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Polyhedral parametrizations of canonical bases & cluster duality

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We establish the relation of Berenstein–Kazhdan's decoration function and Gross–Hacking–Keel–Kontsevich's potential on the open double Bruhat cell in the base affine space $G/\mathcal{N}$ of a simple, simply connected, simply laced algebraic group G. As a byproduct we derive explicit identifications of polyhedral parametrization of canonical bases of the ring of regular functions on $G/\mathcal{N}$ arising from the tropicalizations of the potential and decoration function with the classical string and Lusztig parametrizations. In the appendix we construct maximal green sequences for the open double Bruhat cell in $G/\mathcal{N}$ which is a crucial assumption for Gross–Hacking–Keel–Kontsevich's construction.

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0. Introduction

Let G be a simple, simply connected, simply laced algebraic group over $\mathbb{C}$, $B \subset G$ a Borel subgroup with unipotent radical $\mathcal{N}$.

There is a cluster space $\mathcal{A}$ ([3]) and a dual cluster space $\mathcal{X}$ ([12]) associated to the open double Bruhat cell in the base affine space $\mathrm{G}/\mathcal{N}$. We are interested in the following functions both playing a crucial role in the study of canonical vector space bases of the ring of regular functions $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}})$ on $\mathrm{G}/\mathcal{N}$.

On the one hand, Berenstein–Kazhdan's decoration function f^B defined in [4], a regular function on $\mathcal{A}$. The decoration function is a crucial part of the construction of a decorated geometric crystal and thus intimately connected to the canonical basis $\mathcal{B}_{\mathrm{can}}$ of $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}})$ independently constructed by Kashiwara and Lusztig [28,25].

On the other hand, a remarkable vector space basis $\mathbb{B}$, called theta basis, was recently constructed (up to a natural conjecture, see Remark 5.4) by Gross–Hacking–Keel–Kontsevich [23] using methods in mirror symmetry. An important ingredient in the construction of $\mathbb{B}$ is a regular function W on $\mathcal{X}$ which we call the GHKK-potential.

By identifying a certain p-map in the sense of [22, Chapter 2] we relate the GHKK-potential to the decoration function as follows.

Theorem A. *There exists a regular map* $\mathrm{p} : \mathcal{A} \to \mathcal{X}$ *such that*

$$f^B = W \circ \mathrm{p}\,.$$

The cluster spaces $\mathcal{A}$ and $\mathcal{X}$ are unions of open tori $\mathcal{A} = \bigcup_\Sigma \mathbb{T}_\Sigma$, $\mathcal{X} = \bigcup_\Sigma \widehat{\mathbb{T}}_\Sigma$, which are glued via certain birational transformations, called $\mathcal{A}$- and $\mathcal{X}$-cluster mutations, respectively. The elements Σ in the common index set of the two dual toric systems are called seeds. The families of charts, equip $\mathcal{A}$ and $\mathcal{X}$ with the structure of a positive variety admitting tropicalization as explained in Section 1.4.

The functions f^B and W lead to polyhedral parametrization of $\mathcal{B}_{\mathrm{can}}$ and $\mathbb{B}$, respectively, one for each seed Σ: By [23] the integer polyhedral cone

$$\widehat{\mathcal{C}}_\Sigma = \{x \in [\widehat{\mathbb{T}}_\Sigma]_{\mathrm{trop}} \mid [W|_{\widehat{\mathbb{T}}_\Sigma}]_{\mathrm{trop}}(x) \geq 0\}$$

parametrizes $\mathbb{B}$. By [4] the tropicalization of the decoration function f^B cuts out an integer polyhedral cone

$$\mathcal{C}_\Sigma = \{x \in [\mathbb{T}_\Sigma]_{\mathrm{trop}} \mid [f^B|_{\mathbb{T}_\Sigma}]_{\mathrm{trop}}(x) \geq 0\}$$

which parametrizes Lusztig's canonical basis of the base affine space of the Langlands dual group of G. The explicit constructions in this paper involve the cones $\widehat{\mathcal{C}}_\Sigma = \widehat{\mathcal{C}}_{\mathbf{i}}$ and $\mathcal{C}_\Sigma = \mathcal{C}_{\mathbf{i}}$ attached to seeds $\Sigma = \mathbf{i}$ coming from reduced words $\mathbf{i}$ of w_0.

The construction of the theta basis $\mathbb{B}$ for the ring of regular functions on an $\mathcal{A}$-cluster variety due to [23] relies on certain assumptions. In the course of the proof of

Theorem A we construct in Proposition 5.2 for any frozen vertex of such a cluster space $(\mathcal{A}, \mathcal{X})$ an optimal seed. In [21] Goodearl and Yakimov announced the existence of a maximal green sequence. From this, [3, Proposition 2.6.], [23, Theorem 0.19, Lemma B.7] and Proposition 5.2 the existence of a theta basis for the partial compactification $\mathrm{G}/\mathcal{N}$ of $G^{\mathrm{w}_0,e}$ parametrized by $\widehat{\mathcal{C}}_\Sigma$ follows. For the convenience of the reader we give independently of [21] maximal green sequences for $G^{\mathrm{w}_0,e}$ in the appendix.

Theorem A is deduced by studying the interplay of Gross–Hacking–Keel–Kontsevich's polyhedral parametrization $\widehat{\mathcal{C}}_{\mathbf{i}}$ and the parametrization arising from the tropicalization of the Berenstein–Kazhdan decoration function $\mathcal{C}_{\mathbf{i}}$ with classical polyhedral parametrization of Lusztig's canonical basis obtained by Lusztig and Kashiwara.

Both Lusztig's and Kashiwara's construction yield a family of polyhedral parametrizations, one for each reduced word $\mathbf{i}$ of the longest word w_0 of the Weyl group of G, by the cone of graded $\mathbf{i}$-Lusztig data $\mathrm{gr}\,\mathcal{L}_{\mathbf{i}}$ and the graded string cone $\mathrm{gr}\,\mathcal{S}_{\mathbf{i}}$, respectively. We related $\mathrm{gr}\,\mathcal{L}_{\mathbf{i}}$ and $\mathrm{gr}\,\mathcal{S}_{\mathbf{i}}$ to the functions f^B and W by introducing regular function $\mathfrak{l}_{\mathbf{i}}$ and $\mathfrak{s}_{\mathbf{i}}$ on certain tori $\mathrm{gr}\,\mathbb{L}_{\mathbf{i}}$ and $\mathrm{gr}\,\mathbb{S}_{\mathbf{i}}$, respectively, satisfying

$$\mathrm{gr}\,\mathcal{L}_{\mathbf{i}} = \{x \in [\mathrm{gr}\,\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} \mid [\mathfrak{l}_{\mathbf{i}}]_{\mathrm{trop}}(x) \geq 0\},$$
$$\mathrm{gr}\,\mathcal{S}_{\mathbf{i}} = \{x \in [\mathrm{gr}\,\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \mid [\mathfrak{s}_{\mathbf{i}}]_{\mathrm{trop}}(x) \geq 0\}.$$

We denote the corresponding objects for the Langlands dual group of G by $\mathfrak{l}_{\mathbf{i}}^\vee$, $\mathfrak{s}_{\mathbf{i}}^\vee$, $\mathrm{gr}\,\mathbb{L}_{\mathbf{i}}{}^\vee$, $\mathrm{gr}\,\mathbb{S}_{\mathbf{i}}{}^\vee$, $\mathrm{gr}\,\mathcal{L}_{\mathbf{i}}^\vee$ and $\mathrm{gr}\,\mathcal{S}_{\mathbf{i}}^\vee$.

There are certain toric charts $\mathbb{T}_{\mathbf{i}}$ and $\widehat{\mathbb{T}}_{\mathbf{i}}$ of $\mathcal{A}$ and $\mathcal{X}$, respectively, attached to every reduced word $\mathbf{i}$. Motivated by the *Chamber Ansatz* due to Berenstein–Fomin–Zelevinsky [2] and [5, Equation (4.14)] we introduce explicit torus isomorphisms $\mathrm{gr}\,\mathrm{CA}_{\mathbf{i}} : \mathbb{T}_{\mathbf{i}} \to \mathrm{gr}\,\mathbb{L}_{\mathbf{i}}{}^\vee$, $\mathrm{gr}\,\mathrm{NA}_{\mathbf{i}} : \mathbb{T}_{\mathbf{i}} \to \mathrm{gr}\,\mathbb{S}_{\mathbf{i}}{}^\vee$, $\mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}} : \mathrm{gr}\,\mathbb{S}_{\mathbf{i}} \to \widehat{\mathbb{T}}_{\mathbf{i}}$ and $\mathrm{gr}\,\widehat{\mathrm{NA}}_{\mathbf{i}} : \mathrm{gr}\,\mathbb{L}_{\mathbf{i}} \to \widehat{\mathbb{T}}_{\mathbf{i}}$. The terminology $\mathrm{gr}\,\mathrm{NA}_{\mathbf{i}}$ here stands for *graded Neighbour Ansatz.*

The interplay between the various parametrizations is summarized in the following theorem.

Theorem B *(Theorem 6.5, Theorem 7.5, Lemma 8.1). For every reduced word* $\mathbf{i}$ *we have*

$$\mathfrak{s}_{\mathbf{i}} = W\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}} \circ \mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}}, \qquad \mathfrak{l}_{\mathbf{i}} = W\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}} \circ \mathrm{gr}\,\widehat{\mathrm{NA}}_{\mathbf{i}},$$
$$f^B\big|_{\mathbb{T}_{\mathbf{i}}} = \mathfrak{s}_{\mathbf{i}}^\vee \circ \mathrm{gr}\,\mathrm{NA}_{\mathbf{i}}, \qquad f^B\big|_{\mathbb{T}_{\mathbf{i}}} = \mathfrak{l}_{\mathbf{i}}^\vee \circ \mathrm{gr}\,\mathrm{CA}_{\mathbf{i}}\,.$$

Furthermore, we obtain the following family of commutative diagrams of linear maps indexed by reduced words

$$\begin{array}{ccccc} \mathrm{gr}\,\mathcal{S}_{\mathbf{i}}^\vee & \xleftarrow[\sim]{[\mathrm{gr}\,\mathrm{NA}_{\mathbf{i}}]_{\mathrm{trop}}} & \mathcal{C}_{\mathbf{i}} & \xrightarrow[\sim]{[\mathrm{gr}\,\mathrm{CA}_{\mathbf{i}}]_{\mathrm{trop}}} & \mathrm{gr}\,\mathcal{L}_{\mathbf{i}}^\vee \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{gr}\,\mathcal{S}_{\mathbf{i}} & \xrightarrow[\sim]{[\mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}} & \widehat{\mathcal{C}}_{\mathbf{i}} & \xleftarrow[\sim]{[\mathrm{gr}\,\widehat{\mathrm{NA}}_{\mathbf{i}}]_{\mathrm{trop}}} & \mathrm{gr}\,\mathcal{L}_{\mathbf{i}}\,. \end{array} \tag{1}$$

We obtain Theorem A using Theorem B. By the commutativity of (1) the two candidates for p induced by the maps $\mathbb{T}_{\mathbf{i}} \to \widehat{\mathbb{T}}_{\mathbf{i}}$ obtained by going around the left square and the right square in (1), respectively, coincide.

Another consequence of Theorem B is a lattice isomorphism from the graded string cone to the graded cone of Lusztig's parametrization, recovering a result of Caldero–Marsh–Morier-Genoud.

There are two natural types of inequalities for both $\operatorname{gr}\mathcal{S}_{\mathbf{i}}$ and $\operatorname{gr}\mathcal{L}_{\mathbf{i}}$: One type yields the inequalities for a polyhedral parametrization of a canonical basis of the ring $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ of regular functions on the unipotent radical. We call these inequalities the cone inequalities for the sake of this introduction. The other type of inequalities describes the graded lift of a polyhedral parametrization of $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ to a polyhedral parametrization of a canonical basis of $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}})$, called here grading inequalities.

We show that under Caldero–Marsh–Morier-Genoud's map the cone inequalities of the graded cone of Lusztig's parametrization is mapped to the grading inequalities of the graded string cone and vice versa. We further give an affine unimodular map between the corresponding weight polytopes.

In certain cases, polyhedral parametrizations by a cone of canonical bases of rings of regular functions on a variety X lead to flat degenerations of X to the toric variety defined by the cone. In the case of flag varieties, an overview of many such cases is given in [11].

Theorem B implies that, in the case of the base affine space, the toric fibers appearing in the degeneration construction by Caldero ([7]) and Alexeev–Brion ([1]) also appear in the degenerations constructed by Gross–Hacking–Keel–Kontsevich ([23]). In the special case of $G = \mathrm{SL}_n(\mathbb{C})$ this was proven previously in [6].

Moreover, toric degenerations associated to the graded cone of Lusztig's parametrization and the graded string cone were constructed in [10]. Hence, Theorem B provides further evidence that there should be a natural connection between a subclass of Fang–Fourier–Littelmann's toric degenerations constructed in [10] and Gross–Hacking–Keel–Kontsevich's toric degenerations constructed by cluster duality (see [11, 10.1]).

1. Background and notations

1.1. Simply-laced Lie algebras

Let $\mathfrak{g}$ be simple, simply laced complex Lie algebra of rank n, $I := [n] := \{1, \dots, n\}$, $C = (c_{i,j})_{i,j \in I}$ its Cartan matrix and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. We choose simple coroots $\{h_a\}_{a \in I} \in \mathfrak{h}$ and simple roots $\{\alpha_a\}_{a \in I} \subset \mathfrak{h}^*$ with $\alpha_a(h_b) = c_{a,b}$ and denote by $\Delta^+ \subset \mathfrak{h}^*$ the positive roots associated to the simple roots $\{\alpha_a\}$.

The Weyl group W acts on $\mathfrak{h}^*$ via

$$s_a \mu = \mu - \mu(h_a)\alpha_a \quad a \in I, \mu \in \mathfrak{h}^*.$$

For $a \in I$ we denote by a^* the element of I such that

$$\mathrm{w}_0(\alpha_a) = -\alpha_{a^*} \tag{2}$$

The fundamental weights $\{\omega_a\}_{a\in I} \subset \mathfrak{h}^*$ of $\mathfrak{g}$ are given by $\omega_a(h_j) = \delta_{a,j}$. We denote by $P = \langle \omega_a \mid a \in [n]\rangle_{\mathbb{Z}}$ the weight lattice of $\mathfrak{g}$ and by $P^+ = \langle \omega_a \mid a \in [n]\rangle_{\mathbb{N}} \subset P$ the set of dominant weights.

The Langlands dual Lie algebra ${}^L\mathfrak{g}$ of $\mathfrak{g}$ is the simple, simply laced complex Lie algebra with Cartan matrix C, Cartan subalgebra $\mathfrak{h}^*$, simple roots $\{h_a\}_{a\in I}$, simple co-roots $\{\alpha_a\}_{a\in I}$ and $h_a(\alpha_b) = c_{a,b}$. The fundamental weights of ${}^L\mathfrak{g}$ are $\{\omega_a^\vee\}_{a\in I} \subset \mathfrak{h}$ where $\alpha_a(\omega_b^\vee) = \delta_{a,b}$.

1.2. Weyl groups and reduced words

The Weyl group W of $\mathfrak{g}$ is a Coxeter group generated by the simple reflections s_a $(a \in I)$ with relations

$$\begin{aligned} s_i^2 &= id, \\ s_{i_1}s_{i_2} &= s_{i_2}s_{i_1} \qquad \text{if } c_{i_1,i_2} = 0 \quad \text{(commutation relation)}, \\ s_{i_1}s_{i_2}s_{i_1} &= s_{i_2}s_{i_1}s_{i_2} \qquad \text{if } c_{i_1,i_2} = -1 \quad \text{(braid relation)}. \end{aligned}$$

The group W has a unique longest element w_0 of length $N = \#\Delta^+$. For a reduced expression $s_{i_1}\cdots s_{i_N}$ of w_0 we write $\mathbf{i} := (i_1, \ldots, i_N)$ and call $\mathbf{i}$ a *reduced word* (for w_0). The set of reduced words for w_0 is denoted by $\mathcal{W}(\mathrm{w}_0)$.

We have two operations on the set of reduced words $\mathcal{W}(\mathrm{w}_0)$.

Definition 1.1. A reduced word $\mathbf{j} = (j_1, \ldots, j_N)$ is defined to be obtained from $\mathbf{i} = (i_1, \ldots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ by a 2-*move at position* $k \in [N-1]$ if $i_\ell = j_\ell$ for all $\ell \notin \{k, k+1\}$, $(i_{k+1}, i_k) = (j_k, j_{k+1})$ and $c_{i_k,i_{k+1}} = 0$.

A reduced word $\mathbf{j} = (j_1, \ldots, j_N)$ is defined to be obtained from $\mathbf{i} = (i_1, \ldots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ by a 3-*move at position* $k \in [N-1]$ if $i_\ell = j_\ell$ for all $\ell \notin \{k-1, k, k+1\}$, $j_{k-1} = j_{k+1} = i_k$, $j_k = i_{k-1} = i_{k+1}$ and $c_{i_k,i_{k+1}} = -1$.

We call a total ordering $\leq$ on Δ^+ *convex* if for $\beta_1, \beta_2, \beta_1+\beta_2 \in \Delta^+$ either $\beta_1 \leq \beta_1+\beta_2 \leq \beta_2$ holds or $\beta_2 \leq \beta_1 + \beta_2 \leq \beta_1$. By [35, Theorem p. 662] the set of total convex ordering is in natural bijection with the set of reduced words. Namely, for a reduced word $\mathbf{i} = (i_1, \ldots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ the total ordering

$$\alpha_{i_1} <_{\mathbf{i}} s_{i_1}(\alpha_{i_2}) <_{\mathbf{i}} \ldots <_{\mathbf{i}} s_{i_1}\cdots s_{i_{N-1}}(\alpha_{i_N})$$

on Δ^+ is convex and every convex ordering on Δ^+ arises that way. We write $\Delta^+_{\mathbf{i}} = \{\beta_1, \beta_2, \ldots, \beta_N\}$ for the set of positive roots ordered with respect to the convex ordering $<_{\mathbf{i}}$ and throughout identify $\Delta^+_{\mathbf{i}}$ with $[N]$ via

$$\beta_k \mapsto k. \tag{3}$$

We make use of the following alternative labeling of $\Delta^+_{\mathbf{i}}$ throughout.

Definition 1.2. For $a \in I$ we write $\{\beta_\ell \in \Delta^+_{\mathbf{i}} \mid i_\ell = a\} = \{\beta_{a,1}, \ldots, \beta_{a,m_a}\}$ with $m_a = m_{a,\mathbf{i}} \in \mathbb{N}$ and $\beta_{a,1} <_{\mathbf{i}} \cdots <_{\mathbf{i}} \beta_{a,m_a}$.

