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Geometry of symplectic partially hyperbolic automorphisms on 4-torus

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ABSTRACT

We study topological properties of automorphisms of 4-dimensional torus generated by integer matrices being symplectic either with respect to the standard symplectic structure in \mathbb{R}^4 or w.r.t. a nonstandard symplectic structure generated by an integer skew-symmetric nondegenerate matrix. Such symplectic matrix generates a partially hyperbolic automorphism of the torus, if its eigenvalues are a pair of reals outside the unit circle and a complex conjugate pair on the unit circle. The main classifying element is the topology of a foliation generated by unstable (stable) leaves of the automorphism. There are two different cases, transitive and decomposable ones. For the first case the foliation into unstable (stable) leaves is transitive, for the second case the foliation itself has a sub-foliation into 2-dimensional tori. For both cases the classification is given.

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1. Introduction

We study topological properties of automorphisms of four-dimensional torus generated by integer symplectic transformations of \mathbb{R}^4 . Usually, such transformations are called symplectic automorphisms of the torus. Our main concern is about partially hyperbolic case. There are many results about partially hyperbolic diffeomorphisms [7,13–15] since the basic paper [8]. As for partially hyperbolic automorphisms on a torus \mathbb{T}^n is concerned, the most detailed their study was done in [15] solving the question on their stable ergodicity posed in [16]. Recall that stable ergodicity of a C^r -smooth diffeomorphism f of a manifold M ergodic with respect to a smooth Lebesgue measure on M means the existence of neighbourhood $\mathcal{U} \ni f$ in the space of C^r -smooth diffeomorphisms such that any $g \in \mathcal{U}$ is ergodic. Our goal here is to classify possible types of the orbit behaviour of symplectic automorphisms on \mathbb{T}^4 . We hope this will be useful as good examples.

Consider standard torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ as the factor group of the abelian group \mathbb{R}^n with respect to its discrete subgroup \mathbb{Z}^n of integer vectors. Denote $p : \mathbb{R}^n \rightarrow \mathbb{T}^n$ the related group homomorphism being simultaneously a smooth covering map. The coordinates in the space \mathbb{R}^n will be denoted by $x = (x_1, \dots, x_n)$. Let A be an unimodular matrix with integer entries. Since the linear map $L_A : x \rightarrow Ax$, generated by the matrix A , transforms the

subgroup \mathbb{Z}^n onto itself, such a matrix generates diffeomorphism f_A of the torus \mathbb{T}^n called the automorphism of the torus [1,2,11]. Topological properties of such maps are the classical object of research (see, for example, [1,11,18,23]). Because the torus automorphism also preserves the standard volume element $dx_1 \wedge \dots \wedge dx_n$ on the torus carried over from \mathbb{R}^n , then its ergodic properties have also been the subject of research in many works [5,12,26]. The following classical Halmos theorem holds for automorphisms of a torus [12].

Theorem 1.1 (Halmos): *A continuous automorphism f of a compact abelian group G is ergodic (and mixing) if and only if the induced automorphism on the character group G^* has not finite orbits.*

Recall [24] the group of characters G^* of a topological group G be the set of its homomorphisms $\chi : G \rightarrow S^1$ into the group of rotations of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ endowed with the product of two such homomorphisms χ_1, χ_2 as follows: $(\chi_1 * \chi_2)(g) = \chi_1(g) + \chi_2(g) \pmod{1}$. If $f : G \rightarrow G$ is a group automorphism, then the induced automorphism $f^* : G^* \rightarrow G^*$ on the character group is defined as $[f^* \chi](g) = \chi(f(g))$. If a topological group is compact, its character group is discrete [24].

In the case of the abelian group $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the characters are complex-valued functions $\chi_m(x) = \exp[2\pi i(m, x)]$, where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and (m, x) is the standard inner product. If L_A is the automorphism generated by the matrix A , then the induced automorphism on the character group acts as follows:

$$[L_A^*(\chi_m)](x) = \exp[2\pi i(m, Ax)] = \exp[2\pi i(A^\top m, x)],$$

where A^\top means the transposed matrix. The existence of a finite orbit of the induced automorphism means that for some vector $m_0 \in \mathbb{Z}^n$ and natural k the identity holds

$$\exp[2\pi i((A^\top)^k m_0, x)] \equiv \exp[2\pi i(m_0, x)]$$

for all $x \in \mathbb{T}^n$. This means the integer vector m_0 be an eigen-vector of matrix $(A^\top)^k$ with the eigenvalue 1. Thus, among eigenvalues of A^\top there exists λ such that $\lambda^k = 1$. But eigenvalues of A and A^\top are the same. Conversely, if an integral unimodular matrix A has an eigenvalue λ being a root of unity, $\lambda^k = 1$, then there exists a non-zero vector x such that $(A^k - E)x = 0$. But matrix $A^k - E$ has integer entries and all its minors be integer numbers. Therefore, vector x can be chosen with rational entries and hence can be made an integer vector. Thus, the transposed matrix A^\top has eigenvalue λ with an integer eigenvector m_0 , $(A^\top)^k m_0 = m_0$, and the related induced action on the character group has a finite orbit. Finally, by the Halmos theorem, automorphism L_A is ergodic if and only if the matrix A has not eigenvalues being roots of unity.

In fact, the following assertions are equivalent [19]:

- (1) the automorphism f_A is ergodic with respect to Lebesgue measure;
- (2) the set of periodic points of f_A coincides with the set of points in \mathbb{T}^n with rational coordinates;
- (3) none of the eigenvalues of the matrix A is a root of unity;
- (4) the matrix A has at least one eigenvalue of absolute value greater than one and has not eigenvectors with rational coordinates;

(5) all orbits of the dual map $A^* : \mathbb{Z}^N \rightarrow \mathbb{Z}^N$, other than the trivial zero orbit, are infinite.

