Abundance of entire solutions to nonlinear elliptic equations by the variational method

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A R T I C L E   I N F O

A B S T R A C T

We study entire bounded solutions to the equation $\Delta u - u + u^3 = 0$ in $\mathbb{R}^2$. Our approach is purely variational and is based on concentration arguments and symmetry considerations. This method allows us to construct in an unified way several types of solutions with various symmetries (radial, breather type, rectangular, triangular, hexagonal, etc.), both positive and sign-changing. It is also applicable for more general equations in any dimension.

1. Introduction

Studying solutions to nonlinear elliptic equations is the classical problem arising both in PDE research and applications of PDEs in geometry, physics and material sciences [12,13]. Indeed, many problems of nonlinear physics are related to steady states of nonlinear scalar fields in nonlinear media. Thus, it is necessary to investigate solutions with nontrivial spatial structures (patterns) to nonlinear elliptic PDEs. One of the typical PDEs of this kind is

$$\Delta u + f(u) = 0, \tag{1.1}$$

considered in some domain $D \subset \mathbb{R}^n$ together with some boundary conditions, or in the whole space $\mathbb{R}^n$ with prescribed behavior at infinity. Important solutions with patterns for this equation are those which are localized in one or more variables. This kind of problems can be linked with the study of self-localized solutions in wave-guide optical channels [15], investigations of "particle-like" states of nonlinear fields in some models of elementary particles [43], the description of bi-phase separation in fluids [13] and ordering in binary alloys [5]. The problem of studying patterns in such equations is, in a sense, an analogue of problems from the theory of dynamical systems, where ideally one needs to study and classify all solutions to a given

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system of autonomous differential equations. Unfortunately, nowadays the theory of elliptic PDEs is not as developed as the theory of dynamical systems, where we know at least primary objects to be studied (equilibria, periodic orbits, quasi-periodic orbits, homoclinic and heteroclinic structures etc.). In the absence of a general theory studying certain interesting but more or less "simple" model equations is one of the main problems now. Undoubtedly, Eq. (1.1) is among such models.

Solutions of the equation or the system of equations of type (1.1) also give rise to stationary solutions of the evolution equations such as reaction–diffusion parabolic or wave PDEs. Stationary solutions for these evolution equations are of the primary interest since this is a starting point to study non-stationary solutions close to stationary, various stability problems, asymptotic behavior, etc. The study of so-called stationary waves for nonlinear scalar field equations, like the Klein–Gordon equation, also lead to the study of equations of type (1.1) [8]. Moreover, as it was mentioned in [46], PDEs of type (1.1) do have solutions with interesting patterns, and the structure of their solution sets has remained mostly a mystery. This is especially true for the solutions in the whole space (so-called *entire solutions*) considered in our paper.

Various methods were developed to find solutions of Eqs. (1.1) with some prescribed structures. In particular, for solutions localized in space one method is to search for radial solutions, i.e. those which depend only on the radial variable r. It is extremely interesting that the only possible positive bounded solutions of (1.1) are radial, if some restrictions on nonlinearity are imposed [20]. When searching for radial solutions the problem is reduced to the study of some specific solutions of the related nonautonomous second order ODE with "time" r. For some types of nonlinearities the existence of infinitely many (sign-changing) radial solutions was proved in [21]. Those results were extended to more complicated higher order equations [27,32].

Another method was proposed in [23] and developed by several authors for elliptic equations (systems) with nonlinearities of various types, see, for instance, [7,9,24,33,42]. This method has its roots in theory of center manifolds for ODEs and allows one to construct solutions of elliptic equations in cylinder-like domains that are unbounded in some distinguished direction. The original elliptic PDE can be formally written as an evolutionary equation in the proper functional space but this equation is usually ill-posed in the whole space. However, if one is lucky to find a finite dimensional invariant center submanifold of the proper smoothness, then the restriction of the evolutionary equation to this submanifold gives a finite dimensional flow. Orbits of this flow generate solutions of the original equation.

A different approach to finding solutions is related to bifurcation methods, in particular, the Lyapunov–Schmidt reduction. These methods allow for construction of two-dimensional solutions with triangle symmetry [30] and solutions of Eq. (1.1) in $\mathbb{R}^n$ periodic in one variable and decaying in other variables [16].

A powerful method of constructing multi-peak solutions is based on the implicit function theorem. One can consider a sum of soliton-type solutions (ground states, see [8]) located at several points with pairwise distance sufficiently large. Using the decay of soliton at infinity, one can show that there is a solution of the original equation close to this sum (see, e.g. [31,38,47]). The most difficult step in this approach is to establish that the differential of the operator of the problem (1.1) is non-degenerate. For simple equations this fact is well known (see, e.g. [14]). However, in more complicated cases, in particular for the equations with p-Laplacian, it is not so obvious.

There are several other directions of research in studying elliptic problems. Solutions that were mentioned above are rather regular. A natural problem here is to discover solutions that can be considered as being chaotic. Studying a chaotic behavior is now one of the most rapidly developing field of research in finite dimensional dynamical systems (flows and diffeomorphisms). Till now this direction in PDEs deals with estimates of Hausdorff or Lyapunov dimension of attractors for evolutionary PDEs (see overviews [36,37,45]). For elliptic PDEs this direction of research is represented by the study of so called trajectory attractors, estimates of their upper and lower bounds for the complexity of entire solutions in cylindrical domains (see the overview [37]). Other tools (for instance, a Conley index) are used to construct nontrivial solutions [18],

bounded nonnegative solutions of the two-dimensional elliptic boundary value problem in the positive quadrant of the plane [10], and so forth.

In this paper we construct entire bounded solutions to (1.1) which have various types of symmetries and may also decay in some directions. We use purely variational approach which allows us to construct these solutions using the concentration-compactness principle [29] and symmetry considerations [22].

The paper is organized as follows. In Section 2 we collect the basic known facts used in the proofs as well as some necessary technical lemmata.

The main part of the paper deals with the model equation in $\mathbb{R}^2$

$$\Delta u - u + u^3 = 0. \tag{1.2}$$

In Section 3 we are concerned with periodic solutions. In Section 4 we discuss radially symmetric solutions. Generalization of our results for higher dimensions and more general equations are presented in Section 5.

In Appendix A we give for the comparison a brief explanation of using the central manifold method for our Eq. (1.2). This method was proposed in [4] in order to construct the so-called *breather type* solutions which are localized in one variable and periodic in another variable, see also [1] and [17].

In Appendix B we show another approach of construction of breather type solutions using the Bubnov–Galerkin approximations. This method reduces the problem to finding homoclinic solutions to some saddle equilibrium of a proper Hamiltonian system. However, in this way we obtain only an approximate solution of the original problem. See, in this respect, the paper [2] where the close connection between two-dimensional solutions of the Allen–Cahn equation stabilizing in one variable at infinity and heteroclinic solutions of the related one-dimensional equation was established.

We wish to stress that the variational method used in our paper is rather general, applicable in any dimension and allows us to construct in a unified way several types of solutions (radial, breather type, rectangular, triangular, hexagonal, etc.), both positive and sign-changing. Moreover, this method works without change for wide class of equations, including equations with p-Laplacian.

Let us introduce some notation. We use letter C to denote various positive constants. To indicate that some C depends on a parameter a, we write sometimes $C(a)$.

For a domain $\Omega \subset \mathbb{R}^n$ we denote by Ω_R the set $\{Rx \mid x \in \Omega\}$.

We use the notation $B(x, R)$ for a ball of radius R centered at x.

Notations $o_R(1)$ and $o_\varepsilon(1)$ mean $o(1)$ as $R \to \infty$ and $\varepsilon \to 0$ respectively.

For $1 < p < \infty$ we define

$$p^* = \begin{cases} \dfrac{np}{n - p} & p < n; \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that p^* is the critical Sobolev embedding exponent when $p < n$.

2. Auxiliary statements

2.1. Concentration

Proposition 2.1 (*A Variant of Lemma 1.1 from [29]*). *Suppose that $G(s)$, $s \in \mathbb{R}$ is a positive function. Consider a sequence of functions $u_j(x)$, $x \in \mathbb{R}^n$ and suppose $\int_{\mathbb{R}^n} G(u_j)dx$ is finite for every j. Then (up to a subsequence) one of the two conditions is satisfied:*

1. *(concentration) There exist $\lambda \in (0, 1]$ and a sequence of points $x_j \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exist $\rho > 0$, a sequence $\rho'(j) \to \infty$ and a number j_0 such that for every $j \geqslant j_0$*

$$\left| \int_{B(x_j, \rho)} G(u_j)\, dx - \lambda \int_{\mathbb{R}^n} G(u_j)\, dx \right|$$
$$+ \left| \int_{\mathbb{R}^n \setminus B(x_j, \rho'(j))} G(u_j)\, dx - (1 - \lambda) \int_{\mathbb{R}^n} G(u_j)\, dx \right| < \varepsilon \int_{\mathbb{R}^n} G(u_j)\, dx. \qquad (2.1)$$

In that case x_j is called a concentration sequence of $G(u_j)$ and λ is called a weight of the sequence.

