum}(\Gamma)| = |\mathrm{Aut}(S')||V(\Gamma)|!|E(\Gamma)|!.$$

(b) For a morphism $M \in \mathrm{Hom}(S', S)$ we have

$$|\mathrm{Aut}(M)||\mathrm{Enum}(M)| = |\mathrm{Aut}(S')||\mathrm{Aut}(S)|2^{|\widetilde{E}(M)|}|\widetilde{E}(M)|!|H^{CB}(M)|!.$$

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(c) We have

$$|\mathrm{Aut}(\mathcal{G})||\mathrm{Enum}(\mathcal{G})| = |\mathrm{Aut}(S')||\mathrm{Aut}(S)|2^{|\widetilde{E}(M)|}|\widetilde{E}(M)|!|H^{CB}(M)|!|V(\Gamma)|!|E(\Gamma)|!.$$

We also observe that for a given moduli specification S there are precisely $|\mathrm{Aut}(S)|$ enumerations and they are all isomorphic.

Item (c) of the previous observation allows us to rewrite the RHS of (75) as

$$(76) \qquad \sum_{\mathcal{G}\in\Pi^{-1}(M,\Gamma)} \frac{|\mathrm{Enum}(\mathcal{G})|}{|\mathrm{Aut}(S')||\mathrm{Aut}(S)|2^{|\widetilde{E}(M)|}|\widetilde{E}(M)|!|H^{CB}(M)|!|V(\Gamma)|!|E(\Gamma)|!}\left(\prod_{h\in H^{CB}(\mathcal{G})}\frac{\delta^{graph}(h)}{2u}\right).$$

Note that any enumeration of $\mathcal{G}\in\Pi^{-1}(M,\Gamma)$ induces an enumeration of Γ and of M. When $c = 0$, the opposite is also true: enumerations of Γ and of M allow us to reconstruct a unique $\mathcal{G}\in\Pi^{-1}(M,\Gamma)$ which induces the original enumerations on Γ, M since the only difference between the data of $\mathcal{G}$ and of (M,Γ) is how to identify boundaries of Γ with white vertices of M, and the enumeration gives us such an identification. Thus, when $c = 0$ we have

$$(77) \qquad \sum_{\mathcal{G}\in\Pi^{-1}(M,\Gamma)} |\mathrm{Enum}(\mathcal{G})| = |\mathrm{Enum}(\Gamma)||\mathrm{Enum}(M)|.$$

For a general c we have

$$(78) \qquad \sum_{\mathcal{G}\in\Pi^{-1}(M,\Gamma)} |\mathrm{Enum}(\mathcal{G})|\left(\prod_{h\in H^{CB}(\mathcal{G})}\frac{\delta^{graph}(h)}{2u}\right) = |\mathrm{Enum}(\Gamma)||\mathrm{Enum}(M)|\left(\prod_{h\in H^{CB}(M)}\frac{\delta^{morph}(h)}{2u}\right).$$

The proof is by simple induction. The case $c = 0$ is just (77). If the claim holds for c contracted boundary half-edges, then adding a new contracted boundary half-edge h which belongs to a connected component S'_0 of S' changes the RHS by $\frac{\delta^{morph}(h)}{2u}$. The right-hand side changes, by definition, by the sum of $\frac{|\delta(v_\circ)|}{2u}$, where $v_\circ$ runs over all equators in the connected component Γ_0 of Γ which corresponds to S'_0. This sum is again, by definition, $\frac{\delta^{morph}(h)}{2u}$. Plugging (78) into (76) we obtain

$$(79) \qquad \frac{|\mathrm{Enum}(\Gamma)||\mathrm{Enum}(M)|\left(\prod_{h\in H^{CB}(M)}\frac{\delta^{morph}(h)}{2u}\right)}{|\mathrm{Aut}(S')||\mathrm{Aut}(S)|2^{|\widetilde{E}(M)|}|\widetilde{E}(M)|!|H^{CB}(M)|!|V(\Gamma)|!|E(\Gamma)|!} =$$

$$= |\mathrm{Aut}(S')|\frac{|\mathrm{Enum}(\Gamma)|}{|\mathrm{Aut}(S')||V(\Gamma)|!|E(\Gamma)|!}\frac{|\mathrm{Enum}(M)|\left(\prod_{h\in H^{CB}(M)}\frac{\delta^{morph}(h)}{2u}\right)}{|\mathrm{Aut}(S')||\mathrm{Aut}(S)|2^{|\widetilde{E}(M)|}|\widetilde{E}(M)|!|H^{CB}(M)|!}.$$

Using the first two items of Observation 5.12 this expression gives exactly the LHS of (75). $\qquad\square$

5.4. Agreement in $g = 0$ and for closed topologies.

Proposition 5.13. *Definition 5.9 when restricted to $g = 0$ agrees with the formula of Theorem 3.9.*

Proof. The sums in Definition 5.9, in the $g = 0$ case are of two types. Either there is no contracted boundary half-edge, or there is a single contracted boundary half-edge, but no wavy edges.

The second case corresponds to the second summand in (11), and in both formulas these terms have the same value.

In the first case, in Definition 5.9 there is a sum over BC graphs, which in this case can be identified with trees whose vertices correspond to new faces, edges to wavy edges, each vertex has a non-zero integer degree and a cyclic order of the edges from each vertex is given. On the other hand, in (11) the summation is over the exact same trees, with non-zero integral degrees for vertices, *but without a cyclic order for edges which emanate from a vertex.*
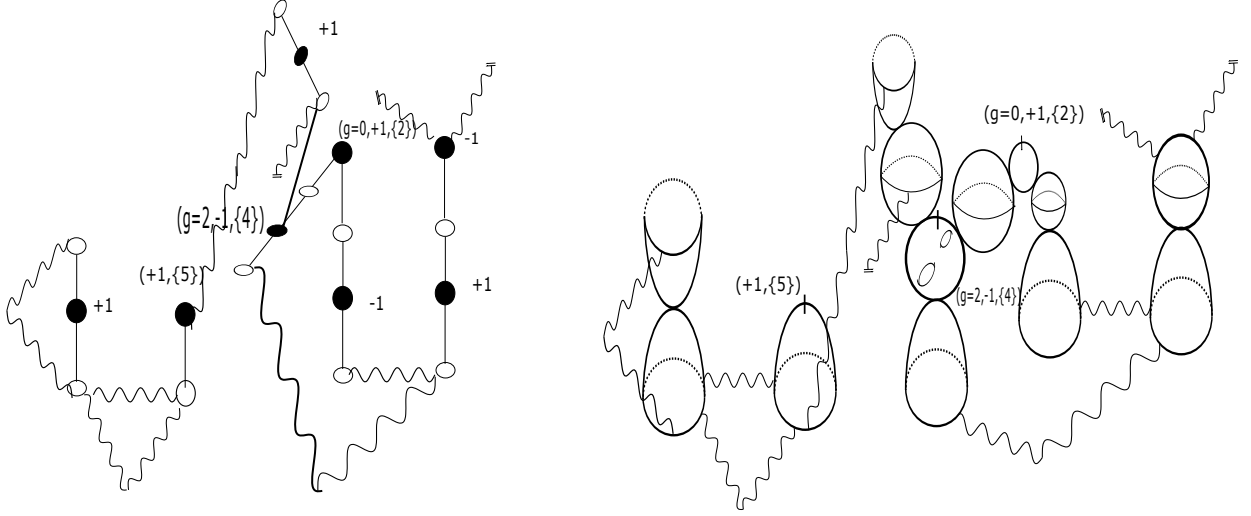
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FIGURE 4. A localization graph which contributes to a high genus invariant and the corresponding geometric stratum. Again for illustration reasons we do not contract degree 0 components. Empty vertices stand for boundary and equator vertices. Next to the remaining vertices we write their genus, if they correspond to contracted component, μ and the markings.

Thus, there is a natural surjection s from the collection of BC graphs which appear in $\mathrm{OGW}(\mathsf{I}, \vec{d})$ and $\mathcal{T}(\mathsf{I}, \vec{d})$, obtained by forgetting the cyclic orders at vertices. The proposition would therefore follow if we could show that for any $T \in \mathcal{T}(\mathsf{I}, \vec{d})$ we have

$$(80) \qquad \sum_{G \in s^{-1}(T)} \mathrm{Vol}_G = \prod_{v \in V} \left(d^+(v) - d^-(v) \right)^{\mathrm{val}(v)}.$$

The sign of each summand in the LHS is $(-1)^{alt}$, where alt is the number of edges between vertices of the tree G which connect a vertex of positive degree to a vertex of negative degree. The sign of the RHS is the sum of valencies of vertices of negative degrees. These two numbers agree modulo 2, hence the signs of the two sides of (80) are the same.