One more result about torus diffeomorphisms is due to Bowen [5] and allows one to calculate the topological entropy of such automorphism.

Theorem 1.2 (Bowen): *If $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map generated by unimodular integer matrix A , then*

$$h_d(f_A) = \sum_{|\lambda_i| > 1} \log |\lambda_i|,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A .

In particular, for a partially hyperbolic automorphism of \mathbb{T}^4 (see below) the topological entropy is always positive and equals to $\log |\lambda|$, $|\lambda| > 1$. So such automorphism always possesses some type of chaoticity.

If the dimension of the torus is even, $n = 2m$, then one can introduce the standard symplectic structure on \mathbb{T}^{2m} using coordinates in \mathbb{R}^{2m} : $\Omega = dx_1 \wedge dx_{m+1} + \dots + dx_m \wedge dx_{2m}$ and consider symplectic automorphisms of the torus which preserve this symplectic structure. A symplectic automorphism f_A is then defined by a symplectic matrix A with integer entries. Such matrices satisfy the identity $A^\top I A = I$, where a skew-symmetric matrix I has the form

$$I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$

The identity above implies the product of two symplectic matrices and the inverse matrix of a symplectic matrix be symplectic, i.e. symplectic matrices form a group denoted by $\text{Sp}(2n, \mathbb{R})$ w.r.t. the operation of matrix multiplication. This group is one of the standard matrix Lie groups [10].

Recall the well-known statement about the characteristic polynomial of a symplectic matrix (see, for instance, [3]).

Proposition 1.1: *The characteristic polynomial of a symplectic matrix is reciprocal $\chi(\lambda) = \lambda^{2n} \chi(1/\lambda)$. If λ is an eigenvalue of a real symplectic matrix A , then the numbers $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ are also its eigenvalues, all they have the same multiplicity and the same structure of elementary divisors. Eigenvalues of $\lambda = \pm 1$ have an even multiplicity, their elementary divisors of odd order, if any, meet in pairs.*

Another (non-standard) symplectic structure on the torus \mathbb{T}^{2m} can also be defined. To this purpose, let us choose a non-degenerate skew-symmetric $2m \times 2m$ matrix $J, J^\top = -J$. Such matrix defines a bilinear 2-form $[x, y] = (Jx, y)$ in \mathbb{R}^{2m} , where (\cdot, \cdot) is the standard coordinate inner product. This form is also called the skew inner product [3]. Then a linear map $S : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ is called symplectic, if the identity $[Sx, Sy] = [x, y]$ holds for any $x, y \in \mathbb{R}^{2m}$. Using the representation of skew inner product via the matrix J and properties of the inner product, we obtain the following identity for the matrix S of the symplectic map: $S^\top J S = J$. This construction allows one to define a symplectic structure on the torus as well, if matrices J, S are unimodular integer ones. The unimodularity of S follows from its symplecticity. Then on the torus a symplectic 2-form is given and map S defines

a symplectic automorphism with respect to this symplectic form. For example, let B be any non-degenerate integer unimodular matrix. Having the standard skew inner product (Ix, y) in \mathbb{R}^{2m} , we can define new skew inner product as $[x, y] = (IBx, By) = (B^T IBx, y)$. Since the matrix $J = B^T IB$ is skew symmetric, integer and non-degenerate, this skew inner product generates symplectic 2-form on the torus.

Let $P(x) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1$ be some reciprocal polynomial with integer coefficients that is irreducible over the field \mathbb{Q} and suppose it to have two complex conjugate roots on the unit circle and two real eigenvalues outside of the unit circle. The companion matrix of this polynomial has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -a & -b & -a \end{pmatrix}.$$

This matrix is not symplectic $A^T IA \neq I$ with respect to the standard symplectic 2-form $[x, y] = (Ix, y)$, $x, y \in \mathbb{R}^4$. Let us show, however, A to be symplectic with respect to a non-standard symplectic structure on \mathbb{R}^4 defined as $[x, y] = (Jx, y)$, $x, y \in \mathbb{R}^4$ with a skew-symmetric integer non-degenerate matrix J . We consider the identity $A^T JA = J$ as the algebraic system $A^T J - JA^{-1} = 0$ for the entries of the matrix J . A solution to this matrix equation gives

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \\ -1 & a & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Thus, the matrix A becomes symplectic with respect to this non-standard symplectic structure. This will be used later on.

The structure of the paper is as follows. In Section 2, we present possible types of symplectic automorphisms on a symplectic torus when matrices A have not eigenvalues ± 1 . In particular, we distinguish among them partially hyperbolic automorphisms. In Section 3, we find invariant foliations related to such automorphisms. In Section 4, we show the existence of partially hyperbolic symplectic automorphisms with transitive unstable foliations. In Section 5, we obtain a classification of symplectic toral automorphisms.

2. Symplectic automorphisms on \mathbb{T}^4

Consider the standard four-dimensional torus $\mathbb{R}^4/\mathbb{Z}^4 = \mathbb{T}^4$. We assume a matrix A to be symplectic, $A \in Sp(4, \mathbb{Z})$. Then, the automorphism f_A of the torus is symplectic with respect to standard symplectic 2-form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ in the standard coordinates (x_1, y_1, x_2, y_2) in \mathbb{R}^4 .

Consider two general classes of real symplectic matrices.