2. *(vanishing) For every $\rho > 0$ the following equality holds:*

$$\lim_{j \to \infty} \sup_{x \in \mathbb{R}^n} \int_{B(x, \rho)} G(u_j)\, dx = 0. \qquad (2.2)$$

Remark 2.1. The condition (2.1) remains true if we substitute the sequence of radii $\rho'(j)$ with a smaller sequence which tends to infinity and ρ with a larger constant.

Remark 2.2. If x_j is a concentration sequence and y_j is a sequence of points such that $|x_j - y_j| \leq d$ for some $d \in \mathbb{R}$ then y_j is a concentration sequence as well. Indeed, it is easy to see that $\rho_y = \rho_x + d$ and $\rho_y'(j) = \rho_x'(j) - d$ makes it satisfy (2.1). In this case sequence y_j is called equivalent to x_j.

2.2. Some lemmata

Proposition 2.2 (*A Variant of Lemma 1.6 from [25]*). *Assume that $1 < p < q < \infty$ and functions $a, b, c \in W_p^1(\Omega) \cap L_q(\Omega)$ have separated supports. Suppose also that $b \not\equiv 0$, $c \not\equiv 0$ and*

$$\frac{\|b\|_{W_p^1}^p}{\|b\|_{L_q}^q} \geq \frac{\|c\|_{W_p^1}^p}{\|c\|_{L_q}^q}. \qquad (2.3)$$

Let

$$u = a + b + c$$

and

$$U = a + \frac{(\|b\|_{L_q}^q + \|c\|_{L_q}^q)^{1/q}}{\|c\|_{L_q}} \cdot c.$$

Then

$$U = a \text{ for } x \in \Omega \setminus (\operatorname{supp} c);$$
$$\int_\Omega U^q\, dx = \int_\Omega u^q\, dx;$$
$$\|U\|_{W_p^1}^p < \|u\|_{W_p^1}^p - C(b, c).$$

Furthermore, the constant $C(b, c)$ depends only on $\|b\|_{L_q(\Omega)}, \|c\|_{L_q(\Omega)}, \|b\|_{W_p^1(\Omega)}$ and $\|c\|_{W_p^1(\Omega)}$.

Proof. The proof is a matter of simple calculation. $\square$

The following lemma was proved in [11] (Lemma 2.9) for the case $p = 2$. The general case is very similar but for convenience's sake we prove it here.

Lemma 2.1.   *Suppose a sequence u_R is bounded in $W_p^1(\mathbb{R}^n)$, $1 < p < q < p^*$ and for some $\rho > 0$*

$$\lim_{R \to \infty} \sup_{x \in \omega} \int_{B(x,\rho)} |u_R|^q \, dx = 0$$

where ω is an open subset in $\mathbb{R}^n$.

Then

$$\int_\omega |u_R|^q \, dx \to 0 \quad as \quad R \to \infty.$$

In case $p < n$ the statement is also true for $q = p^*$.

Proof.   Let d be a positive real number less than $\rho/\sqrt{n}$.

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we write

$$Q_m = [m_1 d, (m_1 + 1)d] \times [m_2 d, (m_2 + 1)d] \times \cdots \times [m_n d, (m_n + 1)d].$$

By the Sobolev inequality, there exists $C > 0$ such that

$$\|u\|_{L_q(Q_m)} \leq C \|u\|_{W_p^1(Q_m)}$$

for all $u \in W_p^1(Q_m)$.

Let M be the set of $m \in \mathbb{Z}^n$ such that $Q_m \cap \omega \neq \emptyset$ and let $\Omega = \bigcup_{m \in M} Q_m$.

For any $u \in W_p^1(\mathbb{R}^n)$ we deduce

$$\begin{aligned}
\|u\|_{L_q(\Omega)}^q &= \sum_{m \in M} \|u\|_{L_q(Q(m))}^q \leq \big(\sup_{m \in M} \|u\|_{L_q(Q_m)}\big)^{q-p} \sum_{m \in M} \|u\|_{L_q(Q_m)}^p \\
&\leq C^p \big(\sup_{m \in M} \|u\|_{L_q(Q_m)}\big)^{q-p} \sum_{m \in M} \|u\|_{W_p^1(Q_m)}^p = C^p \big(\sup_{m \in M} \|u\|_{L_q(Q_m)}\big)^{q-p} \|u\|_{W_p^1(\Omega)}^p \\
&\leq C^p \big(\sup_{m \in M} \|u\|_{L_q(Q_m)}\big)^{q-p} \|u\|_{W_p^1(\mathbb{R}^n)}^p.
\end{aligned} \tag{2.4}$$

By the choice of d, $Q_m \subset B(x,\rho)$ for all $x \in Q_m$. Therefore, for $m \in M$ and $x \in Q_m \cap \omega$ we have

$$\|u\|_{L_q(Q_m)} \leq \|u\|_{L_q(B(x,\rho))},$$

and $\sup_{m \in M} \|u_R\|_{L_q(Q_m)} \to 0$ by hypothesis.

Since u_R is bounded in $W_p^1(\mathbb{R}^n)$ it follows from (2.4) that $\|u_R\|_{L_q(\Omega)} \to 0$. Since $\omega \subset \Omega$ we are done. $\square$

Lemma 2.2 (*An Analogue of Lemma 3.1 from [25]*).   *Suppose a sequence of functions $\{u_R\}$ is normalized in $L_q(\Omega_R)$ and bounded in $W_p^1(\Omega_R)$. Let $\{x_R\}$ be a concentration sequence of $|u_R|^q$, i.e. for every $\varepsilon > 0$ there exists a radius $\rho > 0$ and a sequence of radii $\rho'(R)$ that satisfy concentration condition for $G(s) = |s|^q$. Define σ as a cut-off function that satisfies*

$$\sigma(x) = \begin{cases} 1, & |x - x_R| \leq (11\rho + \rho'(R))/12; \\ 1, & |x - x_R| \geq (\rho + 11\rho'(R))/12; \\ 0, & (5\rho + \rho'(R))/6 \leq |x - x_R| \leq (\rho + 5\rho'(R))/6; \end{cases}$$

$$|\nabla \sigma| \leq \frac{12}{\rho'(R) - \rho}.$$

We claim inequalities

$$\|\sigma u_R\|_{W_p^1(\Omega_R)} \leq \|u_R\|_{W_p^1(\Omega_R)} + o_R(1), \tag{2.5}$$

$$\|\sigma u_R\|_{L_q(\Omega_R)} \geq 1 - o_\varepsilon(1) \tag{2.6}$$

hold true for all sufficiently large R.

Remark 2.3. Suppose we have several concentration sequences. In that case we can use Lemma 2.2 to produce cut-off functions for each sequence and then multiply them to obtain a cut-off function which isolates all of the concentration sequences. Note that (2.5) and (2.6) are still satisfied.

2.3. Concentration theorem

In this section we consider the functional

$$\tilde{J}[u] = \frac{\int_{\Omega_R}(|\nabla u|^p + |u|^p)\, dx}{\left(\int_{\Omega_R}|u|^q\, dx\right)^{p/q}} \tag{2.7}$$

where $1 < p < q < p^*$.

