



Manipulability of Majoritarian Procedures in Two-Dimensional Downsian Model

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Abstract. For the two-dimensional Downsian model the degree of manipulability of 16 known aggregation procedures, based on the majority relation, is evaluated using the Nitzan-Kelly index. Extended preferences for multi-valued choices are used to evaluate the fact of manipulation. Individual manipulability of agents is considered, when manipulating agent moves its ideal point over the plane. The range of possible manipulating positions of the agents is restricted to some rectangle on the two-dimensional coordinate space, within the feasible area of positions of alternatives and agents. The preferences of agents are assumed to be linear orders, constructed by the proximity of the alternatives to the agents, ordered according to Euclidean distance. The computer calculations, using Monte-Carlo simulations has been performed for 3, 4, and 5 alternatives and for even number of agents from 4 to 20. 100 thousands profiles were generated for each number of alternatives – number of agents case. The results of the simulations show that there are groups of procedures with relatively low degree of manipulability for all of the considered multiple-choice extensions.

Keywords: Manipulability · Majoritarian choice procedures · Spatial Downsian model

1 Manipulability Model

The spatial model of voting considered in [9, 10], and in many other works (see e.g. [11]), gives the new insight to understanding voting procedures. We consider here the new model of manipulation in the spatial model. The manipulability in the IC, IAC models is considered in [4], and the manipulability in one-dimensional Downsian model is evaluated in [2]. For the spatial model, in [12] it is inferred, that least manipulable among the positional methods with any number of alternatives is the Borda rule, while the plurality rule is one of the most manipulable of the positional methods.

In the framework of the two-dimensional Downsian model [6] the profiles are generated, representing a set of preferences of agents. There are m alternatives, ($m = 3, 4, \text{ or } 5$) and n ideal points of agents ($n > 3$) on the rectangle in the planar space. For each agent, her preferences are constructed in the following way. The Euclidean

distance from the ideal point of the particular agent to each of the alternatives is calculated. The alternatives are ordered by proximity to the ideal point, the closest alternative is put on the first place, the next on the second place, etc., the farthest alternative is put on the last place.

After positions of agents and alternatives are determined, agents can manipulate by reporting insincere preferences. Agents can shift their ideal position in such a way that provides them better result of collective choice. There are several different methods to do so. First, the agent can only move its position inside the feasible area, in this case, a rectangle. Second, the agent can move throughout the entire space. And third, an agent can reveal any preference, regardless of whether it is acceptable given the current location of the alternatives. In the second version of manipulation, the agent's insincere position, in the case of going beyond the boundaries of the limited space, may not be permissible from the substantive considerations in terms of interpretations of the axes in space. In the third version of manipulation, the presentation of unacceptable preferences can be unambiguously recognized by other participants in the vote as manipulation. Thus, in the evaluations, the first version of manipulation is considered. Tie-breaking situations and degenerated cases are not considered.

In the two-dimensional Downsian model, for example, for 3 alternatives, for the non-degenerated case, in the entire space, all orderings of the alternatives are feasible. However, if the agent for the purpose of manipulation is forced to move far beyond the boundaries of the considered limited space, usually a rectangle, then such orderings are not considered as the possible ones.

Let us consider the example of manipulation for 3 alternatives, 4 agents, for the Leximin extension of preferences for the Minimal dominant set procedure. Let the alternatives and agents be located on the planar space as shown on the left image of Fig. 1. Note that negative positions of the Y axis are also possible.

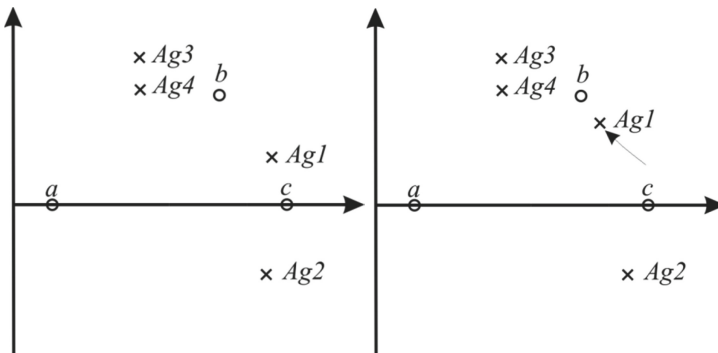


Fig. 1. Example of the manipulation in two-dimensional space.