- The symplectic integer matrix A is hyperbolic, i.e. A has not eigenvalues on the unit circle. Therefore, its eigenvalues form two pairs $\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}$, and $|\lambda_i| \neq 1$. Here two cases are possible: either both pairs are real and $|\lambda_i| < 1$ (the saddle case), or they form a complex quadruple $\rho \exp[\pm i\alpha], \rho^{-1} \exp[\pm i\alpha], 0 < \rho < 1$ (the saddle-focus case). In

any case, the diffeomorphism on the torus is an Anosov automorphism of the torus [1,2,18]. Matrix A defines in \mathbb{R}^4 a linear map L_A with a fixed point at the origin O . This point is a saddle or saddle-focus, respectively. For such fixed point there exist two-dimensional stable and unstable manifolds, here they are two-dimensional subspaces in \mathbb{R}^4 being invariant under the action of L_A , they intersect each other transversally at O . Stable subspace W^s consists of vectors in \mathbb{R}^4 which are contracted exponentially under iterations L_A^n as $n \rightarrow \infty$. Similarly, unstable subspace W^u consists of vectors in \mathbb{R}^4 which are contracted exponentially under iterations L_A^n as $n \rightarrow -\infty$. When projecting on the torus, the subspace W^s and all affine two-dimensional planes, being factor classes of \mathbb{R}^4/W^s , form the foliation of the torus whose leaves are embedded two-dimensional planes. The same is true for unstable subspace W^u : when projecting onto a torus, it generates the unstable foliation on the torus. Each leaf of stable foliation is dense on a torus and intersects transversally each leaf of the unstable foliation, whose leaves are also dense on the torus.

- The symplectic map L_A in \mathbb{R}^4 is partially hyperbolic that corresponds to matrices A with a pair of eigenvalues on the unit circle $\exp[\pm i\alpha]$ and a pair of real numbers λ, λ^{-1} , $|\lambda| < 1$. Such linear map has a fixed point at the origin in \mathbb{R}^4 called 1-elliptic point [22]. This fixed point has a one-dimensional stable eigenspace E^s corresponding to eigenvalue $|\lambda| < 1$, one-dimensional unstable eigenspace E^u corresponding to the eigenvalue $|\lambda^{-1}| > 1$, two-dimensional invariant centre subspace E^c corresponding to the eigenvalues on the unit circle. The projection $p : \mathbb{R}^4 \rightarrow \mathbb{T}^4$ allows, using shifts on the torus, to transfer these subspaces into the tangent spaces at any point of the torus. Then, the tangent space $T_x\mathbb{T}^4$ of the torus at each point x splits into a direct sum of three subspaces (two in the hyperbolic case, stable and unstable ones)

$$T_x\mathbb{T}^4 = E_x^s \oplus E_x^c \oplus E_x^u,$$

where for vectors from E_x^s there is an exponential contraction under iterations of the differential DL_A (in fact, the matrix A), on E_x^u there is an exponential expanding, and vectors in E_x^c are uniformly bounded under iterations with respect to any Riemannian metric on the torus.

The characteristic polynomial of a symplectic matrix has the form $\lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1$. Therefore, if we denote $\mu = \lambda + \lambda^{-1}$, then the characteristic polynomial $\chi(\lambda)$ is expressed as $\chi(\lambda) = \lambda^2(\mu^2 + a\mu + b - 2)$. In the case when symplectic matrix A generates a partially hyperbolic mapping, the value of $\mu_1 = \lambda + 1/\lambda$, $0 < |\lambda| < 1$, is either greater than 2 or lesser than -2 , and the value $\mu_2 = \exp[i\alpha] + \exp[-i\alpha] = 2 \cos \alpha$ does not exceed 2 in modulus. So, in this case one has $|\mu_1| > 2, |\mu_2| < 2, \text{if } \alpha \neq 0, \pi \pmod{2\pi}$.

3. One-dimensional foliations of partially hyperbolic automorphisms

Now, we consider partially hyperbolic automorphisms of the torus. Partially hyperbolic diffeomorphisms on any smooth manifold M were first introduced and studied in [8]. Here, we use a modification of the definition in [15]. Let L be a linear transformation between two normed linear spaces. The norm, respectively conorm, of L are defined as

$$\|L\| := \sup \|Lv\|, \|v\| = 1, \quad m(L) := \inf \|Lv\|, \|v\| = 1.$$

Definition 3.1: A diffeomorphism $f : M \rightarrow M$ is partially hyperbolic, if there is a continuous Df -invariant splitting $TM = E^u \oplus E^c \oplus E^s$ in which E^u and E^s are non-trivial sub-bundles and

$$\begin{aligned} m(D^u f) &> \|D^c f\| \geq m(D^c f) > \|D^s f\|, \\ m(D^u f) &> 1 > \|D^s f\|, \end{aligned}$$

where $D^\sigma f$ is the restriction of Df to E^σ for $\sigma = s, c$ or u .

In the case of a four-dimensional torus, a symplectic partially hyperbolic automorphism is defined by a symplectic matrix having two (simple) real eigenvalues λ, λ^{-1} outside the unit circle and two complex conjugate eigenvalues on the unit circle. The corresponding eigenspaces W^s, W^u and the subspace W^c when projecting on a torus and shifting them at any point give the required decomposition of the definition.

When studying the geometry of partially hyperbolic maps with different dynamics, it is useful to study first the possible behaviour of projections onto a torus of eigenspaces for real eigenvalues λ, λ^{-1} . In the space \mathbb{R}^4 , we have the following orbit structure for the linear map L_A generated by the matrix A . Recall that in the case of a symplectic linear map with a 1-elliptic fixed point O , there is a two-dimensional centre invariant subspace W^c corresponding to a pair of eigenvalues $\nu, \bar{\nu}, \nu = \exp[i\alpha]$ on the unit circle. If $\alpha/2\pi \neq r, r \in \mathbb{Q}$ (the non-resonance case, this is equivalent to $\nu^n \neq 1$ for any $n \in \mathbb{Z}$), then the subspace W^c is foliated into closed invariant curves. This follows from the fact that the restriction of the map L_A on W^c has a quadratic positive definite invariant function (integral) whose level lines are closed invariant curves (in the resonance case $\alpha/2\pi = r$ a quadratic integral also exists but all its level lines consist of periodic points of the same period, due to linearity, therefore one can construct any invariant sets from them). The restriction of the map L_A to any such curve is conjugated to a rotation on the circle $\varphi \rightarrow \varphi + \alpha \pmod{2\pi}$ by the angle α , and the rotation number is independent of the curve due to the linearity of the map. For the case of irrational number $\alpha/2\pi$ these invariant curves are defined correctly and the shift on each such curve is transitive.