Lemma 2.3. *Suppose $u_R \in W_p^1(\Omega_R)$ is a sequence of minimizers of functional (2.7). Then u_R has no more than one concentration sequence.*

Proof. Since $\tilde{J}[tu] = \tilde{J}[u]$ we may assume that $\|u_R\|_{L_q} = 1$. Suppose there are two sequences x_R and y_R with weights λ_1 and λ_2. We are going to show it is more profitable to have only one of them. Let σ be a cut-off function that isolates x_R and y_R (Remark 2.3). Let σ_1 and σ_2 be the components of σ with $x_R \in \operatorname{supp}\sigma_1$ and $y_R \in \operatorname{supp}\sigma_2$. Set $\sigma_0 = \sigma - \sigma_1 - \sigma_2$. Without loss of generality, assume

$$\frac{\|\sigma_1 u_R\|_{W_p^1}^p}{\|\sigma_1 u_R\|_{L_q}^q} \geq \frac{\|\sigma_2 u_R\|_{W_p^1}^p}{\|\sigma_2 u_R\|_{L_q}^q}.$$

We apply Proposition 2.2 for functions $a = \sigma_0 u_R$, $b = \sigma_1 u_R$ and $c = \sigma_2 u_R$ and obtain function v_R which satisfies

$$\|v_R\|_{W_p^1(\Omega_R)} < \|\sigma u_R\|_{W_p^1(\Omega_R)} - \mu,$$
$$\|v_R\|_{L_q(\Omega_R)}^q = \|\sigma u_R\|_{L_q(\Omega_R)}^q > 1 - o_\varepsilon(1).$$

The last inequality is by Lemma 2.2. Thus,

$$\tilde{J}[v_R] < \tilde{J}[\sigma u_R] - \mu_1 \leq \tilde{J}[u_R](1 - \mu_2 + o_\varepsilon(1) + o_R(1))$$

for some $\mu_2 > 0$, independent of ε and R, which contradicts the minimality of u_R. $\square$

Theorem 2.1 (*Concentration Theorem*). *Suppose Ω_R is a sequence of Lipschitz domains in $\mathbb{R}^n$ such that the set of extension operators from $W_p^1(\Omega_R)$ to $W_p^1(\mathbb{R}^n)$ is uniformly bounded in norm (see [44, Chapter 6, Section 3]). Suppose $u_R \in W_p^1(\Omega_R)$ is a sequence of minimizers of functional (2.7). Further, suppose that $\sup_R \|u_R\|_{W_p^1} < \infty$. Then u_R has exactly one concentration sequence of weight 1.*

Proof. We again assume that $\|u_R\|_{L_q} = 1$.

Firstly, by Lemma 2.3 we have no more than one concentration sequence. It remains to prove that it exists and has weight 1. Let us assume the converse. There are two cases: either there is a concentration sequence x_R with weight $\lambda < 1$ or no sequence at all. In the first case consider the cut-off function σ from Lemma 2.2, let σ_1 be the component which isolates x_R and define $\sigma_0 = \sigma - \sigma_1$ (exactly as in the previous Lemma). In the second case we take $\sigma_0 \equiv 1$. Now Proposition 2.1 implies that $\sigma_0 u_R$ satisfies the vanishing condition. Moreover, $\|\sigma_0 u_R\|_{L_q}$ tends to $1 - \lambda$ in the first case and is equal to 1 in the second case.

Let Π_R be the extension operator from $W_p^1(\Omega_R)$ to $W_p^1(\mathbb{R}^n)$. It follows that

$$\|\Pi_R(\sigma_0 u_R)\|_{W_p^1} \leq \|\Pi_R\| \sup_R(\|u_R\|_{W_p^1} + o_R(1)) < C.$$

In other words, $\Pi_R(\sigma_0 u_R)$ is a bounded sequence in $W_p^1(\mathbb{R}^n)$. However,

$$\lim_{R \to \infty} \|\Pi_R(\sigma_0 u_R)\|_{L_q(\mathbb{R}^n)} \geq \lim_{R \to \infty} \|\sigma_0 u_R\|_{L_q(\Omega_R)} > 0,$$

which contradicts Lemma 2.1 since the extension operators preserve the vanishing condition. □

3. Periodic solutions in $\mathbb{R}^2$

Here we construct several families of solutions to (1.2) in dimension 2 with various lattices of periods. To do this, we choose a domain Ω tiling the plane by reflections and solve (1.2) with Neumann (Dirichlet) boundary conditions in Ω_R using classical variational method. Then we use even (respectively, odd) reflection to obtain a solution in the whole plane.

We begin with the general construction of a positive solution in Section 3.1. To recognize the structure of this solution we use the Concentration Theorem 2.1 and find possible positions of the concentration point. Without additional restrictions, for large R the minimal energy solution concentrates at the corner with the smallest angle. This is sufficient to obtain solutions with rectangular and triangular lattices, see Sections 3.2 and 3.3. For solutions with hexagonal symmetries we use the minimization with restrictions and refine the Concentration Theorem for this case, see Section 3.4. To construct breather type solutions in Section 3.5 we deal with unbounded Ω. Finally, in Section 3.6 we consider sign-changing solutions.

3.1. Some general results

Suppose Ω is a domain in $\mathbb{R}^2$ with piecewise smooth boundary. Consider the variational problem for the energy functional

$$J[u] = \frac{\int_{\Omega_R}(|\nabla u|^2 + |u|^2)\, dx}{\left(\int_{\Omega_R} |u|^4\, dx\right)^{1/2}} \to \min. \tag{3.1}$$

Since $J[cu] = J[u]$ it is equivalent to finding a constrained minimum for the problem

$$\int_{\Omega_R}(|\nabla u|^2 + |u|^2)\, dx \to \min; \qquad \int_{\Omega_R} |u|^4\, dx = 1. \tag{3.2}$$

Suppose this minimum is attained (this holds, for instance, in bounded domains, see Remark 3.1). Let v_R be a minimizer. Then it satisfies the Euler–Lagrange equation as well as the natural boundary conditions. Thus it is a weak solution of the problem

$$-\Delta v + v = \lambda v^3 \text{ in } \Omega_R; \qquad \left.\frac{\partial v}{\partial \mathfrak{n}}\right|_{\partial \Omega_R} = 0$$

where $\mathfrak{n}$ stands for the exterior unit normal vector on $\partial \Omega_R$. Notice that the Lagrange multiplier λ coincides with the sought-for minimum of the problem (3.2). By the elliptic regularity theory v_R is a classical solution.

Since $J[|u|] = J[u]$ we can presume $v_R \geq 0$, and after that the maximum principle gives $v_R > 0$ in Ω_R. We multiply v_R by $\sqrt{\lambda}$ and get another minimizer of (3.1) u_R which satisfies

$$-\Delta u + u = u^3 \text{ in } \Omega_R; \qquad \left.\frac{\partial u}{\partial \mathfrak{n}}\right|_{\partial \Omega_R} = 0. \tag{3.3}$$

We note that $\|u_R\|_{L^4} = \sqrt{\lambda}$ evidently depends on R. We now prove it cannot be either too small or too large.

Lemma 3.1. $\|u_R\|_{L^4} = \sqrt{\lambda(R)}$ is bounded and separated from zero as $R \to \infty$.

Proof. First we observe that for u compactly supported $J[u]$ does not depend on R. This implies $\lambda(R)$ is bounded since it is the minimum for problem (3.1).

Further, let $\Pi_R : W_2^1(\Omega_R) \to W_2^1(\mathbb{R}^2)$ stand for the extension operator (see [44, Chapter 6, Section 3]). The sequence of norms $\|\Pi_R\|$ is bounded by some constant as $R \to \infty$. Since the embedding $W_2^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ is bounded (this follows, for instance, from (2.4)), we obtain

$$\|u_R\|_{W_2^1(\Omega_R)} \geq C\|\Pi_R(u_R)\|_{W_2^1(\mathbb{R}^2)} \geq C\|\Pi_R(u_R)\|_{L^4(\mathbb{R}^2)} \geq C\|u_R\|_{L^4(\Omega_R)},$$

so $\lambda = J[u_R] = \|u_R\|_{W_2^1(\Omega_R)}^2 / \|u_R\|_{L^4(\Omega_R)}^2 \geq C$ and we are done. $\square$

Remark 3.1. The functional in the problem (3.2) is coercive and weakly lower semicontinuous in $W_2^1(\Omega_R)$. If Ω is bounded the embedding $W_2^1(\Omega_R) \hookrightarrow L_4(\Omega_R)$ is compact so the set of functions satisfying the condition in the problem (3.2) is weakly closed in $W_2^1(\Omega_R)$. This implies (see [19, Theorems 24.11 and 26.8]) that the minimum in (3.2) is attained and the reasoning above can be applied.

We construct first the simplest families of solutions in $\mathbb{R}^2$ which have a rectangular and triangular lattices of periods.

3.2. Solutions with rectangular symmetry

Suppose Ω_R is a rectangle $(0, R) \times (0, aR)$ where a is a given number. By Remark 3.1 we obtain a positive solution u_R to the problem (3.3) that minimizes the functional (3.1).

By the Concentration Theorem 2.1 in Section 2.3, u_R has exactly one concentration sequence x_R as $R \to \infty$. We consider three possibilities listed below.

1. The distance between concentration sequence and the sets of vertices of respective rectangles is bounded.
2. There is a subsequence with unbounded distance from vertices but bounded distance from $\partial\Omega_R$.
3. There is a subsequence with unbounded distance from $\partial\Omega_R$.

By Remark 2.2 we can assume x_R is a sequence of vertices in the first case and $x_R \in \partial\Omega_R$ in the second case. By Remark 2.1 in the case 2 we can choose $\rho'(R) \to \infty$ such that ρ' is smaller than the distance from x_R to the vertices. Similarly, in the case 3 we can assume $\rho' < \operatorname{dist}(x_R, \partial\Omega_R)$. For R large enough the intersection $B(x_R, \rho'(R)) \cap \Omega_R$ is a quarter-disk in the first case (see Fig. 1), half-disk in the second and full disk in the third (see Fig. 2).