1.3. Simply-connected algebraic groups

Let G be the simple, simply-connected, complex algebraic group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{h}$. For $a \in I$, let $\varphi_a : SL_2 \to G$ be the embedding of SL_2 corresponding to the simple root α_a. We embed the Weyl group $W \simeq \mathrm{Norm}_G(T)/T$ of $\mathfrak{g}$ as a set into $\mathrm{Norm}_G(T)$ via

$$\begin{gathered} s_a \mapsto \bar{s}_a := \varphi_a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Norm}_G(T), \\ \overline{xy} = \bar{x}\bar{y} \text{ if } \mathrm{length}(xy) = \mathrm{length}(x) + \mathrm{length}(y). \end{gathered} \tag{4}$$

We denote by $\mathcal{N}$ and $\mathcal{N}^-$ the maximal unipotent subgroups of G generated by $\{\varphi_a \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mid a \in I, t \in \mathbb{C}\}$ and $\{\varphi_a \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) \mid a \in I, t \in \mathbb{C}\}$, respectively, and set $B = T\mathcal{N}$ and $B^- = T\mathcal{N}^-$.

1.4. Tropicalization

We recall the notion of (min-plus) tropicalization from [23, Section 2]. Let $\mathbb{G}_m$ be the multiplicative group. For a torus $\mathbb{T} = \mathbb{G}_m^k$ we denote by $[\mathbb{T}]_{\mathrm{trop}} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) = \mathbb{Z}^k$ its cocharacter lattice. A positive (i.e. subtraction-free) rational map f on $\mathbb{T}$, $f(x) = \frac{\sum_{u \in J} c_u x^u}{\sum_{u \in K} d_u x^u}$ with $c_u, d_u \in \mathbb{R}_+$, gives rise to a piecewise-linear map

$$[f]_{\mathrm{trop}} : [\mathbb{T}]_{\mathrm{trop}} \to [\mathbb{G}_m]_{\mathrm{trop}} = \mathbb{Z}, \quad x \mapsto \min_{u \in J} \langle x, u \rangle - \min_{u \in K} \langle x, u \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{Z}^k$. We call $[f]_{\mathrm{trop}}$ the *tropicalization* of f. More generally, for a positive rational map

$$f = (f_1, \ldots, f_\ell) : \mathbb{G}_m^k \dashrightarrow \mathbb{G}_m^\ell$$

we define its tropicalization as

$$[f]_{\mathrm{trop}} := ([f_1]_{\mathrm{trop}}, \ldots, [f_\ell]_{\mathrm{trop}}) : [\mathbb{G}_m^k]_{\mathrm{trop}} \to [\mathbb{G}_m^\ell]_{\mathrm{trop}}.$$

The function f is called a *geometric lift of* $[f]_{\mathrm{trop}}$. Note that there are several choices of geometric lifts of a piecewise-linear function.

2. Lusztig's parametrization

2.1. Lusztig's parametrization of the canonical basis

We denote by U_q^- the negative part of the quantized enveloping algebra of $\mathfrak{g}$. Let $\mathbf{i} = (i_1, \dots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ be a reduced word and $\{\beta_1, \dots, \beta_N\} = \Delta_{\mathbf{i}}^+$. In [28, Section 1] a PBW-type basis

$$B_{\mathbf{i}} = \left\{ F_{\mathbf{i},\beta_1}^{(x_{\beta_1})} \cdots F_{\mathbf{i},\beta_N}^{(x_{\beta_N})} \mid (x_{\beta_1}, \dots, x_{\beta_N}) \in \mathbb{N}^{\Delta_{\mathbf{i}}^+} \right\},$$

of U_q^- is defined, where

$$F_{\mathbf{i},\beta_j} = T_{i_1} \cdots T_{i_{j-1}} F_j$$

is given via the braid group action T_i defined in [29, Section 1.3], $X^{(m)}$ is the q-divided power defined by $X^{(m)} := \frac{X^m}{[m][m-1]\cdots[2]}$ and $[m] := q^{m-1} + q^{m-2} + \dots + q^{-m+1}$.

By [28, Proposition 1.1.3] $B_{\mathbf{i}}$ is a basis of U_q^-. The $\mathbb{Z}[q]$-lattice $\mathfrak{L}$, defined as the span of $B_{\mathbf{i}}$, is independent of the choice of reduced expression $\mathbf{i}$, as is the induced basis $B := \pi(B_{\mathbf{i}})$ of $\mathfrak{L}/q\mathfrak{L}$, where $\pi : \mathfrak{L} \to \mathfrak{L}/q\mathfrak{L}$ is the canonical projection. There exists a unique basis $\mathbf{B}$ of $\mathfrak{L}$ whose image under π is B and which is stable under the $\mathbb{Q}$-algebra automorphism preserving the generators of U_q^- and sending q to q^{-1}. We call $\mathbf{B}$ the *canonical basis* of U_q^-.

Definition 2.1. For $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ and $x = (x_1, \dots, x_N) \in \mathbb{N}^N$, we denote the element $F_{\mathbf{i},\beta_1}^{(x_1)} \cdots F_{\mathbf{i},\beta_N}^{(x_N)}$ by F^x and call x its $\mathbf{i}$-*Lusztig datum*. Using identification (3) we write

$$\mathcal{L}_{\mathbf{i}} = \mathbb{N}^{\Delta_{\mathbf{i}}^+} = \mathbb{N}^N$$

for the cone of all $\mathbf{i}$-Lusztig data. We call $\mathcal{L}_{\mathbf{i}}$ the *cone of Lusztig's parametrization* of the canonical basis.

Lusztig's canonical basis has favorable properties. In particular, it projects to a basis of every irreducible finite dimensional U_q^--representation. By specializing $q = 1$ one obtains a canonical basis for every irreducible finite-dimensional G-representation V_λ. From this we obtain a canonical basis of the ring of regular functions $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}}) \simeq \bigoplus_{\lambda \in P^+} V_\lambda$, which by Lemma 2.8 is parametrized by a graded version of $\mathbb{N}^{\Delta_{\mathbf{i}}^+}$ as defined in Section 2.3.

2.2. Transition maps and geometric lifting

Using the identification (3) we associate to the cone $\mathcal{L}_{\mathbf{i}}$ of Lusztig's parametrization the torus

$$\mathbb{L}_{\mathbf{i}} = \mathbb{G}_m^{\Delta_{\mathbf{i}}^+} = \mathbb{G}_m^N.$$

Following [30, 42.2.6] we introduce transition maps.

Definition 2.2. We specify $\Phi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{L}_{\mathbf{i}} \dashrightarrow \mathbb{L}_{\mathbf{j}}$ as follows. If $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ is obtained from $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ by a 3-move at position k then we set for $x = (x_1, \ldots, x_N)$

$$\Phi_{\mathbf{j}}^{\mathbf{i}} x = \left(x_1, \ldots, x_{k-2}, \frac{x_k x_{k+1}}{x_{k-1} + x_{k+1}}, x_{k-1} + x_{k+1}, \frac{x_{k-1} x_k}{x_{k-1} + x_{k+1}}, x_{k+2}, \ldots, x_N\right).$$

If $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ is obtained from $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ by a 2-move at position k we set

$$\Phi_{\mathbf{j}}^{\mathbf{i}} x = (x_1, \ldots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \ldots, x_N).$$

For arbitrary $\mathbf{i}, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ we define $\Phi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{L}_{\mathbf{i}} \dashrightarrow \mathbb{L}_{\mathbf{j}}$ as the composition of the transition maps corresponding to a sequence of 2- and 3-moves transforming $\mathbf{i}$ into $\mathbf{j}$.

Using Definition 1.2 and the identification (3) we define for $a \in I$ and $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ the positive regular map $\mathfrak{l}_{\mathbf{i},\mathrm{a}}$ on $\mathbb{L}_{\mathbf{i}}$ by

$$\mathfrak{l}_{\mathbf{i},\mathrm{a}}(x) = \sum_{r=1}^{m_a} x_{a,r}.$$

Recalling from Section 1.4 that $[\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} = \mathbb{Z}^N$ we obtain that Lusztig's parametrizations $\mathcal{L}_{\mathbf{i}} \subset \mathbb{Z}^N$ are cut out by $[\mathfrak{l}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}$:

Lemma 2.3. *For reduced words* $\mathbf{i}, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ *we have:*

(1) $\mathfrak{l}_{\mathbf{j},\mathrm{a}} = \mathfrak{l}_{\mathbf{i},\mathrm{a}} \circ \Phi_{\mathbf{i}}^{\mathbf{j}}$.
(2) $\mathcal{L}_{\mathbf{i}} = \{x \in [\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} \mid \forall a \in I : [\mathfrak{l}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}(x) \geq 0\}$.

Proof. Statement (1) is a straightforward computation and Statement (2) follows directly from the definition. □

We emphasize that Lemma 2.3 is simply a reformulation, adapted to our setup, of well-known facts about Lusztig's parametrizations obtained in [30,5].

2.3. Lusztig's graded parametrization

In this section we provide a geometric lift of the defining inequalities of the graded cone of Lusztig's parametrization of the canonical basis of the ring of regular functions $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}}) = \oplus V_\lambda$. For this we extend the functions $\mathfrak{l}_{\mathbf{i},\mathrm{a}}$ from $\mathbb{L}_{\mathbf{i}}$ to

$$\mathrm{gr}\,\mathbb{L}_{\mathbf{i}} := \mathbb{G}_m^I \times \mathbb{L}_{\mathbf{i}}$$

via the canonical projection $\mathrm{gr}\,\mathbb{L}_{\mathbf{i}} \twoheadrightarrow \mathbb{L}_{\mathbf{i}}$ and additionally introduce

Definition 2.4. Using (2) we denote by $\{\mathfrak{l}_{\mathbf{i},\text{-a}}\}_{a\in[n]}$ the positive rational functions on $\operatorname{gr}\mathbb{L}_{\mathbf{i}}$ satisfying:

(1) For $(\lambda, x) \in \operatorname{gr}\mathbb{L}_{\mathbf{i}}$ one has $\mathfrak{l}_{\mathbf{i},-i_N}(\lambda, x) = \lambda_{i_N^*} x_N{}^{-1}$.
(2) For $\mathbf{i},\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ one has $\mathfrak{l}_{\mathbf{j},\text{-a}} = \mathfrak{l}_{\mathbf{i},\text{-a}} \circ (\mathrm{id} \times \Phi_{\mathbf{i}}^{\mathbf{j}})$ with $\Phi_{\mathbf{i}}^{\mathbf{j}}$ as in (2.2).

Example 2.5. For $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1,2,1)$ we have (note that $i_N = i_3 = 1,\ i_N^* = 2$)

$$\mathfrak{l}_{\mathbf{i},-1}(\lambda_1, \lambda_2, x_1, x_2, x_3) = \lambda_2 x_3^{-1}.$$

For $\mathbf{j} = (2,1,2)$ we have

$$\mathfrak{l}_{\mathbf{i},-2}(\lambda_1, \lambda_2, x_1, x_2, x_3) = \mathfrak{l}_{\mathbf{j},-2} \circ (\mathrm{id} \times \Phi_{\mathbf{j}}^{\mathbf{i}})(\lambda_1, \lambda_2, x_1, x_2, x_3) = \lambda_1 \frac{x_1 x_2}{x_1 + x_3}.$$

We also introduce the analogue of $\mathfrak{l}_{\mathbf{i},\mathrm{a}}$ for ${}^{\mathrm{L}}\mathfrak{g}$ as follows:

Definition 2.6. For $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ and $a \in I$ we set $\check{\mathfrak{l}}_{\mathbf{i},\mathrm{a}} := \mathfrak{l}_{\mathbf{i},\mathrm{a}}$ and define the positive rational functions $\{\check{\mathfrak{l}}_{\mathbf{i},\text{-a}}\}_{a\in[n]}$ on $\operatorname{gr}\mathbb{L}_{\mathbf{i}}$ by requiring:

(1) For $(\lambda, x) \in \operatorname{gr}\mathbb{L}_{\mathbf{i}}$ one has $\check{\mathfrak{l}}_{\mathbf{i},-i_N}(\lambda, x) = \prod_{b\in I} \lambda_{b^*}^{c_{i_N,b}} x_N{}^{-1}$.
(2) For $\mathbf{i},\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ one has $\check{\mathfrak{l}}_{\mathbf{j},\text{-a}} = \check{\mathfrak{l}}_{\mathbf{i},\text{-a}} \circ (\mathrm{id} \times \Phi_{\mathbf{i}}^{\mathbf{j}})$ with $\Phi_{\mathbf{i}}^{\mathbf{j}}$ as in (2.2).

Remark 2.7. In Corollary 6.6 we show that $\mathfrak{l}_{\mathbf{i},\text{-a}}$ and $\check{\mathfrak{l}}_{\mathbf{i},\text{-a}}$ are regular.

The functions $[\mathfrak{l}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}$ cut out *Lusztig's graded parametrization*

$$\operatorname{gr}\mathcal{L}_{\mathbf{i}} = \{(\lambda, x) \in [\operatorname{gr}\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} \mid \forall a \in -[n] \cup [n] : [\mathfrak{l}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}(\lambda, x) \geq 0\}. \tag{5}$$

Similarly, we define Lusztig's graded parametrization associated to ${}^{\mathrm{L}}\mathfrak{g}$ as

$$\operatorname{gr}\mathcal{L}_{\mathbf{i}}^{\vee} := \left\{(\lambda, x) \in [\operatorname{gr}\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} \,\middle|\, \forall a \in -[n] \cup [n] : [\check{\mathfrak{l}}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}(\lambda, x) \geq 0\right\}.$$

We have:

Proposition 2.8.

(1) *For* $\mathbf{i},\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ *one has*

$$\operatorname{gr}\mathcal{L}_{\mathbf{j}} = [id \times \Phi_{\mathbf{j}}^{\mathbf{i}}]_{\mathrm{trop}} \operatorname{gr}\mathcal{L}_{\mathbf{i}}.$$

(2) *For* $\lambda \in \mathbb{N}^I$ *and* $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ *the set*

$$\mathcal{L}_{\mathbf{i}}(\lambda) := \{x \in [\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} \mid (\lambda, x) \in \operatorname{gr}\mathcal{L}_{\mathbf{i}}\}$$

parametrizes Lusztig's canonical basis of the irreducible representation $V(\sum_{a\in I}\lambda_a\omega_a)$.

Before giving the proof, lets us recall that by [28] (see also [5, Proposition 3.6. (i)]) $\mathcal{L}_{\mathbf{i}}$ has a crystal structure isomorphic to $B(\infty)$ in the sense of [25]. We denote the Kashiwara involution defined in [25, Section 8.3] by $*$ and set

$$\varepsilon_a^*(x) := \varepsilon_a(x^*) := \max_{k\in\mathbb{N}}\{\tilde{e}_a^k x^* \in \mathcal{L}_{\mathbf{i}}\},$$

where $\tilde{e}_a$ is the Kashiwara crystal operator ([25, Section 7.2] corresponding to the simple root α_a. Since $\tilde{e}_a$ is defined on the canonical basis $\mathbf{B}$ (see [30,5]) we have for $\mathbf{i},\mathbf{j}\in\mathcal{W}(\mathrm{w}_0)$ and $x\in\mathcal{L}_{\mathbf{i}}$

$$\varepsilon_a\left(\left[\Phi_{\mathbf{j}}^{\mathbf{i}}\right]_{\mathrm{trop}} x\right) = \varepsilon_a(x). \tag{6}$$

Furthermore, by [5, Proposition 3.3 (ii)] we have for $x\in\mathcal{L}_{\mathbf{i}}$

$$\varepsilon_{i_1}(x_1,\ldots,x_N) = x_1. \tag{7}$$

Proof of Proposition 2.8. Statement (1) is a direct consequence of Lemma 2.3, the definition of the cone of Lusztig's graded parametrization given in (5) and Definition 2.4.

For statement (2) note that by [25, Proposition 8.2], Lusztig's canonical basis of the irreducible representation $V(\sum_{a\in I}\lambda_a\omega_a)$ is parametrized by $\{x\in\mathcal{L}_{\mathbf{i}} \mid \forall a\in I : \varepsilon_a^*(x)\le\lambda_a\}$. It thus suffices to show that for $a\in I$

$$\varepsilon_{a^*}^*(x)\le\lambda_{a^*} \Leftrightarrow \left[\mathfrak{l}_{\mathbf{i},\text{-a}}\right]_{\mathrm{trop}}(x)\ge 0. \tag{8}$$

For $\mathbf{i}=(i_1,\ldots,i_N)\in\mathcal{W}(\mathrm{w}_0)$, we define $\mathbf{i}^{\star\mathrm{op}} := (i_N^*,\ldots,i_1^*)$. By [5, Proposition 3.3 (iii)] (see also [28]) we have for $x=(x_1,\ldots,x_N)\in\mathcal{L}_{\mathbf{i}}$ that

$$x^* = \left[\Phi_{\mathbf{i}}^{\mathbf{i}^{\star\mathrm{op}}}\right]_{\mathrm{trop}}(x_N,\ldots,x_1). \tag{9}$$

By (6), (7) and (9) we obtain

$$\varepsilon_{i_N^*}^*(x) = \varepsilon_{i_N^*}(x^*) = \varepsilon_{i_N^*}\left(\left[\Phi_{\mathbf{i}^{\star\mathrm{op}}}^{\mathbf{i}}\right]_{\mathrm{trop}} x^*\right) = \varepsilon_{i_N^*}(x_N,\ldots x_1) = x_N. \tag{10}$$

From (10) and Definition 2.4 we deduce (8). □

Remark 2.9. In general there is no closed explicit description of $[\mathfrak{l}_{\mathbf{i},\text{-a}}]_{\mathrm{trop}}$. For arbitrary reduced words in type A the explicit form of $[\mathfrak{l}_{\mathbf{i},\text{-a}}]_{\mathrm{trop}}$ is obtained in [17, Theorem 2.16] by combinatorial means. For two special classes of reduced words it is obtained in [36] and in [37]: In [36, Proposition 7.4] reduced words adapted to the Dynkin quiver Q of $\mathfrak{g}$ with vertex set I satisfying a certain homological condition are treated. By [38, Corollary 3.23] these are precisely the reduced words $\mathbf{i}$ adapted to Q such that ω_a spans a minuscule

$\mathfrak{g}$-representation for every sink $a \in I$. For a different class of reduced words satisfying a combinatorial condition called "simply-braided for $a \in I$" (see [37, Definition 4.1]), the function $[\mathfrak{l}_{\mathbf{i},\text{-a}}]_{\text{trop}}$ can explicitly be obtained from the "bracketing rules" in [37, Theorem 4.5].