In addition to the indicated invariant lines and the centre plane, there are two more three-dimensional invariant subspaces in \mathbb{R}^4 spanned, respectively, by vectors from subspaces W^c and l^s (the centre-stable 3-plane W^{cs}) and vectors from W^c and l^u (the centre-unstable 3-plane W^{cu}). The factor-classes \mathbb{R}^4/W^{cu} and \mathbb{R}^4/W^{cs} define two invariant 3-foliations in \mathbb{R}^4 into three-dimensional affine planes. Here, the invariance is understood in the following sense: the image with respect to L_A of a leaf of the foliation coincides with some leaf (possibly with another one) of the same foliation. All orbits of L_A not lying in the union $W^{cs} \cup W^{cu}$, go to infinity for both positive and negative iterations of L_A . In particular, the mapping L_A has not any other three-dimensional invariant subspaces except those two W^{cs} and W^{cu} . If $\exp[i\alpha]$ is not a root of unity, then subspaces W^{cs}, W^{cu} are foliated into two-dimensional cylinders being stable (respectively, unstable) invariant manifolds of invariant curves on the central plane W^c .

The projection of the eigen-lines l^s, l^u onto the torus can lead to different situations. To understand this, it is necessary to clarify what is a projection of a subspace from \mathbb{R}^4 on the torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$. Since the torus is a commutative Lie group, the tangent space at zero possesses the structure of the commutative Lie algebra, this is identified with

\mathbb{R}^4 in the standard way. Then the projection p is an exponential map of the Lie algebra onto the Lie group. One-dimensional subspace l^u (or l^s) coincides with the one-parameter subgroup $t\gamma^u$ generated by the vector γ^u , and its projection is the image under the exponential mapping of the algebra into the group. This subgroup is included into orbits of the constant vector field on \mathbb{T}^4 invariant under shifts. In coordinates θ on \mathbb{T}^4 induced by coordinates in \mathbb{R}^4 we get the vector field $\dot{\theta} = \gamma^u$. Its orbit structure depends on the number of rationally independent integer solutions of the equation $(\xi, \gamma^u) = 0$. This number can take values 0, 1, 2, 3. In the first case, as is known, the related orbits of the constant vector field are transitive in \mathbb{T}^4 (it is a partial case of the Kronecker theorem, see, for instance, [9]).

The one-parameter subgroup in \mathbb{T}^4 generated by γ^u is an invariant subset with respect to the automorphism f_A (this is the strongly unstable curve of the fixed point \hat{O}). Therefore, its closure is also an invariant subset, that is a smooth invariant torus of some dimension in \mathbb{T}^4 [6]. As was said above, the dimension of this torus depends on the number of an integer linear independent relations of the form $(m, \gamma^u) = 0$: (1) none integer relations of the form $(m, \gamma^u) = 0, n \in \mathbb{Z}^4$; (2) vector γ^u satisfies an only integer relation (up to multiplication at a constant) $(n, \gamma^u) = 0, n \in \mathbb{Z}^4$; (3) there are two such rationally independent relations $(n, \gamma^u) = 0, (m, \gamma^u) = 0$, vectors $n, m \in \mathbb{Z}^4$, are linearly independent; (4) there are three such relations with linearly independent vectors $m, n, k \in \mathbb{Z}^4$. In the last three cases, vector γ^u is called resonant, in the first case it is called incommensurate or non-resonant.

In the second case, the linear three-dimensional subspace in \mathbb{R}^4 defined by the equation $(n, x) = 0$ is projected onto a 3-torus in \mathbb{T}^4 , and the straight-line spanned by vector γ^u does not pass through the points of its integer sub-lattice. Therefore, this line is projected into a transitive immersed line in this 3-torus.

Similarly, for the third case, the corresponding subspace is two-dimensional, it is projected onto 2-torus in \mathbb{T}^4 . The straight-line, spanned by vector γ^u , is projected into a transitive immersed line in this 2-torus. In the fourth case, the straight line necessarily passes through the integer point and therefore it is projected onto a simple closed curve in \mathbb{T}^4 without self-intersections.

First, obviously the eigen-line l^u corresponding to $\lambda > 1$ (l^s for $\lambda < 1$) cannot intersect the integer lattice \mathbb{Z}^4 . Indeed, if so, the projection in the torus of this straight line is a closed invariant curve for the map f_A and the restriction of this map on this circle gives a diffeomorphism on the circle with one unstable (respectively, stable) fixed point that is impossible. Any trajectory of the vector field $\dot{\theta} = \gamma^u$ coincides with a leaf of the unstable (stable) invariant foliation on the torus which consists of projections of all straight lines corresponding to the factor-classes \mathbb{R}^4/l^u (respectively, \mathbb{R}^4/l^s).

The second, the closure of any trajectory of the constant vector field on the torus is a smooth torus of some dimension, its dimension is equal to $4-q$, where q is the number of rationally independent linear relations for the components of the vector $\gamma^u : (m, \gamma^u) = 0, m \in \mathbb{Z}^4$. If such relations are absent at all, the vector γ^u is incommensurate, and then each trajectory of the vector field is dense on the torus, due to the Kronecker theorem.

If there is the unique such relation (up to the multiplication at a rational number), the torus \mathbb{T}^4 is foliated into three-dimensional invariant tori, on each of them any trajectory of the constant vector field is everywhere dense. If there are two such independent rationally relations, the torus \mathbb{T}^4 is foliated into two-dimensional tori, on each of them the trajectory

is everywhere dense. The case of three rationally independent relations leads to the foliation by the circle which in our case is impossible, as was mentioned above.