Now we claim that the case 3 is impossible. Indeed, choose $\varepsilon > 0$. Let σ be a cut-off function from Lemma 2.2 in Section 2.2. We consider the component h_R of σu_R which contains $B(x_R, \rho)$. By Lemma 2.2 $J[h_R] \leq J[u_R] + o_\varepsilon(1)$ for R large enough.

Now let $\tilde{h}_R$ be the symmetric rearrangement of h_R. It is well known that $\|h_R\|_{W_2^1} \geq \|\tilde{h}_R\|_{W_2^1}$ and $\|h_R\|_{L^4} = \|\tilde{h}_R\|_{L^4}$ (see [22, Section II.9, Corollary 2.35]). Therefore, $J[h_R] \geq J[\tilde{h}_R]$. We then consider a trial function v_R which is a quarter-disk of $\tilde{h}_R$ placed in the corner of the rectangle.

It is easy to see $J[\tilde{h}_R] = 2J[v_R]$. Therefore, $J[v_R] \leq J[u_R]/2 + o_\varepsilon(1)$. This contradicts the minimality of $J[u_R]$ if R is large enough and ε small enough, and the claim follows.

The case 2 is likewise impossible.

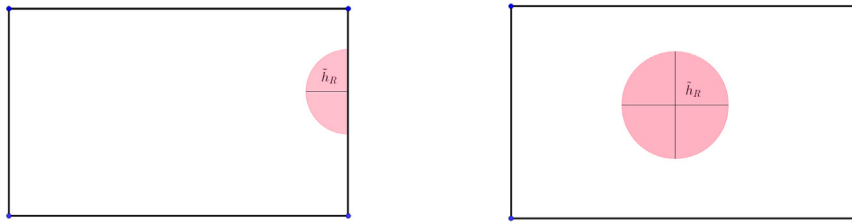We conclude that u_R is concentrated in the corner.

Using even reflection, we extend u_R to the rectangle $(-R, R) \times (-aR, aR)$ and then extend it to $\mathbb{R}^2$ periodically. Thus, we obtain solutions of (1.2) in $\mathbb{R}^2$ with $(2R, 2aR)$ rectangular lattice of periods (see Fig. 3).

Therefore, for every $a \in \mathbb{R}_+$ for sufficiently large R our periodic solutions are non-trivial and distinct.
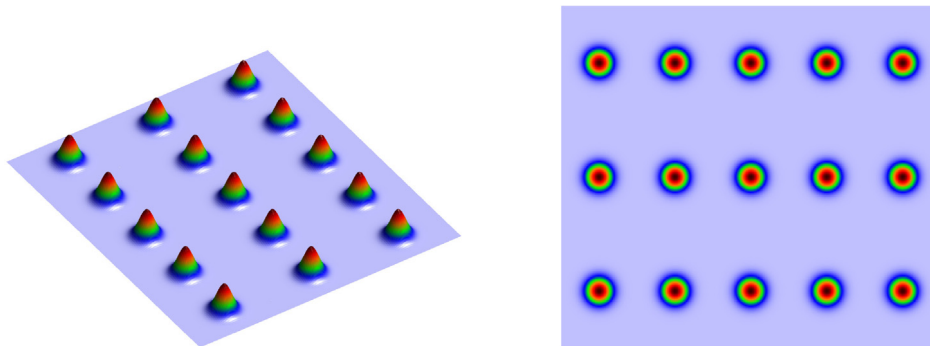
Fig. 1.

Fig. 2.

Fig. 3.

Corollary 3.1. *Given $a \geq 1$ and $N \in \mathbb{N}$ there exist $R^*(a, N) \in \mathbb{R}$ such that for $R > R^*(a, N)$ Eq. (1.2) in $\mathbb{R}^2$ has at least N different nontrivial positive $(2R, 2aR)$-periodic solutions.*

Proof. We proved that there exists R_0 such that u_R is concentrated in the corner for $R > R_0$. For $R > 2R_0$ we can consider the problem (3.1) in $\Omega_R = (0, R) \times (0, aR)$ and in $\Omega_R = (0, R/2) \times (0, aR/2)$. By the previous argument this gives different solutions, both are nontrivial, positive and $(2R, 2aR)$-periodic. The case of arbitrary N is managed similarly. □

3.3. Solutions with triangular symmetry

Suppose now Ω is the equilateral triangle with sides of length 1. By the same argument as in Section 3.2 the minimizer u_R of the variational problem (3.1) is a strong solution of (3.3) and is concentrated in a corner of triangle for R sufficiently large.

Using even reflection we extend u_R to the hexagon with side R (see Fig. 4) and then extend it periodically to $\mathbb{R}^2$. We obtain a solution of (1.2) with triangular lattice with side $2R$ (see Fig. 5).
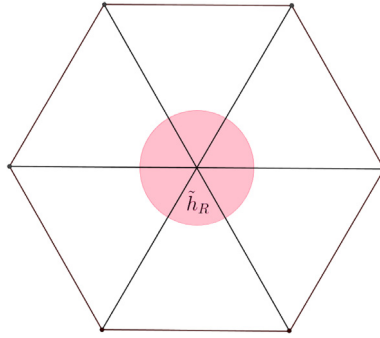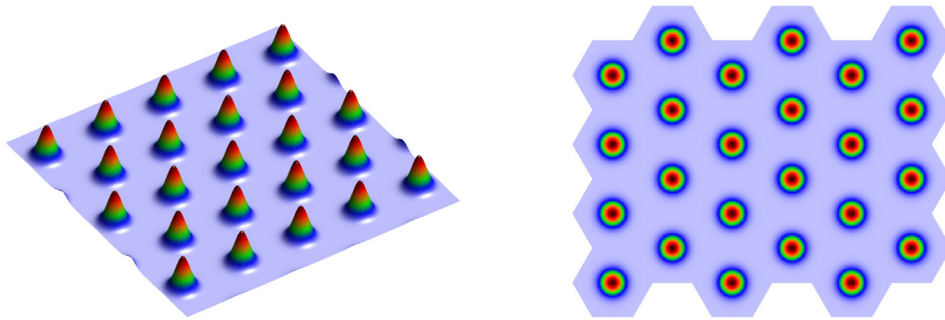
Fig. 4.

Fig. 5.

Exactly as in the previous case the Concentration Theorem 2.1 gives us the following corollary.

Corollary 3.2. *Giver $N \in \mathbb{N}$ there exist $R^*(N) \in \mathbb{R}$ such that for $R > R^*(N)$ Eq. (1.2) in $\mathbb{R}^2$ has at least N different nontrivial positive R-triangular-periodic solutions.*

3.4. Solutions with hexagonal symmetry

For the case of hexagonal lattice of periods a little more intricate argument is required. Let Ω be a triangle with angles $\pi/2$, $\pi/3$, $\pi/6$ and hypotenuse of length 1. Denote Ω_R by XYZ where $\angle X = \pi/3$, $\angle Y = \pi/6$, $\angle Z = \pi/2$ and denote sector $\Omega_R \cap B(Y, R/2)$ by A_R. If we consider the usual problem (3.2) in Ω_R the minimizer is concentrated near Y as it is the corner with the least angle. After extending it to $\mathbb{R}^2$ we get a solution with a triangular lattice of periods similar to Section 3.3. To prevent this we study the variational problem (3.2) with an additional condition

$$\int_{A_R} |u|^4 \, dx \leq 1/4. \tag{3.4}$$

Note that we can still apply some of the reasoning from Section 3.1 and deduce that the minimum is attained and the solution is non-negative. We now prove a variant of the Concentration Theorem for this case.

Theorem 3.1. *Let u_R is a sequence of minimizers for the problem (3.2) with the additional condition (3.4). Then u_R has exactly one concentration sequence of weight 1 concentrated near X (see Fig. 6).*
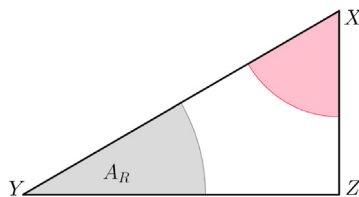
Fig. 6.