Let agent Ag1 manipulate, shifting her position towards alternative b as shown on the right image of Fig. 1.

For each agent, the ordering of alternatives is built by proximity to the ideal point of the agent. For example, for Ag1, the ordering is as follows: $c > b > a$.

The profile consisting of the preferences of 4 agents for 3 alternatives is presented in Table 1.

Table 1. The profile for 3 alternatives, 4 agents for sincere positions on Fig. 1

| Ag1 | Ag2 | Ag3 | Ag4 |
|-----|-----|-----|-----|
| c | c | b | b |
| b | b | a | a |
| a | a | c | c |

The majority relation matrix has the following form (see definition of the majority relation below).

Table 2. Majority matrix for the profile in Table 1.

| | a | b | c |
|---|---|---|---|
| a | 0 | 0 | 0 |
| b | 1 | 0 | 0 |
| c | 0 | 0 | 0 |

In this matrix in Table 2 the number “1” means that the alternative b dominates the alternative a. “0” means that the pair of alternatives does not dominate each other.

The choice according to the Minimal dominant set is {a, b, c} (see the definitions of the aggregation procedures below). If the ordering of the agent Ag1 according to sincere preferences was $c > b > a$, then after manipulating the agent reports the following ordering of alternatives $b > c > a$. The profile after manipulation of the agent Ag1 has the form (Table 3):

Table 3. Example profile after manipulation.

| Ag1 | Ag2 | Ag3 | Ag4 |
|----------|-----|-----|-----|
| b | c | b | b |
| c | b | a | a |
| a | a | c | c |

The corresponding majority relation matrix has the form

Table 4. Majority matrix after manipulation.

| | a | b | c |
|---|---|---|---|
| a | 0 | 0 | 0 |
| b | 1 | 0 | 1 |
| c | 0 | 0 | 0 |

In the matrix in Table 4 the number “1” means that the alternative in the row dominates the alternative in the column, that is $b > a$ and $b > c$. The number “0” means that there is no domination between the pair of the alternatives.

The choice made by the Minimal dominant set after manipulation by the agent Ag1 is {b}.

In accordance with the Leximin extension, for the Ag1 agent, the choice resulting from manipulation is more preferable than that with sincere preferences. Thus, the profile shown in Fig. 1. is manipulable for the Minimum Dominant Set procedure, for the Leximin extension.

Manipulability is evaluated for the following aggregation procedures based on the majoritarian relation.

Minimal Dominant Set (MDS), Minimal Undominated Set (MUS), Minimal Weakly Stable Set (MWSS), Fishburn’s rule (F), Uncovered Set I, II (US1, US2), Richelson’s rule (R), Copeland’s rule I, II, III (C1, C2, C3), k-stable set, $k = 1, 2, 3$, (kSS), k-stable set II, $k = 1, 2, 3$, (kSSII).

The description of the procedures, except for the k-stable set rule, is given in [5]. Different versions of the k-stable set were introduced in [3]. In [1], extended preferences were introduced, in which many alternatives are ordered based on the single-valued preferences of the agents. To evaluate the degree of manipulability of aggregation procedures, the Nitzan – Kelly index is used, which was introduced in [7, 8]. The index is defined as

$$NK = \frac{d_0}{d_{total}},$$

where d_0 - number of manipulable profiles, d_{total} - total number of profiles.

2 The Aggregation Procedures

We present definitions of the choice procedures, for which the degree of manipulability is evaluated.

The majority relation μ is defined as follows. For some alternatives x and y from the set of alternatives A it is said, that alternative x dominates the alternative y via majority relation μ ($x\mu y$) if the following condition holds. The number of agents for which x better, than y exceeds the number of agents with the opposite preferences (y is better, than x). The Upper counter set of an alternative x in the relation P is the set $D(x)$ such that $D(x) = \{y \in A \mid yPx\}$. The Lower counter set of x in the relation P is the set $L(x)$ such that $L(x) = \{y \in A \mid xPy\}$.

Now we list the considered rules and discuss the choice of the sample profile. We follow the definitions and notations of [3, 5].

1. (MDS) Minimal dominant set. The set Q is called dominant set if any alternative in Q dominates each alternative outside Q via majority relation μ . The dominant set Q is said to be the minimal if no proper subset is a dominant set. The choice is equal to Q . If there are several of them, then the union is taken.