In fact, for the case we study, the following statement is valid.

Proposition 3.1: *The closure of the unstable leaf passing through a fixed point \hat{O} is either a two-dimensional torus or the whole \mathbb{T}^4 , that is, in the latter case the leaf is transitive.*

Proof: Assume the closure of the unstable manifold of a fixed point \hat{O} on \mathbb{T}^4 to form a three-dimensional torus \mathbb{T}^3 . This means the eigenvector γ^u to satisfy the unique integer relation $(m, \gamma^u) = 0, m \in \mathbb{Z}^4$. Consider three-dimensional hyperplane $(m, x) = 0$ in \mathbb{R}^4 defined by the co-vector m . The three-dimensional torus is the projection of this hyperplane.

Let us show in this case the stable one-dimensional manifold of the fixed point \hat{O} be transversal to the torus \mathbb{T}^3 . The torus \mathbb{T}^3 contains the fixed point \hat{O} . Since the torus is smooth and invariant with respect to the map f_A (as the closure of an invariant set), the tangent space to the torus at the fixed point is invariant with respect to differential Df_A . The torus \mathbb{T}^3 is obtained by the projection of the hyperplane in \mathbb{R}^4 passing through the fixed point O of the map L_A . This plane is invariant with respect to L_A , but linear partially hyperbolic map L_A has no other invariant 3-planes through O except for W^{cs}, W^{cu} . Only the second of them contains the straight line spanned by the vector γ^u . So, the torus \mathbb{T}^3 is a projection of the centre-unstable plane. But then the stable eigenvector is transversal to this plane, therefore the projection onto \mathbb{T}^4 of the stable eigen-line is a smooth curve transversal at the point \hat{O} to the torus \mathbb{T}^3 . Since \mathbb{T}^3 is a smooth closed submanifold in \mathbb{T}^4 , there is a neighbourhood V of the point \hat{O} such that all points of the stable curve $W^s(\hat{O})$ in V do not belong to \mathbb{T}^3 .

Due to transversality of the torus \mathbb{T}^3 and a stable invariant curve at \hat{O} , they must intersect each other at more than one point in \mathbb{T}^4 since this curve is not closed. Stable invariant curve is given as $t\gamma^s \pmod{1}$, it contains a point z other than \hat{O} which belongs to \mathbb{T}^3 . Since z belongs to W^s , its backward iterations $f_A^{-n}(z)$ must lie on the stable curve near the point \hat{O} for positive n large enough. As a consequence, these points do not belong to \mathbb{T}^3 for such iterations. On the other hand, the torus is invariant with respect to f_A , therefore, all iterations of z should lie on it. This contradiction proves that the closure of an unstable curve is not a 3-torus.

The case of two rationally independent relations for partially hyperbolic matrix is possible. It is sufficient to choose, for example, a block-diagonal integer matrix A composed of two integer 2×2 -blocks, one of which has eigenvalues $\lambda, \lambda^{-1}, |\lambda| < 1$, and the second block has two complex conjugate eigenvalues $\exp[i\alpha], \exp[-i\alpha]$ on the unit circle. Note in this case, that the characteristic polynomial of such a matrix is the product of two monic polynomials of second degree with integer coefficients, i.e. it is reduced over the field of rational numbers and the numbers $\exp[i\alpha]$ are roots of unity (of degree 3,4,6). ■

For the case when the closure of the unstable curve is a two-dimensional torus, the following assertion is valid.

Proposition 3.2: *If the closure of the unstable (stable) invariant curve of \hat{O} is a two-dimensional torus, then the characteristic polynomial of A is reducible over rational numbers*

\mathbb{Q} , i.e. it is the product of two monic polynomials with integer coefficients. In particular, the eigenvalues of the matrix A lying on the unit circle are roots of unity.

Proof: Let the closure of an unstable (stable) invariant curve of \hat{O} be a two-dimensional torus T . This torus is a smooth invariant manifold for f_A containing a fixed point \hat{O} . Tangent plane to T at \hat{O} is the invariant 2-plane w.r.t. differential Df_A . In the covering space \mathbb{R}^4 the pre-image through the origin O w.r.t. the projection p of this 2-plane is invariant 2-plane w.r.t. L_A . There are only two such 2-planes: W^c and the plane W^* spanned by two eigen-vectors γ_s, γ_u . Only the second of them contains γ_u , so W^* is this plane. There exist two linear independent integer vectors in W^* , since the projection of this plane to \mathbb{T}^4 is the invariant two-dimensional torus T . The lattice generated by these two integer vectors does not intersect the eigen straight-line, because its projection is everywhere dense on the torus. This implies the restriction of f_A to this torus be an Anosov map which is generated by the restriction of L_A onto invariant 2-plane W^* . Eigenvalues of this restriction are λ, λ^{-1} with their eigen-vectors γ_s, γ_u . In contrast to a transitive case, where in \mathbb{R}^4 the invariant w.r.t. L_A such 2-plane also exists, here this plane contains an integer sub-lattice with two independent integer vectors.

Consider the minimal polynomial corresponding to the invariant 2-plane W^* in \mathbb{R}^4 , its degree is 2. This polynomial can be generated by one integer vector $v \in W^*$, the independent integer vector Av , therefore $A^2v = av + bv$, due to the invariance of the plane, $a, b \in \mathbb{R}$. Let us consider this vector equality as the over-defined system w.r.t. unknowns a, b . Equality says that the solution of the system exists. Integer vectors v, Av are independent, hence there is a non-zero integer 2-minor in the related coefficient matrix. Vector A^2v is also an integer. Thus existing solution (a, b) of the system consists of rational numbers.