Proof. Suppose there are two concentration sequences x_R and y_R. We use Remark 2.3 to construct a cut-off function which isolates them. Let σ_1 and σ_2 be the components of σ with $x_R \in \operatorname{supp}\sigma_1$ and $y_R \in \operatorname{supp}\sigma_2$ respectively. Assume first that both of the components σ_1 and σ_2 lie within A_R. In that case functions v_R constructed in the proof of Lemma 2.3 still satisfy condition (3.4) which contradicts the minimality of u_R. The same argument can be applied if both components lie in $\Omega_R \setminus A_R$. Further, suppose the support of one of the components, say, σ_1, intersects $(\partial A_R) \cap \Omega_R$. By the construction of σ the diameter of σ_1 is not more than $2(5\rho + \rho'(R))/6$. The width of the annulus around σ_1 where σ equals zero is $4(\rho'(R) - \rho)/6$. Therefore, for R large enough we can move the "bubble" $\sigma_1 u_R$ fully into $\Omega_R \setminus A_R$ which eliminates this case. Thus, we can presume $\operatorname{supp}\sigma_1 \subset \Omega_R \setminus A_R$ and $\operatorname{supp}\sigma_2 \subset A_R$.

The proof of the Concentration Theorem 2.1 shows that the combined weight of the concentration sequences is 1. After that, the reasoning from Section 3.2 applied for each sequence shows that we can assume $x_R = X$ and $y_R = Y$. Next we are going to show that it is more profitable to have all of the weight concentrated near X.

Remark 2.1 allows us to take radii ρ and $\rho'(R)$ equal for x_R and y_R. Using symmetrization if needed we can assume that $\sigma_1 u_R(x) = h_1(|x_R - x|)$, $\sigma_2 u_R(x) = h_2(|y_R - x|)$ where $h_1(t)$ and $h_2(t)$ are decreasing functions vanishing for $t > \rho'(R)$.

Consider the function

$$g(x) = \frac{\|\sigma_1 u_R\|_{L_4}}{\|\sigma_2 u_R\|_{L_4}} \cdot \left(\frac{\pi/6}{\pi/3}\right)^{1/4} \cdot h_2(|x_R - x|).$$

It is easy to see that $\|g\|_{L_4} = \|\sigma_1 u_R\|_{L_4}$. Therefore, replacing $\sigma_1 u_R$ with g preserves the L_4 norm of the function. Since u_R is a minimizer it follows from (2.5) that

$$\|\sigma_1 u_R\|_{W_2^1} \le \|g\|_{W_2^1} - o_R(1) = \frac{\|\sigma_1 u_R\|_{L_4}}{\|\sigma_2 u_R\|_{L_4}} \cdot 2^{1/4} \cdot \|\sigma_2 u_R\|_{W_2^1} - o_R(1).$$

Combining (3.4) and (2.6) gives us

$$\|\sigma_2 u_R\|_{L_4}^4 \le 1/4; \qquad \|\sigma_1 u_R\|_{L_4}^4 \ge 3/4 - o_\varepsilon(1).$$

Hence

$$\frac{\|\sigma_1 u_R\|_{W_2^1}^2}{\|\sigma_1 u_R\|_{L_4}^4} \le \frac{\|\sigma_1 u_R\|_{L_4}^2}{\|\sigma_2 u_R\|_{L_4}^2} \cdot 2^{1/2} \cdot \frac{\|\sigma_2 u_R\|_{W_2^1}^2}{\|\sigma_1 u_R\|_{L_4}^4} - o_R(1)$$

$$\le \left(\frac{2 \cdot 1/4}{3/4 - o_\varepsilon(1)}\right)^{1/2} \cdot \frac{\|\sigma_2 u_R\|_{W_2^1}^2}{\|\sigma_2 u_R\|_{L_4}^4} - o_R(1) < \frac{\|\sigma_2 u_R\|_{W_2^1}^2}{\|\sigma_2 u_R\|_{L_4}^4}$$

for R large enough and ε small enough.

Finally, this shows the condition (2.3) is satisfied for functions $b = \sigma_2 u_R$, $c = \sigma_1 u_R$ and $a = (\sigma - \sigma_1 - \sigma_2)u_R$. Proposition 2.2 then gives us a function U satisfying the condition (3.4) which contradicts the minimality of u_R.

Thus, there is at most one concentration sequence. As it was mentioned, the combined weight of concentration sequences is 1 and the statement follows. $\square$
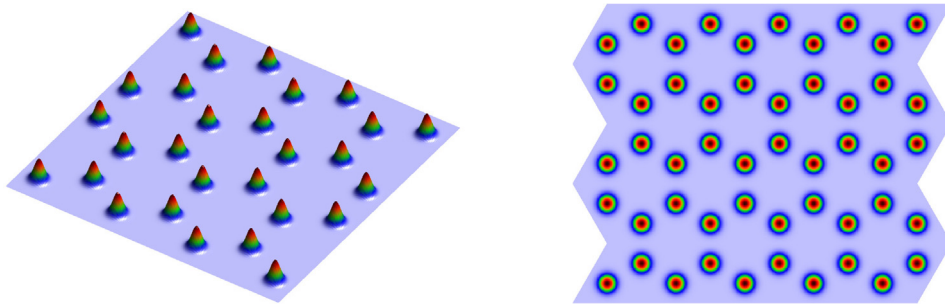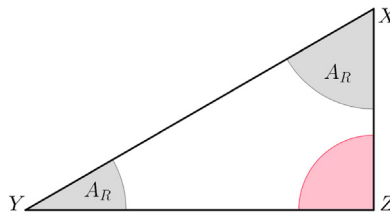
Fig. 7.

Fig. 8.

[Theorem 3.1](#) implies that

$$\int_{A_R} |u_R|^4 \, dx \to 0$$

as $R \to \infty$ since u_R is concentrated around X with weight 1. Therefore, for R large enough the restriction in (3.4) is non-active and the Euler–Lagrange equation

$$-\Delta u_R + u_R = \lambda u_R^3 \quad \text{in} \quad \Omega_R.$$

is derived the usual way.

Similar to Section 3.1 we multiply u_R by $\sqrt{\lambda}$ and obtain a solution of (1.2) concentrated near X. Finally, we use even reflection to extend it to a hexagon and then extend it to $\mathbb{R}^2$ periodically. The constructed solution has a hexagonal lattice of periods (see [Fig. 7](#)).

Remark 3.2. Using the same technique with two constrictions (see [Fig. 8](#))

$$\int_{B(Y,R/4)} |u|^4 \, dx \le 1/8, \quad \int_{B(X,R/4)} |u|^4 \, dx \le 1/8$$

we can force the solution to concentrate near Z. This gives us yet another lattice, depicted below (see [Fig. 9](#)).

3.5. Breather type solutions

Here we construct a family of solutions periodic in one variable and rapidly decaying in another. To do this we consider problem (3.2) in the strip $\Omega_R = (0, R) \times \mathbb{R}$. Since the embedding of $W_2^1(\Omega_R)$ into $L_4(\Omega_R)$ is not compact we cannot apply the argument from Section 3.1 directly and need to refine it.
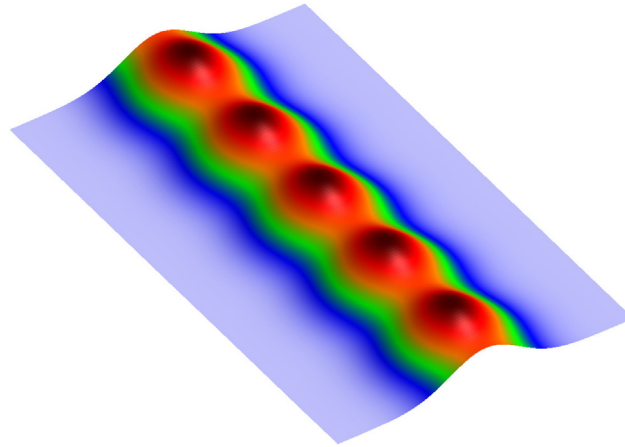
Let v_n be a minimization sequence for the problem (3.2):

$$\|v_n\|_{L^4} = 1; \qquad \|v_n\|_{W_2^1} \searrow \min \quad \text{as } n \to \infty.$$

Extending the functions to $\mathbb{R}^2$ and using [Lemma 2.1](#) shows there is no vanishing. Therefore, v_n must satisfy the concentration condition.
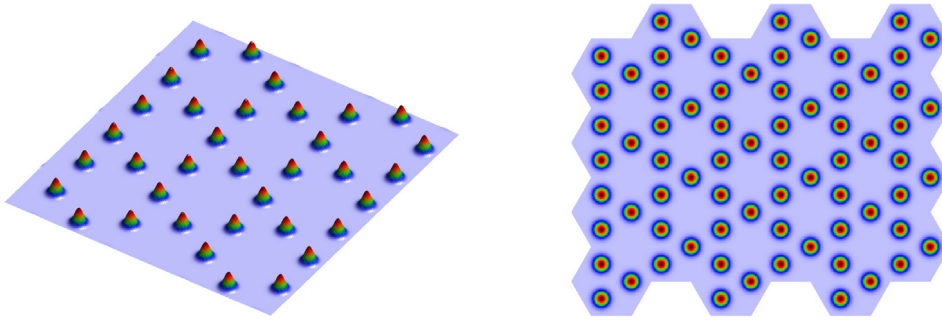
Fig. 9.