2. (MUS) Minimal Undominated Set. The set Q is called undominated set if no alternative outside Q dominates any alternative inside Q via majority relation μ . The undominated set Q is a minimal one, if no proper subset is an undominated set. The choice is equal to Q . If the set is not unique, then the union of such sets is taken.
3. (MWSS) Minimal Weakly Stable Set. The alternative x belongs to the weakly stable set Q if the following condition holds. If there exists an alternative $y \in A \setminus Q$ which dominates x via majority relation μ , then there exists some alternative $z \in Q$, which dominates y , i.e., $z\mu y$. A set is called the minimal weakly stable if it does not contain proper weakly stable subsets. The choice consists of the alternatives of the set Q . If the set is not unique, then the union of all such sets is taken.
4. (F) Fishburn's Rule. The binary relation γ is defined as follows $x\gamma y \Leftrightarrow D(x) \subset D(y)$, where $D(x)$ is the upper contour set of the alternative x in the majority relation μ . The choice contains the undominated alternatives via γ .
5. (US1) Uncovered Set I. We construct a binary relation δ in the following way $x\delta y \Leftrightarrow L(x) \supset L(y)$, where $L(x)$ is the lower contour sets of the alternative x in the majority relation μ . The undominated alternatives on δ are chosen.
6. (US2) Uncovered Set II. The alternative x B -dominates some alternative y (xBy) if $x\mu y$ and $D(x) \subseteq D(y)$, where $D(x)$ is the upper contour set of x in μ . The choice consists of B -undominated alternatives.
7. (R) Richelson's Rule. For the majority relation μ upper and lower contour sets ($D(x)$ and $L(x)$) for each alternative $x \in A$ in the relation μ are constructed. Then the binary relation σ is defined as follows:

$$x\sigma y \Leftrightarrow [L(x) \supseteq L(y) \wedge D(x) \subseteq D(y) \wedge ([L(x) \supset L(y)] \vee [D(x) \subset D(y)])]$$

The choice consists of the alternatives undominated via σ .

8. (C1) Copeland's rule I. For each alternative x the value function $u(x)$ is constructed as the difference of the cardinalities of lower and upper contour sets of the alternative in majority relation μ . The choice consists of the alternatives with maximum value of u .
9. (C2) Copeland's rule II. Function $u(x)$ is equal to the cardinality of the lower contour set of alternative x in majority relation μ . The alternatives with maximum of u are chosen.
10. (C3) Copeland's rule III. Function $u(x)$ is equal to the cardinality of the upper contour set of the alternative x in majority relation μ . The alternatives with minimum of u are chosen.
11. (kSS) k -stable set.
The alternative x belongs to the k -stable set for $k = 1, 2, 3$, if one of following two conditions hold:
 - a) the alternative x is undominated by alternatives outside the k -stable set via majority relation μ , or
 - b) if $\exists y \notin kSS : y\mu x$ then $\exists z_1, \dots, z_k, z_1 \in kSS$ and $z_1\mu z_2\mu \dots z_k\mu y$. In general, $k = 1, 2$, or 3 . The k -stable set is minimal, if no proper subsets of it are k -stable. If there are several different k -stable sets, then the union is taken.
12. (kSSII) k -stable set II.
The k -stable set II consists of the alternatives, that are not dominated by alternatives outside the set. And for each alternative outside the set, there is an alternative

in the set, which dominates it via majority relation in not more than k steps. k steps here means that $\exists z_1, \dots, z_k, z_1 \in kSS$ and $z_1 \mu z_2 \mu \dots \mu z_k \mu y$.

3 Extended Preferences

For the case of multiple choice, the sets of alternatives also should be compared as an extension to the agents' preferences. In total, as introduced and discussed in [1], there are 4 extensions for 3 alternatives, 10 extensions for 4 alternatives, and 12 extensions for 5 alternatives. In the following formulae the extensions differ in underlined parts.