Due to its minimality, the polynomial $z^2 - az - b$ divides the characteristic polynomial of A . So, the characteristic polynomial of A is the product of two polynomials of degree two with rational coefficients, i.e.

$$\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \alpha\lambda + 1 = (\lambda^2 + r\lambda + 1)(\lambda^2 + s\lambda + 1),$$

with integer α, β , where $r, s \in \mathbb{Q}$. Then, equating the coefficients at the same powers, we get the system $r + s = \alpha, rs = \beta - 2$. Thus r is a root of the monic quadratic polynomial $t^2 - \alpha t + \beta - 2$ with integer coefficients. Hence, its roots are either integer or irrational numbers. So, r is an integer number and s is as well. Therefore, the characteristic polynomial splits into the product of two polynomials with integer coefficients. One of these polynomials, say $\lambda^2 + s\lambda + 1 = 0$ with integer s corresponds to the pair of roots on the unit circle. This means $|s| < 2$ and so $s = -1, 0, 1$. Then we get the numbers $\exp[i\alpha]$ be roots of unity of degree 3,4,6.

Now, we shall show that the projection onto the torus \mathbb{T}^4 of the invariant subspace W^c also gives an invariant two-dimensional torus. To this end, one needs to find two independent integer vectors in W^c . Matrix A has eigenvalues being roots of unity of degree $k = 3, 4, 6, \lambda^k = 1$. Then A^k has the double eigenvalue 1. Therefore, the equation for finding related eigenvectors is of the form $(A^k - E)x = 0$. Matrix $A^k - E$ is integer with all its minors being integer numbers. Therefore, vector x can be chosen with rational coefficients and hence can be made integer vector. Thus, matrix A^k has an eigenvalue 1 with an integer eigenvector $m_0, A^k m_0 = m_0$. Notice that W^c is the invariant plane for A , hence $m_0 \in W^c$.

The vector Am_0 is also integer and m_0 is not an eigenvector, hence $Am_0 \neq \gamma m_0$ and vectors m_0, Am_0 form the basis of W^c . So, the projection of this plane onto \mathbb{T}^4 is an invariant two-dimensional torus. ■

In view of this theorem, we shall call *decomposable* the case, when the closure of the unstable leaf of \hat{O} is a two-dimensional torus. Recall the theorem on the rational canonical form [17] (Frobenius normal form).

Theorem 3.1: *Every matrix A with rational entries is similar over \mathbb{Q} to a block diagonal matrix with blocks being companion matrices of its elementary divisors.*

It follows that the matrix A of a linear operator L_A is similar to the block matrix B with integer coefficients consisting of two (2×2) blocks. As was shown above, in the decomposable case the characteristic polynomial of the linear operator L_A splits into the product of two second degree polynomials with integer coefficients.

Proposition 3.3: *If the decomposable case is realized, then the matrix A is similar to B via some integer similarity matrix T .*

Proof: By the definition of matrix similarity, the relation $B = TAT^{-1}$ is valid, where T is the transform matrix. We rewrite this relation in the form $BT - TA = 0$. This matrix equation in T is equivalent to a system of 16 linear homogeneous equations w.r.t. 16 unknown T -entries. Finding a transforming matrix is reduced to a solution of this system. Moreover, we have to find among the set of all solutions such that $\det T \neq 0$. Such a solution exists since the matrices A and B are similar, but we need to find an integer T .

The set of solutions of a linear homogeneous system for T is always infinite given as $T = UT_1$, where T_1 is one of transforming matrices, and U is an arbitrary matrix that is permutable with B . This allows one to make the matrix T integer. Since matrices A and B are integer, the system $BT - TA = 0$ has integer coefficients. As free variables, we can always choose integers, then the solution is either integer or rational. Thus, the entries of the matrix T are either integer or rational numbers. In the first case, the statement is proved. In the second case, we use the formula $T = UT_1$, where T_1 is a matrix with rational coefficients, U is a scalar matrix μE , where μ the common denominator of matrix elements for T_1 . Matrix U of this kind always permutes with B . Then the matrix T is integer. ■

4. Partially hyperbolic automorphism of a 4-torus with transitive unstable foliation

To order to construct examples of symplectic partially hyperbolic automorphisms of a torus with different dynamical properties, we need to find a matrix in $Sp(4, \mathbb{Z})$ which have two complex conjugate eigenvalues on the unit circle and two real eigenvalues λ and λ^{-1} , $0 < |\lambda| < 1$. Moreover, we would like to obtain an automorphism of the torus whose one-dimensional foliations corresponding to real eigenvalues are transitive. To obtain such a matrix, we start, following [20], with an irreducible over rational numbers quadratic polynomial $P(z)$ with integer coefficients having two roots, one is greater than 2 and the second is lesser than 2 in modulus. We make a change of variable $z = x + x^{-1}$ in this polynomial

to obtain a polynomial of the fourth degree which serves as the characteristic polynomial of the matrix with the desired properties.

For example, we start with the polynomial $P = z^2 - 3z + 1$ that gives the polynomial $Q = x^4 - 3x^3 + 3x^2 - 3x + 1$ which is irreducible over the field \mathbb{Q} . Then, the companion matrix of this polynomial Q [17] has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -3 & 3 \end{pmatrix}.$$

This matrix possesses the required properties of its eigenvalues. However, this matrix is not symplectic with respect to the standard symplectic 2-form $[x, y] = (Ix, y)$, $x, y \in \mathbb{R}^4$: $A^\top IA \neq I$. We show, however, A to be symplectic with respect to a non-standard symplectic structure on \mathbb{R}^4 defined as $[x, y] = (Jx, y)$, $x, y \in \mathbb{R}^4$ via a skew-symmetric integer non-degenerate matrix J .

To get symplecticity $(JAx, Ax) = (A^\top JAx, x)$, we need the equality $A^\top JA = J$ to hold. As above, we consider this equality as the algebraic system $A^\top J - JA^{-1} = 0$ for the entries of the matrix J . The solution to this matrix equation gives

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ -1 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Thus, the matrix A becomes symplectic with respect to this non-standard symplectic structure.