3. String parametrization

3.1. String parametrization of the canonical basis

Let $B(\infty)$ be the crystal of U_q^- in the sense of [25]. Recall that the *string parametrization* of the canonical basis corresponding to the reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ is given by the set of $\mathbf{i}$-string data of the elements in $B(\infty)$. Here the $\mathbf{i}$-*string datum* $\mathrm{str}_{\mathbf{i}}(b) \in \mathbb{N}^N$ of $b \in B(\infty)$ is determined inductively by

$$\begin{aligned} x_1 &= \max\{k \in \mathbb{N} \mid \tilde{e}_{i_1}^k b \in B(\infty)\}, \\ x_2 &= \max\{k \in \mathbb{N} \mid \tilde{e}_{i_2}^k \tilde{e}_{i_1}^{x_1} b \in B(\infty)\}, \\ &\vdots \\ x_N &= \max\{k \in \mathbb{N} \mid \tilde{e}_{i_N}^k \tilde{e}_{i_{N-1}}^{x_{N-1}} \cdots \tilde{e}_{i_1}^{x_1} b \in B(\infty)\}. \end{aligned}$$

Following [5,27] we call

$$\mathcal{S}_{\mathbf{i}} := \{\mathrm{str}_{\mathbf{i}}(b) \mid b \in B(\infty)\} \subset \mathbb{N}^N$$

the *string cone associated to* $\mathbf{i}$.

3.2. Transition maps and geometric lifting

Using the identification (3) we associate to the string cone $\mathcal{S}_{\mathbf{i}}$ the torus

$$\mathbb{S}_{\mathbf{i}} = \mathbb{G}_m^{\Delta_{\mathbf{i}}^+} = \mathbb{G}_m^N.$$

Following [5] we further introduce positive rational functions $\Psi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{S}_{\mathbf{i}} \dashrightarrow \mathbb{S}_{\mathbf{j}}$ such that the tropicalization $[\Psi_{\mathbf{j}}^{\mathbf{i}}]_{\text{trop}}$ gives the transition map between the string cones associated to reduced words $\mathbf{i}, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$:

Definition 3.1. We specify $\Psi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{S}_{\mathbf{i}} \dashrightarrow \mathbb{S}_{\mathbf{j}}$ as follows. If $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ is obtained from $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ by a 3-move at position k then we set for $x = (x_1, \ldots, x_N)$

$$\Psi_{\mathbf{j}}^{\mathbf{i}} x = \left(x_1, \ldots, x_{k-2}, \frac{x_k x_{k+1}}{x_{k-1}x_{k+1} + x_k}, x_{k-1}x_{k+1}, \frac{x_{k+1}x_{k-1} + x_k}{x_{k+1}}, x_{k+2}, \ldots, x_N\right).$$

If $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ is obtained from $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ by a 2-move at position k we set

$$\Psi_{\mathbf{j}}^{\mathbf{i}} x = (x_1, \ldots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \ldots, x_N)\,.$$

For arbitrary $\mathbf{i}, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ we define $\Psi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{S}_{\mathbf{i}} \dashrightarrow \mathbb{S}_{\mathbf{j}}$ as the composition of the transition maps corresponding to a sequence of 2- and 3-moves transforming $\mathbf{i}$ into $\mathbf{j}$.

Recall that $[\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} = \mathbb{Z}^N$. By [5,27], we have on $B(\infty)$

$$\mathrm{str}_{\mathbf{j}} = [\Psi_{\mathbf{j}}^{\mathbf{i}}]_{\mathrm{trop}} \circ \mathrm{str}_{\mathbf{i}}\,.$$

In the remainder of this subsection we introduce certain positive functions $\mathfrak{s}_{\mathbf{i},\mathrm{a}}$ on $\mathbb{S}_{\mathbf{i}}$ and show that the string cone $\mathcal{S}_{\mathbf{i}} \subset [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} = \mathbb{Z}^N$ is cut out by the functions $[\mathfrak{s}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}$.

Definition 3.2. We denote by $\{\mathfrak{s}_{\mathbf{i},\mathrm{a}}\}_{a\in I}$ the positive rational functions on $\mathbb{S}_{\mathbf{i}}$ satisfying:

(1) For $x \in \mathbb{S}_{\mathbf{i}}$ one has $\mathfrak{s}_{\mathbf{i},i_N}(x) = x_N$.
(2) For $\mathbf{i}, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ one has $\mathfrak{s}_{\mathbf{j},\mathrm{a}} = \mathfrak{s}_{\mathbf{i},\mathrm{a}} \circ \Psi_{\mathbf{i}}^{\mathbf{j}}$ with $\Psi_{\mathbf{i}}^{\mathbf{j}}$ as in (2.2).

Remark 3.3. We show in Corollary 7.6 that $\mathfrak{s}_{\mathbf{i},\mathrm{a}}$ is regular.

Remark 3.4. By Theorem 6.5 and Theorem 7.5 the function $\mathfrak{s}_{\mathbf{i},\mathrm{a}}$ is closely related to the function $\mathfrak{l}_{\mathbf{i},-\mathrm{a}}$ given in Definition 2.4.

Proposition 3.5. *For* $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ *we have*

$$\mathcal{S}_{\mathbf{i}} = \{x \in [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \,|\, [\mathfrak{s}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}(x) \geq 0 \text{ for all } a \in I\}\,. \tag{11}$$

Remark 3.6. In general there is no closed explicit description of the function $[\mathfrak{s}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}$. Explicit inequalities for the string cone $\mathcal{S}_{\mathbf{i}}$ are obtained in [27] for a special class of reduced words and in [20] for all reduced words in type A (also in [5] for arbitrary reduced words but in a less explicit form). In [17] we show that the functions $[\mathfrak{s}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}$ recover the string cone inequalities from [20].

Before proving Proposition 3.5 we recall from [24,34] that $[\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}}$ has the structure of a free crystal in the sense of [9] given as follows. For $x = (x_1, \ldots, x_N) \in \mathbb{Z}^N = [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}}$, $k \in [N]$ and $a \in I$ we set

$$\begin{aligned} \nu^k(x) &:= x_k + \sum_{\ell=k+1}^{N} c_{k,\ell}\, x_\ell \in \mathbb{Z}, \\ \varepsilon_a^*(x) &:= \max\{\nu^k(x) \mid k \in [N],\, i_k = a\} \in \mathbb{Z}, \\ f_a^*(x) &:= (x_\ell + \delta_{\ell,k(x)})_{\ell\in[N]} \in [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}}, \end{aligned} \tag{12}$$

where $k(x) \in [N]$ is the smallest k with $i_k = a$ and $\nu^k(x) = \varepsilon_a^*(x)$. The maps f_a^* and ε_a^* satisfy

$$\varepsilon_a^* = \varepsilon_a^* \circ [\Psi_{\mathbf{j}}^{\mathbf{i}}]_{\mathrm{trop}}, \tag{13}$$

$$[\Psi_{\mathbf{j}}^{\mathbf{i}}]_{\mathrm{trop}} \circ f_a^* = f_a^* \circ [\Psi_{\mathbf{j}}^{\mathbf{i}}]_{\mathrm{trop}}. \tag{14}$$

By [34, Theorem 2.5 and its proof] we have for $a \in I$

$$f_a^* \, \mathcal{S}_{\mathbf{i}} \subset \mathcal{S}_{\mathbf{i}} \,. \tag{15}$$

Lemma 3.7. *Denoting the set on the right hand side of (11) by $\mathcal{P}_{\mathbf{i}}$, we have*

(1) $\mathcal{P}_{\mathbf{i}} = \{x \in [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \mid \forall k \in [N], \forall \, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0) \, : \, (\Psi_{\mathbf{j}}^{\mathbf{i}}(x))_k \geq 0\}$,
(2) $(f_a^*)^{-1}\{x \in \mathcal{P}_{\mathbf{i}} \mid \varepsilon_a^*(x) > 0\} \subset \mathcal{P}_{\mathbf{i}}$,
(3) *for all* $x \in \mathcal{P}_{\mathbf{i}}$ *and* $a \in I \, : \, \varepsilon_a^*(x) \geq 0$,
(4) *for all* $x \in \mathcal{P}_{\mathbf{i}} \, : \, (\forall a \in I \, : \, \varepsilon_a^*(x) = 0) \iff x = (0, 0, \ldots, 0)$.

Proof. The proof of claim (1) is along the lines of [20, Proof of Theorem 5.4] adapted to our setup. Note that $\mathcal{P}_{\mathbf{i}}$ consists of all $x \in [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}}$ such that $(\Psi_{\mathbf{j}}^{\mathbf{i}}(x))_N \geq 0$ for all $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$. We show that any such x satisfies $(\Psi_{\mathbf{j}}^{\mathbf{i}}(x))_k \geq 0$ for all $k \in [N]$ and for all $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$.

Suppose this is not the case and let

$$k = \max\left\{\ell \in [N] \mid \exists \, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0) : (\Psi_{\mathbf{j}}^{\mathbf{i}}(x))_\ell < 0\right\}.$$

We choose $\mathbf{j}$ with $(\Psi_{\mathbf{j}}^{\mathbf{i}}(x))_k < 0$ and write $x' = \Psi_{\mathbf{j}}^{\mathbf{i}}(x)$. Up to 2-moves and 3-moves not affecting x'_k there exists a reduced word $\mathbf{j}' \in \mathcal{W}(\mathrm{w}_0)$ which is obtained from $\mathbf{j}$ by a 3-move at position j with $k \in \{j-1, j\}$. Since, by our assumption on k the inequality $x'_{j+1} \geq 0$ holds, we have by Definition 3.1

$$\left(\Psi_{\mathbf{j}'}^{\mathbf{i}}(x)\right)_{j+1} = \left(\Psi_{\mathbf{j}'}^{\mathbf{j}}(x')\right)_{j+1} = \min\left\{x'_{j-1}, x'_j - x'_{j+1}\right\} < 0$$

contradicting the maximality of k.

Let us now prove claim (2). Assume for $x \in \mathcal{P}_{\mathbf{i}}$ that we can find $a \in I$ such that $\varepsilon_a^*(x) > 0$ and $(f_a^*)^{-1}(x) \notin \mathcal{P}_{\mathbf{i}}$. In other words there exists $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ such that $(\Psi_{\mathbf{j}}^{\mathbf{i}}(f_a^*)^{-1}x))_N < 0$. Using (14) we see that $((f_a^*)^{-1} \, \Psi_{\mathbf{j}}^{\mathbf{i}} \, x)_N < 0$. Thus $a = i_N$, $(\Psi_{\mathbf{j}}^{\mathbf{i}} \, x)_N = 0$ and $0 = \varepsilon_a^*(\Psi_{\mathbf{j}}^{\mathbf{i}} \, x) = \varepsilon_a^*(x)$ where the last equality uses (13). This contradicts our assumption $\varepsilon_a^*(x) > 0$.

For claim (3) note that the definition of $\mathcal{P}_{\mathbf{i}}$ implies, together with (13), $\varepsilon_a^*(x) \geq 0$.

Next we assume in contradiction to claim (4) that there exists $x \in \mathcal{P}_{\mathbf{i}}$ with $\varepsilon_a^*(x) = 0$ for all $a \in I$ and $x_k \neq 0$. We further assume that k is maximally chosen with this property. By claim (1) we have $\mathcal{P}_{\mathbf{i}} \subset \mathbb{Z}_{\geq 0}^N$ and hence $x_k > 0$. We thus obtain the contradiction $\varepsilon_{i_k}(x) \geq \nu^k(x) = x_k > 0$. □

Proof of Proposition 3.5. Denoting the set on the right hand side of (11) by $\mathcal{P}_{\mathbf{i}}$ we have $\mathcal{S}_{\mathbf{i}} \subset \mathcal{P}_{\mathbf{i}}$ by definition.

Let $x \in \mathcal{P}_{\mathbf{i}} \backslash \mathcal{S}_{\mathbf{i}}$ minimize $\sum_{k \in [N]} x_k$ on $\mathcal{P}_{\mathbf{i}} \backslash \mathcal{S}_{\mathbf{i}}$. We show

$$\forall a \in I : \varepsilon_a^*(x) = 0 \tag{16}$$

as follows. If there exists an $a \in I$ with $\varepsilon_a^*(x) > 0$ we obtain $y := (f_a^*)^{-1}(x) \in \mathcal{P}_{\mathbf{i}}$ by Lemma 3.7(2). Since $\sum_{k \in [N]} y_k = \sum_{k \in [N]} x_k - 1$ we conclude from the minimality assumption on x that $y \in \mathcal{S}_{\mathbf{i}}$. Using (15) we obtain the contradiction $x = f_a^*(y) \in \mathcal{S}_{\mathbf{i}}$. Thus (16) holds. But now Lemma 3.7(4) tells us that $x = (0, 0, \ldots, 0) \in \mathcal{S}_{\mathbf{i}}$ contradicting $x \in \mathcal{P}_{\mathbf{i}} \backslash \mathcal{S}_{\mathbf{i}}$. □

Remark 3.8. The reason we use for $a \in I$ the notation ε_a^*, f_a^* is that we may interpret this crystal structure on $\mathbb{S}_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$) as a realization of the $*$-crystal structure on $B(\infty)$, where $*$ denotes the Kashiwara involution [25, Section 8.3]. By doing this we get an identification for the string polytope $\mathbb{S}_{\mathbf{i}}(\lambda)$ ($\lambda \in P^+$) defined in (46) as

$$B(\lambda) = \{x \in B(\infty) \mid \varepsilon_a(x) \leq \langle \alpha_a, \lambda \rangle \ \ \forall a \in I\}$$

in the spirit of [25] (see also Remark 3.10). This is addressed in more detail in [18,19].

Remark 3.9. In the special case that $\mathfrak{g}$ is of type A, a proof of the equality $\mathcal{S}_{\mathbf{i}} = \mathcal{P}_{\mathbf{i}}$ was obtained in [20] using explicit defining inequalities for $\mathcal{S}_{\mathbf{i}}$ derived in [20].

3.3. Graded string cones

Following [27], we define the *graded string cone*

$$\operatorname{gr} \mathcal{S}_{\mathbf{i}} := \left\{ (\lambda, x) \in \mathbb{Z}^I \times \mathcal{S}_{\mathbf{i}} \,\middle|\, \lambda_a \geq \max_{i_k = a} \left\{ x_k + \sum_{\ell = k+1}^{N} c_{i_\ell, i_k} x_\ell \right\} \text{ for all } a \in I \right\},$$

which parametrizes a basis of $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}})$ by [27, Proposition 1.5].

Remark 3.10. By [25,34] the string cone $\mathcal{S}_{\mathbf{i}}$ has a crystal structure isomorphic to $B(\infty)$ with $\varepsilon_a^*(x)$ given by (12) (see also Remark 3.8). Furthermore, by [25, Proposition 8.2] Lusztig's canonical basis of the irreducible representation $V(\sum_{a \in I} \lambda_a \omega_a)$ is parametrized by the set of $x \in \mathcal{S}_{\mathbf{i}}$ such that $\varepsilon_a^*(x) \leq \lambda_a$ for all $a \in I$. This gives an alternative proof that $\operatorname{gr} \mathcal{S}_{\mathbf{i}}$ parametrizes Lusztig's canonical basis of $H^0(\mathrm{G}/\mathcal{N}, \mathcal{O}_{\mathrm{G}/\mathcal{N}})$.

In the following we introduce positive functions on

$$\operatorname{gr} \mathbb{S}_{\mathbf{i}} := \mathbb{G}_m^I \times \mathbb{S}_{\mathbf{i}},$$

whose tropicalization cut out $\operatorname{gr} \mathcal{S}_{\mathbf{i}}$:

Definition 3.11. With the notation of Definition 3.2 we specify for $a \in I$ the positive functions on $\operatorname{gr}\mathbb{S}_{\mathbf{i}}$

$$\mathfrak{s}_{\mathbf{i},\mathrm{a}}(\lambda, x) = \mathfrak{s}_{\mathbf{i},\mathrm{a}}(x),$$
$$\mathfrak{s}_{\mathbf{i},\text{-a}}(\lambda, x) = \lambda_a \sum_{\substack{k\in[N]\\ i_k=a}} \left(x_k^{-1} \prod_{\ell=k+1}^{N} x_\ell^{-c_{i_\ell,i_k}} \right)$$

Example 3.12. For $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1,2,1)$ we have (note that $i_N = i_3 = 1$)

$$\mathfrak{s}_{\mathbf{i},1}(\lambda_1, \lambda_2, x_1, x_2, x_3) = x_3,$$
$$\mathfrak{s}_{\mathbf{i},-1}(\lambda_1, \lambda_2, x_1, x_2, x_3) = \lambda_1 x_1^{-1} x_2 x_3^{-2} + \lambda_1 x_3^{-1}.$$

For $\mathbf{j} = (2,1,2)$ we have

$$\mathfrak{s}_{\mathbf{i},2}(\lambda_1, \lambda_2, x_1, x_2, x_3) = \mathfrak{s}_{\mathbf{j},2} \circ (\mathrm{id} \times \Psi_{\mathbf{j}}^{\mathbf{i}})(\lambda_1, \lambda_2, x_1, x_2, x_3) = \frac{x_3 x_1 + x_2}{x_3},$$
$$\mathfrak{s}_{\mathbf{i},-2}(\lambda_1, \lambda_2, x_1, x_2, x_3) = \lambda_2 x_2^{-1}.$$

Proposition 3.13. *For reduced words* $\mathbf{i}, \mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ *we have*

(1) $\mathfrak{s}_{\mathbf{j},\text{-a}} = \mathfrak{s}_{\mathbf{i},\text{-a}} \circ \Psi_{\mathbf{j}}^{\mathbf{i}}$,
(2) $\operatorname{gr}\mathcal{S}_{\mathbf{i}} = \{(\lambda, x) \in [\operatorname{gr}\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \mid \forall a \in -[n] \cup [n] : [\mathfrak{s}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}(\lambda, x) \geq 0\}$.

Proof. Statement (1) is a straightforward computation. Statement (2) follows from Proposition 3.5. ◻

We introduce the analogues of $\mathfrak{s}_{\mathbf{i},\mathrm{a}}$ and $\operatorname{gr}\mathcal{S}_{\mathbf{i}}$ for ${}^L\mathfrak{g}$ as follows.