For 3 alternatives, the extended preferences look as follows, (agent's single-valued ordering is assumed to be $a > b > c$):

Leximin (3 alts): $\{a\} \succ \{a, b\} \succ \underline{\{b\}} \succ \{a, c\} \succ \{a, b, c\} \succ \{b, c\} \succ \{c\}$

Leximax (3 alts): $\{a\} \succ \{a, b\} \succ \underline{\{a, b, c\}} \succ \{a, c\} \succ \{b\} \succ \{b, c\} \succ \{c\}$

PWorst (3 alts): $\{a\} \succ \{a, b\} \succ \underline{\{b\}} \succ \{a, b, c\} \succ \{a, c\} \succ \{b, c\} \succ \{c\}$

PBest (3 alts): $\{a\} \succ \{a, b\} \succ \underline{\{a, c\}} \succ \{a, b, c\} \succ \{b\} \succ \{b, c\} \succ \{c\}$

For 4 alternatives let us preset one out of ten extensions (single-valued ordering of the agent is assumed to be $a > b > c > d$).

PWorst (4 alts):

$$\begin{aligned} & \{a\} \succ \{a, b\} \succ \{b\} \succ \{a, b, c\} \succ \{a, c\} \succ \{b, c\} \succ \{a, b, c, d\} \\ & \succ \{a, b, d\} \succ \{a, c, d\} \succ \{b, c, d\} \succ \{a, d\} \succ \{b, d\} \succ \{c, d\} \succ \{d\} \end{aligned}$$

For 5 alternatives let us preset only one out of twelve extensions (single-valued ordering of the agent is assumed to be $a > b > c > d > e$).

Rank increasing power Leximin (5 alts):

$$\begin{aligned} & \{a\} \succ \{a, b\} \succ \underline{\{b\}} \succ \{a, c\} \succ \{a, b, c\} \succ \{a, b, d\} \succ \\ & \underline{\{b, c\}} \succ \underline{\{a, d\}} \succ \{a, b, c, d\} \succ \underline{\{a, c, d\}} \succ \underline{\{a, b, e\}} \succ \{a, b, c, e\} \succ \\ & \{c\} \succ \underline{\{b, d\}} \succ \underline{\{a, e\}} \succ \underline{\{b, c, d\}} \succ \underline{\{a, c, e\}} \succ \\ & \{a, b, d, e\} \succ \{a, b, c, d, e\} \succ \\ & \{a, c, d, e\} \succ \underline{\{b, c, e\}} \succ \underline{\{a, d, e\}} \succ \\ & \underline{\{c, d\}} \succ \underline{\{b, e\}} \succ \underline{\{b, c, d, e\}} \succ \{b, d, e\} \succ \\ & \underline{\{d\}} \succ \underline{\{c, e\}} \succ \underline{\{c, d, e\}} \succ \{d, e\} \succ \{e\} \end{aligned}$$

The preference extensions are defined axiomatically. Formal definitions of all extended preferences and detailed discussion is presented in [1].

4 The Scheme of Calculation

The model, in general, is based on the following intuition. Let us place alternatives and agents on the two-dimensional space. Let us also construct a profile of preferences, in which each agent's ordering is induced by the distances of the alternatives to this agent. Then we move the position of each agent individually within some area, not too far from the positions of the alternatives and other agents. The changes in the profiles yield different choices, obtained via the aggregation procedures. If the result of such shift of the position is better for the agent, then the profile is manipulable. The goal is to evaluate the share of such manipulable profiles for each aggregation procedure considered.

The calculation is performed in the following way. On the two-dimensional coordinate space, the rectangle, centered at the point $[0; 0]$, with side X to side Y ratio 1:2 is defined. The restricted rectangle area can be formally defined as set of points with coordinates $0 \leq x \leq 1$ and $-1 \leq y \leq 1$.

There are m alternatives ($m = 3, 4, 5$) and n agents ($n = 4-20$), even number, generated randomly with X coordinates from 0 to 1 and Y from -1 to 1. Without loss of generality we can assume that in any profile there are two alternatives that are farthest from each other, for example, a and e in the case of 5 alternatives, are placed in fixed positions: a in $[0,0]$, and e in $[1,0]$ and all alternatives are sorted alphabetically along the X axe. For each agent, the ordering of sincere preferences is constructed according to the distance from the agent's ideal point to alternatives. For the generated profile, the choice is calculated for each of the considered aggregation procedures.

Individual manipulation by agents is considered. The evaluations are performed in the following way. For each agent individually, on the grid of L steps, the agent's manipulating point with the coordinates (l_x, l_y) for l_x and l_y with a step of $1/L$ is positioned. For the presented results, $L = 100$. The agent's insincere preferences are constructed, as if the manipulating point was the ideal point of the agent. Alternatives are ordered by proximity to this point. For a profile constructed in such a way with insincere preferences of the manipulating agent, the choice is evaluated according to all considered aggregation procedures.