Recall Definitions 5.4, 5.6. When the tree T is trivial, equation (80) clearly holds. Otherwise, any $G \in s^{-1}(T)$ has no loops and has a single old face whose perimeter is the sum of perimeters of the new faces. If f_v is the new face of G which corresponds to the vertex v of T, and if we write $e \in f_v$ to specify that an oriented edge e belongs to the new face f_v, then the space W_G is defined as the space

$$\left\{ (x_e) \in \mathbb{R}^{\vec{E}} \,\middle|\, \forall v \sum_{e \in f_v} x_e = d(f_v) \text{ and } x_e d(f_v) > 0 \right\}.$$

Thus, W_G is identified with a product of simplices of the form

$$\{(x_1, \ldots, x_r) | \sum x_i = d \text{ and all } x_i \text{ have the same sign}\}.$$

If we take $\mathcal{E}$ as in Definition 5.4 as a set whose intersection with $\{e | e \in f_v\}$ is of cardinality $|\{e | e \in f_v\}| - 1$ for any v, we see that

$$\left| \int_{\overline{W}_G} \Omega_G \right| = \prod_v |d(f_v)|^{|\{e | e \in f_v\}| - 1} / (|\{e | e \in f_v\}| - 1)!$$

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as the product of volumes of simplices. Multiplying by $\prod_{f \in F^{new}} d(f) = \prod_v d(f(v))$, and summing over all $\prod_v (|\{e|e \in f_v\}| - 1)! = \prod_v (\mathrm{val}(v) - 1)!$ elements of $s^{-1}(T)$ we obtain

$$\sum_{G \in s^{-1}(T)} |\mathrm{Vol}_G| = \prod_{f \in F^{new}} |d(f)|^{|\{e|e \in f\}|} = \left| \prod_{v \in V} (d^+(v) - d^-(v))^{\mathrm{val}(v)} \right|,$$

which completes the proof of equation (80). $\qquad \square$

Remark 5.14. The genus 0 case and the positive genus cases share the feature that the contribution of wavy edges (the Vol_G term) decouples from that of the fixed-point graphs. Still, the genus 0 case is simpler than the general case as the wavy edge contribution factorizes as a product over wavy edges of the wavy edge term

$$(81) \qquad \frac{(d^+(w_1) - d^-(w_1))(d^+(w_2) - d^-(w_2))}{-2u},$$

where the wavy edge e connects w_1 and w_2. It turns out that if one considers less refined intersection numbers, obtained by summing $\mathrm{OGW}(S, \vec{a}, \vec{\epsilon})$ over all moduli specifications with fixed $g, \mathsf{l}, \vec{d}$ (but varying number of boundaries and boundary degrees), one obtains a formula which is a sum over graphs which are similar to localization graphs, but without cyclic orders. In this case the wavy edge contribution again factorizes as a product of edge weights, where wavy edges which connect different vertices have the weight (81), while wavy edges which connect the same white vertex w have the weight

$$-\frac{1}{2u} \binom{|d^+ w - d^- w|}{2}.$$

More details, including a proof of this claim, will appear in the sequel.

Another important observation is that in the special case of closed moduli specifications, Definition 5.9 agrees with the geometric definition of stationary descendent integrals. More precisely, we have

Theorem 5.15. *For a closed connected moduli specification S we have*

$$\mathrm{OGW}(S, \vec{a}, \vec{\epsilon}) = \int_{\overline{\mathcal{M}}_{g_s(S), \mathsf{l}(S)} (|\vec{d}(S)|)} \prod_{i \in \mathsf{l}(S)} \psi_i^{a_i} \mathrm{ev}_i^* \mathrm{pt}.$$

Proof. This follows from the virtual localization formula [15]. $\qquad \square$

5.5. **Main conjecture.** Theorem 5.15 and Proposition 5.13 provide evidences to our main conjecture.

Conjecture 1. *There exists a* geometric definition *for* $\mathrm{OGW}(S, \vec{a}, \vec{\epsilon})$ *for all moduli specifications S. In particular, we have*

$$\mathrm{OGW}(S, \vec{a}, \vec{\epsilon}) = 0$$

for all underdetermined $(S, \vec{a}, \vec{\epsilon})$.

A pair $(S, \vec{a})$ is *underdetermined* if the degree of the integrand,

$$\deg(S) = \sum_{i \in \mathsf{l}} 2(a_i + 1),$$

is less than the expected dimension of the moduli,

$$\mathrm{vdim}_{\mathbb{R}}(S) = 2d + 2g - 2 + 2|\mathsf{l}|.$$

In the sequel we will sketch a proof that under ideal, though non-existent, transversality assumptions, the conjecture holds. The sketch generalizes in a non-trivial way the proof of Theorem 3.9, and we believe that when open virtual localization will be rigorously defined, it will be automatic to promote the sketch of proof to a full proof.

6. Canonical orientations for the moduli spaces and their properties

This section is devoted to defining and understanding the properties of the orientations we have used throughout the paper. We will analyze the orientation of the moduli of maps from disks to $(\mathbb{CP}^1, \mathbb{RP}^1)$ using a natural decomposition of this moduli into chambers of full dimension. The chambers will be described in terms of certain types of maps. These maps will also play a role in the map decomposition property of $\mathrm{OGW}(S)$, that will be discussed in the sequel. The section is built as follows. We first describe the maps we need. We then use them to define an orientation for each chamber. We show that the orientations glue to the whole moduli, and then, again using the language of maps, analyze the behavior of the orientations with respect to degenerations.

6.1. **Decorated maps.** For a moduli specification S write $\Sigma(S)$ for the topological marked surface whose label set is $\mathsf{I}(S)$ and whose topological type is the one dictated by S.

Definition 6.1. Let S be a moduli specification, and $\mathsf{k} \subset \Upsilon$ a finite set. A *rigid decorated* (S, k)-*map* $\mathfrak{G}$ (or a rigid decorated S–map when $\mathsf{k} = \varnothing$) is the following data.

(a) A collection of smooth simple *loops and arcs* drawn on $\Sigma = \Sigma(S)$ which cut Σ into open *regions*. We call the boundary components of $\Sigma(S)$ the *boundary loops*. The other loops are called the *internal loops*. The internal loops and arcs are pairwise non-intersecting.
(b) The internal loops do not intersect $\partial\Sigma(S)$, the arcs intersect $\partial\Sigma$ exactly in their two endpoints.
(c) Any region $\mathcal{R}$ has a well defined *sign* $s(\mathcal{R}) \in \{\pm 1\}$. Neighboring regions, which are regions whose closures intersect, have opposite signs. A region is *positive* if its sign is positive and otherwise is *negative*.
(d) Any boundary component of the closure of a region is endowed a perimeter (circumference) $p \in \mathbb{N}$, in particular any internal loop $\mathcal{C}$ is endowed a perimeter $p(\mathcal{C})$. If $\mathcal{C}$ is a boundary component of $\Sigma(S)$ which touches no arc, then the product of its perimeter and the sign of the unique region it touches equals the degree of the corresponding boundary of S.
(e) The *degree* of a region is $(q, 0) \in \mathbb{Z}^2$ for a positive region and $(0, q) \in \mathbb{Z}^2$ for a negative one, where q is the sum of perimeters of the region's boundary components. The sum of degrees of regions in the connected component of Σ which corresponds to the connected moduli specification S' is $\vec{d}(S')$.
(f) The internal markings are distinct and do not lie on arcs or internal loops, and for each marking it is specified to which region it belongs.
(g) There are $|\mathsf{k}|$ distinct boundary markings, labeled by k on the boundary components. They differ from the endpoints of the arcs.

We call $\Sigma = \Sigma(S)$ the underlying surface of the map.

An isomorphism of rigid decorated maps $\mathfrak{G}_1, \mathfrak{G}_2$ is a homeomorphism of their underlying surfaces, which induces bijections between regions of $\mathfrak{G}_1$ and regions of $\mathfrak{G}_2$, loops of $\mathfrak{G}_1$ and loops of $\mathfrak{G}_2$, arcs of $\mathfrak{G}_1$ and arcs of $\mathfrak{G}_2$, boundary segments of the underlying surface of $\mathfrak{G}_1$ and boundary segments of the underlying surface of $\mathfrak{G}_2$, and endpoints of arcs of $\mathfrak{G}_1$ and endpoints of arcs of $\mathfrak{G}_2$, which respects the (internal and boundary) markings and such that inclusion relations and all discrete parameters: sign, perimeter, orientation (for a boundary segment) are preserved.

A *decorated* $((S, \mathsf{k})$- *or* $S-)$ *map* is an isomorphism class of rigid decorated $((S, \mathsf{k})$- or $S-)$ maps. We shall sometimes use the term 'decorated map' for a rigid representative of a decorated map, and we will use the same notations for maps as for rigid decorated maps.

Write $\mathrm{Maps}_{S,\mathsf{k}}$ for the collection of decorated (S, k)-maps and Maps_S for the collection of decorated S–maps. We write $\mathrm{Maps}_{0,k,l,\vec{d}}$ for $\mathrm{Maps}_{S,\mathsf{k}}$, where S is the disk moduli specification $([l], \vec{d})$ and $\mathsf{k} = [k]$.

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Throughout this section it will sometimes be convenient to work with $\mathsf{k} = [k]$ or $\mathsf{l} = [l]$. In such cases we write $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$ for $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d})$, $\overline{\mathcal{M}}_{0,0,0}$ for $\overline{\mathcal{M}}_{0,\varnothing,\varnothing}(\vec{d})$ etc. Moduli points will be denoted by (Σ, u), where Σ is a nodal marked disk and u is the map to $\mathbb{CP}^1$.

6.2. **Chambers and orientation.** It is well known that for any relative class $\vec{d} = (d_+, d_-) \neq \vec{0}$ the moduli space $\overline{\mathcal{M}}_{0,0,0}(\vec{d})$ is orientable (see [10] for a much more general theorem; see also [13]). A relatively simple consequence is

Theorem 6.2. *Suppose* $k, l, \vec{d} = (d_+, d_-)$ *satisfy the stability constraint* (16), *then the moduli* $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$ *is orientable.*

Indeed, $\overline{\mathcal{M}}_{0,k,l}(\vec{0}) \simeq \mathbb{RP}^1 \times \overline{\mathcal{M}}_{0,k,l}$, where $\overline{\mathcal{M}}_{0,k,l}$ is the moduli of stable marked disk. $\overline{\mathcal{M}}_{0,k,l}$ is orientable. See, for example, [31, Section 2.5]. One possible proof for the case $\vec{d} \neq \vec{0}$ is as follows. $\mathcal{M}_{0,k,l}(\vec{d})$ is orientable, as can be seen by iteratively forgetting the markings and noting that the fibers of the forgetful maps are either punctured disks or punctured circles, both are orientable. $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$ is obtained from $\mathcal{M}_{0,k,l}(\vec{d})$ first by adding strata of codimension at least 2 which are not the topological boundary of the moduli. Adding them cannot affect orientability. We then add the boundary, which contains codimension 1 strata, but adding boundary does not affect orientation as well.