Fig. 10.

Notice that the Steiner symmetrization with respect to y and monotonous rearrangement with respect to x do not increase the functional (3.1) (see, e.g., [22, II.7]). Taking into account that $J[|v|] = J[v]$, we can presume that v_n are nonnegative, symmetrically decreasing in y and monotonous (without loss of generality, decreasing) in x. Then they can only concentrate around $(0,0)$.

Next, we extract a subsequence v_{n_k} which converges weakly in $W_2^1(\Omega_R)$ to some function u_R. Let $T > 0$ and consider $\Omega_{R,T} = (0,R) \times (-T,T)$. The sequence v_{n_k} (restricted to $\Omega_{R,T}$) converges weakly in $W_2^1(\Omega_{R,T})$. Since the embedding of $W_2^1(\Omega_{R,T})$ into $L_4(\Omega_{R,T})$ is compact it converges strongly in $L_4(\Omega_{R,T})$. Choose $\varepsilon > 0$. The sequence v_{n_k} is concentrated around $(0,0)$ which shows there is $T > 0$ such that

$$\|v_{n_k}\|_{L_4(\Omega_R)} \geq \|v_{n_k}\|_{L_4(\Omega_{R,T})} > (1-\varepsilon)\|v_{n_k}\|_{L_4(\Omega_R)}.$$

Therefore,

$$\liminf \|v_{n_k}\|_{L_4(\Omega_R)} \geq \|u_R\|_{L_4(\Omega_{R,T})} \geq \limsup(1-\varepsilon)\|v_{n_k}\|_{L_4(\Omega_R)}.$$

Thus $\|u_R\|_{L_4(\Omega_R)} = \lim_{T \to \infty} \|u_R\|_{L_4(\Omega_{R,T})} = 1$. Finally, since u_R is the weak limit of v_{n_k}, $\|u_R\|_{W_2^1} \leq \liminf \|v_{n_k}\|_{W_2^1}$ and u_R is, therefore, a minimizer. By construction, u_R is non-constant in x provided R is large enough.

Now we extend u_R to the whole plane by even reflection and periodic expansion. This gives us a required solution of (1.2) in $\mathbb{R}^2$ (see Fig. 10).

As a corollary, given $N \in \mathbb{N}$, for $R > R^*(N)$ we obtain at least N different nontrivial positive solutions of (1.2) in $\mathbb{R}^2$, $2R$-periodic in x and symmetrically decreasing in y.
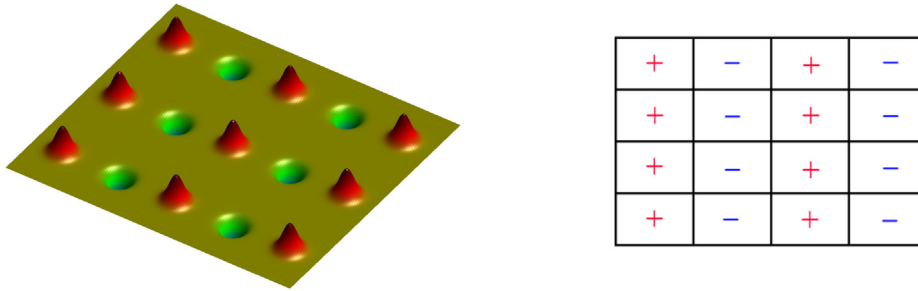
+	–	+	–
+	–	+	–
+	–	+	–
+	–	+	–

Fig. 11.

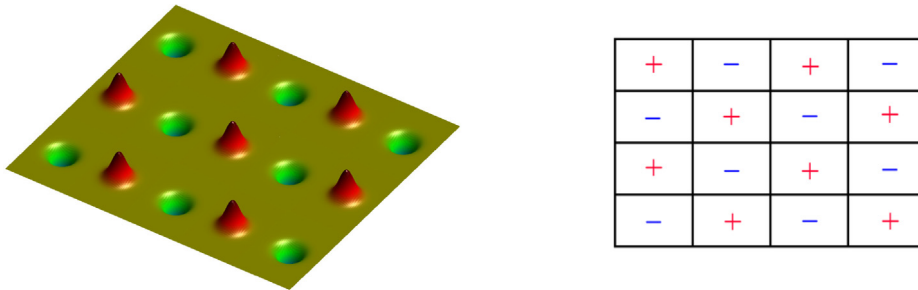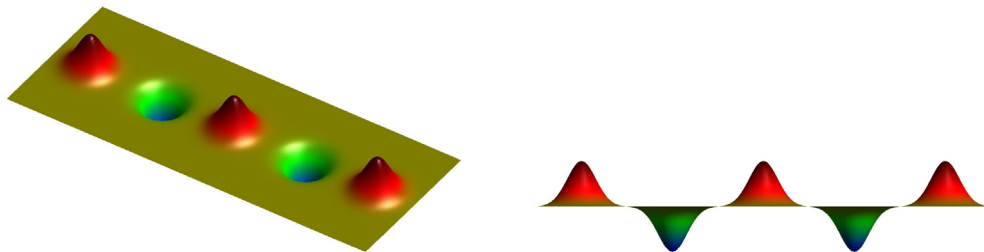| +	–	+	–
–	+	–	+
+	–	+	–
–	+	–	+

Fig. 12.

Fig. 13.

3.6. Sign-changing solutions

Now we consider problem (3.1) in the rectangle $(0, R) \times (0, aR)$ with the additional condition of $u = 0$ on $\{0, R\} \times (0, aR)$ (the vertical sides of the rectangle). Similarly to the previous sections we deduce the solution is concentrated in a half-circle adherent to the horizontal side. We again extend it to the strip $(0, R) \times \mathbb{R}$ by even reflection and then to the strip $(0, 2R) \times \mathbb{R}$ by odd reflection. Next we extend the function to $\mathbb{R}^2$ periodically which gives us a solution with alternating signs (see Fig. 11).

We can also consider the boundary condition $u = 0$ on all the boundary of the rectangle which will give us a solution concentrated in a circle. We extend it to $(-R, R) \times (-aR, aR)$ oddly and to $\mathbb{R}^2$ periodically. The resulting signs are in staggered order (see Fig. 12). Breather type case is analogous (see Fig. 13).

The equilateral triangular case is a little different. If the condition $u = 0$ holds for only one side of the triangle then the minimizer is concentrated in the opposite corner. We extend it to the hexagon and the function is positive there. We could try to extend it to the whole plane but the partitioning of $\mathbb{R}^2$ into hexagons have three hexagons which are pairwise adherent and each pair must have different signs which is impossible.
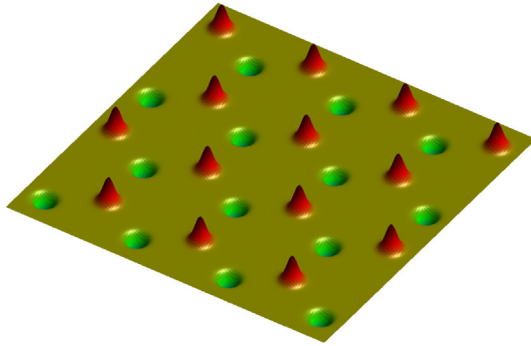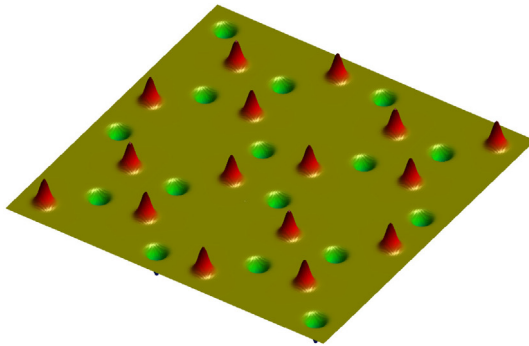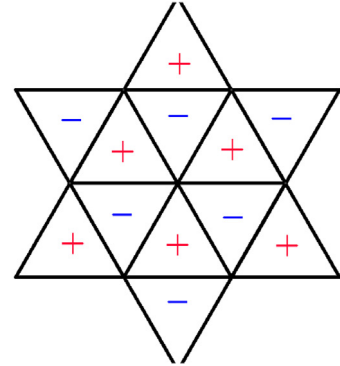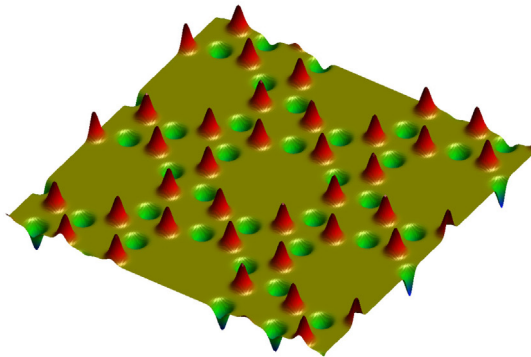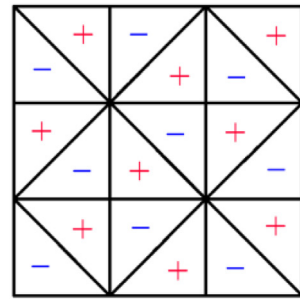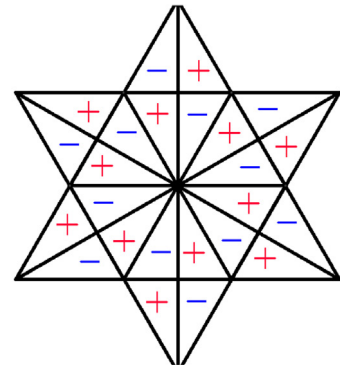
Fig. 14.