For a manipulating agent, for each choice procedure, the choices obtained by sincere preferences and by insincere preferences are compared using all considered preferences extensions. (The results are presented for Leximin, Leximax, PWorst, and PBest extensions for 3 alternatives, for PWorst extension for 4 alternatives, and for Rank increasing power leximin for 5 alternatives.) If the collective choice obtained on the profile, which consists of insincere preferences of the manipulating agent is better for the manipulating agent than a choice obtained from the sincere preferences, then manipulation is considered to be a successful one for a given profile, a given rule and the type of extended preferences.

For each rule and extended preferences the number of manipulable profiles is calculated, and the Nitzan-Kelly manipulability index is calculated as the ratio of the number of manipulated profiles to the total number of generated profiles.

5 Calculation Results

For even number of agents, the procedures MDS, F, US1, R, C1, C2 have high degree of manipulability for most extensions in the cases for 3, 4, and 5 alternatives.

We present the results of the evaluation of the degree of individual manipulability for 3, 4, and 5 alternatives in two-dimensional space for even number of agents from 4 to 20.

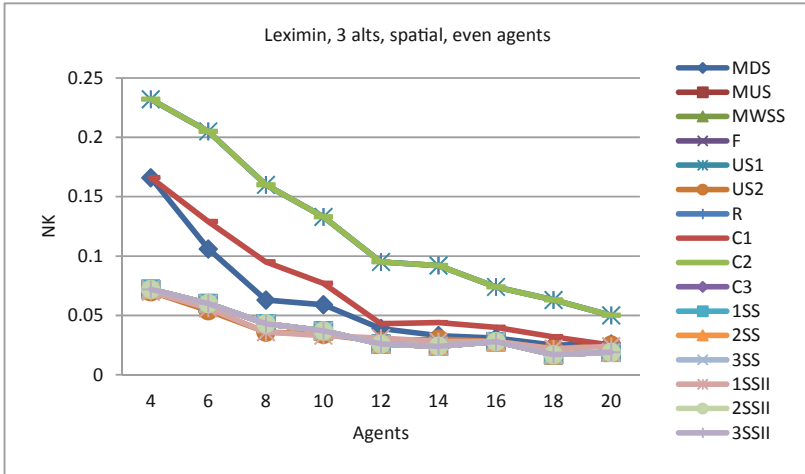


Fig. 2. Manipulability of aggregation procedures for 3 alternatives, Leximin extension, even number of agents.

Table 5. NK index, 3 alts, Leximin.

| | Agents | | | | | | | | |
|-------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| MDS | 0.166 | 0.106 | 0.063 | 0.059 | 0.039 | 0.033 | 0.031 | 0.025 | 0.026 |
| MUS | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| MWSS | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| F | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| US1 | 0.232 | 0.205 | 0.16 | 0.133 | 0.095 | 0.092 | 0.074 | 0.063 | 0.05 |
| US2 | 0.07 | 0.054 | 0.036 | 0.034 | 0.028 | 0.03 | 0.028 | 0.022 | 0.025 |
| R | 0.232 | 0.205 | 0.16 | 0.133 | 0.095 | 0.092 | 0.074 | 0.063 | 0.05 |
| C1 | 0.166 | 0.129 | 0.095 | 0.077 | 0.043 | 0.044 | 0.04 | 0.032 | 0.025 |
| C2 | 0.232 | 0.205 | 0.16 | 0.133 | 0.095 | 0.092 | 0.074 | 0.063 | 0.05 |
| C3 | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| 1SS | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| 2SS | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| 3SS | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| 1SSII | 0.07 | 0.056 | 0.036 | 0.033 | 0.031 | 0.028 | 0.028 | 0.022 | 0.024 |
| 2SSII | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |
| 3SSII | 0.072 | 0.06 | 0.043 | 0.037 | 0.026 | 0.024 | 0.028 | 0.017 | 0.019 |

For example, for 3 alternatives for the Leximin extension, it can be seen from the Fig. 2. and Table 5, that there are 3 groups of procedures with more or less same level of manipulability. The high-manipulable group consists of US1, R, and C2. There are MDS and C1 procedures, which have middle level of manipulability for low number of agents and then their level of manipulability decreases. The third group with small level of manipulability consists of MUS, MWSS, F, US2, C3, 1SS, 2SS, 3SS, and 1SSII, 2SSII, and 3SSII.