Recall that an orientable manifold or orbifold M with boundary or corners induces orientation on its boundary. Let $o_M \in \det(M)^\times/\mathbb{R}_+$ denote the orientation of M. For any $p \in \partial M$, which does not belong to any corner of codimension at least 2 we have the short exact sequence

(82) $$0 \to \iota^* T_p \partial M \to T_p M \to N \to 0 \Rightarrow \det(N) \otimes \det(T\partial M) \simeq \det(M),$$

where N is the normal and $\iota : \partial M \to M$ the inclusion. The *induced orientation* on ∂M at p is the element $o_{\partial M} \in \det(T\partial M)/\mathbb{R}_+$ such that under the isomorphism (82) we have

$$o_N \otimes o_{\partial M} = \iota^* o_M,$$

where o_N is the orientation on N as an outward pointing normal.

Definition 6.3. A generic $[u : (\Sigma, \partial\Sigma) \to (\mathbb{CP}^1, \mathbb{RP}^1)] \in \overline{\mathcal{M}}_{0,k,l}(\vec{d})$ defines an *associated decorated map* $u^{-1}(\mathbb{RP}^1) \subseteq \Sigma$. This decorated map belongs to $\mathrm{Maps}_{0,k,l,\vec{d}}$ and has a canonical *metric* for boundary segments, loops and arcs, $\ell = \ell_{(\Sigma,u)}$ defined by $\ell(\gamma) = |\int_\gamma u^* d\theta|$, where γ is a boundary segment, loop or an arc.

Notation 6.4. For $\mathfrak{G} \in \mathrm{Maps}_{0,k,l,\vec{d}}$, denote by $\mathcal{M}_{0,k,l}(\vec{d})^{\mathfrak{G}}$ the subset of the coarse moduli space of $\mathcal{M}_{0,k,l}(\vec{d})$ made of elements whose associated decorated map is $\mathfrak{G}$. The *chamber* of $\mathfrak{G}$ is the closure $\overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}}$.

The next observation follows from standard complex analysis and dimensional arguments.

Observation 6.5. For $\mathfrak{G} \in \mathrm{Maps}_{0,k,l,\vec{d}}$, $\mathcal{M}_{0,k,l}(\vec{d})^{\mathfrak{G}}$ is an open subspace of full dimension of the coarse $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$. Moreover, the space

$$\overline{\mathcal{M}}_{0,k,l}(\vec{d}) \backslash \bigcup_{\mathfrak{G} \in \mathrm{Maps}_{0,k,l,\vec{d}}} \mathcal{M}_{0,k,l}(\vec{d})^{\mathfrak{G}}$$

is covered by finitely many real suborbifolds of codimension 1.

Example 6.6. The moduli of degree $(d, 0)$ maps consists of a single chamber – the one of the decorated map without loops or arcs, except the boundary of the underlying disk, whose perimeter is d. See Figure 5, left picture, for the case $d = 2$.
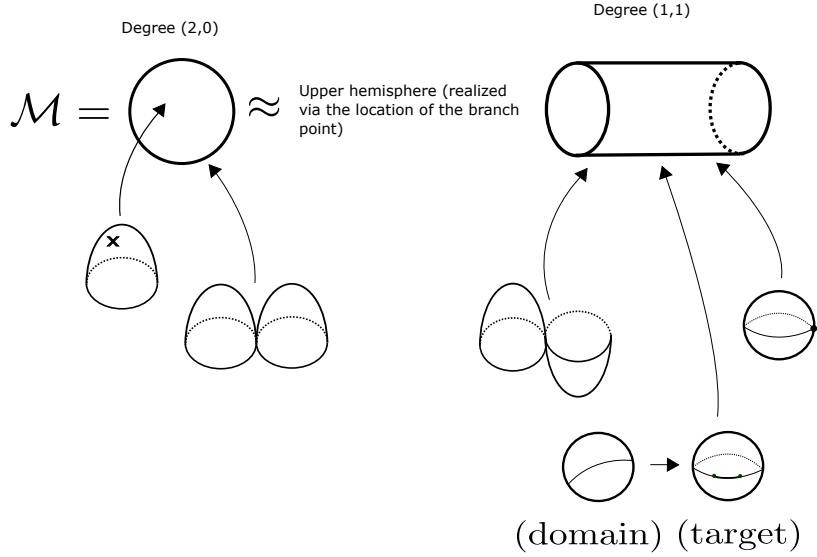
FIGURE 5. An illustration of the moduli spaces for genus 0, degrees $(2,0)$, $(1,1)$ and no internal markings. $\mathcal{M}_{0,0,0}(2,0)$ is made of a single chamber, without arcs and loops. The moduli is equivalent to a disk via the location of the branch point. $\mathcal{M}_{0,0,0}(1,1)$ is equivalent to a cylinder. There is a single chamber, which corresponds to a decorated map with a single arc and no loops.

The moduli of maps of degree $(1,1)$ consists of a single chamber – of the decorated map with a single arc and no loops, such that one region is positive, the other is negative and the perimeters of their boundaries are 1. This moduli is topologically a cylinder $\mathbb{RP}^1 \times [0, 2\pi]$, see the right picture in Figure 5. A map is completely determined by the images its two ramification points on $\mathbb{RP}^1$, $u_{\pm}$, chosen in a way that in the domain curve the oriented boundary arc between the preimage of u_- to that of u_+ is in the boundary of the positive (the one which maps to the upper hemisphere).

One boundary of the moduli (the exceptional boundary) is obtained when $u_- \to u_+$ from the left, which results in a contracted boundary component, or equivalently, a degree one sphere with a single marked point constrained to lay in $\mathbb{RP}^1$. The other boundary is obtained when $u_+ \to u_-$ from the left. The result is a map from a pair of disks, one maps to upper hemisphere (degree $(1,0)$) and the other to lower hemisphere (degree $(0,1)$). The moduli is parameterized by the image of the node, which is an arbitrary point of $\mathbb{RP}^1$. Thus, the two boundaries are indeed topologically S^1.

The moduli of maps of degree $(2,1)$ consists of two chambers. One has the boundary loop and an internal loop, both of perimeter 1. These loops are both boundaries of a region which is a positive cylinder. The other region is a negative disk. The second chamber corresponds to a decorated map with a single arc but no loops. It divides the disk into a positive topological disk region with two corners whose perimeter is 2, and a negative one of perimeter 1.

We now turn to define the orientation of $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$. Although Theorem 6.2 guarantees the existence of such an orientation to each moduli separately, in order to define invariants we must choose a specific orientation for each moduli, and our calculations required to understand the relations between different orientations. Our approach resembles the approach of [33], where the orientation of the moduli space of graded surfaces with boundary is constructed using Strebel's stratification and its boundary behaviour is explored. We assume $\vec{d} \neq \vec{0}$. We define the orientation by explicitly writing an expression for the orientations of the chambers of decorated maps. There are alternative ways to define the orientation, but this definition is best suited for our needs.

Observe that a generic $u : (\Sigma, \partial\Sigma) \to (\mathbb{CP}^1, \mathbb{RP}^1)$ of degree $\vec{d}$ has only simple ramification points with distinct images under u, called the *branch points* and moreover it has l_u internal ramification points and k_u boundary ramification points where

(83) $$2l_u + k_u = 2(g + d_+ + d_-) - 2.$$

Indeed, the doubled map $u_{\mathbb{C}} : \Sigma_{\mathbb{C}} \to \mathbb{CP}^1$ generically has $2l_u + k_u$ ramification points on the one hand, and by Riemann-Hurwitz this number equals $2(g + d_+ + d_-) - 2$. See [13], Section 5, for more details.

In fact, the same argument shows the following slightly stronger claim.

Observation 6.7. The number of (complex) branch points inside a region $\mathcal{R}$ of genus g (the genus is the doubled genus of $\mathcal{R}$ thought of as a topological surface with boundary) which has b boundary components, the ith one covers $\mathbb{RP}^1$ d_i times is

$$g - 1 + \sum_{i=1}^{b} |d_i|.$$

Other ramification types or nodes appear in positive codimension. Since any automorphism of a stable map must take branch points to branch points it is straightforward to verify that the locus of orbifold points in the moduli is of real codimension at least 2.