Real eigenvalues of matrix A are

$$\lambda = \frac{3 + \sqrt{5} + \sqrt{6\sqrt{5} - 2}}{4} > 1, \lambda^{-1}.$$

The eigenvector corresponding to λ is $\gamma^u = (1, \lambda, \lambda^2, \lambda^3)^\top$. Factor-classes \mathbb{R}^4/l^u form an invariant foliation under L_A into affine lines. These lines are projected onto \mathbb{T}^4 as trajectories of the vector field

$$\dot{x}_1 = 1, \dot{x}_2 = \lambda, \dot{y}_1 = \lambda^2, \dot{y}_2 = \lambda^3.$$

To prove the orbits of this vector field to be transitive in \mathbb{T}^4 , we need to verify the vector γ^u to be incommensurate, i.e. $(m, \gamma^u) \neq 0$ for any integer vector $m \in \mathbb{Z}^4$. The number λ is an algebraic number being the root of the polynomial with integers coefficients.

Recall a number $\xi \in \mathbb{C}$ be *algebraic* if it is a root of a polynomial with rational coefficients, the number ξ is called *integer algebraic*, if it is a root of a polynomial with integer coefficients whose leading coefficient equals to unity (a monic polynomial). With any algebraic number, the notion of its degree is associated. Namely, if the number ξ is the root of a polynomial of some degree, then multiplying this polynomial at another polynomial with rational coefficients we obtain a polynomial of a greater degree with the root ξ . The polynomial of the least degree with rational coefficients having the root ξ is called the *minimal*

polynomial of the algebraic number ξ , and its degree is called the degree of the number ξ . In particular, all rational numbers are algebraic numbers of degree one. The polynomial Q above has degree 4 due to the following theorem ([17], theorem 3.3.14, p. 147)

Theorem 4.1: *Every monic polynomial is both the minimal polynomial and the characteristic polynomial of its companion matrix.*

Theorem 4.2: *Let ξ be an algebraic number over field \mathbb{Q} and p its minimal monic polynomial. Then*

- (1) *the polynomial p is irreducible over \mathbb{Q} ;*
- (2) *the polynomial p is unique;*
- (3) *if ξ is a root of some polynomial f over the field \mathbb{Q} then $f:p$.*

From the last statement of this theorem, it follows that if the number λ is a root of the irreducible polynomial Q of the fourth order, it could not be a root of a polynomial of smaller degree with rational (integer) coefficients.

Now, we can prove the lemma.

Lemma 4.1: *The vector γ is incommensurate.*

Proof: Suppose vector γ be commensurate. Then an integer vector (m_1, m_2, m_3, m_4) exists such that the equality $m_1 + m_2\lambda + m_3\lambda^2 + m_4\lambda^3 = 0$ holds, i.e. λ is a root of the polynomial P of the third (or lesser) degree with integer coefficients. So P divides Q and Q is reducible. ■

Next proposition is well known for a hyperbolic case, its proof for the partially hyperbolic case follows the lines of [18].

Proposition 4.1: *The set of periodic points of the map L_A is dense. If unstable the one-dimensional foliation of the automorphism f_A is transitive, then f_A is a topologically mixing transformation.*

Before the proof, recall the definition of topologically mixing map f of the metric space $M, f : M \rightarrow M$ [18].

Definition 4.1: A map $f : M \rightarrow M$ is called topologically mixing, if for any two non-empty open sets $U, V \subset M$ there exists a positive integer $N = N(U, V)$ such that for every $n > N$ the intersection $f^n(U) \cap V$ is non-empty.

Proof: The fact that points with rational coordinates are the only periodic points for L_A and hence the density of periodic orbits is similar to the proof in the hyperbolic case (see, for instance, [18]). Now, we prove the topological mixing for map f_A , if its unstable foliation is transitive. Consider any two non-empty open subsets of $U, V \subset \mathbb{T}^4$. Let $p \in U, q \in V$ be periodic points and n their common period. Consider three-dimensional leaf of the centre-stable foliation through the point p . This leaf is f_A^n -invariant. We also consider a

one-dimensional leaf of the unstable foliation for a point q ; it is also f_A^n -invariant. By the proposition, the leaf of the unstable foliation is everywhere dense in \mathbb{T}^4 . In the covering space \mathbb{R}^4 subspaces l^u and W^{cs} are transversal and intersect each other at only point O , therefore the one-dimensional unstable and three-dimensional centre-stable leaves on the torus are also transversal. Thus there exists a neighbourhood G of the point p , in which the unstable foliation generates a smooth flow box into segments of leaves, and the centre-stable foliation generates a smooth foliation into 3-disks, and leaves of these two foliations are transversal. The intersection with the neighbourhood G of the leaf through q forms a dense set of segments. Segments of this leaf intersect the three-dimensional disk of the leaf through p at the dense set of points. Therefore, in the 3-disk there are points of the one-dimensional leaf accumulating to p . So, there exists a sufficiently large number k such that $f_A^{kn}(U) \cap V$ is non-empty. ■

In the example above, the stable one-dimensional foliation will also be transitive, because it is generated by the eigenvector $(1, \lambda^{-1}, \lambda^{-2}, \lambda^{-3})$ and the integer relation $m_0 + m_1\lambda^{-1} + m_2\lambda^{-2} + m_3\lambda^{-3} = 0$ is impossible. This fact is not occasional.

Proposition 4.2: *If f_A is a symplectic automorphism of \mathbb{T}^4 with the dense unstable foliation, then its stable foliation is also dense.*

Proof: Indeed, let γ_u, γ_s be related eigen-vectors and the projection of the line $t\gamma_u$ into \mathbb{T}^4 is dense. Then characteristic polynomial of A is irreducible. Suppose the closure of the projection of $t\gamma_s$ is a two-dimensional invariant torus in \mathbb{T}^4 . As above, it implies the characteristic polynomial of A is reducible, that is a contradiction. ■

It would be useful to get some general statement concerning the foliation generated by the centre plane W^c for the transitive case. The projection of W^c onto the torus \mathbb{T}^4 is evidently cannot be an invariant 2-torus. The foliation of W^c into closed invariant curves homotopic to zero implies that near fixed point \hat{O} we also get closed invariant curves homotopic to zero. Embedding of W^c to \mathbb{T}^4 can give an immersed plane or an immersed cylinder. We think the following hypothesis is valid.