Let us present the NK index for the rest of extended preferences, Leximax, PWorst, and PBest (Tables 6, 7 and 8).

Table 6. NK index, 3 alts, Leximax.

| | Agents | | | | | | | | |
|-------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| MDS | 0.114 | 0.122 | 0.11 | 0.094 | 0.081 | 0.062 | 0.05 | 0.051 | 0.037 |
| MUS | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| MWSS | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| F | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| US1 | 0.24 | 0.197 | 0.154 | 0.129 | 0.091 | 0.088 | 0.073 | 0.062 | 0.048 |
| US2 | 0.1 | 0.069 | 0.044 | 0.038 | 0.029 | 0.034 | 0.028 | 0.023 | 0.024 |
| R | 0.24 | 0.197 | 0.154 | 0.129 | 0.091 | 0.088 | 0.073 | 0.062 | 0.048 |
| C1 | 0.19 | 0.144 | 0.098 | 0.082 | 0.046 | 0.05 | 0.047 | 0.041 | 0.03 |
| C2 | 0.24 | 0.197 | 0.154 | 0.129 | 0.091 | 0.088 | 0.073 | 0.062 | 0.048 |
| C3 | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| 1SS | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| 2SS | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| 3SS | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| 1SSII | 0.194 | 0.14 | 0.1 | 0.087 | 0.069 | 0.061 | 0.053 | 0.035 | 0.04 |
| 2SSII | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |
| 3SSII | 0.1 | 0.067 | 0.041 | 0.033 | 0.025 | 0.027 | 0.027 | 0.021 | 0.021 |

Table 7. NK index, 3 alts, PWorst.

| | Agents | | | | | | | | |
|------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| MDS | 0.185 | 0.131 | 0.089 | 0.085 | 0.055 | 0.044 | 0.045 | 0.041 | 0.034 |
| MUS | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| MWSS | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| F | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| US1 | 0.232 | 0.205 | 0.157 | 0.132 | 0.095 | 0.088 | 0.074 | 0.062 | 0.049 |
| US2 | 0.1 | 0.069 | 0.044 | 0.038 | 0.029 | 0.034 | 0.028 | 0.023 | 0.024 |
| R | 0.232 | 0.205 | 0.157 | 0.132 | 0.095 | 0.088 | 0.074 | 0.062 | 0.049 |
| C1 | 0.166 | 0.129 | 0.095 | 0.077 | 0.043 | 0.044 | 0.04 | 0.032 | 0.025 |

(continued)

Table 7. (continued)

| | Agents | | | | | | | | |
|-------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| C2 | 0.232 | 0.205 | 0.157 | 0.132 | 0.095 | 0.088 | 0.074 | 0.062 | 0.049 |
| C3 | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| 1SS | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| 2SS | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| 3SS | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| 1SSII | 0.158 | 0.102 | 0.062 | 0.051 | 0.036 | 0.032 | 0.031 | 0.025 | 0.025 |
| 2SSII | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |
| 3SSII | 0.1 | 0.067 | 0.044 | 0.034 | 0.023 | 0.025 | 0.027 | 0.017 | 0.018 |

Table 8. NK index, 3 alts, PBest.

| | Agents | | | | | | | | |
|-------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| MDS | 0.247 | 0.186 | 0.132 | 0.113 | 0.092 | 0.072 | 0.058 | 0.048 | 0.048 |
| MUS | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| MWSS | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| F | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| US1 | 0.24 | 0.197 | 0.157 | 0.13 | 0.091 | 0.092 | 0.073 | 0.063 | 0.049 |
| US2 | 0.07 | 0.054 | 0.036 | 0.034 | 0.028 | 0.03 | 0.028 | 0.022 | 0.025 |
| R | 0.24 | 0.197 | 0.157 | 0.13 | 0.091 | 0.092 | 0.073 | 0.063 | 0.049 |
| C1 | 0.19 | 0.144 | 0.098 | 0.082 | 0.046 | 0.05 | 0.047 | 0.041 | 0.03 |
| C2 | 0.24 | 0.197 | 0.157 | 0.13 | 0.091 | 0.092 | 0.073 | 0.063 | 0.049 |
| C3 | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| 1SS | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| 2SS | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| 3SS | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| 1SSII | 0.106 | 0.096 | 0.077 | 0.07 | 0.065 | 0.059 | 0.052 | 0.037 | 0.041 |
| 2SSII | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |
| 3SSII | 0.072 | 0.06 | 0.04 | 0.036 | 0.028 | 0.026 | 0.028 | 0.021 | 0.022 |