It is a classical fact that the branch points form local holomorphic coordinates to an open dense subset of the moduli of maps of degree d from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ ([9], page 17, proof of Proposition 2, shows an all genus claim). The same holds for maps of degree $\vec{d}$ from a disk to $(\mathbb{CP}^1, \mathbb{RP}^1)$. Let U be an open subset of $\mathcal{M}_{0,k,l}(\vec{d})$, all of whose points correspond to maps with only simple ramifications and no automorphisms. Assume that U is of the form $V \cap \text{For}_{marking}^{-1}(U')$, where $V \subseteq \mathcal{M}_{0,k,l}(\vec{d})$, $U' \subset \mathcal{M}_{0,0,0}(\vec{d})$ are open, U' is contractible and $\text{For}_{marking}$ is the map which forgets all markings. Local coordinates for U can be taken to be

$$\left((W_i)_{i=1}^{l_u}, (Y_i)_{i=1}^{k_u}, (Z_i)_{i=1}^{l}, (X_i)_{i=1}^{k} \right) : U \to (\mathbb{CP}^1 \backslash \mathbb{RP}^1)^{l_u} \times (\mathbb{RP}^1)^{k_u} \times \mathbb{H}^l \times \partial\mathbb{H}^k,$$

where $W_i \in \mathbb{CP}^1$ is the ith complex branch point, $Y_i \in \mathbb{RP}^1$ is the ith real branch point, and Z_i, X_j are defined as follows. We assume that U' is small enough so that there are smooth functions

$$P : U' \to \mathbb{RP}^1, \quad Q : U' \to \mathbb{CP}^1 \backslash \mathbb{RP}^1$$

which satisfy that $P(\Sigma, u), Q(\Sigma, u)$ are never branch points of

$$u : (\Sigma, \partial\Sigma) \to (\mathbb{CP}^1, \mathbb{RP}^1).$$

Such functions can be found for all U' small enough, and any U' as above can be covered by smaller open sets for which such functions P, Q can be found. Define the smooth maps $\widetilde{P}, \widetilde{Q}$ from U' to the universal curve restricted to U' by the requirement

$$\widetilde{P}(\Sigma, u) \in \partial\Sigma, \quad u(\widetilde{P}(\Sigma, u)) = P(\Sigma, u), \quad \widetilde{Q}(\Sigma, u) \in \Sigma \backslash \partial\Sigma, \quad u(\widetilde{Q}(\Sigma, u)) = Q(\Sigma, u).$$

There is a unique holomorphic biholomorphism $\phi_{(\Sigma, u)}$ which takes $(\Sigma, \partial\Sigma)$ to $(\overline{\mathbb{H}}, \partial\mathbb{H})$, where $\mathbb{H}$ is the upper hemisphere, and which takes $\widetilde{P}$ to 0 and $\widetilde{Q}$ to $+\sqrt{-1} \in \mathbb{H}$. The images of the internal and boundary markings under $\phi_{[(\Sigma, u)]}$ are the functions X_i, Z_j respectively. We call them the *locations* of the markings in the domain disk (identified with $\overline{\mathbb{H}}$).

We call these coordinates the *branching and marking coordinates (B&M coordinates)*.

Remark 6.8. Even when $k, l = \varnothing$ the branch points give only a local description of the moduli, and are not sufficient for reconstructing a map to $(\mathbb{CP}^1, \mathbb{RP}^1)$. For instance, even in the generic case an additional discrete data is needed for reconstruction: how do the different sheets which are preimages of neighborhoods of the branch point glue to give the whole surface.

For a decorated map $\mathfrak{G}$ let $s_{\mathfrak{G}}$ be -1 when $k_u = 0$ and $d_- > d_+$. Otherwise put $s_{\mathfrak{G}} = 1$. We say that Y_i is *negative* if the region on the left of the *ramification* point $y_i \in u^{-1}(Y_i)$ is negative, otherwise it is *positive*. We similarly call the corresponding ramification points $y_i \in u^{-1}(Y_i)$ positive or negative. The points $y_i \in u^{-1}(Y_i)$ are ordered along the boundary of the disk in a way that a negative point follows a positive one, and a positive point follows a negative one.

The B&M coordinates and an order π on the boundary marked points determine orientations $\mathfrak{o}_{\mathfrak{G}}^{\pi}$ on $\overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}}$ given by

$$(84) \qquad \mathbb{R}_+ s_{\mathfrak{G}} \bigwedge \frac{\partial}{\partial X_i} \otimes \bigwedge \frac{\partial}{\partial Y_i} \otimes \bigwedge \frac{\sqrt{-1}}{2} \frac{\partial}{\partial Z_i} \wedge \frac{\partial}{\partial \bar{Z}_i} \otimes \bigwedge \frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_i} \wedge \frac{\partial}{\partial \bar{W}_i},$$

where $\frac{\partial}{\partial X_i}$ are ordered in reverse π order (from the last to the first, with respect to π), and $\frac{\partial}{\partial Y_i}$ are ordered according to the cyclic order of the ramification points y_i on the boundary of the domain disk, *starting from a positive point*. The orientations of all 1–forms are the canonical ones, defined by the oriented domain and target surfaces, and the positive orientation on $\mathbb{RP}^1$, the one induced by the upper hemisphere. When $k \leq 1$ we omit π from the notation.

Remark 6.9. In expression (84) we consider W_i to be complex and Y_i to be real, via the embedding $(\mathbb{C}, \mathbb{R}) \hookrightarrow (\mathbb{CP}^1, \mathbb{RP}^1)$, and use the upper half-plane model for the domain disk. Later, for some purposes it will be more convenient to consider $\mathbb{RP}^1$ as the boundary of the unit disk in $\mathbb{C} \subset \mathbb{CP}^1$, with its angular parametrization. The transition between the two coordinate system changes (84) by an irrelevant positive function of the Y_i variables. Thus, even in this case we will consider the explicit expression (84) as the orientation expression.

Some straightforward corollaries of the definitions are

Observation 6.10. The above orientation is independent of the choices of P, Q.

Indeed, changing the functions P,Q amounts to changing the coordinate orientation expression by a positive function.

Under $z \to 1/z$, positive and negative regions are interchanged and if there are no real branchings also $s_{\mathfrak{G}}$ is flipped, therefore

Observation 6.11. Fix an order π on the boundary markings. The map $z \to 1/z$ on the target $\mathbb{CP}^1$ induces a map

$$\overline{\mathcal{M}}_{0,k,l}(\vec{d}) \to \overline{\mathcal{M}}_{0,k,l}(d_-, d_+)$$

by composition. This map is orientation reversing, with respect to the orientations of (84).

In addition, since also in the closed case the branch points form local holomorphic coordinates on an open dense subset, we have

Observation 6.12. In the closed case, the orientation (84) agrees with the standard complex orientation of the moduli of maps.

Let $\mathfrak{G}'$ be the decorated map obtained from $\mathfrak{G}$ by erasing the last internal or boundary marking. Let π' be the order on the remaining boundary points induced from π. Consider the fiber of the forgetful map

$$\text{For} : \mathcal{M}_{0,k,l}(\vec{d})^{\mathfrak{G}} \to \mathcal{M}_{0,k-\varepsilon_k, l-\varepsilon_l}(\vec{d})^{\mathfrak{G}'},$$

where $\varepsilon_k = 1, \varepsilon_l = 0$ if we forget the last boundary marking, and otherwise $\varepsilon_k = 1, \varepsilon_l = 0$. The fiber is identified with an interval embedded in the boundary or the whole boundary if $\varepsilon_k = 1$, and otherwise it is a surface embedded in the domain disk. In both cases, if $U' \subseteq \mathcal{M}_{0,k-\varepsilon_k, l-\varepsilon_l}(\vec{d})^{\mathfrak{G}'}$ is an open set with B&M coordinates, then on $U = \text{For}^{-1}(U')$ one may define B&M coordinates by $W_i^U = W_i^{U'} \circ \text{For}$, $Y_i^U = Y_i^{U'} \circ \text{For}$ and using $\widetilde{P}^U = \widetilde{P}^{U'} \circ \text{For}$, $\widetilde{Q}^U = \widetilde{Q}^{U'} \circ \text{For}$. Thus, for all Z_i except

Z_l, when $\varepsilon_l = 1$, it holds that $Z_i^U = Z_i^{U'} \circ \mathrm{For}$. Similarly for X_i. The additional coordinate Z_l or X_k may be taken as a coordinate of the fiber of For. With these identifications of coordinates, and the orientation (84), the following holds.

Observation 6.13. The orientation of the fiber, o_{Fiber}, induced by the equation

$$o_{Fiber} \otimes \mathrm{For}^* \, \mathfrak{o}_{\mathfrak{G}'}^{\pi'} = \mathfrak{o}_{\mathfrak{G}}^{\pi},$$

agrees with the one induced from the embedding of the fiber in the domain disk or its boundary.

The final result of this section is

Proposition 6.14. $\mathfrak{o}_{\mathfrak{G}}^{\pi}$ *glue to give a consistent orientation* $\mathfrak{o}^{\pi}$ *on* $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$.

Proof. Consider $\mathfrak{G} \in \mathrm{Maps}_{0,k,l,\vec{d}}$, and let U be as above. Then $U \cap \overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}}$ is an open dense subset of $\overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}}$. The ramification picture allows us to understand codimension 1 boundaries of the moduli and more generally of the specific chambers. *Walls* or codimension 1 boundaries of the chambers which are not boundaries of the moduli, occur when there is a change in the topology of the decorated maps which does not involve a change in the topology of the underlying surface. In codimension 1 it happens in one of the following cases. The simplest case is when a boundary marking moves from one region to another. It is straightforward, from the explicit dependence of the orientations on the boundary markings, to see that the orientations of the corresponding chambers glue.

The more interesting cases are when either two regions meet, or when one region splits into two, or when a region hits the boundary. These cases can be equivalently described in terms of ramification points, as can be seen directly from the decorated map picture, or by counting arguments using Observation 6.7: from this point of view a wall is described by a ramification point which hits a boundary of a region, or by contracting the boundary interval between two boundary ramification points. In the first case the point will actually hit two boundaries (which are different at least locally) and will map to $\mathbb{RP}^1$. In the second the two ramification points will unite to create a single ramification point of order 3. When the internal ramification point hits the boundary of the surface, if the result is a true wall between chambers (and not a boundary component of the moduli), then after crossing that wall the internal ramification point splits into two boundary ramification points. The local pictures appear in Figure 6.