Fig. 15.

Fig. 16.

However if we set $u = 0$ on the whole boundary of the triangle we do not have this problem. We use odd reflection and extend the function to $\mathbb{R}^2$ like on the picture (see Fig. 14). Doing the same with triangles that have angles $\pi/2, \pi/4, \pi/4$ and $\pi/2, \pi/3, \pi/6$ gives us two more types of solutions (see Figs. 15 and 16).

4. Radially symmetric solutions

In this section we prove the existence of a positive radially symmetric solution using the standard method and the existence of a countable family of radially symmetric solutions using Lusternik–Schnirelmann

Theorem. Notice that the radial case is well studied by many authors, see, e.g. [8,21]. We give here the proofs for the sake of completeness.

4.1. Positive solution

Here we consider the problem (3.2) in $\Omega = \Omega_R = \mathbb{R}^2$. As in the breather type case we can find a minimization sequence v_n which is radially symmetric since symmetrical rearrangement does not increase the functional (3.1) (see, e.g., [22, II.9]). We again note that v_n must concentrate around $(0,0)$. By considering the problem in balls $B(0,T)$ and taking T to infinity we prove the nonzero limit exists similarly to Section 3.5. In the end this gives us a positive, symmetrically decreasing solution of the problem (1.2) in $\mathbb{R}^2$.

4.2. A countable family

To prove that problem (1.2) has a countable number of radial solutions we need the following statement (the Lusternik–Schnirelmann theorem, see, e.g., [40, Chapter 8]).

Proposition 4.1. *Suppose H is a Hilbert space and $I : H \to \mathbb{R}$ is a (nonlinear) functional such that:*

1. *I is weakly continuous and smooth (namely, $I \in C^{1,1}_{loc}$),*
2. *I is even and $I[0] = 0$,*
3. *$I[u] > 0$ and $\|I'[u]\| > 0$ for $u \neq 0$ (non-degenerate).*

Then I has at least a countable number of critical points on the sphere $S_a = \{x \in H \mid \|x\| = a\}$ for every $a > 0$.

We take the subspace of radial functions in $W_2^1(\mathbb{R}^2)$ as H and put $I[u] = \int_{\mathbb{R}^2} u^4$. We now show that I satisfies the conditions of Proposition 4.1. The conditions 2 and 3 are evident. The map $u \mapsto I'[u]$ is Lipschitz on any bounded set in H so the functional is $C^{1,1}_{loc}$. It remains to prove that I is weakly continuous. We pass to the polar coordinates and write

$$\|u\|_H^2 = 2\pi \int_0^\infty r(|u|^2 + |u'|^2)\,dr.$$

$$\|u\|_{L_4}^4 = 2\pi \int_0^\infty r|u|^4\,dr.$$

Using an obvious inequality

$$\|u\|_{L_4(R,R+1)} \leq C\|u\|_{W_2^1(R,R+1)}$$

we obtain for $R \geq 1$

$$\left(\int_R^{R+1} r|u|^4\,dr \right)^{1/2} \leq (R+1)^{1/2} C R^{-1} \int_R^{R+1} r(|u|^2 + |u'|^2)\,dr$$

$$= C\left(\frac{R+1}{R^2} \right)^{1/2} \int_R^{R+1} r(|u|^2 + |u'|^2)\,dr.$$

This implies, similar to Lemma 2.1,

$$\int_R^\infty r|u|^4\,dr = \sum_{k=0}^\infty \int_{R+k}^{R+k+1} r|u|^4\,dr$$

$$\leq \left(\sup_k \int_{R+k}^{R+k+1} r|u|^4\,dr \right)^{1/2} \sum_{k=0}^\infty \left(\int_{R+k}^{R+k+1} r|u|^4\,dr \right)^{1/2}$$

$$\leq C\Big(\sup_k \int_{R+k}^{R+k+1} r|u|^4\, dr\Big)^{1/2} \sum_{k=0}^{\infty} \Big(\frac{R+k+1}{(R+k)^2}\Big)^{1/2} \int_{R+k}^{R+k+1} r(|u|^2 + |u'|^2)\, dr$$

$$\leq C\frac{R+1}{R^2}\Big(\int_R^{\infty} r(|u|^2 + |u'|^2)\, dr\Big)^2. \tag{4.1}$$

It is well known that the embeddings $W_2^1(B(0,R)) \hookrightarrow L_4(B(0,R))$ are compact. It follows that the mappings $u \mapsto u|_{B_R(0)}$ from H to $L_4(\mathbb{R}^2)$ are compact. By (4.1) the embedding $H \hookrightarrow L_4(\mathbb{R}^2)$ is the norm limit of compact operators hence compact which proves the weak continuity of I.

The critical points of I on the sphere S_a are those where

$$\int_{\mathbb{R}^2} u^3 h = \lambda \int_{\mathbb{R}^2} (\nabla u \nabla h + uh) \tag{4.2}$$

for some $\lambda \in \mathbb{R}$ and every $h \in H$.

By the principle of symmetric criticality, see [41], the relation (4.2) holds for any $h \in W_2^1(\mathbb{R}^2)$. Taking $h = u$ shows that $\lambda > 0$. Therefore we can multiply u by $\sqrt{\lambda}$ and get a solution of (1.2) in $\mathbb{R}^2$. Thus, Lusternik–Schnirelmann theorem implies that there is a countable number of radial solutions.

5. Some generalizations

Our arguments are valid if we consider Eq. (1.2) in $\mathbb{R}^3$. By choosing an appropriate domain Ω we can get the following:

1. solutions periodic in x, y, z;
2. solutions triangular-periodic in x, y and periodic in z;
3. solutions periodic in x, y and symmetrically decreasing in z;
4. solutions triangular-periodic in x, y and symmetrically decreasing in z;
5. solutions periodic in x and symmetrically decreasing in y, z;
6. radial solutions, etc.

More generally, consider the equation

$$\Delta_p u - |u|^{p-2}u + |u|^{q-2}u = 0 \quad \text{in} \quad \mathbb{R}^n \tag{5.1}$$

Here $1 < p < \infty$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}u)$ is a p-Laplacian while $q \in (p, p^*)$. The corresponding variational problem is the minimization of the functional $J[u] = \|u\|_{W_p^1}/\|u\|_{L_q}$. Since the Concentration Theorem (Theorem 2.1 holds true, argument similar to the one in Section 2 can be applied again. In that way we obtain positive solutions of (5.1) which have various periodic lattices in some variables and are symmetrically decaying in other variables. The sign-changing solutions with various periodic structures could be obtained as well.

Using the classical Nehari method, it is possible to apply our machinery for a more general equation

$$\Delta_p u - |u|^{p-2}u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n \tag{5.2}$$

with an odd function f satisfying some natural assumptions. Roughly speaking, $f(s)$ is assumed to be "more convex" than s^p for $s > 0$ and to have subcritical growth at infinity.

For instance, the requirements for f can be given as follows:

$$sf'(s) > (p-1)f(s) \text{ for almost any } s \geq 0;$$

$$\liminf_{s\to\infty} \frac{sf(s)}{\int_0^s f(t)\, dt} > p;$$

$$\lim_{s \to 0} \frac{f(s)}{s^{p-1}} = 0;$$

$$\lim_{s \to \infty} \frac{sf(s)}{\Phi(s)} = 0,$$

where

$$\Phi(s) = s^{p^*}, \qquad\qquad p < n,$$

$$\Phi(s) = s^q \text{ for any } q \in (p, \infty), \quad p \geq n.$$

The method used to prove the existence of solutions is well known (see e. g. [28,39]). After the solution is found the concentration theory can be applied (with appropriate modifications) and the results follow.