For 4 alternatives, the procedures MUS, MWSS, F, C3, 1SS, 2SS, 3SS, 2SSII, 3SSII have low manipulability for all extended preference types. For some of the extensions, the manipulability of US2 and 1SSII procedures is close to the minimal one (Table 9, Fig. 3).

Table 9. NK index, 4 alts, PWorst.

| | Agents | | | | | | | | |
|-------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| MDS | 0.232 | 0.164 | 0.107 | 0.089 | 0.053 | 0.041 | 0.036 | 0.034 | 0.032 |
| MUS | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.014 | 0.017 |
| MWSS | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.014 | 0.017 |
| F | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.014 | 0.017 |
| US1 | 0.379 | 0.251 | 0.223 | 0.165 | 0.124 | 0.09 | 0.074 | 0.053 | 0.059 |
| US2 | 0.155 | 0.11 | 0.068 | 0.05 | 0.04 | 0.027 | 0.023 | 0.022 | 0.021 |
| R | 0.379 | 0.253 | 0.223 | 0.165 | 0.124 | 0.091 | 0.074 | 0.053 | 0.06 |
| C1 | 0.271 | 0.165 | 0.124 | 0.098 | 0.072 | 0.046 | 0.042 | 0.031 | 0.032 |
| C2 | 0.353 | 0.239 | 0.205 | 0.154 | 0.113 | 0.082 | 0.066 | 0.049 | 0.055 |
| C3 | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.018 | 0.016 | 0.017 |
| 1SS | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.014 | 0.017 |
| 2SS | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.013 | 0.018 |
| 3SS | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.013 | 0.018 |
| 1SSII | 0.192 | 0.131 | 0.081 | 0.057 | 0.043 | 0.028 | 0.023 | 0.022 | 0.021 |
| 2SSII | 0.153 | 0.09 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.014 | 0.017 |
| 3SSII | 0.153 | 0.089 | 0.059 | 0.041 | 0.029 | 0.021 | 0.017 | 0.014 | 0.017 |

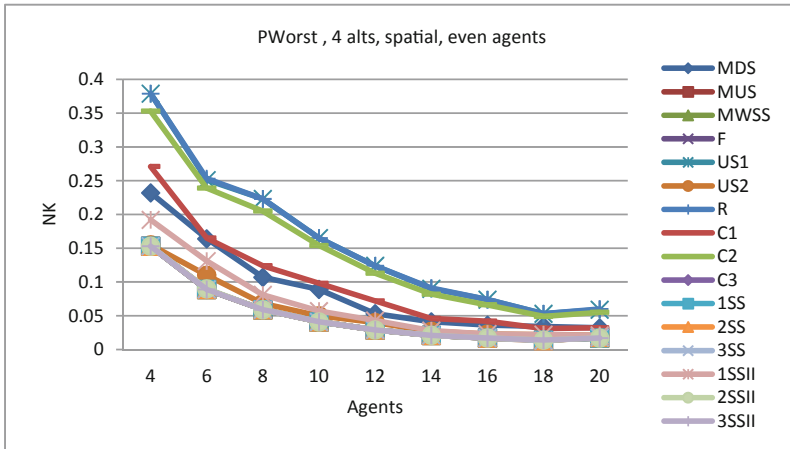


Fig. 3. Manipulability of choice procedures for 4 alternatives, PWorst extension, even number of agents.

For 5 alternatives, the same procedures: MUS, MWSS, F, US2, C3, 1SS, 2SS, 3SS, 1SSII, 2SSII, 3SSII show low level of manipulability (Table 10, Fig. 4).