Hypothesis 4.1: In a transitive case, the image of $W^c, p(W^c)$, is a transitive surjection. This means that p is an inclusion (no two points whose image is the same) and the closure of $p(W^c)$ coincides with \mathbb{T}^4 .

5. Classification of partially hyperbolic automorphisms

At the study of partially hyperbolic symplectic automorphisms a natural question arises: when two such automorphisms are topologically conjugate. Recall that two homeomorphisms f_1, f_2 of a metric space M are called topologically conjugate, if there exists a homeomorphism $h : M \rightarrow M$ such that $h \circ f_1 = f_2 \circ h$.

Obviously, two automorphisms, one of which has transitive unstable foliation and another one for which the unstable foliation generates the foliation \mathbb{T}^4 into two-dimensional tori are topologically non-equivalent. However, two partially hyperbolic symplectic automorphisms of the torus having both transitive unstable foliations can also be non-equivalent.

Note, that classification of ergodic automorphisms of the torus from a measure theory point of view is given by their entropy [21]. This follows from the Ornstein isomorphism theorem [25] and the fact that every ergodic automorphism of a torus is Bernoulli one with respect to the Lebesgue measure [21]. But we want to obtain a topological classification. The existence of a conjugating homeomorphism $h : \mathbb{T}^4 \rightarrow \mathbb{T}^4$ for two automorphisms $f_A, f_{A'}$ implies the relation $H \circ A = A' \circ H$ in the fundamental group \mathbb{Z}^4 of the torus, where H is the linear homomorphism in \mathbb{Z}^4 generated by h . Thus, matrices A, A' are similar by H . Matrix H is integer with its determinant ± 1 . This is not occasional due to the following Arov's theorem [4]. We formulate the consequence of the Arov's theorem

Theorem 5.1: *Two ergodic automorphisms T, P of compact commutative group X are topologically conjugate if and only if they are isomorphic, that is, an isomorphism $Q : X \rightarrow X$ exists such that $Q \circ T = P \circ Q$.*

Thus, the conjugacy problem for two transitive (i.e. ergodic) partially hyperbolic symplectic automorphisms $f_A, f_{A'}$ is reduced to the problem: suppose two integer symplectic partially hyperbolic matrices A, A' generate both transitive automorphisms of \mathbb{T}^4 , when are A, A' integer unimodularly conjugate? A necessary condition is the equality of their integer characteristic polynomials. Thus, companion matrices of their characteristic polynomials should coincide. Conversely, if A, A' have the same characteristic polynomials, then, the characteristic polynomial of matrix A is irreducible integer, and, as was proven above, A is similar to the companion matrix of this polynomial by means of some integer unimodular matrix. The same is true for A' . Hence, we get similarity of A, A' and the conjugacy of f_A and $f_{A'}$ via some automorphism of \mathbb{T}^4 . So, the following classification theorem for the transitive case is valid.

Theorem 5.2: *Let A be a partially hyperbolic integer unimodular symplectic matrix generating a transitive automorphism f_A on \mathbb{T}^4 . Then f_A is conjugate to the automorphism f_B , where B is the companion matrix of the characteristic polynomial of the matrix A . This f_B is a symplectic diffeomorphism on \mathbb{T}^4 w.r.t. a non-standard symplectic structure.*

It follows from the above considerations, the theorem on the structure of a decomposable symplectic automorphism.

Theorem 5.3: *If f_A is a decomposable symplectic automorphism of \mathbb{T}^4 , then f_A is conjugated by an automorphism f_S generated by an integer unimodular matrix S in \mathbb{R}^4 with the direct product of two automorphisms f_H and f_I on $\mathbb{T}^2 \times \mathbb{T}^2$ given by a hyperbolic matrix H and a periodic matrix I such that $I^k = E$ for $k \in \{3, 4, 6\}$.*

6. Conclusion

In the paper, we presented a classification of automorphisms f_A of four-dimensional torus \mathbb{T}^4 generated by symplectic integer matrices $A \in Sp(4, \mathbb{Z})$. These automorphisms can be transitive or decomposable. In the first case two transitive automorphisms are conjugate, if they have the same characteristic polynomials. In the second case, a decomposable f_A is conjugate to the direct product on $\mathbb{T}^2 \times \mathbb{T}^2$ of two two-dimensional automorphisms, one of which is an Anosov automorphism and another one is periodic.

Here a natural question arises on an extension of these results to the multi-dimensional case. The problem becomes harder because many different opportunities exist, since the dimensions of a centre sub-bundle and stable/unstable sub-bundles can vary even for the fixed dimension of the torus. For instance, for symplectic automorphisms on \mathbb{T}^6 we have symplectic integer matrices with

- $\dim W^c = 4, \dim W^s = 1, W^u = 1;$
- $\dim W^c = 2, \dim W^s = 2, W^u = 2.$

The first case is similar, in a sense, to what was done here. We again can study the one-dimensional foliations generated by unstable and stable eigenlines. The related foliation of the tori within \mathbb{T}^6 can be transitive, decomposable into 4-tori, or decomposable into 2-tori. Each case requires a special investigation for its classification. The second case generates stable and unstable foliations with two-dimensional leaves (similar, in a sense, to a four-dimensional symplectic Anosov automorphism). Such an automorphism can generate either transitive case or a decomposable case with a foliation into four-dimensional tori. In greater dimensions situation is more involved. We hope to return to these cases in future work.

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