Definition 3.14. For $a \in I$ we specify on $\operatorname{gr}\mathbb{S}_{\mathbf{i}}$ the positive functions $\check{\mathfrak{s}}_{\mathbf{i},\mathrm{a}} := \mathfrak{s}_{\mathbf{i},\mathrm{a}}$ and

$$\check{\mathfrak{s}}_{\mathbf{i},\text{-a}}(\lambda, x) = \left(\prod_{b\in I} \lambda_b^{c_{a,b}} \right) \sum_{\substack{k\in[N]\\ i_k=a}} \left(x_k^{-1} \prod_{\ell=k+1}^{N} x_\ell^{-c_{i_\ell,i_k}} \right).$$

Definition 3.15. We introduce the graded string cone of ${}^L\mathfrak{g}$ as

$$\operatorname{gr}\mathcal{S}_{\mathbf{i}}^{\vee} := \{(\lambda, x) \in [\operatorname{gr}\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \mid \forall a \in -[n] \cup [n] : [\check{\mathfrak{s}}_{\mathbf{i},\mathrm{a}}]_{\mathrm{trop}}(\lambda, x) \geq 0\}.$$

4. The cluster spaces of the base affine space

4.1. Generalized minors and the open double Bruhat cell

In the following we identify the weight lattice of $\mathfrak{g}$ with the group of multiplicative characters on T. For a dominant weight $\lambda : T \to \mathbb{G}_m$, we define the *principal minor* $\Delta_\lambda : G \to \mathbb{A}^1$ to be the function defined on the open subset $\mathcal{N}^- T \mathcal{N} \subset G$ by

$$\Delta_\lambda(u^- t u^+) := \lambda(t) \quad u^- \in \mathcal{N}^-, t \in T, u^+ \in \mathcal{N}.$$

Let γ, δ be extremal weights such that $\gamma = w_1 \lambda$, $\delta = w_2 \lambda$ for some $w_1, w_2 \in W$, $\lambda \in P^+$. Recall the embedding of sets (4) of W into $\mathrm{Norm}_G(T)$. The *generalized minor* associated to γ and δ is

$$\Delta_{\gamma,\delta}(g) := \Delta_\lambda(\overline{w}_1^{-1} g \overline{w}_2), \quad g \in G.$$

The base affine space $\mathrm{G}/\mathcal{N}$ is the partial compactification of the *open double Bruhat cell*

$$G^{\mathrm{w}_0, e} := B\, \mathrm{w}_0\, B \cap B_-$$

obtained by allowing the generalized minors $\Delta_{\omega_a,\omega_a}$ and $\Delta_{\mathrm{w}_0\,\omega_a,\omega_a}$ to vanish (see [3, Section 2.6]).

4.2. The cluster ensemble of the open double Bruhat cell

4.2.1. Cluster seeds

Following Fomin–Zelevinsky, Berenstein–Fomin–Zelevinsky and Fock–Goncharov [14, 3,15,12] we recall the definition of the cluster ensemble associated to $G^{\mathrm{w}_0,e}$. We start by introducing the notion of a seed.

Definition 4.1. A seed is a datum $\Sigma = (\Lambda, \{\cdot,\cdot\}, \{e_k\}_{k\in M}, M_0)$, where

(i) Λ is a lattice,
(ii) $\{\cdot,\cdot\}$ is a skew-symmetric $\mathbb{Z}$-valued bilinear form on Λ,
(iii) $M_0 \subset M$ is a finite set,
(iv) $\{e_k\}_{k\in M}$ is a basis of Λ.

We associate to a seed $\Sigma = (\Lambda, \{\cdot,\cdot\}, \{e_k\}_{k\in M}, M_0)$ a quiver Γ_Σ as follows. The vertices $\{v_k\}_{k\in M}$ of Γ_Σ are indexed by M. If $\{e_k, e_\ell\} > 0$ then there are $\{e_k, e_\ell\}$ arrows with source v_k and target v_ℓ in Γ_Σ. A vertex v_k is called *frozen* if $k \in M_0$ and *mutable* if $k \in M \setminus M_0$.

Let $\Sigma = (\Lambda, \{\cdot,\cdot\}, \{e_k\}_{k\in M}, M_0)$ be a seed. For each $k \in M \setminus M_0$ we define the seed $\mu_k(\Gamma_\Sigma) = (\Lambda, \{e_k, e_\ell\}, \{e'_k\}_{k\in M}, M_0)$, called the *mutation of* Σ *at* k, by setting

$$e'_j = \begin{cases} e_j + \max(0, \{e_j, e_k\})\, e_k & \text{if } j \neq k \\ -e_k & \text{if } j = k. \end{cases}$$

The quiver $\mu_k(\Gamma_\Sigma) := \Gamma_{\mu_k \Sigma}$, called the *mutation of* Γ_Σ *at* k, is obtained as follows. The vertices and frozen vertices of Γ_Σ and $\mu_k(\Gamma_\Sigma)$ coincide. Furthermore $\mu_k(\Gamma_\Sigma)$ has the same arrows as Γ_Σ, except:

(i) All arrows of Γ_Σ with source or target v_k get replaced in $\mu_k(\Gamma_\Sigma)$ by the reversed arrow.

(ii) For every pair of arrows $(h_1, h_2) \in \Gamma_\Sigma \times \Gamma_\Sigma$ with

$$v_k = \text{target of } h_1 = \text{source of } h_2$$

we add to $\mu_k(\Gamma_\Sigma)$ an arrow from the source of h_1 to the target of h_2.

(iii) If a 2-cycles was obtained during (i) and (ii), the arrows of this 2-cycle get canceled in $\mu_k(\Gamma_\Sigma)$.

(iv) Finally we erase all arrows between frozen vertices.

To a seed $\Sigma = (\Lambda, \{\cdot,\cdot\}, \{e_k\}_{k\in M}, M_0)$ we assign the $\mathcal{A}$- and $\mathcal{X}$-cluster tori

$$\mathbb{T}_\Sigma := \mathrm{Spec}\mathbb{Z}[A_k^{\pm 1} \mid k \in M] \quad \text{and} \quad \widehat{\mathbb{T}}_\Sigma := \mathrm{Spec}\mathbb{Z}[X_k^{\pm 1} \mid k \in M].$$

We also consider the tori

$$\mathbb{T}_\Sigma^{\mathrm{uf}} := \mathrm{Spec}\mathbb{Z}[A_k^{\pm 1} \mid k \in M\backslash M_0] \quad \text{and} \quad \widehat{\mathbb{T}}_\Sigma^{\mathrm{uf}} := \mathrm{Spec}\mathbb{Z}[X_k^{\pm 1} \mid k \in M\backslash M_0]$$

corresponding to the mutable vertices along with the natural maps

$$\iota_\Sigma : \mathbb{T}_\Sigma^{\mathrm{uf}} \hookrightarrow \mathbb{T}_\Sigma \quad \text{and} \quad \pi_\Sigma : \widehat{\mathbb{T}}_\Sigma \twoheadrightarrow \widehat{\mathbb{T}}_\Sigma^{\mathrm{uf}}.$$

We introduce birational $\mathcal{A}$-cluster transformations $\mu_k : \mathbb{T}_\Sigma \dashrightarrow \mathbb{T}_{\mu_k \Sigma}$ and $\mathcal{X}$-cluster transformations $\widehat{\mu}_k : \widehat{\mathbb{T}}_\Sigma \dashrightarrow \widehat{\mathbb{T}}_{\mu_k \Sigma}$:

$$\mu_k^* A_\ell = \begin{cases} \displaystyle\prod_{j:\{e_j,e_k\}>0} \frac{A_j^{\{e_j,e_k\}}}{A_k} + \prod_{j:\{e_j,e_k\}<0} \frac{A_j^{-\{e_j,e_k\}}}{A_k} & \text{if } \ell = k, \\ A_\ell & \text{else,} \end{cases} \tag{17}$$

$$\widehat{\mu}_k^* X_\ell = \begin{cases} X_k^{-1} & \text{if } \ell = k, \\ X_\ell \left(1 + X_k^{-\operatorname{sgn}\{e_\ell,e_k\}}\right)^{-\{e_\ell,e_k\}} & \text{else.} \end{cases} \tag{18}$$

For seeds Σ and Σ' obtained by a sequence of mutations we define $\mu^{\Sigma}_{\Sigma'} : \mathbb{T}_{\Sigma} \dashrightarrow \mathbb{T}_{\Sigma'}$ and $\widehat{\mu}^{\Sigma}_{\Sigma'} : \widehat{\mathbb{T}}_{\Sigma} \dashrightarrow \widehat{\mathbb{T}}_{\Sigma'}$ by composition.

The *cluster space* $\mathcal{A}$ and the *dual cluster space* $\mathcal{X}$ are the schemes obtained by gluing the tori $(\mathbb{T}_{\Sigma})$ and $(\widehat{\mathbb{T}}_{\Sigma})$ via (17) and (18), respectively. The spaces $\mathcal{A}$ and $\mathcal{X}$ are related by a set of maps, which are referred to as p-maps in the literature. We recall the definition following [22, Chapter 2]:

Definition 4.2. A *p-map* is a family $\mathrm{p} = (\mathrm{p}_{\Sigma}) : \mathcal{A} \to \mathcal{X}$ of morphisms

$$\mathrm{p}_{\Sigma} : \mathbb{T}_{\Sigma} \to \widehat{\mathbb{T}}_{\Sigma},$$

which is compatible with mutations, i.e. $\mathrm{p}_{\Sigma'} = \widehat{\mu}^{\Sigma}_{\Sigma'} \circ \mathrm{p}_{\Sigma} \circ \mu^{\Sigma}_{\Sigma'}$, and satisfies

$$(\mathrm{p}_{\Sigma} \circ \iota_{\Sigma}(A))_k = \prod_{\ell \in M \setminus M_0} A_{\ell}^{\{e_k, e_\ell\}} \quad \text{for } k \in M, \tag{19}$$

$$(\pi_{\Sigma} \circ \mathrm{p}_{\Sigma}(A))_k = \prod_{\ell \in M} A_{\ell}^{\{e_k, e_\ell\}} \quad \text{for } k \in M \setminus M_0. \tag{20}$$

If $M_0 = \varnothing$ holds, i.e. if there are no frozen vertices then there exists a unique p-map. For an arbitrary cluster datum as in Definition 4.1 the situation is as follows. For a seed Σ we write

$$\mathbb{T}_{\Sigma} := \mathbb{T}^{\mathrm{uf}}_{\Sigma} \times \mathbb{T}^{\mathrm{f}}_{\Sigma}, \quad \widehat{\mathbb{T}}_{\Sigma} := \widehat{\mathbb{T}}^{\mathrm{uf}}_{\Sigma} \times \widehat{\mathbb{T}}^{\mathrm{f}}_{\Sigma}.$$

Written in the torus coordinates of a fixed seed Σ_0 any two p-maps $\mathrm{p}, \mathrm{p}' : \mathcal{A} \to \mathcal{X}$ defer by a morphism $\delta_{\Sigma_0} : \mathbb{T}^{\mathrm{f}}_{\Sigma_0} \to \widehat{\mathbb{T}}^{\mathrm{f}}_{\Sigma_0}$, i.e.

$$\mathrm{p}_{\Sigma_0} = (\mathrm{id} \times \delta_{\Sigma_0}) \cdot \mathrm{p}'_{\Sigma_0} : \mathbb{T}_{\Sigma_0} = \mathbb{G}_m^{\{v \text{ mutable}\}} \times \mathbb{T}^{\mathrm{f}}_{\Sigma_0} \to \mathbb{G}_m^{\{v \text{ mutable}\}} \times \widehat{\mathbb{T}}^{\mathrm{f}}_{\Sigma_0} = \widehat{\mathbb{T}}_{\Sigma_0}. \tag{21}$$

Here $\cdot$ denotes the multiplication in $\widehat{\mathbb{T}}_{\Sigma_0}$.

Conversely, any mutation compatible family of maps $(\mathbb{T}_{\Sigma} \dashrightarrow \widehat{\mathbb{T}}_{\Sigma})$, with Σ running over the set of seeds, is a p-map if it satisfies equations (19) and (20) for a fixed seed Σ_0 with $\mathrm{p}_{\Sigma_0} \in \mathrm{Hom}(\mathbb{T}_{\Sigma_0}, \widehat{\mathbb{T}}_{\Sigma_0})$:

Lemma 4.3. *Let $\mathrm{p}_{\Sigma_0} \in \mathrm{Hom}(\mathbb{T}_{\Sigma_0}, \widehat{\mathbb{T}}_{\Sigma_0})$ satisfy (19) and (20) for $\Sigma = \Sigma_0$. Then*

$$\mathrm{p} := \left(\widehat{\mu}^{\Sigma_0}_{\Sigma} \circ \mathrm{p}_{\Sigma_0} \circ \mu^{\Sigma}_{\Sigma_0}\right) : \mathcal{A} = (\mathbb{T}_{\Sigma}) \to \left(\widehat{\mathbb{T}}_{\Sigma}\right) = \mathcal{X}$$

is a p-map. In particular, the map p is regular.

Proof. Associating to the seed $\Sigma_0 = (\Lambda, \{\cdot,\cdot\}, \{e_k\}_{k \in M}, M_0)$ the seed $\Sigma_0^{df} = (\Lambda, \{\cdot,\cdot\}, \{e_k\}_{k \in M}, \varnothing)$ we obtain the cluster spaces $\mathcal{A}^{df}$ and $\mathcal{X}^{df}$. In other words, Σ_0^{df} is obtained from Σ_0 by defining all $k \in M$ to be mutable. Thus, we have $\mathbb{T}_{\Sigma_0} = \mathbb{T}_{\Sigma_0^{df}}$ and $\widehat{\mathbb{T}}_{\Sigma_0} = \widehat{\mathbb{T}}_{\Sigma_0^{df}}$.

Let p′ denote the unique p-map between $\mathcal{A}^{df}$ and $\mathcal{X}^{df}$. Then p′ satisfies (19) and (20). Consequently, there exists $\delta_{\Sigma_0} : \mathbb{T}^{\mathrm{f}}_{\Sigma_0} \to \widehat{\mathbb{T}}^{\mathrm{f}}_{\Sigma_0}$, satisfying (21).

From (18) we deduce

$$\widehat{\mu}^{\Sigma_0}_{\Sigma} \circ \mathrm{p}_{\Sigma_0} \circ \mu^{\Sigma}_{\Sigma_0} = \widehat{\mu}^{\Sigma_0}_{\Sigma} \circ \left(\mathrm{id} \times \delta_{\Sigma_0} \cdot \mathrm{p}'_{\Sigma_0}\right) \circ \mu^{\Sigma}_{\Sigma_0} = (\mathrm{id} \times \delta_{\Sigma_0}) \cdot \left(\widehat{\mu}^{\Sigma_0}_{\Sigma} \circ \mathrm{p}'_{\Sigma_0} \circ \mu^{\Sigma}_{\Sigma_0}\right).$$

The claim now follows, since regularity as well as the validity of (19) and (20) is not affected by multiplying with $\mathrm{id} \times \delta_{\Sigma_0}$. □

4.2.2. Seeds associated to reduced words

Following [3] we associate to every reduced word $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ a seed $\Sigma_{\mathbf{i}}$. We throughout identify

$$\Sigma_{\mathbf{i}} \text{ and } \mathbf{i},$$

i.e. we denote the seed $\Sigma_{\mathbf{i}}$ also by $\mathbf{i}$. Consequently, we write $\mathbb{T}_{\mathbf{i}}$, $\widehat{\mathbb{T}}_{\mathbf{i}}$, $\mu^{\mathbf{i}}_{\mathbf{j}}$ and $\widehat{\mu}^{\mathbf{i}}_{\mathbf{j}}$ for $\mathbb{T}_{\Sigma_{\mathbf{i}}}$, $\widehat{\mathbb{T}}_{\Sigma_{\mathbf{i}}}$, $\mu^{\Sigma_{\mathbf{i}}}_{\Sigma_{\mathbf{j}}}$ and $\widehat{\mu}^{\Sigma_{\mathbf{i}}}_{\Sigma_{\mathbf{j}}}$, respectively.

The quiver $\Gamma_{\mathbf{i}}$ can be described as follows. We denote the vertices of $\Gamma_{\mathbf{i}}$ by $\{v_k \mid k \in M\}$, where

$$M := \{-1, \ldots, -n\} \cup \{1, \ldots, N\}. \tag{22}$$

Using Definition 1.2 we write $v_\ell = v_{a,r}$ if $\beta_\ell = \beta_{a,r}$ and set $v_{a,0} := v_{-a}$. The frozen vertices of $\Gamma_{\mathbf{i}}$ are

$$\{w_{-a} := v_{a,0} \mid a \in I\} \cup \{w_a := v_{a,m_a} \mid a \in I\}. \tag{23}$$

In order to define the arrows in $\Gamma_{\mathbf{i}}$ we introduce the following notion. For $k \in [-n]$ we set $i_k = -k$. For $k \in M$ we denote by $k^+ = k^+_{\mathbf{i}}$ the smallest $\ell \in M$ such that $k < \ell$ and $i_\ell = i_k$. If no such ℓ exists, we set $k^+ = N+1$. For $k \in [N]$, we further let k^- be the largest index $\ell \in M$ with $\ell < k$ and $i_\ell = i_k$.

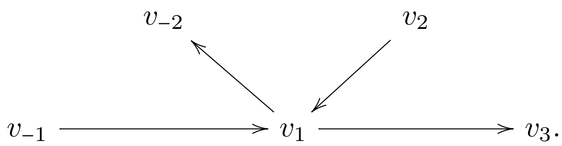
There is an edge connecting v_k and v_ℓ with $k < \ell$ if at least one of the two vertices is mutable and one of the following conditions is satisfied:

(1) $\ell = k^+$,
(2) $\ell < k^+ < \ell^+$, $c_{k,\ell} < 0$ and $k, \ell \in [N]$.

Edges of type (1) are called *horizontal* and are directed from k to ℓ. Edges of type (2) are called *inclined* and are directed from ℓ to k.

Remark 4.4. The quiver $\Gamma_{\mathbf{i}}$ associated to a reduced word $\mathbf{i}$ has at most one edge between any two vertices v, w. We denote this edge by $[v, w]$ if it exists.

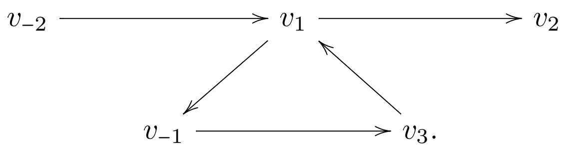
Example 4.5. Let $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1,2,1)$. Then $\Gamma_{\mathbf{i}}$ looks as follows:



Lemma 4.6. *Let $\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ be obtained from $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ by a 2-move or a 3-move in position k. Then the swapping of the vertex v_k with the vertex v_{k+1} induces an isomorphism of quivers $\Gamma_{\mathbf{j}} \simeq \mu_{k-1}\Gamma_{\mathbf{i}}$.*

Proof. The statement follows from the construction of $\Gamma_{\mathbf{i}}$ and [39, Theorem 3.5]. $\square$

Example 4.7. Consider the graph $\Gamma_{\mathbf{i}}$ from Example 4.5. Then $\mu_1\Gamma_{\mathbf{i}}$ looks as follows:



Swapping v_3 and v_2 yields the graph Γ_{212}.