In order to show that $\mathfrak{o}^{\pi}$ exists it is enough to show that when two chambers share a common wall the orientation varies continuously between them. By the definition of orientation and Observation 6.13, the claim holds for $\overline{\mathcal{M}}_{0,k,l}(\vec{d})$ if and only if it holds for $\overline{\mathcal{M}}_{0,0,0}(\vec{d})$. Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two decorated maps which represent neighboring chambers. In case they have the same number of arcs, then the passage between $\overline{\mathcal{M}}_{0,0,0}(\vec{d})^{\mathfrak{G}_1}, \overline{\mathcal{M}}_{0,0,0}(\vec{d})^{\mathfrak{G}_2}$ is the result of a ramification point which moves from one region to another one and crosses a loop or an arc. See again Figure 6, upper row. Locally near a generic point of the wall between the chambers we can use the same set of B&M coordinates (in fact, only branching coordinates as $k = l = 0$) to see that the orientation expression does not change.

Suppose now that $\mathfrak{G}_1, \mathfrak{G}_2$ differ in their number of arcs. The difference must be one, so suppose $\mathfrak{G}_1$ has m arcs, and $\mathfrak{G}_2$ has $m+1$. Geometrically we are in the scenario that one complex ramification point, which belongs to a region of $\mathfrak{G}_1$ touching the boundary, hits the boundary. The local picture is as in Figure 6, lower row.

Suppose this is the point with coordinate $W = W_j$. Assume first that W is in the upper hemisphere, which we now consider in the upper half-plane model. Let $W = x + iy$, $x, y \in \mathbb{R}$. We have

$$\mathfrak{o}_{\mathfrak{G}_1} = \frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_1} \wedge \frac{\partial}{\partial \bar{W}_1} \wedge \tilde{\mathfrak{o}} = -\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x} \wedge \tilde{\mathfrak{o}},$$
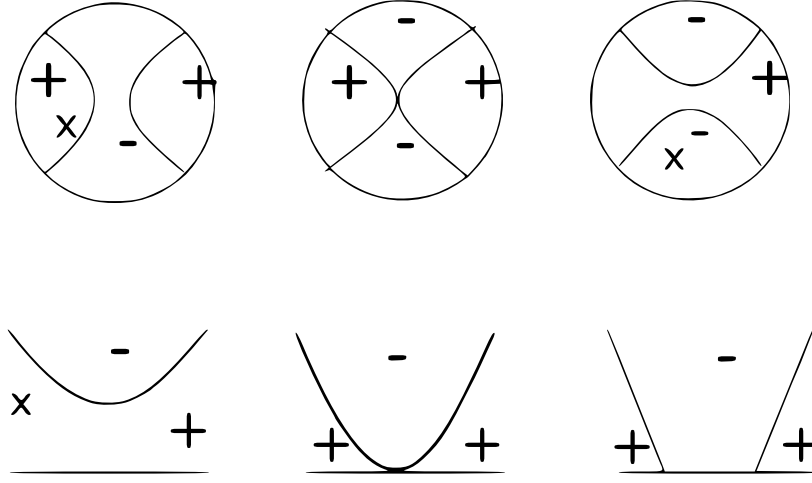
FIGURE 6. Local picture of the change in the topology of decorated maps. In the upper row we zoom in the neighborhood of an internal ramification point (denoted by ×). In the left picture, this point approaches a boundary of a positive region. In the middle it hits that boundary, and simultaneously other boundary hits it as well, in the same point. On the right the ramification point moves to a neighboring negative region, and the topology of the decorated map change. In the lower row an internal ramification point × approaches the boundary of the underlying surface. The middle picture describes the moment it hits, the right picture shows how it "splits" into two boundary ramification points.
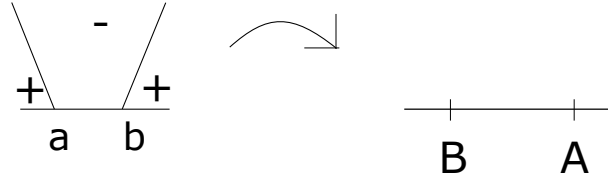

FIGURE 7. Local picture near consecutive boundary ramification points, in the domain (left) and the target (right). Under the map $a \to A, b \to B$. The horizontal segments in the left picture, which stand for boundary segments in the domain, are mapped to the horizontal segment from B to A (in $\mathbb{RP}^1$) on the right. The two arcs on the left map to the segments on the left of B and on the right of A.

where $\tilde{\mathfrak{o}}$ is the remaining expression for the orientation. The outward normal to the corresponding boundary of $\overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}_1}$ is $-\frac{\partial}{\partial y}$. The induced orientation on the wall is therefore $\frac{\partial}{\partial x} \wedge \tilde{\mathfrak{o}}$. After hitting the boundary, the ramification point "splits" into two real *ramification* points, which we denote by $a = Y_{2m+1}, b = Y_{2m+2}$, where a is to the left of b, and a is positive (the region to its left is positive), while b is negative. After applying the map, the corresponding *branch points*, A, B are in opposite order $- B$ is to the left of A. See Figure 7. We can thus write

$$\mathfrak{o}_{\mathfrak{G}_2} = \frac{\partial}{\partial A} \wedge \frac{\partial}{\partial B} \wedge \tilde{\mathfrak{o}} = -\left(\frac{\partial}{\partial B} - \frac{\partial}{\partial A}\right) \wedge \frac{\partial}{\partial A} \wedge \tilde{\mathfrak{o}},$$

where $\tilde{\mathfrak{o}}$ is as before. The outward normal for the boundary, from the $\overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}_2}$ side is $\left(\frac{\partial}{\partial B} - \frac{\partial}{\partial A}\right)$, since $A - B$ becomes smaller as we get closer to the wall $B = A$. Thus, the orientation on the wall, induced by the orientation of the chamber of $\mathfrak{G}_2$ is $-\frac{\partial}{\partial A} \wedge \tilde{\mathfrak{o}}$. Since along the wall $\frac{\partial}{\partial A}$ can be identified with $\frac{\partial}{\partial x}$, we see that the two chambers induce opposite orientations on the wall, and thus agree.
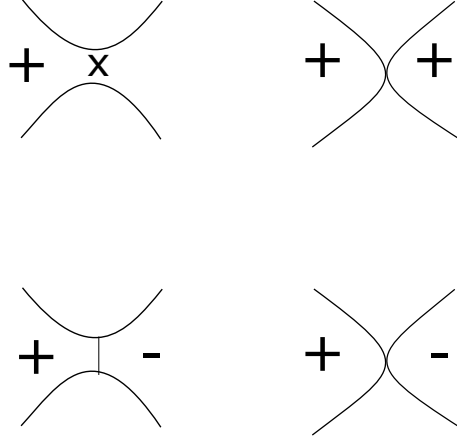
FIGURE 8. Local picture of the creation of boundary nodes. In the upper row a ramification point, denoted ×, approaches two (locally) different boundary components of the underlying domain. This results in a boundary node between two regions of the same sign. The lower row describes an arc which contracts to a point and creates a boundary nodes between regions of different signs.

The case that W is in the lower hemisphere follows similarly, or can be deduced from this case by applying inversion $z \to 1/z$ on the target, which interchanges positive and negative regions. $\qquad\square$

Remark 6.15. The loci in the moduli where a ramification point hits the boundary, are either a wall between two chambers, as considered in Proposition 6.14, or a true boundary of the moduli. The way to distinguish between these two cases is as follows. By following the number of complex ramification points in each region, using Observation 6.7, we see that whenever a ramification point hits the boundary either a boundary point of another *region* collides with the boundary of the disk (and in the same point), or two boundary points of the disk collide. The former case gives rise to a change in chambers, the latter give rise to the codimension 1 boundary discussed below. See also Figure 8 for the second case.

6.3. **Properties of the orientation $\mathfrak{o}^\pi$.** The goal of this section is to understand the behaviour of the orientation $\mathfrak{o}^\pi$ when restricted to nodal strata. The section is divided into three parts. The first considers the induced orientation on real codimension one standard boundary strata, which parameterizes maps from nodal disks with a boundary node. The second considers strata of nodal disks with an internal node. The last handles the exceptional boundary stratum of the moduli, where the boundary of the disk contracts to a point which is mapped to $\mathbb{RP}^1$.