Table 10. NK index, 5 alts, Rank increasing power leximin.

| | Agents | | | | | | | | |
|-------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| MDS | 0.385 | 0.31 | 0.222 | 0.178 | 0.147 | 0.126 | 0.106 | 0.084 | 0.086 |
| MUS | 0.215 | 0.156 | 0.102 | 0.085 | 0.073 | 0.055 | 0.052 | 0.044 | 0.045 |
| MWSS | 0.215 | 0.156 | 0.102 | 0.085 | 0.073 | 0.055 | 0.051 | 0.044 | 0.045 |
| F | 0.215 | 0.156 | 0.102 | 0.085 | 0.073 | 0.055 | 0.052 | 0.044 | 0.044 |
| US1 | 0.514 | 0.444 | 0.34 | 0.267 | 0.221 | 0.202 | 0.182 | 0.138 | 0.123 |
| US2 | 0.246 | 0.18 | 0.135 | 0.11 | 0.092 | 0.069 | 0.069 | 0.053 | 0.051 |
| R | 0.528 | 0.449 | 0.344 | 0.268 | 0.221 | 0.203 | 0.182 | 0.138 | 0.123 |
| C1 | 0.352 | 0.292 | 0.223 | 0.155 | 0.13 | 0.107 | 0.099 | 0.069 | 0.069 |
| C2 | 0.449 | 0.396 | 0.321 | 0.253 | 0.209 | 0.19 | 0.176 | 0.133 | 0.118 |
| C3 | 0.215 | 0.156 | 0.103 | 0.086 | 0.073 | 0.055 | 0.052 | 0.044 | 0.046 |
| 1SS | 0.215 | 0.156 | 0.102 | 0.085 | 0.073 | 0.055 | 0.051 | 0.044 | 0.045 |
| 2SS | 0.215 | 0.156 | 0.101 | 0.086 | 0.072 | 0.054 | 0.051 | 0.043 | 0.043 |
| 3SS | 0.215 | 0.156 | 0.101 | 0.086 | 0.073 | 0.055 | 0.051 | 0.043 | 0.044 |
| 1SSII | 0.283 | 0.232 | 0.172 | 0.138 | 0.116 | 0.088 | 0.078 | 0.072 | 0.068 |
| 2SSII | 0.217 | 0.157 | 0.102 | 0.087 | 0.073 | 0.056 | 0.051 | 0.044 | 0.044 |
| 3SSII | 0.215 | 0.156 | 0.102 | 0.085 | 0.073 | 0.055 | 0.052 | 0.044 | 0.045 |

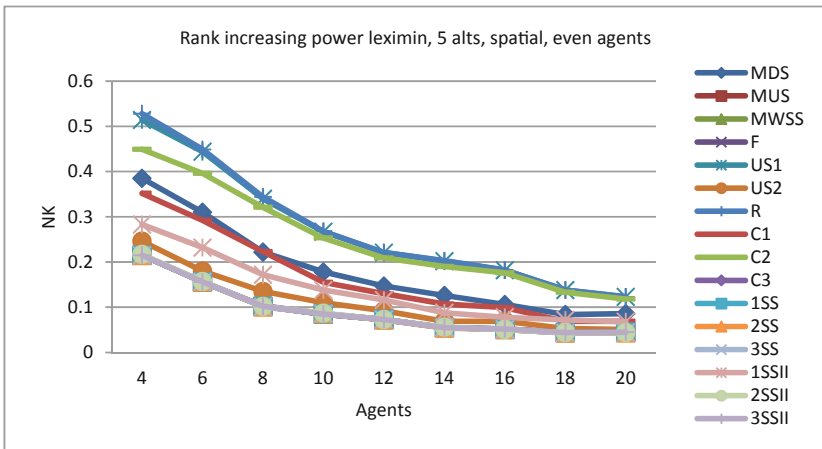


Fig. 4. Manipulability of aggregation procedures for 5 alternatives, Rank increasing power leximin, even number of agents.

6 Conclusion

We have considered a new model of manipulability of main majoritarian procedures in the Downsian model with extended preferences and show that almost all of them in this model have rather low level of manipulability. It is interesting to note that the following procedures have the level of manipulability close to minimum: Minimal Undominated Set, Minimal Weakly Stable Set, Fishburn's rule, Uncovered Set II, k -stable set, and k -stable set II, for $k = 1, 2, 3$.

Next interesting question is how manipulable are different scoring procedures. In the current model the rectangle area of feasible points of alternatives and agents is restricted in y coordinates by $-1 \leq y \leq 1$. It would be interesting to evaluate the sensitivity of the manipulability level depending on the size of the area.

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