We consider the families $(\mathbb{T}_\Sigma)$ and $(\widehat{\mathbb{T}}_\Sigma)$ of all tori corresponding to seeds Σ which can be obtained from a seed $\Sigma = \mathbf{i}$ corresponding to a reduced word $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ by a sequence of mutations. By [3] the open double Bruhat cell $G^{\mathrm{w}_0,e}$ is covered up to codimension 2 by $(\mathbb{T}_\Sigma)$ via

$$\begin{aligned} G^{\mathrm{w}_0,e} &\dashrightarrow \mathbb{T}_\Sigma, \\ g &\mapsto \left(\Delta_{s_{i_1}\cdots s_{i_k}\omega_{i_k},\omega_{i_k}}(g)\right)_{k\in M}. \end{aligned} \tag{24}$$

The associated gluing maps are given in Section 4.2.2. We call the pair $(\mathcal{A},\mathcal{X})$ the cluster ensemble associated to $G^{\mathrm{w}_0,e}$.

Remark 4.8. We defer from the convention in [3] as follows. The reduced word $\mathbf{i}$ for $G^{\mathrm{w}_0,e}$ is denoted by $-\mathbf{i}$ in [3]. The quiver $\Gamma_{\mathbf{i}}$ is isomorphic to the quiver associated to $\mathbf{i}$ in [3], i.e. obtained from the quiver associated to $-\mathbf{i}$ in [3] by turning around arrows.

We use the following fact later.

Lemma 4.9. *For $a \in I$ and reduced words $\mathbf{i},\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ we have the equality of functions in $\mathcal{O}(\widehat{\mathbb{T}}_{\mathbf{i}})$*

$$\prod_{r=0}^{m_{a,\mathbf{i}}} X_{a,r} = \prod_{r=0}^{m_{a,\mathbf{j}}} X_{a,r} \circ \widehat{\mu}_{\mathbf{j}}^{\mathbf{i}}.$$

Proof. Without loss of generality we can assume that $\mathbf{j}$ is obtained from $\mathbf{i}$ by a 3-move at position ℓ with $v_\ell = v_{a,s}$. The claim then follows since we have for $r \in [m_a^{\mathbf{j}}]$

$$X_{a,r} \circ \widehat{\mu}_{\mathbf{j}}^{\mathbf{i}} = \begin{cases} X_{a,r} & \text{if } r < s-1, \\ X_{a,r}(1+(X_{a,s})^{-1})^{-1} & \text{if } r = s-1, \\ X_{a,r-1}(1+X_{a,s}) & \text{if } r = s, \\ X_{a,r-1} & \text{if } r > s. \quad \square \end{cases}$$

5. Gross-Hacking-Keel-Kontsevich potential and Berenstein-Kazhdan decoration function

5.1. Gross-Hacking-Keel-Kontsevich potential

Recall from Section 4.1 that the base affine space $\mathrm{G}/\mathcal{N}$ is the partial compactification of the open double Bruhat cell $G^{\mathrm{w}_0,e}$:

$$\mathrm{G}/\mathcal{N} = G^{\mathrm{w}_0,e} \cup \bigcup_{\pm a \in [n]} D_a,$$

where D_a is the divisor given by the vanishing locus of the functions $\Delta_{\omega_a,\omega_a}$ for $a < 0$ and $\Delta_{\mathrm{w}_0\,\omega_a,\omega_a}$ for $a > 0$, corresponding to the frozen vertices of $\Gamma_{\mathbf{i}}$ by (24).

In [23] a Landau-Ginzburg potential W on the dual cluster space $\mathcal{X}$ associated to $\mathrm{G}/\mathcal{N}$ is defined as the sum $W = \sum_{\pm a \in [n]} W_a$ of certain global monomials W_a attached to the divisors D_a. We are interested in $W|_{\widehat{\mathbb{T}}_{\mathbf{i}}}$ since the cone

$$\widehat{\mathcal{C}}_\Sigma := \left\{ x \in [\widehat{\mathbb{T}}_\Sigma]_{\mathrm{trop}} \,\middle|\, [W|_{\widehat{\mathbb{T}}_\Sigma}]_{\mathrm{trop}}(x) \geq 0 \right\}$$

cut out by the tropicalization of $W|_{\widehat{\mathbb{T}}_{\mathbf{i}}}$, up to a natural conjecture (see Remark 5.4), parametrizes a canonical basis for the ring of regular functions on the partial compactification $\mathrm{G}/\mathcal{N}$ of $G^{\mathrm{w}_0,e}$.

Using (23) we have the following definition of W_a, which in [23, Corollary 9.17] is shown to be well-defined.

Definition 5.1. If there is no arrow in Γ_Σ from the frozen vertex w_a to a mutable vertex we call Σ *optimized for* w_a and have $W_a|_{\widehat{\mathbb{T}}_\Sigma} = X_{w_a}^{-1}$.

For certain toric charts we have a closed explicit description of $W_a|_{\widehat{\mathbb{T}}_{\mathbf{i}}}$:

Proposition 5.2. *Every frozen vertex w_a has an optimized seed. Furthermore, for $a \in I$ and $\mathbf{i} = (i_1, \ldots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ we have*

$$W_{i_N}\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}}(X) = X_N^{-1}, \tag{25}$$

$$W_{-a}\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}}(X) = \sum_{k=0}^{m_a-1} \prod_{\ell=0}^{k} X_{a,\ell}^{-1}. \tag{26}$$

Proof. By definition the quiver $\Gamma_{\mathbf{i}}$ is optimized for the frozen vertex v_{i_N} and (25) follows. It remains to show (26) and that for $a \in I$ the vertex w_{-a} has an optimized seed.

Let $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$, $j \in [m_a - 1]$ and let $\Gamma_{\mathbf{i}}^{(j)}$ be the resulting quiver after applying the sequence of mutations at the vertices $v_{a,1}, v_{a,2}, \ldots, v_{a,j}$ to $\Gamma_{\mathbf{i}}$.

Between two vertices there is at most one arrow in $\Gamma_{\mathbf{i}}^{(j)}$. Furthermore, we have for $j \geq 2$

$$[v_{0,a}, v_{a,j+1}],\ [v_{a,j+1}, v_{a,j}] \in \Gamma_{\mathbf{i}}^{(j)}. \tag{27}$$

We prove that $[v_{0,a}, v_{a,j+1}]$ is the only arrow in $\Gamma_{\mathbf{i}}^{(j)}$ with source $v_{0,a}$.

Let j be minimal such that there exist an arrow $[v_{0,a}, w]$ in $\Gamma_{\mathbf{i}}^{(j)}$ with $w \neq v_{a,j+1}$ and w mutable. Then $[v_{a,j-1}, w]$ is an arrow of $\Gamma_{\mathbf{i}}^{(j-1)}$. Note that $[v_{a,j-1}, w]$ has to be an inclined arrow of $\Gamma_{\mathbf{i}}$. Since w and $v_{a,j-1}$ are mutable, there exists an inclined arrow $[w, v_{a,r}]$ in $\Gamma_{\mathbf{i}}$ with $r < j-1$. This arrow stays unchanged under the sequence of mutations at $v_{a,1}, v_{a,2}, \ldots, v_{a,r-1}$, creates an arrow $[w, v_{a,r-1+s}]$ in $\Gamma_{\mathbf{i}}^{(r+s)}$ for $r + s < j$ and cancels the arrow $[v_{0,a}, w]$ in $\Gamma_{\mathbf{i}}^{(j)}$ yielding a contradiction.

Hence, the seed Σ corresponding to $\Gamma_{\mathbf{i}}^{(m_a-1)}$ is optimized for $v_{a,0} = w_{-a}$. Furthermore, from (27) we recursively compute (26). □

Example 5.3. Continuing Example 4.5 we have for $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1, 2, 1)$

$$W_{-1}\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}}(X) = X_{-1}^{-1} + X_{-1}^{-1}X_1^{-1}, \qquad W_1\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}}(X) = X_3^{-1},$$
$$W_{-2}\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}}(X) = X_{-2}^{-1}, \qquad W_2\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}}(X) = X_2^{-1} + X_1^{-1}X_2^{-1}.$$

Remark 5.4. In [23] a canonical basis for the ring of regular functions on an $\mathcal{A}$-cluster variety, called theta basis, is constructed under the assumptions given in [23, Definition 0.6]. This assumption is called the full Fock–Goncharov conjecture and ensures in particular, that the theta basis is naturally identified with the tropical points of the corresponding $\mathcal{X}$-cluster variety.

The full Fock-Goncharov conjecture for a partially compactified $\mathcal{A}$-cluster variety follows from the work of Gross–Hacking–Keel–Kontsevich (specifically [23, Proposition 8.24, Proposition 8.25, Proposition 8.27, Lemma 9.10]) assuming

(1) the existence of a green-to-red sequence,
(2) that every frozen vertex has an optimized seed,
(3) the surjectivity of the composition of the projection onto the mutable part with a p-map, i.e. the map referred to as p_2 in 5.1.

In the case of double Bruhat cells, Goodearl and Yakimov announced in [21] (see [23, Example 0.15]) the existence of a maximal green sequence. For the convenience of the reader in the appendix we independently give maximal green sequences for seeds of $G^{\mathrm{w}_0,e}$ associated to certain reduced words. By Proposition 5.2 every frozen vertex of $\Gamma_{\mathbf{i}}$ has an optimized seed and by [3, Proposition 2.6.] assumption (3) holds.

In conclusion, the full Fock–Goncharov conjecture for $G^{\mathrm{w}_0,e}$ holds and there exists a theta basis for the partial compactification $\mathrm{G}/\mathcal{N}$ of $G^{\mathrm{w}_0,e}$ parametrized by $\widehat{\mathcal{C}}_\Sigma$.

5.2. Berenstein-Kazhdan decorations

In [4, Corollary 1.25], Berenstein and Kazhdan introduced as part of the datum of a G-decorated geometric crystal the *decoration function* $f^B = \sum_{a\in\pm I} f^B_a$ on G, where for $a \in I$

$$f^B_{-a}(g) = \frac{\Delta_{\mathrm{w}_0\,\omega_a, s_a\omega_a}(g)}{\Delta_{\mathrm{w}_0\,\omega_a,\omega_a}(g)}, \qquad f^B_a(g) = \frac{\Delta_{\mathrm{w}_0\,s_a\omega_a,\omega_a}(g)}{\Delta_{\mathrm{w}_0\,\omega_a,\omega_a}(g)}.$$

For certain toric charts we have a closed explicit description of f^B_a:

Proposition 5.5. *For $a \in I$, $\mathbf{i} = (i_1, \dots, i_N) \in \mathcal{W}(\mathrm{w}_0)$ and $A = (A_k)_{k\in M} \in \mathbb{T}_{\mathbf{i}}$ we have*

$$f^B_{i_N}\big|_{\mathbb{T}_{\mathbf{i}}}(A) = A_N^{-1} A_{N^-}, \tag{28}$$

$$f^B_{-a}\big|_{\mathbb{T}_{\mathbf{i}}}(A) = \sum_{\substack{k\in[N]\\ i_k=a}} \left(A_{k^-} A_k \prod_{\substack{\ell\in M\\ \ell<k<\ell^+}} A_\ell^{c_{i_\ell,a}} \right)^{-1}. \tag{29}$$

Proof. Recall from (24) that we may identify for $k \in M$ the function $\Delta_{s_{i_1}\cdots s_{i_k}\omega_{i_k},\omega_{i_k}}$ on $G^{w_0,e}$ with the k-th coordinate function A_k in the ring of regular functions on $\mathbb{T}_{\mathbf{i}}$.

Equality (28) follows directly. For equality (29) we first assume $i_N = a$. By [5, Equation (5.8)] and [13, Proposition 2.7] we get

$$f^B_{-a}(g) = \frac{\Delta_{\mathrm{w}_0\,\omega_a, s_a\omega_a}(g)}{\Delta_{\mathrm{w}_0\,\omega_a,\omega_a}(g)} = \sum_{\substack{k\in[N]\\ i_k=a}} \frac{\prod_{b\in I\setminus\{a\}} \Delta^{-c_{a,b}}_{s_{i_1}\cdots s_{i_k}\omega_b,\omega_b}(g)}{\Delta_{s_{i_1}\cdots s_{i_k}\omega_a,\omega_a}(g)\Delta_{s_{i_1}\cdots s_{i_{k-1}}\omega_a,\omega_a}(g)}. \tag{30}$$

The claim follows from (30) using identification (24) since we have for $k \in [N]$ and $a, b \in I$ with $a \neq b$

$$s_a\omega_b = \omega_b,$$
$$\prod_{\substack{\ell\in M\\ \ell<k<\ell^+}} \Delta^{-c_{a,i_\ell}}_{s_{i_1}\cdots s_{i_\ell}\omega_b,\omega_b} = \prod_{\substack{b\in I\\ b\neq a}} \Delta^{-c_{a,b}}_{s_{i_1}\cdots s_{i_{k-1}}\omega_b,\omega_b}.$$

In order to establish equation (29) for general $\mathbf{i}$, it is enough to show that the right hand side of (29) is invariant under 3-moves. If $\mathbf{j}$ is obtained from $\mathbf{i}$ by a 3-move at position $k \in [N]$ replacing (i_{k-1}, i_k, i_{k+1}) by (i_k, i_{k-1}, i_k) with $i_k = a$ then by Lemma 4.6 and (17)

$$\mu_k^*\left(A_{k^-}^{-1}A_k^{-1}\prod_{\substack{\ell\in M\\ \ell<k<\ell^+}} A_\ell^{-c_{i_\ell,a}} + A_k^{-1}A_{k^+}^{-1}\prod_{\substack{\ell\in M\\ \ell<k^+<\ell^+}} A_\ell^{-c_{i_\ell,a}}\right) = A_{k^-}^{-1}A_k^{-1}\prod_{\substack{\ell\in M\\ \ell<k<\ell^+}} A_\ell^{-c_{i_\ell,a}}.$$

If $i_k \neq a$ with $c_{i_k} \neq 0$ the claim follows by symmetry. For $c_{i_k} = 0$ there is nothing to show. □

Example 5.6. Continuing Example 4.5 we have for $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1,2,1)$

$$f_{-1}^B|_{\mathbb{T}_\mathbf{i}}(A) = \frac{A_{-2}}{A_{-1}A_1} + \frac{A_2}{A_1A_3}, \qquad f_1^B|_{\mathbb{T}_\mathbf{i}}(A) = \frac{A_1}{A_3},$$
$$f_{-2}^B|_{\mathbb{T}_\mathbf{i}}(A) = \frac{A_1}{A_{-2}A_2}, \qquad f_2^B|_{\mathbb{T}_\mathbf{i}}(A) = \frac{A_{-1}}{A_1} + \frac{A_{-2}A_3}{A_1A_2}.$$

We denote by $\mathcal{C}_\Sigma$ the cone cut out by the decoration function f^B:

$$\mathcal{C}_\Sigma = \left\{A \in [\mathbb{T}_\Sigma]_{\mathrm{trop}} \;\middle|\; \left[f^B|_{\mathbb{T}_\Sigma}\right]_{\mathrm{trop}}(A) \geq 0\right\}.$$

6. Lusztig parametrization via cluster varieties

In this section we relate the cone $\mathrm{gr}\,\mathcal{L}_\mathbf{i}$ of Lusztig's graded parametrization to the cone $\widehat{\mathcal{C}}_\mathbf{i}$ cut out by the tropicalization of the Gross–Hacking–Keel–Kontsevich potential function W introduced in Section 5.1. Dually, we relate the cone $\mathrm{gr}\,\mathcal{L}_\mathbf{i}^\vee$ associated to the Langlands dual Lie algebra ${}^L\mathfrak{g}$ to the cone $\mathcal{C}_\mathbf{i}$ cut out by the tropicalization of the decoration function f^B due to Berenstein–Kazhdan defined in Section 5.2.

Motivated by the *Chamber Ansatz* due to Berenstein-Fomin-Zelevinsky [2] as well as by [5, Equation (4.14)] and [17, Section 5] we introduce the following coordinate transformations using the notations of Section 4.2.2 and (2). First, we set for $k, \ell \in M = -[n] \cup [N]$

$$[\![k,\ell]\!] := -c_{i_k,i_\ell}\cdot\begin{cases}1 & \text{if } k<\ell<k^+,\\ \frac{1}{2} & \text{if } \ell = k \text{ or } \ell = k^+\\ 0 & \text{else.}\end{cases} \tag{31}$$

Definition 6.1. We specify $\mathrm{gr}\,\widehat{\mathrm{NA}}_\mathbf{i} \in \mathrm{Hom}(\mathrm{gr}\,\mathbb{L}_\mathbf{i}, \widehat{\mathbb{T}}_\mathbf{i})$ and $\mathrm{gr}\,\mathrm{CA}_\mathbf{i} \in \mathrm{Hom}(\mathbb{T}_\mathbf{i}, \mathrm{gr}\,\mathbb{L}_\mathbf{i})$ ($A = (A_k)_{k\in M} \in \mathbb{T}_\mathbf{i}, (x,\lambda) \in \mathrm{gr}\,\mathbb{L}_\mathbf{i}$):

$$(\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}(\lambda,x))_k = \begin{cases} x_{k^+}^{-1} & \text{if } k<0, \\ x_k x_{k^+}^{-1} & \text{if } k,k^+\in[N], \\ x_k \lambda_{i_k^*}^{-1} & \text{if } k^+ = N+1, \end{cases}$$

$$\operatorname{gr}\mathrm{CA}_{\mathbf{i}}(A) = \left(\left(A_{a^*,m_{a^*}}^{-1}\right)_{a\in I}, \left(\prod_{\ell\in M} A_\ell^{[\![\ell,k]\!]}\right)_{k\in[N]}\right).$$

Example 6.2. Let $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1,2,1)$. Then we have

$$\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}\,(\lambda_1,\lambda_2,x_1,x_2,x_3) = \left(\frac{1}{x_1},\frac{1}{x_2},\frac{x_1}{x_3},\frac{x_2}{\lambda_1},\frac{x_3}{\lambda_2}\right),$$

$$\operatorname{gr}\mathrm{CA}_{\mathbf{i}}\,(A_{-1},A_{-2},A_1,A_2,A_3) = \left(\frac{1}{A_2},\frac{1}{A_3},\frac{A_{-2}}{A_{-1}A_1},\frac{A_1}{A_{-2}A_2},\frac{A_2}{A_1A_3}\right).$$

Remark 6.3. In Lemma 8.1 we show that $[\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}]_{\mathrm{trop}}$ and $[\operatorname{gr}\mathrm{CA}_{\mathbf{i}}]_{\mathrm{trop}}$ and consequently also $\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}$ and $\operatorname{gr}\mathrm{CA}_{\mathbf{i}}$ are isomorphisms.