6.3.1. *Induced orientation to standard boundary strata of codimension* 1. Suppose now $k = 0$ and consider a real codimension 1 boundary component

$$\mathcal{B} = \mathcal{M}_{0,\star_1,\mathsf{l}_1}\left(\vec{d}_1\right) \times_L \mathcal{M}_{0,\star_2,\mathsf{l}_2}\left(\vec{d}_2\right) \xrightarrow{i_\mathcal{B}} \partial\overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}\left(\vec{d}\right)$$

The component $\mathcal{B}$ is endowed with an orientation $\mathfrak{o}_\mathcal{B}$ coming from the orientation on $\mathcal{M}_{0,\varnothing,\mathsf{l}}\left(\vec{d}\right)$ and also with a map

$$f_\mathcal{B} = (f_1 \times f_2) \circ g : \mathcal{B} \to \mathcal{M}_1^\diamond \times \mathcal{M}_2^\diamond,$$

where f_i is the map of (33), defined by forgetting the node, $\mathcal{M}_i^\diamond = \mathcal{M}_{0,\varnothing,\mathsf{l}_i}\left(\vec{d}_i\right)$, and $g : \mathcal{B} \to \mathcal{M}_{0,\star_1,\mathsf{l}_1}\left(\vec{d}_1\right) \times \mathcal{M}_{0,\star_2,\mathsf{l}_2}\left(\vec{d}_2\right)$ is the structure map. Moreover, $\mathcal{B}$ is endowed with an evaluation map at the node, $\mathrm{ev}_\star : \mathcal{B} \to \mathbb{RP}^1$. Its generic fiber F is of real codimension 1 and the map $\mathrm{ev}_\star|_F$ is a

submersion away from a finite number of points. F is endowed with two orientations. One is the ratio $\mathfrak{o}_F = \frac{\mathfrak{o}_{\mathcal{B}}}{f_{\mathcal{B}}^*(\mathfrak{o}_{0,\varnothing,\mathsf{l}_1;\vec{d}_1} \boxtimes \mathfrak{o}_{0,\varnothing,\mathsf{l}_2;\vec{d}_2})}$, defined using the canonical isomorphism

$$\det T_p F \otimes f_{\mathcal{B}}^* \det(T_{f_{\mathcal{B}}(p)}(\mathcal{M}_1 \times \mathcal{M}_2)) \simeq \det T_p \mathcal{B},$$

which follows from the exact sequence

$$0 \to T_p F \to T_p \mathcal{B} \to f_{\mathcal{B}}^* T_{f_{\mathcal{B}}(p)}(\mathcal{M}_1 \times \mathcal{M}_2) \to 0.$$

The other orientation is defined away from points where $\mathrm{ev}_*|_F$ is not submersion by

$$d\mathrm{ev}_* \mathfrak{o}_F' = \frac{\partial}{\partial \theta}.$$

One may compare the two orientations.

Assume now that $\vec{d}_1, \vec{d}_2 \neq 0$ and let $(\Sigma, u) \in \mathcal{B}$. Generically the node is not a ramification point for either of the disks. Therefore it touches a single region in each disk which has a definite sign, defined as usual according to whether this region maps to the positive or negative hemisphere.

Lemma 6.16. *With the above notations, consider a $(\Sigma, u) \in \mathcal{B}$, where the node of Σ is not a ramification point. The orientations $\mathfrak{o}_F, \mathfrak{o}_F'$ induced on the fiber of the map which forgets the node agree, locally near (Σ, u), precisely if the two regions which touch the half nodes $\star_i$ have the same sign.*

Proof. Let U be a neighborhood of (Σ, u) small enough such that $U \subseteq \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(\vec{d})^{\mathfrak{G}}$ for a fixed decorated map $\mathfrak{G}$. Denote by $\mathfrak{G}_i$ the associated decorated map of the projection of $U \cap \mathcal{B}$ to the $\mathcal{M}_i^{\diamond}$ component. Consider first the case that the signs of the two regions near the node are the same. The topological picture near $U \cap \mathcal{B}$ is that some ramification point approaches the boundary, and its corresponding branch point approaches $\mathbb{RP}^1$, see the upper row of Figure 8.

By using Observation 6.11 it is enough to handle the case where the two regions touching the node are positive. Let W_1 be the coordinate of the branch point which hits the boundary and thus creates the node. One may write

(85) $$\mathfrak{o}_{\mathfrak{G}} = \frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_1} \wedge \frac{\partial}{\partial \bar{W}_1} \wedge \tilde{\mathfrak{o}},$$

where $\tilde{\mathfrak{o}}$ is the rest of the orientation expression (84) in the B&M coordinates. We can write

$$\frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_1} \wedge \frac{\partial}{\partial \bar{W}_1} = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} = -\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1},$$

where W_1 is considered in the upper half-plane, $x_1 = Re(W_1)$, $y_1 = Im(W_1)$. As W_1 approaches the boundary, $-\frac{\partial}{\partial y_1}$ can be taken as the external normal, while $\frac{\partial x_1}{\partial \theta} > 0$. From equations (84),(85) it is evident that

$$(f_{\mathcal{B}})_* (\tilde{\mathfrak{o}}|_{\mathcal{B} \cap U}) = (\mathfrak{o}_{0,\varnothing,\mathsf{l}_1;\vec{d}_1} \boxtimes \mathfrak{o}_{0,\varnothing,\mathsf{l}_2;\vec{d}_2})|_{f_{\mathcal{B}}(\mathcal{B} \cap U)}.$$

Thus, in this case $\mathfrak{o}_F = \mathfrak{o}_F'$.

We now compare the two orientations in case the signs of the two regions touching the node are opposite. Assume that there are $2r$ real ramification points. We show that in this case the ratio is -1. By the usual inversion trick we see that proving this claim for (d_+, d_-) is equivalent for proving for the pair (d_-, d_+). We may assume $d_+ \geq d_-$, so that $s_{\mathfrak{G}} = 1$. From the point of view of topology of decorated maps, the case under consideration occurs when some arc between two real ramification points contracts to a point. Recall from Definition 6.3 that arcs and boundary segments between ramification points have an intrinsic length. Let y_i be the ramification point in the preimages of the real branch point Y_i. We choose the indexing so that for $i < 2r$ y_{i+1} follows

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y_i cyclically according to the orientation of the domain, and y_{2i} are negative points for $i \in [r]$. We will use angular parametrization for $\mathbb{RP}^1$

(86)
$$\alpha \to w_0 e^{2\pi i \alpha}$$

for some w_0 with $|w_0| = 1$, and consider Y_i to belong to the interval $[0, 2\pi)$. Assume it is the arc α from y_1, which is positive, to y_{2s}, which is negative, that contracts to yield a node, as in the lower row of Figure 8. Consider $(\Sigma, u) \in \mathcal{B}$, and a neighborhood U of it such that $U \cap \overline{\mathcal{M}}_{0,k,l}(\vec{d}) \subset \overline{\mathcal{M}}_{0,k,l}(\vec{d})^{\mathfrak{G}}$. Suppose U is small enough that we can choose the coordinates Y_i in a way that for $i < 2r$

(87)
$$Y_i - Y_{i+1} = \begin{cases} \{\ell(\overline{y_i y_{i+1}})\}, & \text{if } y_i \text{ is positive,} \\ -\{\ell(\overline{y_i y_{i+1}})\}, & \text{if } y_i \text{ is negative,} \end{cases}$$

where $\overline{y_i y_{i+1}}$ is the positively oriented boundary segment from y_i to y_{i+1}, and $\{\}$ stands for fractional part (the requirement on U being small enough amount to the existence of w_0 as in (86) which makes the sequence $(Y_i)_{i \in [2r]} \subset [0, 2\pi)$ monotonically increasing. That would guarantee property (87)). In this case, since boundary components of regions have integer perimeters, it is easy to see that the length $\ell(\alpha)$ equals, modulo 1, to

$$-\left(Y_1 - Y_{2s} + \sum_{i \neq 1, 2s} a_i Y_i \right) \mod 1,$$

for some integers a_i, see the example in Figure 9. $\ell(\alpha)$ is a coordinate for the inward normal, therefore the outward normal can locally be written as

$$\frac{\partial}{\partial N} = \frac{\partial}{\partial Y_1} - \frac{\partial}{\partial Y_{2s}} + \sum_{i \neq 1, 2s} a_i \frac{\partial}{\partial Y_i}.$$

Now, the orientation $\mathfrak{o}_{0,\varnothing,l;\vec{d}}$ can be locally written as

$$\frac{\partial}{\partial Y_1} \wedge \frac{\partial}{\partial Y_{2s}} \bigwedge_{i \neq 1, 2s} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}' \wedge \mathfrak{o}'' = \left(\frac{\partial}{\partial Y_1} - \frac{\partial}{\partial Y_{2s}} + \sum_{i \neq 1, 2s} a_i \frac{\partial}{\partial Y_i} \right) \wedge \frac{\partial}{\partial Y_{2s}} \bigwedge_{i \neq 1, 2s} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}' \wedge \mathfrak{o}'',$$

where the order of taking the wedge is just the numerical order, $\mathfrak{o}', \mathfrak{o}''$ stand for the wedge of terms which come from complex branch points or internal marked points, in the two disk components. We assume that the disk with the first l_1 points is the one which touches the node in a positive region. The induced orientation on $\mathcal{B}$ is therefore

$$\frac{\partial}{\partial Y_{2s}} \bigwedge_{i \neq 1, 2s} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}' \wedge \mathfrak{o}''.$$

As $\frac{\partial}{\partial Y_{2s}}$ and $\frac{\partial}{\partial \theta}$ point to the same direction, it remains to check that

(88)
$$(f_{\mathcal{B}})_* \left(\bigwedge_{i \neq 1, 2s} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}' \wedge \mathfrak{o}'' \right) = -\mathfrak{o}_{0,\varnothing,l_1;\vec{d}_1} \boxtimes \mathfrak{o}_{0,\varnothing,l_2;\vec{d}_2}.$$