6. Conclusions

For the elliptic equation (1.2) and its generalizations we have constructed nontrivial solutions of several types: periodic with various lattices of periods, breather type and radial. Of course, many questions remain open. It seems that using methods of [47] it is possible to obtain solutions with finite number of humps located at different points on the plane with pairwise distances large enough. Also it would be interesting to prove the existence of solutions which are localized in both variables, invariant under the rotation by the angle $\pi/2$ and being not rotationally invariant. Simulations with Eq. (1.2) showed their existence (see Figure 3 in [3]).

Acknowledgments

Appendix A. Breather type solutions and center manifold reduction

The variational approach of previous sections provides families of breather type solutions (see Section 3.5) with large enough periods. It is of interest to find a lower bound for these periods. An approach to this problem was proposed in [4]. It relies on the bifurcation theory and an extension of the center manifold theory to elliptic equations in cylindric domains (see [23] and more rigorously in [33]). To explain this approach, let us rewrite Eq. (1.2) formally as a dynamical system w.r.t. the "time" y:

$$u_y = v, \quad v_y = -u_{xx} + u - u^3. \tag{A.1}$$

Let us stress that here we consider the "time" y as future periodic variable. In the next section we change their roles and take variable x as the "time".

The system (A.1) has the plane wall type solution

$$u_0(x) = \pm\sqrt{2}/\cosh x, \quad v_0(x) = 0,$$

which formally is "the equilibrium state" of this system. Let us linearize the system at this solution and study the related spectral problem. We obtain a linear system which is equivalent to the Sturm–Liouville equation

$$\mathcal{L}\phi \equiv -\frac{d^2\phi}{dx^2} + (1 - 3u_0^2(x))\phi = \lambda^2\phi.$$

This equation is invariant with respect to change of variable $x \to -x$, so we may consider it in the space of even in x functions in $L^2(\mathbb{R})$. The function $-3u_0^2(x)$ (the potential, if one interprets this equation as the Schrödinger equation) is rapidly decaying as $|x| \to \infty$. Thus the spectrum of the operator $\mathcal{L} - 1$ consists of finitely many negative eigenvalues and the continuous spectrum $[0, \infty)$ [26, Ch. IX]. For the equation in question the unique eigenvalue is $\lambda^2 = -3$ with the eigenfunction $h(x) = c/\cosh^2(x)$ and the continuous spectrum $[1, \infty)$ is separated from the eigenvalue.

For the system linearized at the equilibrium its spectrum consists of the pair of pure imaginary eigenvalues $\pm i\sqrt{3}$ and two rays of continuous spectrum $\lambda \geq 1$ and $\lambda \leq -1$. Thus, if the center manifold theorem from [33,35] is valid for this case, then in the whole phase space one gets a smooth local two dimensional center manifold through the equilibrium [34] which corresponds to eigenvalues $\pm i\sqrt{3}$. The restriction of the system (A.1) to this manifold generates a two-dimensional Hamiltonian system in "time" variable y with the equilibrium – center – whose neighborhood is filled with periodic orbits. These orbits provide periodic in y solutions of the initial elliptic equation. As the amplitudes tend to zero, their periods tend the period of the linearized oscillations, which is equal to $2\pi/\sqrt{3}$.

In [4] this scheme was presented with the necessary calculations for the related operators. But the needed estimates on operators that would allow one to apply the center manifold theorem from [33] were not proved there. To apply results of [6,33,35] and others one needs to introduce the Banach space $(u, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ as a phase space. The system (A.1) becomes a differential equation in this space, and we consider it near the equilibrium. The conditions of [35] on the linearized operator and nonlinearities can be verified which gives the needed center manifold. As mentioned in [35], in general the center manifold is not smooth enough to apply the theory of Hamiltonian systems directly but in the case under consideration this is not a problem since this manifold is two-dimensional and is filled with periodic orbits.

Appendix B. Breather type solutions and homoclinic orbits

To demonstrate another view on breather type solutions, namely, their connection with homoclinic orbits of some dynamical system, we use the Fourier expansion in y variable. Denote by l the period in y of such a solution, $u(x, y) \equiv u(x, y + l)$. We also assume $u(x, y) \to 0$ as $x \to \pm\infty$.

After plugging the Fourier series into the equation we get an infinite system of ODEs for the Fourier coefficients. Using the Bubnov–Galerkin procedure, we truncate this system keeping only three modes: zero mode and two complex conjugated modes $\exp[\pm i2\pi y/l]$. Thus the ansatz

$$u(x, y) = (U_1(x) - iV_1(x))e^{-i\frac{2\pi}{l}y} + U_0(x) + (U_1(x) + iV_1(x))e^{i\frac{2\pi}{l}y} \tag{B.1}$$

gives an approximation solution. After plugging (B.1) into (1.2), projecting on the subspace of related modes and scaling we arrive at the following Euler–Lagrange type system of second order ODEs:

$$\begin{cases} U_1'' - \left(\frac{4\pi^2}{l^2} + 1\right)U_1 + 6U_1U_0^2 + 3(U_1^2 + V_1^2)U_1 = 0, \\ V_1'' - \left(\frac{4\pi^2}{l^2} + 1\right)V_1 + 6V_1U_0^2 + 3(U_1^2 + V_1^2)V_1 = 0, \\ U_0'' - U_0 + 2U_0^3 + 6(U_1^2 + V_1^2)U_0 = 0. \end{cases} \tag{B.2}$$

It can be transformed to the Hamiltonian form with Hamiltonian

$$H = \frac{p_0^2 + p_1^2 + q_1^2}{2} - \frac{U_0^2 + \left(\frac{4\pi^2}{l^2} + 1\right)(U_1^2 + V_1^2)}{2} + \frac{U_0^4}{2} + 3U_0^2(U_1^2 + V_1^2) + \frac{3}{4}(U_1^2 + V_1^2)^2. \tag{B.3}$$

It is easy to verify that this system has an additional integral $K = p_1V_1 - q_1U_1$.

Solutions corresponding to the plane walls belong to the invariant 2-plane $U_1 = p_1 = V_1 = q_1 = 0$, and they form two homoclinic loops of the saddle equilibrium $U_0 = p_0 = 0$ in this plane.

Breather type solutions we are searching for correspond to homoclinic orbits of the zero equilibrium situated out of this 2-plane. The equilibrium in the whole space is a saddle, its eigenvalues are three nonzero real pairs

$$\pm 1, \quad \pm\sqrt{1 + 4\pi^2/l^2}, \quad \pm\sqrt{1 + 4\pi^2/l^2}.$$

Hence, the equilibrium has three-dimensional smooth stable and unstable invariant manifolds, both of which lie in the level $H = 0$. Homoclinic orbits to the equilibrium belong to the intersection of these two manifolds, hence they are doubly asymptotic as $x \to \pm\infty$ to the equilibrium. Since the system has integral K, the value of K is also preserved along the homoclinic orbit. Therefore homoclinic orbits should belong to the joint level of two integrals $H = K = 0$.

For the Hamiltonian system obtained the following assertion is valid.

Proposition B.1. *The system has two homoclinic orbits of the zero equilibrium in the invariant 2-plane $U_1 = p_1 = V_1 = q_1 = 0$ and a one parameter family of homoclinic orbits in the invariant 4-plane $U_0 = p_0 = 0$.*

The proof is done by means of a direct integration using two integrals. For the one parameter family expressions for U_1, V_1 are as follows

$$U_1 = r\cos\theta = \frac{-2\lambda}{e^{\lambda x} + \frac{3}{2}e^{-\lambda x}}\cos\theta, \quad V_1 = r\sin\theta = \frac{-2\lambda}{e^{\lambda x} + \frac{3}{2}e^{-\lambda x}}\sin\theta$$

where $\lambda^2 = \frac{4\pi^2}{l^2} + 1$. After that one can find expressions for all functions U_0, U_1, V_1 and construct approximate solutions $U(x, y)$.

This approach can be extended capturing $2n + 1$ symmetrically chosen modes. Again, using the Bubnov–Galerkin truncation we get (after scaling) a Hamiltonian system with $2n+1$ degrees of freedom with a saddle type equilibrium at the origin and homoclinic loops.

But we also can consider the problem as follows. Take as a functional space the set of $(2\pi/l)$-periodic in y functions, more exactly, the space $H^1(S_l^1) \times L^2(S_l^1)$, where S_l^1 denote the circle with y-coordinate being $(2\pi/l)$-periodic. In this space we consider the system (A.1) where roles of x and y are interchanged. The system has the equilibrium at the origin but its spectrum is $|\lambda| \geq 1$. Here the equilibrium formally has infinite dimensional stable and unstable manifolds and solutions we seek correspond to homoclinic orbits of this equilibrium lying at their intersection. In fact, their existence is known from other considerations presented in Sections above.

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