The families $(\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}})$ and $(\operatorname{gr}\mathrm{CA}_{\mathbf{i}})$ have the following transformation behavior.

Lemma 6.4. *For* $\mathbf{i},\mathbf{j}\in\mathcal{W}(\mathrm{w}_0)$ *the following diagrams commutes.*

$$\begin{array}{ccccc}
\mathbb{T}_{\mathbf{i}} & \xrightarrow{\operatorname{gr}\mathrm{CA}_{\mathbf{i}}} & \operatorname{gr}\mathbb{L}_{\mathbf{i}} & \xrightarrow{\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}} & \widehat{\mathbb{T}}_{\mathbf{i}} \\
\downarrow{\scriptstyle \mu_{\mathbf{j}}^{\mathbf{i}}} & & \downarrow{\scriptstyle id\times\Phi_{\mathbf{j}}^{\mathbf{i}}} & & \downarrow{\scriptstyle \widehat{\mu}_{\mathbf{j}}^{\mathbf{i}}} \\
\mathbb{T}_{\mathbf{j}} & \xrightarrow{\operatorname{gr}\mathrm{CA}_{\mathbf{j}}} & \operatorname{gr}\mathbb{L}_{\mathbf{j}} & \xrightarrow{\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{j}}} & \widehat{\mathbb{T}}_{\mathbf{j}}.
\end{array}$$

Proof. Without loss of generality we can assume that $\mathbf{j}$ is obtained from $\mathbf{i}$ by a 3-move at position k with $i_k = a$. Then by Lemma 4.6, the tori $\widehat{\mathbb{T}}_{\mathbf{j}}$ and $\mathbb{T}_{\mathbf{j}}$ are obtained from $\widehat{\mathbb{T}}_{\mathbf{i}}$ and $\mathbb{T}_{\mathbf{i}}$ by $\mathcal{X}$-cluster and $\mathcal{A}$-cluster transformation at the 4-valent vertex $k-1$ of $\Gamma_{\mathbf{i}}$, respectively. The commutativity of the diagram then can be checked by direct computation. □

We relate the GHKK-potential and the BK-decoration function to the functions $\mathfrak{l}_{\mathbf{i},\text{-a}}$ and $\check{\mathfrak{l}}_{\mathbf{i},\text{-a}}$ introduced in Definition 2.4 and 2.6, respectively:

Theorem 6.5. *For* $a\in\pm I$ *and* $\mathbf{i}\in\mathcal{W}(\mathrm{w}_0)$ *we have*

$$\mathfrak{l}_{\mathbf{i},\text{-a}} = W_a\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}} \circ \operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}, \tag{32}$$

$$f_a^B\big|_{\mathbb{T}_{\mathbf{i}}} = \check{\mathfrak{l}}_{\mathbf{i},\text{-a}} \circ \operatorname{gr}\mathrm{CA}_{\mathbf{i}}\,. \tag{33}$$

Proof. For $a < 0$ equalities (32) and (33) follow directly from Proposition 5.2 and Proposition 5.5, respectively. For $a > 0$ equality (32) follows directly from Lemma 6.4 and Proposition 5.2. In order to show (33) we compute using Lemma 6.4 and Proposition 5.5 for $a = i_N$, $A \in \mathbb{T}_{\mathbf{i}}$:

$$\check{\mathfrak{l}}_{\mathbf{i},\text{-a}} \circ \operatorname{gr}\mathrm{CA}_{\mathbf{i}}(A) = \left(\prod_{b\in I} A_{b,m_b}^{-c_{a,b}}\right) A_{a,m_a-1}A_{a,m_a}\prod_{b\neq a} A_{b,m_b}^{c_{a,b}} = \frac{A_{N^-}}{A_N} = f_a^B\big|_{\mathbb{T}_{\mathbf{i}}}(A). \quad \square$$

As a direct corollary of Theorem 6.5 we obtain:

Corollary 6.6. *For $a \in I$ the functions $\mathfrak{l}_{\mathbf{i},\text{-a}}$ and $\check{\mathfrak{l}}_{\mathbf{i},\text{-a}}$ are regular.*

Theorem 6.5 has the following implications for the interplay of parametrizations of canonical bases.

Corollary 6.7. *We have the following equalities of cones:*

$$\widehat{\mathcal{C}_{\mathbf{i}}} = [\operatorname{gr}\widehat{\mathrm{NA}_{\mathbf{i}}}]_{\mathrm{trop}}(\operatorname{gr}\mathcal{L}_{\mathbf{i}}),$$
$$\operatorname{gr}\mathcal{L}_{\mathbf{i}}^{\vee} = [\operatorname{gr}\mathrm{CA}_{\mathbf{i}}]_{\mathrm{trop}}(\mathcal{C}_{\mathbf{i}}).$$

Proof. The statement follows by the tropicalization of Theorem 6.5. $\square$

7. String parametrization via cluster varieties

In analogy to Section 6 we relate in this section the graded string cone $\operatorname{gr}\mathcal{S}_{\mathbf{i}}$ to the cone $\widehat{\mathcal{C}_{\mathbf{i}}}$ cut out by the tropicalization of the GHKK-potential function W. Dually, we relate the cone $\operatorname{gr}\mathcal{S}_{\mathbf{i}}^{\vee}$ associated to the Langlands dual Lie algebra ${}^{\mathrm{L}}\mathfrak{g}$ to the cone $\mathcal{C}_{\mathbf{i}}$ cut out by the tropicalization of the BK-decoration function f^B.

Motivated by the *Chamber Ansatz* due to Berenstein–Fomin–Zelevinsky [2] as well as by [5, Equation (4.14)] and [17, Section 5] we introduce the following coordinate transformations using the notations of Section 4.2.2, (31) and (2).

Definition 7.1. We specify $\operatorname{gr}\widehat{\mathrm{CA}_{\mathbf{i}}} \in \operatorname{Hom}(\operatorname{gr}\mathbb{S}_{\mathbf{i}}, \widehat{\mathbb{T}_{\mathbf{i}}})$ and $\operatorname{gr}\mathrm{NA}_{\mathbf{i}} \in \operatorname{Hom}(\mathbb{T}_{\mathbf{i}}, \operatorname{gr}\mathbb{S}_{\mathbf{i}})$ ($(\lambda, x) \in \operatorname{gr}\mathbb{S}_{\mathbf{i}}, X = (X_k)_{k\in M} \in \widehat{\mathbb{T}_{\mathbf{i}}}$):

$$(\operatorname{gr}\widehat{\mathrm{CA}_{\mathbf{i}}}(\lambda, x))_k = \begin{cases} \lambda_{-k}^{-1} \displaystyle\prod_{\ell\in[N]} x_\ell^{c_{i_\ell,i_k}+[\![k,\ell]\!]} & \text{if } k < 0, \\ \displaystyle\prod_{\ell\in[N]} x_\ell^{[\![k,\ell]\!]} & \text{else,} \end{cases}$$

$$\operatorname{gr}\mathrm{NA}_{\mathbf{i}}(X) = \left(\left(X_{a,m_a}^{-1}\right)_{a\in I}, \left(X_k^{-1}X_{k^-}\right)_{k\in[N]}\right).$$

Remark 7.2. In Lemma 8.1 we show that $[\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}$ and $[\operatorname{gr}\mathrm{NA}_{\mathbf{i}}]_{\mathrm{trop}}$ and consequently also $\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}$ and $\operatorname{gr}\mathrm{NA}_{\mathbf{i}}$ are isomorphisms.

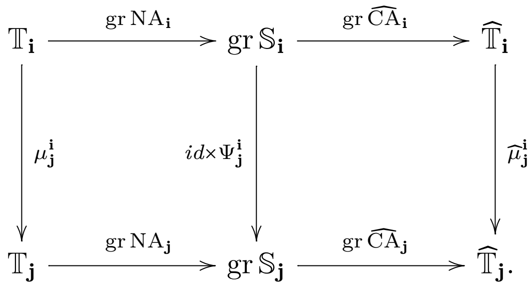
Example 7.3. For $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1,2,1)$ we have

$$\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}\,(\lambda_1,\lambda_2,x_1,x_2,x_3) = \left(\frac{x_1x_3^2}{\lambda_1 x_2}, \frac{x_2}{\lambda_2 x_3}, \frac{x_2}{x_1x_3}, \frac{x_3}{x_2}, \frac{1}{x_3}\right),$$

$$\operatorname{gr}\mathrm{NA}_{\mathbf{i}}\,(X_{-1},X_{-2},X_1,X_2,X_3) = \left(\frac{1}{X_3}, \frac{1}{X_2}, \frac{X_{-1}}{X_1}, \frac{X_{-2}}{X_2}, \frac{X_1}{X_3}\right).$$

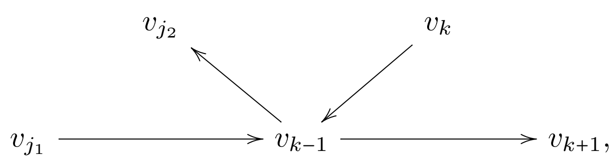
The families $(\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}})$ and $(\operatorname{gr}\mathrm{NA}_{\mathbf{i}})$ have the following transformation behavior.

Lemma 7.4. *For* $\mathbf{i},\mathbf{j} \in \mathcal{W}(\mathrm{w}_0)$ *the following diagrams commutes.*



Proof. Without loss of generality we can assume that $\mathbf{j}$ is obtained from $\mathbf{i}$ by a 3-move at position k with $i_k = a$. Then by Lemma 4.6, the tori $\widehat{\mathbb{T}}_{\mathbf{j}}$ and $\mathbb{T}_{\mathbf{j}}$ are obtained from $\widehat{\mathbb{T}}_{\mathbf{i}}$ and $\mathbb{T}_{\mathbf{i}}$ by $\mathcal{X}$-cluster and $\mathcal{A}$-cluster transformation at the 4-valent vertex $k-1$ of $\Gamma_{\mathbf{i}}$, respectively. The commutativity of the left diagram then can be checked by direct computation.

The commutativity of the right diagram is obtained as follows. Note first that the graph $\Gamma_{\mathbf{i}}$ looks locally around $k-1$ as follows:



for some $j_1, j_2 \in M \smallsetminus \{k-1,k,k+1\}$. Let $(\lambda,x) \in \operatorname{gr}\mathbb{S}_{\mathbf{i}}$. For $m \in [N]$ we define $y_m = \prod_{\ell\in[N]\smallsetminus\{k-1,k,k+1\}} x_\ell^{[\![m,\ell]\!]}$. Let $X' = (\widehat{\mu}^{\mathbf{i}}_{\mathbf{j}} \circ \operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}})(\lambda,x)$ and $X'' = (\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{j}} \circ \mathrm{id}\times\Psi^{\mathbf{i}}_{\mathbf{j}})(\lambda,x)$. For $m \in [N] \smallsetminus \{j_1,j_2,k-1,k,k+1\}$ we clearly have

$$X'_m = X''_m. \tag{34}$$

Noting the swapping of k and $k+1$ in Lemma 4.6 and the definition of $[\![\cdot,\cdot]\!]$ in (31) we have

$$X''_{k-1} = x_k^{-1} x_{k-1} x_{k+1} = X'_{k-1}$$
$$X''_k = y_k x_{k-1}^{-1} x_{k+1}^{-2} (x_{k+1} x_{k-1} + x_k) = X'_k$$
$$X''_{k+1} = y_{k+1} x_{k+1} (x_{k+1} x_{k-1} + x_k)^{-1} = X'_{k+1}.$$

If $j_1 \geq 0$, $j_2 \geq 0$, we have

$$X'_{j_2} = y_{j_2} x_k^{-1} x_{k-1} (1 + x_{k-1}^{-1} x_{k+1}^{-1} x_k) = X''_{j_2},$$
$$X'_{j_1} = y_{j_1} x_{k-1}^{-1} (1 + X_{k-1} X_{k+1} X_k^{-1})^{-1} = X''_{j_1}.$$

It remains to show (34) for $m \in \{-n, \dots, -1\}$. Using Definition 1.2 and $v_{a,0} = v_{-a}$, the following equality holds for any $b \in I$ and any reduced word $\mathbf{i}' \in \mathcal{W}(\mathrm{w}_0)$:

$$\prod_{r=0}^{m_{b,\mathbf{i}'}} \left(\mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}'}(\lambda, x)\right)_{b,r} = \lambda_b^{-1}. \tag{35}$$

By Lemma 4.9 we deduce from (35)

$$\prod_{r=0}^{m_{b,\mathbf{j}}} \left(\widehat{\mu}_{\mathbf{j}}^{\mathbf{i}} \circ \mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}}(\lambda, x)\right)_{b,r} = \lambda_b^{-1}. \tag{36}$$

Setting $-m = a \in I$ we obtain

$$X''_m = \lambda_a^{-1} \prod_{r=1}^{m_{a,\mathbf{j}}} \left(\left(\mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{j}} \circ \mathrm{id} \times \Psi_{\mathbf{j}}^{\mathbf{i}}\right)(\lambda, x)\right)_{a,r}^{-1}$$
$$= \lambda_a^{-1} \prod_{r=1}^{m_{a,\mathbf{j}}} \left(\widehat{\mu}_{\mathbf{j}}^{\mathbf{i}} \circ \mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}}(\lambda, x)\right)_{a,r}^{-1} = X'_m,$$

where the first equality follows from (35), the third from (36) and the second from what has been shown above. □

We relate the GHKK-potential and the BK-decoration function to the functions $\mathfrak{s}_{\mathbf{i},\text{-a}}$ and $\check{\mathfrak{s}}_{\mathbf{i},\text{-a}}$ introduced in Definition 3.11 and 3.14, respectively:

Theorem 7.5. *For $a \in \pm I$ and $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$ we have*

$$\mathfrak{s}_{\mathbf{i},\mathrm{a}} = W_a\big|_{\widehat{\mathbb{T}}_{\mathbf{i}}} \circ \mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}}, \tag{37}$$
$$f_a^B\big|_{\mathbb{T}_{\mathbf{i}}} = \check{\mathfrak{s}}_{\mathbf{i},\mathrm{a}} \circ \mathrm{gr}\,\mathrm{NA}_{\mathbf{i}}\,. \tag{38}$$

Proof. For $a > 0$ equalities (37) and (38) follow directly from Lemma 7.4 together with Proposition 5.2 and Proposition 5.5, respectively. We define $k(a,s)$ by $v_{k(a,s)} := v_{a,s}$. For $a < 0$ equality (37) follows from Proposition 5.2 and $(X \in \widehat{\mathbb{T}}_{\mathbf{i}})$

$$\prod_{r=0}^{s}\prod_{\ell\in[N]} X_\ell^{[\![k(-a,r),\ell]\!]} = X_{-a,s+1}\prod_{\ell\le k(-a,s+1)} X_\ell^{-c_{i_\ell,-a}}.$$

To prove (38) we compute using Proposition 5.5

$$\check{\mathfrak{s}}_{\mathbf{i},\mathrm{a}}\circ \operatorname{gr}\mathrm{NA}_{\mathbf{i}}(A) = \left(\prod_{b\in I} A_{b,m_b}^{-c_{-a,b}}\right)\sum_{\substack{k\in[N]\\ i_k=-a}}\frac{A_k}{A_{k^-}}\prod_{\ell=k+1}^{N}\left(\frac{A_{\ell^-}}{A_\ell}\right)^{-c_{i_\ell,-a}} = f_a^B\big|_{\mathbb{T}_{\mathbf{i}}}(A). \quad \square$$

As a direct corollary of Theorem 7.5 we obtain:

Corollary 7.6. *For $a\in I$ the functions $\mathfrak{s}_{\mathbf{i},\mathrm{a}}$ and $\check{\mathfrak{s}}_{\mathbf{i},\mathrm{a}}$ are regular.*

Theorem 7.5 has the following implications for the interplay of parametrizations of canonical bases.

Corollary 7.7. *We have the following equality of cones:*

$$\widehat{\mathcal{C}_{\mathbf{i}}} = [\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}(\operatorname{gr}\mathcal{S}_{\mathbf{i}}), \tag{39}$$
$$\operatorname{gr}\mathcal{S}_{\mathbf{i}}^{\vee} = [\operatorname{gr}\mathrm{NA}_{\mathbf{i}}]_{\mathrm{trop}}\mathcal{C}_{\mathbf{i}}.$$

Proof. The statement follows by tropicalization Theorem 7.5 and Proposition 3.13. $\square$

Remark 7.8. For the special case that $\mathfrak{g}$ is of type A, equality (39) appeared in [31] for the lexicographically minimal reduced word and in [6] for an arbitrary reduced word.

8. Unimodularity of cones and polytopes

In this section we view all cones as subsets of $\mathbb{R}^{n+N}$ and will refer to them by the same name as their integral analogs.

Let $m\in\mathbb{N}$ and $C_1, C_2\subset\mathbb{R}^m$ be two polyhedral cones. We call a bijection $f: C_1\to C_2$ a *unimodular isomorphism* of C_1 and C_2 if there exists a lattice isomorphism $g:\mathbb{Z}^m\to\mathbb{Z}^m$ such that $f = g\big|_{C_1}$.

Lemma 8.1. *We have the following unimodular isomorphisms:*

$$[\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}]_{\mathrm{trop}}:\operatorname{gr}\mathcal{L}_{\mathbf{i}}\to\widehat{\mathcal{C}}_{\mathbf{i}}, \tag{40}$$
$$[\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}:\operatorname{gr}\mathcal{S}_{\mathbf{i}}\to\widehat{\mathcal{C}}_{\mathbf{i}}, \tag{41}$$
$$[\operatorname{gr}\mathrm{CA}_{\mathbf{i}}]_{\mathrm{trop}}:\mathcal{C}_{\mathbf{i}}\to\operatorname{gr}\mathcal{L}_{\mathbf{i}}^{\vee}, \tag{42}$$
$$[\operatorname{gr}\mathrm{NA}_{\mathbf{i}}]_{\mathrm{trop}}:\mathcal{C}_{\mathbf{i}}\to\operatorname{gr}\mathcal{S}_{\mathbf{i}}^{\vee}. \tag{43}$$

Proof. Reordering the coordinates on $x \in [\operatorname{gr}\mathbb{L}_{\mathbf{i}}]_{\mathrm{trop}} \otimes_{\mathbb{Z}} \mathbb{R}$ as

$$(x_1, \dots, x_N, \lambda_1, \dots, \lambda_n)$$

the definition of $[\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}]_{\mathrm{trop}}$ yields that the corresponding matrix has integer entries, is lower triangular and all diagonal entries are equal to -1, whereas the definition of $[\operatorname{gr}\mathrm{CA}_{\mathbf{i}}]_{\mathrm{trop}}$ yields that the corresponding matrix has integer entries, is upper triangular and all diagonal entries are equal to -1. Hence, $[\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}]_{\mathrm{trop}}$ and $[\operatorname{gr}\mathrm{CA}_{\mathbf{i}}]_{\mathrm{trop}}$ are lattice automorphisms of $\mathbb{Z}^{n+N}$. Using Corollary 6.7, claim (40) and (42) follow.