We can write

$$\bigwedge_{i \neq 1, 2s} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}' \wedge \mathfrak{o}'' = \left(\bigwedge_{i=2}^{2s-1} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}'' \right) \wedge \left(\bigwedge_{i > 2s} \frac{\partial}{\partial Y_i} \mathfrak{o}' \right),$$

the wedge is taken again in the numerical order. Now, with $\mathrm{pr}_i : \mathcal{M}_1^{\Diamond} \times \mathcal{M}_2^{\Diamond} \to \mathcal{M}_i^{\Diamond}$, we have

$$(\mathrm{pr}_1 \circ f_{\mathcal{B}})_* \left(\bigwedge_{i > 2s} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}' \right) = \mathfrak{o}_{0,\varnothing,l_1;\vec{d}_1},$$
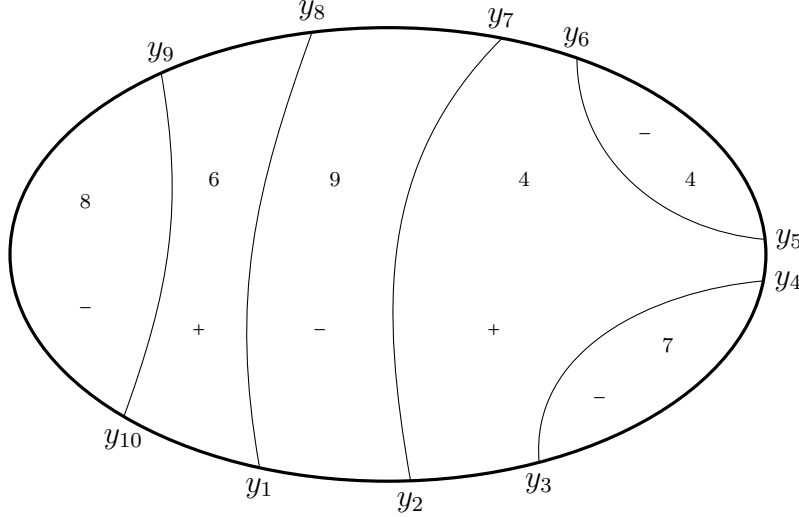
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FIGURE 9. We consider a map with several arcs which divide the disk into domains, some positive, and some negative. The perimeter of each domain also appears in the figure. Lengths of arcs, modulo 1, can be calculated using the lengths of boundary segments between the ramification points. Y_i is the branching coordinate of y_i. For example, modulo 1, the length of boundary segment from y_3 to y_4 is $Y_3 - Y_4$, the length of the arc from y_3 to y_4 is $Y_4 - Y_3$, and the length of the arc from y_1 to y_8 is $-Y_1 + Y_8 - Y_7 + Y_2 - (-(Y_3 - Y_2 + Y_5 - Y_4 + Y_7 - Y_6 - (Y_4 - Y_3 + Y_6 - Y_5)))$.

since if there are Y_i with $i > 2s$, they are taken in the correct order, as Y_{2s+1} is positive, while if not, then it must be that $d_1' > d_2'$, and again we find agreement, since the node touches the corresponding disk component in a positive region. Similarly,

$$(\mathrm{pr}_2 \circ f_{\mathcal{B}})_* \left(\bigwedge_{i=2}^{2s-1} \frac{\partial}{\partial Y_i} \wedge \mathfrak{o}'' \right) = -\mathfrak{o}_{0,\varnothing,\mathsf{l}_2;\vec{d}_2},$$

since if $s > 1$, the wedge is taken starting from a negative point Y_2, while if $s = 1$ an extra sign $s_{\mathfrak{G}'}$ appears in the definition of $\mathfrak{o}_{0,\varnothing,\mathsf{l}_2;\vec{d}_2}^{\mathfrak{G}'}$ for any decorated map $\mathfrak{G}'$ without real branches. Equation (88) follows. $\square$

Still working with the same notations, we have the following simple proposition. Let $\mathcal{B}$ parameterize stable maps from a domain made of two nodal disks, one is of degree 0 and contains the ith internal marking, and the other of non-zero degree, and contains the other markings. Thus, we can identify $\mathcal{M}_{0,\star_1,\{i\}}(\vec{0})$ with a point, and then

$$\mathcal{B} = \mathcal{M}_{0,\star_1,\{i\}}(\vec{0}) \times_L \mathcal{M}_{0,\star_2,\mathsf{l}\setminus\{i\}}(\vec{d})$$

canonically inherits a canonical S^1-bundle structure

$$f_2 : \mathcal{B} \to \mathcal{M}_2^\diamond = \mathcal{M}_{0,\varnothing,\mathsf{l}\setminus\{i\}}(\vec{d})$$

whose fiber at (Σ_2, u_2) is $\partial \Sigma_2$. Let $\mathfrak{o}_F$ be the orientation of the fiber which agrees, via this identification, with the orientation of the boundary of the *domain* disk Σ_2.

Proposition 6.17. *The orientations $\mathfrak{o}_{\mathcal{B}}$, $\mathfrak{o}_F \wedge f_2^* \mathfrak{o}_{\mathcal{M}_2^\diamond}$ agree. Hence, for any fiber F of $f_2 : \mathcal{B} \to \mathcal{M}_2^\diamond$, oriented by $\mathfrak{o}_F$, we have*

$$\int_F \mathrm{ev}_{\star_2}^* d\theta = d_+ - d_-.$$

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Proof. The proof of the first part is similar to the proof of Lemma 6.16, the case when a ramification point hits the boundary, we only mention the differences: this time it is a marked point whose coordinate is Z_i which hits the boundary, rather than a branch point. Writing $Z_i = x_i + \sqrt{-1}y_i$, we see that $-y_i$ corresponds to the external normal, and x_i is the location of the node in the domain, seen from the second component (of non-zero degree). Hence $\frac{\partial}{\partial x_i}$ is oriented as $\mathfrak{o}_F$.

The degree of $\mathrm{ev}_{\star_2}|_F : F \to \mathbb{RP}^1$ is $\pm(d_+ - d_-)$. In order to determine the sign, observe that when the node is in a positive region, $\mathrm{ev}_{\star_2}|_F$ takes positively oriented vectors, with respect to $\mathfrak{o}_F$, to positively oriented vectors, with respect to $\frac{\partial}{\partial\theta}$. In the negative regions positively oriented vectors are taken negatively oriented vectors. The sign is thus $+$ and the 'Hence' part follows. $\qquad\square$

6.3.2. *Induced orientation to strata of one internal node.* Let $i_{\mathcal{I}} : \mathcal{I} \hookrightarrow \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d})$ be the stratum

$$\mathcal{I} = \mathcal{M}_{0,\mathsf{l}_c\cup\{\star_c\}}(d_c) \times_{\mathbb{CP}^1} \mathcal{M}_{0,\mathsf{k},\mathsf{l}_o\cup\{\star_o\}}(\vec{d}_o) =: \mathcal{M}_c \times_{\mathbb{CP}^1} \mathcal{M}_o.$$

we assume $\vec{d}_o = (d_o^+, d_o^-) \neq (0,0)$. Then $\mathcal{N}_{\mathcal{I}}$, the normal bundle of $\mathcal{M}_{\mathcal{I}}$ in $\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d})$, is canonically

$$(89) \qquad\qquad\qquad\qquad \mathcal{N}_{\mathcal{I}} \simeq \mathbb{L}_c^\vee \times \mathbb{L}_o^\vee,$$

where $\mathbb{L}_c^\vee$ is the tangent at the node $\star_c$ in the closed component and $\mathbb{L}_o^\vee$ is the tangent at $\star_o$ in the open component, see, for example, [14]. $\mathcal{I}$ is induced an orientation $\mathfrak{o}_{\Gamma}^{\pi}$ from the orientations $\mathfrak{o}_o^{\pi}, \mathfrak{o}_c$ on $\mathcal{M}_o, \mathcal{M}_c$ respectively (which, as mentioned above, agree with the standard complex orientation on the moduli $\overline{\mathcal{M}}_c$), and the complex orientation of $\mathbb{CP}^1$ by

$$(90) \qquad\qquad 0 \to T_{(\Sigma,u)}\mathcal{I} \to T_{(\Sigma_c,u_c)}\mathcal{M}_c \boxplus T_{(\Sigma_o,u_o)}\mathcal{M}_o \to T_{u(\star)}\mathbb{CP}^1 \to 0,$$

for any $(\Sigma, u) \in \mathcal{I}$ with a sphere component (Σ_c, u_c), a disk component (Σ_o, u_o), and a node $\star$.

Lemma 6.18. *The orientation $o_{\mathcal{N}_{\mathcal{I}}}^{\pi}$ induced on $\mathcal{N}_{\mathcal{I}}$ by the sequence*

$$0 \to T\mathcal{I} \to i_{\mathcal{I}}^* T\overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d}) \to \mathcal{N}_{\mathcal{I}} \to 0$$

and the orientations $\mathfrak{o}_{\mathcal{I}}^{\pi}, \mathfrak{o}_{0,\mathsf{k},\mathsf{l};\vec{d}}$ agrees with the complex orientation of $\mathcal{N}_{\mathcal{I}}$.

Proof. In case we work in moduli space of maps to $\mathbb{CP}^1$ from Riemann surfaces without boundary, the corresponding claim is known to hold. For the open case, we work in the B&M coordinates. Assume first $\vec{d}_c \neq 0$. It will be convenient to assume $\mathsf{k} = [k], \mathsf{l} = [l]$.

By the definition of the orientations we have that the induced orientation on the fiber of For_l, the map forgetting the lth internal marked point, agrees with the complex orientation induced from the domain surface (or equivalently from the universal curve). If there are markings on the closed components, using the commutativity of

$$(91) \qquad\qquad \begin{array}{ccc} \mathcal{I} & \longrightarrow & \overline{\mathcal{M}}_{0,k,l}(\vec{d}) \\ \downarrow & & \downarrow \\ \mathcal{I}' & \longrightarrow & \overline{\mathcal{M}}_{0,k,l-1}(\vec{d}), \end{array}$$

where $\mathcal{I}' = \mathrm{For}_l(\mathcal{I})$, and the fact that, since For_l does not contract the closed component, (89) is compatible with For_l, we see that the claim for l points is equivalent to the one with $l-1$ points. So we assume there are no markings on the closed component.