Reordering the coordinates on $x \in [\operatorname{gr}\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \otimes_{\mathbb{Z}} \mathbb{R}$ as

$$(\lambda_1, \dots, \lambda_n, x_1, \dots, x_N)$$

the definition of $[\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}$ yields that the corresponding matrix has integer entries, is upper triangular and all diagonal entries are equal to -1. Hence, $[\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}$ is a lattice automorphism of $\mathbb{Z}^{n+N}$. Using Corollary 7.7 claim (41) follows.

Reordering the coordinates on $y \in [\mathbb{S}_{\mathbf{i}}]_{\mathrm{trop}} \otimes_{\mathbb{Z}} \mathbb{R}$ as

$$(x_1, \dots, x_N, \lambda_n, \lambda_{n-1}, \dots, \lambda_1)$$

the definition of $[\operatorname{gr}\mathrm{NA}_{\mathbf{i}}]_{\mathrm{trop}}$ yields that the corresponding matrix has integer entries, is upper triangular and all diagonal entries in $\{-1, 1\}$. Hence, $[\operatorname{gr}\mathrm{NA}_{\mathbf{i}}]_{\mathrm{trop}}$ is a lattice automorphism of $\mathbb{Z}^{n+N}$. Using Corollary 7.7 claim (43) follows. □

Using Lemma 8.1 we deduce a unimodular isomorphism of the graded string cone and the graded cone of Lusztig's parametrization which can be found in the literature combining [33, Corollaire 3.5], [8, Lemma 6.3].

Proposition 8.2.

(1) *The map* $[\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}^{-1} \circ \operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}$ *is a unimodular isomorphism of* $\operatorname{gr}\mathcal{S}_{\mathbf{i}}$ *and* $\operatorname{gr}\mathcal{L}_{\mathbf{i}}$. *Explicitly, it is given by*

$$\begin{aligned} (\lambda', x') &= [\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}^{-1} \circ \operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}(\lambda, x), \\ \lambda'_a &= \lambda_{a^*}, \\ x'_k &= \lambda_{i_k} - x_k - \sum_{\ell > k} c_{i_k, i_\ell} x_\ell. \end{aligned} \tag{44}$$

(2) *The map* $[\operatorname{gr}\mathrm{CA}_{\mathbf{i}} \circ \operatorname{gr}\mathrm{NA}_{\mathbf{i}}^{-1}]_{\mathrm{trop}}$ *is a unimodular isomorphism of* $\operatorname{gr}\mathcal{S}_{\mathbf{i}}^{\vee}$ *and* $\operatorname{gr}\mathcal{L}_{\mathbf{i}}^{\vee}$. *Explicitly, it is given by*

$$\begin{aligned}(\lambda', x') &= [\operatorname{gr}\mathrm{CA}_{\mathbf{i}} \circ \operatorname{gr}\mathrm{NA}_{\mathbf{i}}{}^{-1}]_{\mathrm{trop}}(\lambda, x),\\ \lambda'_a &= \lambda_{a^*},\\ x'_k &= \left(\sum_{a\in I} c_{i_k,a}\lambda_a\right) - x_k - \sum_{\ell>k} c_{i_k,i_\ell} x_\ell.\end{aligned} \tag{45}$$

Proof. The unimodularity follows from Lemma 8.1. It remains to show the explicit description of the maps. We define $k(a,s)$ by $v_{k(a,s)} := v_{a,s}$. Then (44) follows by tropicalizing the identities

$$\begin{aligned}x'_{a,r} &= \prod_{r=0}^{s-1}([\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}(\lambda,x))^{-1}_{a,r} = \lambda_a x^{-1}_{a,s} \prod_{\ell>k(a,s)} x_\ell^{-c_{i_\ell,a}},\\ \lambda'_a &= \prod_{r=0}^{m_{a^*}}([\operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}(\lambda,x))^{-1}_{a^*,r} = \lambda_{a^*}.\end{aligned}$$

Using $\operatorname{gr}\mathrm{NA}_{\mathbf{i}}{}^{-1}(\lambda,x) = \lambda_a^{-1}\prod_{s>r} x_{a,s}$ equality (45) follows by applying $\operatorname{gr}\mathrm{CA}_{\mathbf{i}}$ and tropicalizing. □

We call two polytopes $P_1, P_2 \subset \mathbb{R}^m$ *affine unimodular isomorphic* if there exists a lattice isomorphism $g : \mathbb{Z}^m \to \mathbb{Z}^m$ and a vector $v \in \mathbb{Z}^m$ such that $g(P_1) + v = P_2$.

For $\lambda \in \mathbb{N}^I$ the polytope

$$\mathcal{S}_{\mathbf{i}}(\lambda) = \{x \in \mathbb{N}^N \mid (\lambda, x) \in \operatorname{gr}\mathcal{S}_{\mathbf{i}}\} \tag{46}$$

is called the *string polytope of weight* λ. For $\lambda \in \mathbb{N}^I$ the polytope

$$\mathcal{L}_{\mathbf{i}}(\lambda) = \{x \in \mathbb{N}^N \mid (\lambda, x) \in \operatorname{gr}\mathcal{L}_{\mathbf{i}}\}$$

is called *Lusztig's polytope of weight* λ.

Corollary 8.3. *For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^N$ and $\lambda^* := (\lambda_{1^*}, \ldots, \lambda_{n^*})$, the polytopes $\mathcal{S}_{\mathbf{i}}(\lambda)$ and $\mathcal{L}_{\mathbf{i}}(\lambda^*)$ are affine unimodular isomorphic.*

Proof. By Proposition 8.2, we have

$$[\operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}}^{-1} \circ \operatorname{gr}\widehat{\mathrm{CA}}_{\mathbf{i}}]_{\mathrm{trop}}(\{\lambda\} \times \mathcal{S}_{\mathbf{i}}(\lambda)) = \{\lambda^*\} \times \mathcal{L}_{\mathbf{i}}(\lambda^*)$$

and the claim follows. □

9. Proof of Theorem A

In this section we deduce Theorem A from Theorem 6.5. We set

$$\mathrm{p}_{\mathbf{i}} := \operatorname{gr}\widehat{\mathrm{NA}}_{\mathbf{i}} \circ \mathcal{D} \circ \operatorname{gr}\mathrm{CA}_{\mathbf{i}} : \mathbb{T}_{\mathbf{i}} \to \widehat{\mathbb{T}}_{\mathbf{i}},$$

where $\mathcal{D}: \mathbb{G}_m^I \times \mathbb{G}_m^N \to \mathbb{G}_m^I \times \mathbb{G}_m^N$ denotes the regular map given by $\mathcal{D}(\lambda, x) = (\lambda', x)$ and

$$\lambda'_a = \prod_{b \in I} \lambda_b^{c_{a,b}}.$$

By Theorem 6.5 we have for $\mathbf{i} \in \mathcal{W}(\mathrm{w}_0)$

$$f^B\big|_{\mathbb{T}_\mathbf{i}} = W\big|_{\widehat{\mathbb{T}}_\mathbf{i}} \circ \mathrm{p}_\mathbf{i}\,.$$

Thus, globally $f^B = W \circ \mathrm{p}$ holds, where $\mathrm{p} = (\mathrm{p}_\Sigma) : \mathcal{A} \to \mathcal{X}$ is defined as

$$\mathrm{p}_\Sigma := \widehat{\mu}_\Sigma^{\Sigma_\mathbf{i}} \circ \mathrm{p}_\mathbf{i} \circ \mu_{\Sigma_\mathbf{i}}^\Sigma : \mathbb{T}_\Sigma \to \widehat{\mathbb{T}}_\Sigma. \tag{47}$$

A priori p_Σ is a rational function. It thus remains to show that p_Σ is regular. We establish the stronger statement that $\mathrm{p} = (\mathrm{p}_\Sigma)$ is a p-map in the sense of [22, Chapter 2] (see Definition 4.2):

Proposition 9.1. *The map* $\mathrm{p} = (\mathrm{p}_\Sigma)$ *defined by* (47) *is a* p*-map.*

Proof. By Lemma 4.3 it is enough to verify (19) and (20) for a single seed Σ. We therefore assume that $\Sigma = \Sigma_\mathbf{i}$ is associated to a reduced word $\mathbf{i}$ as described in Section 4.2.2. In particular, in (22) we identified the vertices of the quiver associated to $\Sigma_\mathbf{i}$ with the set $M = -[n] \cup [N]$. For

$$k \in M \backslash M_0 = \{k \in [N] \,|\, k^+ \in [N]\}$$

Equations (19) and (20) hold since

$$\begin{aligned}(\mathrm{p}_\mathbf{i}\, x)_k &= \left(\mathrm{gr}\,\widehat{\mathrm{NA}}_\mathbf{i} \circ \mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x\right)_k = \frac{(\mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x)_k}{(\mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x)_{k^+}} = \frac{(\mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x)_k}{(\mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x)_{k^+}} \\ &= \prod_{\ell \in M} x_\ell^{[\![\ell,k]\!] - [\![\ell,k^+]\!]} = \frac{x_{k^+}}{x_{k^-}} \prod_{\substack{\ell \in M \\ \ell < k < \ell^+ < k^+ \\ c_{i_\ell, i_k} = -1}} x_\ell \prod_{\substack{\ell \in M \\ k < \ell < k^+ < \ell^+ \\ c_{i_\ell, i_k} = -1}} x_\ell^{-1} = \prod_{\ell \in M} x_\ell^{\{e_k, e_\ell\}}.\end{aligned}$$

It remains to show (19) for $k \in M_0$. Suppose that $k \in -[n]$. We have

$$\begin{aligned}(\mathrm{p}_\mathbf{i}\, x)_k &= \left(\mathrm{gr}\,\widehat{\mathrm{NA}}_\mathbf{i} \circ \mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x\right)_k = (\mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x)_{k^+}^{-1} = (\mathrm{gr}\,\mathrm{CA}_\mathbf{i}\, x)_{k^+}^{-1} \\ &= \prod_{\ell \in M} x_\ell^{-[\![\ell,k^+]\!]} = x_k\,(x_{k^+}) \prod_{\substack{\ell \in M \\ \ell < k \\ k^+ < \ell^+ \\ c_{i_\ell, i_k} = -1}} x_\ell^{-1} \prod_{\substack{\ell \in M \\ k < \ell < k^+ < \ell^+ \\ c_{i_\ell, i_k} = -1}} x_\ell^{-1}.\end{aligned}$$

Assuming that k^+ is not contained in M_0 we thus obtain for the restriction $\mathrm{p}_\mathbf{i} \circ \iota_\mathbf{i}$ of $\mathrm{p}_\mathbf{i}$ to the torus corresponding to the vertices $\ell \in M \backslash M_0$

$$(\mathrm{p}_{\mathbf{i}} \circ \iota_{\mathbf{i}})_k = (x_{k^+}) \prod_{\substack{\ell \in M \setminus M_0 \\ k<\ell<k^+<\ell^+ \\ c_{i_\ell, i_k}=-1}} x_\ell^{-1} = \prod_{\ell \in M \setminus M_0} x_\ell^{\{e_k, e_\ell\}}.$$

If k^+ is contained in M_0 we obtain

$$(\mathrm{p}_{\mathbf{i}} \circ \iota_{\mathbf{i}})_k = \prod_{\substack{\ell \in M \setminus M_0 \\ k<\ell<k^+<\ell^+ \\ c_{i_\ell, i_k}=-1}} x_\ell^{-1} = \prod_{\ell \in M \setminus M_0} x_\ell^{\{e_k, e_\ell\}}.$$

It remains to verify (19) for $k \in [N]$ with $k^+ \notin [N]$:

$$\begin{aligned}(\mathrm{p}_{\mathbf{i}}\, x)_k &= \left(\mathrm{gr}\,\widehat{\mathrm{NA}}_{\mathbf{i}} \circ \mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_{\mathbf{i}}\, x\right)_k = (\mathcal{D} \circ \mathrm{gr}\,\mathrm{CA}_{\mathbf{i}}\, x)_k \prod_{b \in I} x_{b^*, m_{b^*}}^{c_{i_k^*, b^*}} \\ &= \prod_{\ell \in M} x_\ell^{[\![\ell, k]\!]} \prod_{b \in I} x_{b, m_b}^{c_{i_k^*, b}} = (x_{k^-})^{-1} x_k^{-1} \prod_{\substack{\ell \in M \\ \ell<k<\ell^+ \\ c_{i_\ell, i_k}=-1}} x_\ell .\end{aligned}$$

Assuming that k^- is not contained in M_0 we thus obtain for the restriction $\mathrm{p}_{\mathbf{i}} \circ \iota_{\mathbf{i}}$ of $\mathrm{p}_{\mathbf{i}}$ to the torus corresponding to the vertices $\ell \in M \setminus M_0$

$$(\mathrm{p}_{\mathbf{i}} \circ \iota_{\mathbf{i}})_k = (x_{k^-})^{-1} \prod_{\substack{\ell \in M \setminus M_0 \\ \ell<k<\ell^+<k^+ \\ c_{i_\ell, i_k}=-1}} x_\ell = \prod_{\ell \in M \setminus M_0} x_\ell^{\{e_k, e_\ell\}}.$$

If k^- is contained in M_0 we obtain

$$(\mathrm{p}_{\mathbf{i}} \circ \iota_{\mathbf{i}})_k = \prod_{\substack{\ell \in M \setminus M_0 \\ \ell<k<\ell^+<k^+ \\ c_{i_\ell, i_k}=-1}} x_\ell = \prod_{\ell \in M \setminus M_0} x_\ell^{\{e_k, e_\ell\}}. \quad \square$$

From explicit construction of $\mathrm{p}_{\mathbf{i}}$ for seeds $\Sigma = \Sigma_{\mathbf{i}}$ associated to reduced words we deduce

Proposition 9.2. *The map* $\mathrm{p} : \mathcal{A} \to \mathcal{X}$ *is a finite cover. The degree of* p *equals the determinant of the Cartan-Matrix* $C = (c_{a,b})$.

Proof. By Proposition 8.2 the maps $\mathrm{gr}\,\widehat{\mathrm{NA}}_{\mathbf{i}}$ and $\mathrm{gr}\,\mathrm{CA}_{\mathbf{i}}$ are isomorphisms. The claim follows by inspecting the definition of $\mathcal{D}$. $\square$

Example 9.3. For $\mathfrak{g} = \mathrm{sl}_3(\mathbb{C})$ and $\mathbf{i} = (1, 2, 1)$ we have

$$\mathrm{p}\,(A_{-1}, A_{-2}, A_1, A_2, A_3) = \left(\frac{A_2}{A_{-1}}, \frac{A_{-1}A_3}{A_{-2}A_2}, \frac{A_{-2}A_2}{A_1 A_3}, \frac{A_{-2}, A_3}{A_{-1}}, \frac{A_{-1}A_1}{A_{-2}}\right).$$

By Theorem 7.5 we obtain that the map $\mathrm{p}' : \mathcal{A} \to \mathcal{X}$ defined by

$$\mathrm{p}'_{\mathbf{i}} := \mathrm{gr}\,\widehat{\mathrm{CA}}_{\mathbf{i}} \circ \mathcal{D} \circ \mathrm{gr}\,\mathrm{NA}_{\mathbf{i}} : \mathbb{T}_{\mathbf{i}} \to \widehat{\mathbb{T}}_{\mathbf{i}}$$

also satisfies $f_B = W \circ \mathrm{p}'$. We deduce from a computation analogous to the proof of Proposition 9.1 that the functions p' and p coincide.

Remark 9.4. In [16] the first author gives another construction of the map p in type A as a canonical choice of a p-map $\mathrm{p} : \mathcal{A} \to \mathcal{X}$ without referring to the Chamber Ansatz. In [16] the map p occurs as a crucial ingredient in the explicit definition of a $B(\infty)$-crystal structure on the analogues of $\mathcal{C}_{\mathbf{i}} \subset [\mathbb{T}_{\mathbf{i}}]_{\mathrm{trop}}$ and $\widehat{\mathcal{C}_{\mathbf{i}}} \subset [\widehat{\mathbb{T}}_{\mathbf{i}}]_{\mathrm{trop}}$ for the reduced double Bruhat cell.

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Appendix A. Maximal green sequences

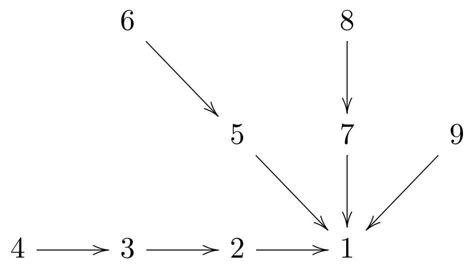
We refer to [26] for an overview over the topic of maximal green sequences. Goodearl and Yakimov [21] announced that for G simple, simply connected the cluster algebra of the double Bruhat cell $B^{\mathrm{w}_0,e}$ possesses a maximal green sequence. For $B^{\mathrm{w}_0,e}$ of type A a maximal green sequence was constructed in [32]. Following a suggestion of the anonymous referee we get the existence of maximal green sequences for double Bruhat cells of type D and E which we demonstrate here, details of proofs and more general results appear in another paper.

Recall, that for a quiver Q without frozen vertices, the *principal extension* is a quiver Q^{prin} such that for each vertex $v \in Q$ we add a frozen vertex v^d and an arrow $v \to v^d$. A vertex v of a quiver is *green* if in the principal extension all arrows joining the vertex v and frozen vertices are directed from the vertex v to the frozen and *red* if the all arrows joining the vertex v and frozen vertices are directed from the frozen vertex to v. A *maximal green sequence* is a sequence of mutations at green vertices which starts at Q^{prin} and terminates at a quiver where all mutable vertices are red.

To get maximal green sequences for the types D_n and E, we consider a more general situation.

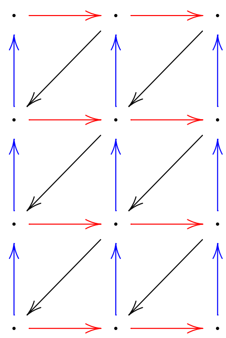
We need some notations. For natural numbers $k_1 \geq \ldots \geq k_s \geq 2$ we denote by $T_{k_1,\ldots,k_s}$ a quiver which is a *bouquet* of equioriented quivers of types $A_{k_1}, \ldots, A_{k_s}$ such that each line quiver has one sink and all these sinks are glued together at the root of the bouquet. We label the vertices of the bouquet $T_{k_1,\ldots,k_s}$ as follows. The unique sink is labeled by 1. We then traverse from the sink to the source of A_{k_1} and assign gradually labels $2, \ldots, k_1$. We continue with A_{k_2} from the sink and assign gradually labels $k_1 + 1, \ldots, k_1 + k_2 - 1$; $k_1 + k_2, \ldots, k_1 + k_2 + k_3 - 2$; and so on until A_{k_s} with the labels $k_1 + \ldots + k_{s-1} - (s-2) + 1, \ldots, k_1 + \ldots + k_s - (s-1)$.

Here we depicted $T_{4,3,3,2}$.



A *banner* $B_{m,l}$ of size $m \times \ell$ ($(m, \ell) \in \mathbb{Z}^2_{\geq 2}$) is a quiver which is the product of equioriented quivers of type A_m and type A_ℓ enriched with arrows in each type $A_2 \times A_2$-subquiver, such that the added arrows divide the squares into two cyclically ordered triangles.

Here is an example of $B_{3,4}$, where each copy of type A_3 is colored red, each copy of type A_4 is colored blue and the arrows diving the squares are colored black. (For interpretation of the colors in the diagrams, the reader is referred to the web version of this article.)

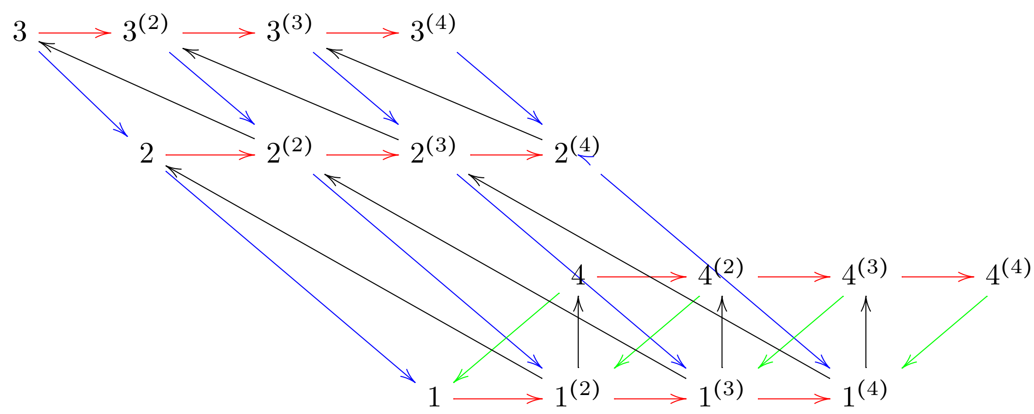


A *multi-banner* $T_{k_1,\ldots,k_s} \square A_\ell$ is defined as follows. We take the equioriented quiver of type A_ℓ with a single source and single sink and enrich the product $T_{k_1,\ldots,k_s} \times A_\ell$ by replacing each product of type $A_{k_i} \times A_\ell$ with the banner $B_{k_i,\ell}$. In other words, we take the amalgamation of the banners $B_{k_1,\ell}, \ldots, B_{k_s,\ell}$ along A_ℓ.

We assign the following labels to the vertices of the multi-banner $T_{k_1,\ldots,k_s} \square A_\ell$. Firstly, we place the original bouquet $T_{k_1,\ldots,k_s}$ at the source of A_ℓ. We place the copy $T^{(2)}_{k_1,\ldots,k_s}$

at the vertex next to the source and so on, the last copy $T^{(\ell)}_{k_1,\dots,k_s}$ at the sink vertex. We label the vertices of the t-the copy $T^{(t)}_{k_1,\dots,k_s}$, $2 \le t \le \ell$, by $r^{(t)}$, where r denotes the label of the original vertex of the bouquet $T_{k_1,\dots,k_s}$, $1 \le r \le k_1 + \ldots + k_s - (s-1)$.

Here is an example of $T_{3,2} \square A_4$, where the quivers of type A_3 in each $T_{3,2}$ copy are colored blue, the quivers of type A_2 green, while the arrows corresponding to the type A_4-quivers in the product are colored red. The arrows we add to the products in the banner are colored black.



A maximal green sequence for the above example is given by successive mutations at the following sequence of vertices read from left to right:

$$\begin{aligned}&(4^{(4)},3^{(4)},2^{(4)},1^{(4)},4^{(3)},3^{(3)},2^{(3)},1^{(3)},4^{(4)},3^{(4)},2^{(4)},1^{(4)},4^{(2)},3^{(2)},2^{(2)},\\&1^{(2)},4^{(3)},3^{(3)},2^{(3)},1^{(3)},4^{(4)},3^{(4)},2^{(4)},1^{(4)},4^{(1)},3^{(1)},2^{(1)},1^{(1)},4^{(2)},3^{(2)},\\&2^{(2)},1^{(2)},4^{(3)},3^{(3)},2^{(3)},1^{(3)},4^{(4)},3^{(4)},2^{(4)},1^{(4)}).\end{aligned}$$

We denote by $\mathfrak{T}_r$ $(r \in [\ell])$ the composition of mutations at the following sequence of vertices in the multi-banner $T_{k_1,\dots,k_s} \square A_\ell$:

$$\mathfrak{T}_r := ((k_1 + k_2 + \ldots + k_s - s + 1)^{(r)}, (k_1 + k_2 + \ldots + k_s - s)^{(r)}, \ldots, 1^{(r)})$$

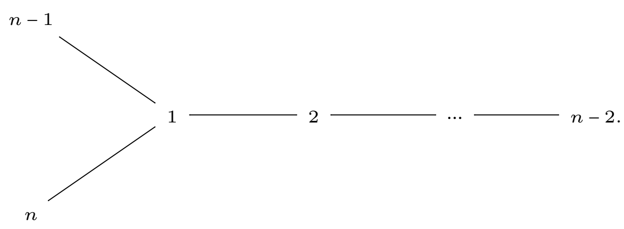
Proposition A.1. *For a multi-banner $T_{k_1,\dots,k_s} \square A_\ell$, the sequence*

$$\mathfrak{T}_n\,(\mathfrak{T}_{n-1}\mathfrak{T}_n)\ldots(\mathfrak{T}_1\mathfrak{T}_2\ldots\mathfrak{T}_n)$$

is a maximal green sequence.

From this proposition we get maximal green sequences for type D_n $(n \ge 4)$, E_6, E_7, and E_8 as follows.

For type D_n, we consider the following labeling of simple roots depicted in the Dynkin diagram:

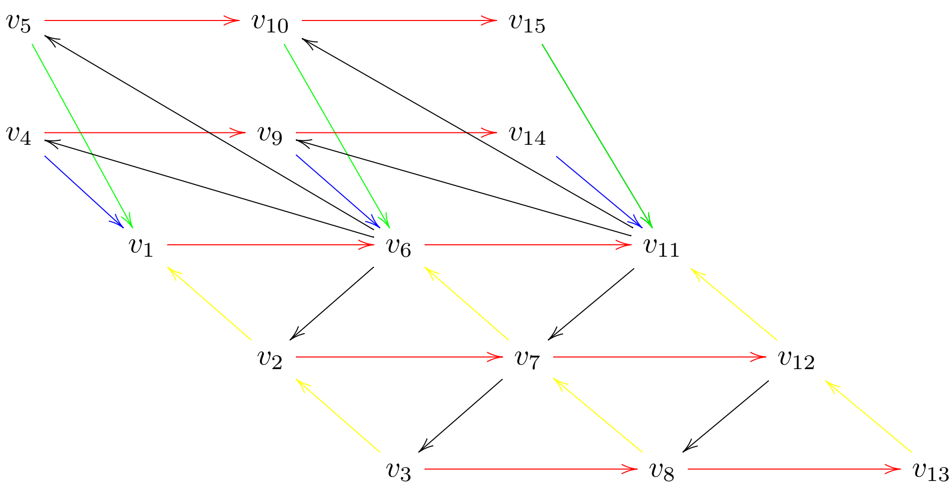


Then, for the reduced word

$$\mathbf{i}_D := (12\dots(n-2)(n-1)n)^{n-1} \in \mathcal{W}(\mathrm{w}_0)$$

the quiver $\Gamma_{\mathbf{i}_D}$ with frozen vertices cut off is the multi-banner $T_{n-2,2,2} \square A_{n-1}$. Therefore the maximal green sequence of Proposition A.1 for $T_{n-2,2,2} \square A_{n-2}$ is a maximal green sequence for the D_n-type double Bruhat cell $B^{\mathrm{w}_0,e}$.

We give an example of the graph $\Gamma_{\mathbf{i}_D}$ for D_5 below. This is the multi-banner $T_{3,2,2} \square A_3$. In every copy of $T_{3,2,2}$ we color the corresponding quiver of type A_3 yellow and the quivers of type A_2 green and blue, respectively. The type A_3 quiver along which the amalgamation is taken is colored red. The labels of the vertices are the labels used in the definition of $\Gamma_{\mathbf{i}_D}$.



A maximal green sequence for the quiver $\Gamma_{\mathbf{i}_D}$ is in the example above given by the sequence of mutations at the vertices $(15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 15, 14, 13, 12, 11, 5, 4, 3, 2, 1, 10, 9, 8, 7, 6, 15, 14, 13, 12, 11)$.

For general D_n a maximal green sequence is given by successive mutation on the vertices:

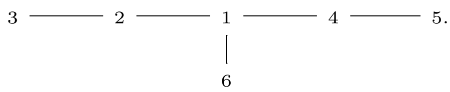
$$\mathfrak{S}(1), \mathfrak{S}(2), \dots, \mathfrak{S}\left(\frac{N}{n} - 1\right), \tag{48}$$

where we set

$$\mathfrak{I}(k) := (N - kn, N - kn - 1, \dots, N - (k+1)n + 1),$$
$$\mathfrak{S}(k) := \mathfrak{I}(k), \mathfrak{I}(k-1), \dots, \mathfrak{I}(1).$$

The reader who prefers not to rely on Proposition A.1 will encounter no difficulty in checking that (48) is a maximal green sequence by induction over n.

For type E_6, we consider the following labeling of simple roots:



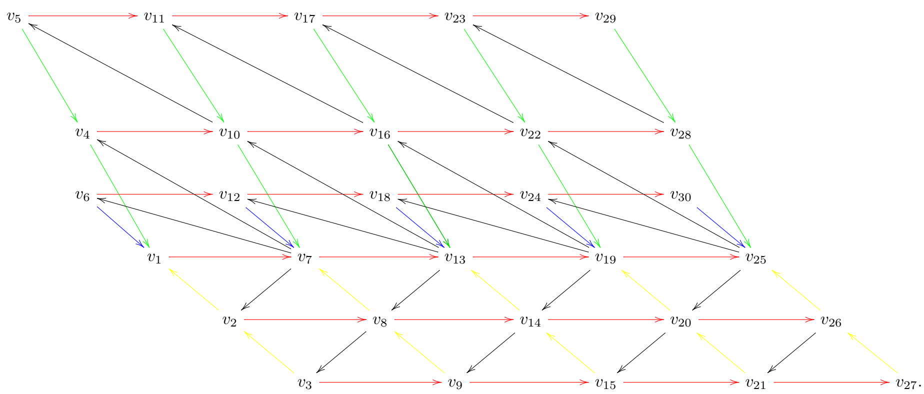
Then, for the reduced word

$$\mathbf{i}_{E_6} = (123456)^6 \in \mathcal{W}(\mathrm{w}_0)$$

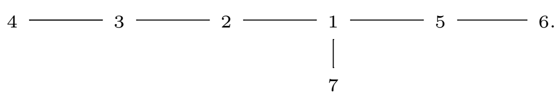
the quiver $\Gamma_{\mathbf{i}_{E_6}}$ with frozen vertices cut off is the multi-banner $T_{3,3,2} \square A_5$. We get the following maximal green sequence from Proposition A.1:

$$(30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 30, 29, 28, 27, 26, 25, 18, 17, 16, 15, 14, 13, 24, 23, 22, 21, 20, 19, 30, 29, 28, 27, 26, 25, 12, 11, 10, 9, 8, 7, 18, 17, 16, 15, 14, 13, 24, 23, 22, 21, 20, 19, 30, 29, 28, 27, 26, 25, 6, 5, 4, 3, 2, 1, 12, 11, 10, 9, 8, 7, 18, 17, 16, 15, 14, 13, 24, 23, 22, 21, 20, 19, 30, 29, 28, 27, 26, 25).$$

We depict the unfrozen part of $\Gamma_{\mathbf{i}_{E_6}}$ below for the convenience of the reader:



For type E_7, we consider the following labeling of simple roots:



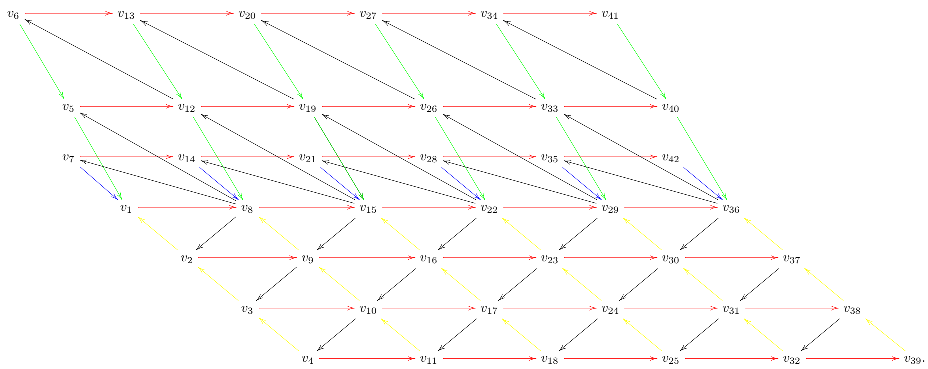
Then for the reduced word

$$\mathbf{i}_{E_7} = (1234567)^7 \in \mathcal{W}(\mathrm{w}_0)$$

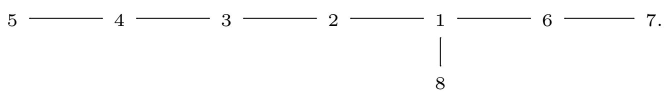
the quiver $\Gamma_{\mathbf{i}_{E_7}}$ with frozen vertices cut off is the multi-banner $T_{4,3,2} \,\square\, A_6$. Hence we obtain from Proposition A.1 the maximal green sequence

$$\begin{aligned}&(42,41,40,39,38,37,36,35,34,33,32,31,30,29,42,41,40,39,38,37,36,\\&28,27,26,25,24,23,22,35,34,33,32,31,30,29,42,41,40,39,38,37,36,\\&21,20,19,18,17,16,15,28,27,26,25,24,23,22,35,34,33,32,31,30,29,\\&42,41,40,39,38,37,36,14,13,12,11,10,9,8,21,20,19,18,17,16,15,28,\\&27,26,25,24,23,22,35,34,33,32,31,30,29,42,41,40,39,38,37,36,7,6,\\&5,4,3,2,1,14,13,12,11,10,9,8,21,20,19,18,17,16,15,28,27,26,25,24,\\&23,22,35,34,33,32,31,30,29,42,41,40,39,38,37,36)\end{aligned}$$

We depict the unfrozen part of $\Gamma_{\mathbf{i}_{E_7}}$ below for the convenience of the reader:



For type E_8 we consider the following labeling of simple roots:



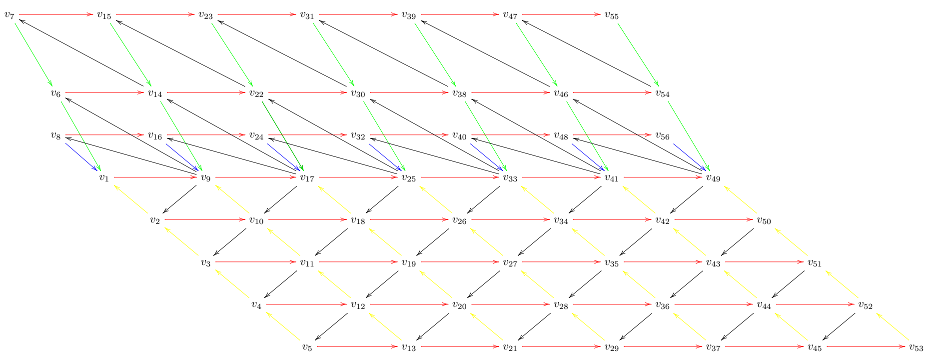
Then, for the reduced word

$$\mathbf{i}_{E_8} = (12345678)^8 \in \mathcal{W}(\mathrm{w}_0)$$

the quiver $\Gamma_{\mathbf{i}_{E_8}}$ with frozen vertices cut off is the multi-banner $T_{5,3,2} \,\square\, A_7$. Thus from Proposition A.1 we obtain the maximal green sequence

$$(56, 55, 54, 53, 52, 51, 50, 49, 48, 47, 46, 45, 44, 43, 42, 41, 56, 55, 54, 53, 52,$$
$$51, 50, 49, 40, 39, 38, 37, 36, 35, 34, 33, 48, 47, 46, 45, 44, 43, 42, 41, 56, 55,$$
$$54, 53, 52, 51, 50, 49, 32, 31, 30, 29, 28, 27, 26, 25, 40, 39, 38, 37, 36, 35, 34,$$
$$33, 48, 47, 46, 45, 44, 43, 42, 41, 56, 55, 54, 53, 52, 51, 50, 49, 24, 23, 22, 21,$$
$$20, 19, 18, 17, 32, 31, 30, 29, 28, 27, 26, 25, 40, 39, 38, 37, 36, 35, 34, 33, 48,$$
$$47, 46, 45, 44, 43, 42, 41, 56, 55, 54, 53, 52, 51, 50, 49, 16, 15, 14, 13, 12, 11,$$
$$10, 9, 24, 23, 22, 21, 20, 19, 18, 17, 32, 31, 30, 29, 28, 27, 26, 25, 40, 39, 38,$$
$$37, 36, 35, 34, 33, 48, 47, 46, 45, 44, 43, 42, 41, 56, 55, 54, 53, 52, 51, 50, 49,$$
$$8, 7, 6, 5, 4, 3, 2, 1, 16, 15, 14, 13, 12, 11, 10, 9, 24, 23, 22, 21, 20, 19, 18, 17,$$
$$32, 31, 30, 29, 28, 27, 26, 25, 40, 39, 38, 37, 36, 35, 34, 33, 48, 47, 46, 45, 44,$$
$$43, 42, 41, 56, 55, 54, 53, 52, 51, 50, 49).$$

We depict the unfrozen part of $\Gamma_{\mathbf{i}_{E_8}}$ for the convenience of the reader:



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