By using Observation 6.11, we may assume without loss of generality that the node is in the upper hemisphere. An internal bubbling is then the result of a collision between two branch points in the upper hemisphere. In fact, a more accurate statement holds. Fix a map (Σ, u), and let $W_1, W_2 \in \mathbb{CP}^1$ be two very close branch points in the upper hemisphere. Let $V \subset \mathbb{CP}^1$ be a small common contractible neighborhood containing no other branch points. Consider $u^{-1}(V) \subset \Sigma$, and $w_i = u^{-1}(W_i)$ the corresponding ramification points.

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Let $\gamma : [0,1] \to V$ be a path with endpoints $\gamma(0) = W_1$, $\gamma(1) = W_2$. Define

$$\Gamma : [0,1] \to U \subset \overline{\mathcal{M}}_{0,\mathsf{k},\mathsf{l}}(\vec{d}),$$

where U is an open neighborhood of (Σ, u) for which B&M coordinates can be found, to be the path in the moduli with $\Gamma(0) = (\Sigma, u)$, and $\Gamma(t)$ characterized by having all B&M coordinates the same as those of (Σ, u) except for $W_1(\Gamma(t)) = \gamma(t)$. Let $(\Sigma', u') = \Gamma(1)$. When V is small enough there is no difficulty in finding such γ, U, Γ.

If w_1, w_2 are not in the same connected component of $u^{-1}(V)$, then the moduli point (Σ', u') has a ramification of type $(12)(34)$ over W_1 (see [28] for definitions of the different ramification types, [18] is another helpful reference). Suppose now that they are in the same connected component of $u^{-1}(V)$. Let $\tilde{\gamma}$ be a path in $u^{-1}(V)$, connecting w_1 to w_2, and we may take $\gamma = u \circ \tilde{\gamma}$ to be its projection to V. Recall that a for a generic $P \in \mathbb{CP}^1$ in the upper hemisphere $|u^{-1}(P)| = d_+$. If the number of connected components of $u^{-1}(\gamma)$ is $d_+ - 2$, then (Σ', u') has a ramification of type (123) at W_1. The complementary case, when the number of connected components of $u^{-1}(\gamma)$ is $d_+ - 1$, is the one which gives a nodal map (Σ', u'), when W_1, W_2 collide.

With this topological observation, the proof of the lemma, in case $d_c \neq 0$, is now simple. Let W_1, W_2 be the coordinates of the branch points that collide and create the node. We identify the upper hemisphere with the upper half-plane. One may write

$$\mathfrak{o}_{\mathfrak{G}} = \left(\frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_1} \wedge \frac{\partial}{\partial \bar{W}_1} \right) \wedge \left(\frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_2} \wedge \frac{\partial}{\partial \bar{W}_2} \right) \wedge \tilde{\mathfrak{o}},$$

where $\tilde{\mathfrak{o}}$ is the rest of the orientation expression (84) in the B&M coordinates. Locally $\mathcal{I}$ is the locus $W_1 = W_2$. Thus locally the normal bundle has a complex trivialization given by the section

$$\frac{\partial}{\partial W_1} - \frac{\partial}{\partial W_2},$$

and the complex orientation induced by this trivialization agrees with the complex orientation of $\mathcal{N}_{\mathcal{I}}$ defined by (89). In addition,

$$\left. \left(\frac{\sqrt{-1}}{2} \frac{\partial}{\partial W_2} \wedge \frac{\partial}{\partial \bar{W}_2} \wedge \tilde{\mathfrak{o}} \right) \right|_{\mathcal{I} \cap U}$$

is identified with $\mathfrak{o}_{\mathcal{I}}^{\pi}$, by (90) and the definition of the canonical orientations. This settles the case $d_c \neq 0$.

Suppose now $d_c = 0$. In this case there are at least two markings on the closed component. By applying the same trick of forgetting markings on the closed component, we see that it is enough to prove the claim when the closed component has exactly two markings. Assume one of them is the lth marking, and the other marking is the $(l-1)^{th}$. Locally near a generic point of $\mathcal{I}$ we can consider B&M coordinates. Locally a dense open subset of $\mathcal{I}$ is specified by the equation $Z_l = Z_{l-1}$, and can be described by the same B&M coordinates, only without Z_l. Denote by $\tilde{\mathfrak{o}}$ the orientation expression (84) in these new B&M coordinates. The normal bundle is locally trivialized by (the image of)

$$\frac{\partial}{\partial Z_l} - \frac{\partial}{\partial Z_{l-1}}.$$

Comparing orientation expressions we see that $\tilde{\mathfrak{o}}$ will agree with the orientation induced from the fibered product

$$\mathcal{M}_{0,\{l-1,l,\star_c\}}(0) \times_{\mathbb{CP}^1} \mathcal{M}_{0,\mathsf{k},[l-2]\cup\{\star_o\}}(\vec{d}) \simeq \mathcal{M}_{0,\mathsf{k},[l-2]\cup\{\star_o\}}(\vec{d}),$$

precisely if the orientation on the normal bundle defined using (89), where $\mathbb{L}_o^\vee, \mathbb{L}_c^\vee$ are given their natural complex orientations, equals the complex orientation defined on the normal using the local

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section

$$\frac{\partial}{\partial Z_l} - \frac{\partial}{\partial Z_{l-1}}.$$

But, by identifying a small neighborhood of the node $\star_o$, located at Z_{l-1}, with a small neighborhood of 0 in its tangent line, we see that $\frac{\partial}{\partial Z_l} - \frac{\partial}{\partial Z_{l-1}}$ can also be identified as a complex trivialization of $\mathbb{L}_o^\vee$ at the node. In addition $\mathbb{L}_c^\vee$ is a trivial complex line. Putting together we see that the two orientations for the normal bundle agree. $\qquad\square$

6.3.3. *Induced orientation to the exceptional boundary.* Suppose $\mathsf{k} = \varnothing$ and $d_+ = d_- = d$. Consider the exceptional boundary component

$$\mathcal{E} \hookrightarrow \overline{\mathcal{M}}_{0,\varnothing,\mathsf{l}}(d,d)$$

which parameterizes maps from domains with a contracted boundary, elements of $\mathcal{E}$ are equivalent to closed maps of degree (d,d) with markings $\mathsf{l} \sqcup \star$, the last one $\star$, is constrained to $\mathbb{RP}^1$, so we can also write

$$\mathcal{E} = \mathrm{ev}_\star^{-1}(L) \subset \overline{\mathcal{M}}_{0,\mathsf{l}\sqcup\star}(d).$$

The generic fiber of the forgetful map $f = \mathrm{For}_\star$, is canonically a d–cover of $\mathbb{RP}^1$, and is induced an orientation $\frac{\partial}{\partial\phi} = (dev_\star)^{-1}\frac{\partial}{\partial\theta}$ defined by pulling back the canonical orientation of $\mathbb{RP}^1$ along this cover.

Lemma 6.19. *The orientation induced on $\mathcal{E}$ by $\mathfrak{o}_{0,\mathsf{l},(d,d)}$ is*

$$-f^*\mathfrak{o}_c \otimes \frac{\partial}{\partial\phi},$$

where $\mathfrak{o}_c$ is the complex orientation of the moduli $\overline{\mathcal{M}}_{0,\mathsf{l}}(d)$.

In particular, integrating $d\theta$ along the fiber of f, with its induced orientation, gives $-d$.

Proof. Chambers whose closure intersects $\mathcal{E}$ in codimension 0 are those whose associated decorated maps have a single arc, from the positive ramification point x to the negative y. Choose such a chamber, and let X, Y be the coordinates of the corresponding branch points in angular parametrization as in Lemma 6.16. We choose the angular parametrization so that near $\mathcal{E}$ it holds that

$$|X - Y| = \ell(\overline{xy}),$$

the length of the boundary segment from x to y. The orientation near $\mathcal{E}$ can be written as $\frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Y} \wedge \tilde{\mathfrak{o}} = -(\frac{\partial}{\partial Y} - \frac{\partial}{\partial X}) \wedge \frac{\partial}{\partial X} \wedge \tilde{\mathfrak{o}}$, where $\tilde{\mathfrak{o}}$ is the wedge of the terms correspond to the remaining complex branch points. Near $\mathcal{E}$ the length of the arc between x, y is close to some positive integer $\leq d$, while the length of the boundary segment between them is close to 0. Thus, only a small fraction of $\mathbb{RP}^1$ is covered by the boundary, and is in fact covered twice, with opposite orientations. The covered segment, in the image, is oriented from Y to X, meaning that X is its right endpoint, hence near $\mathcal{E}$ its length is given by $X - Y$. Thus, $\frac{\partial}{\partial Y} - \frac{\partial}{\partial X}$ is an outward normal. The induced orientation on $\mathcal{E}$ is therefore $-\frac{\partial}{\partial X} \wedge \tilde{\mathfrak{o}}$. Since on $\mathcal{E}$ the vectors $dev_\star\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial\theta}$ point to the same direction, the main statement of the lemma follows.

The 'In particular' follows by integration: for a generic $(\Sigma, u) \in \overline{\mathcal{M}}_{0,\mathsf{l}}(d)$, the fiber $f^{-1}(\Sigma, u)$ is canonically identified with the subset $u^{-1}(\mathbb{RP}^1) \subset \Sigma$, which is a union of loops. The degree of

$$u|_{u^{-1}(\mathbb{RP}^1)} : u^{-1}(\mathbb{RP}^1) \to \mathbb{RP}^1$$

is $\pm d$. The sign is determined by the previous paragraph to be -1. $\qquad\square$

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