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Drinfeld–Gaitsgory–Vinberg interpolation Grassmannian and geometric Satake equivalence

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with Appendix by Dennis Gaitsgory

To Ernest Borisovich Vinberg with admiration

Abstract

Let G be a reductive complex algebraic group. We fix a pair of opposite Borel subgroups and consider the corresponding semi-infinite orbits in the affine Grassmannian Gr_G. We prove Simon Schieder's conjecture identifying his bialgebra formed by the top compactly supported cohomology of the intersections of opposite semi-infinite orbits with $U(\mathfrak{n}^\vee)$ (the universal enveloping algebra of the positive nilpotent subalgebra of the Langlands dual Lie algebra $\mathfrak{g}^\vee$). To this end we construct an action of Schieder bialgebra on the geometric Satake fiber functor. We propose a conjectural construction of Schieder bialgebra for an arbitrary symmetric Kac–Moody Lie algebra in terms of Coulomb branch of the corresponding quiver gauge theory.

1. *Introduction*

1.1

Let $\Lambda = \bigoplus_n \Lambda^n$ be the ring of symmetric functions equipped with the base of Schur functions s_λ. It also carries a natural coproduct. The classical Schubert calculus is the based isomorphism of the bialgebra Λ with $H^\bullet(\mathrm{Gr}, \mathbb{Z})$ (doubling the degrees) taking s_λ to the fundamental class of the corresponding Schubert variety σ_λ. Here Gr is the infinite Grassmannian $\mathrm{Gr} = \lim_\to \mathrm{Gr}(k, m) \simeq BU(\infty)$, and the coproduct on $H^\bullet(\mathrm{Gr}, \mathbb{Z})$ comes from the H-space structure on the classifying space $BU(\infty)$.

Here is a more algebraic geometric construction of the coproduct on $H^\bullet(\mathrm{Gr}, \mathbb{Z})$. We have $H^{2n}(\mathrm{Gr}, \mathbb{Z}) = H^{2n}(\overline{\mathrm{Sch}}_n, \mathbb{Z}) = H^{2n}_c(\overline{\mathrm{Sch}}_n, \mathbb{Z}) = H^{2n}_c(\mathrm{Sch}_n, \mathbb{Z})$, where $\mathrm{Sch}_n \subset \mathrm{Gr}$ (respectively, $\overline{\mathrm{Sch}}_n \subset \mathrm{Gr}$) stands for the union of all n-dimensional (respectively, $\leqslant n$-dimensional) Schubert cells (with respect to a fixed flag).

Recall the Calogero–Moser phase space $\mathcal{C}_n$: the space of pairs of $n \times n$-matrices (X, Y) such that $[X, Y] + \mathrm{Id}$ has rank 1, modulo the simultaneous conjugation of X, Y. The integrable system $\pi_n \colon \mathcal{C}_n \to \mathbb{A}^{(n)}$ takes (X, Y) to the spectrum of X. Wilson [26] has discovered the following two key properties of the Calogero–Moser integrable system.

(a) For $n_1 + n_2 = n$, a *factorization* isomorphism

$$\mathcal{C}_n \times_{\mathbb{A}^{(n)}} (\mathbb{A}^{(n_1)} \times \mathbb{A}^{(n_2)})_{\mathrm{disj}} \xrightarrow{\sim} (\mathcal{C}_{n_1} \times \mathcal{C}_{n_2}) \times_{(\mathbb{A}^{(n_1)} \times \mathbb{A}^{(n_2)})} (\mathbb{A}^{(n_1)} \times \mathbb{A}^{(n_2)})_{\mathrm{disj}}.$$

(b) For $x \in \mathbb{A}^1$, an isomorphism $\pi_n^{-1}(n \cdot x) \xrightarrow{\sim} \mathrm{Sch}_n$.

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Now the desired coproduct

$$\Delta = \bigoplus_{n_1+n_2=n} \Delta_{n_1,n_2} \colon H_c^{2n}(\mathrm{Sch}_n, \mathbb{Z}) \to \bigoplus_{n_1+n_2=n} H_c^{2n_1}(\mathrm{Sch}_{n_1}, \mathbb{Z}) \otimes H_c^{2n_2}(\mathrm{Sch}_{n_2}, \mathbb{Z})$$

is nothing but the cospecialization† morphism for the compactly supported cohomology of the fibers of π_n restricted to the subfamily $\pi_n^{-1}(n_1 \cdot x + n_2 \cdot y) \subset \mathcal{C}_n$ (from the fibers over the diagonal $x = y$ to the off-diagonal fibers $x \neq y$), cf. [**13**].

1.2

Given a reductive complex algebraic group G, Schieder [**22**] constructed a bialgebra $\mathcal{A}$ playing the role of $\bigoplus_n H_c^{2n}(\mathrm{Sch}_n, \mathbb{C})$ for the affine Grassmannian Gr_G in place of Gr. In order to explain his construction, we set up the basic notations for G and Gr_G.

We fix a Borel and a Cartan subgroup $G \supset B \supset T$, and denote by W the Weyl group of (G, T). Let N denote the unipotent radical of the Borel B, and let N_- stand for the unipotent radical of the opposite Borel B_-. Let Λ (respectively, $\Lambda^\vee$) be the coweight (respectively, weight) lattice, and let $\Lambda^+ \subset \Lambda$ (respectively, $\Lambda^{\vee+} \subset \Lambda^\vee$) be the cone of dominant coweights (respectively, weights). Let also $\Lambda^{\mathrm{pos}} \subset \Lambda$ (respectively, $\Lambda^{\vee,\mathrm{pos}} \subset \Lambda^\vee$) be the submonoid spanned by the simple coroots (respectively, roots) $\alpha_i,\ i \in I$ (respectively, $\alpha_i^\vee,\ i \in I$). We denote by $G^\vee \supset T^\vee$ the Langlands dual group, so that Λ (respectively, $\Lambda^\vee$) is the weight (respectively, coweight) lattice of $G^\vee$.

Let $\mathcal{O}$ denote the formal power series ring $\mathbb{C}[[z]]$, and let $\mathcal{K}$ denote its fraction field $\mathbb{C}((z))$. The affine Grassmannian $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$ is an ind-projective scheme, the union $\bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_G^\lambda$ of $G_\mathcal{O}$-orbits. The closure of Gr_G^λ is a projective variety $\overline{\mathrm{Gr}}_G^\lambda = \bigsqcup_{\mu \leqslant \lambda} \mathrm{Gr}_G^\mu$. The fixed point set Gr_G^T is naturally identified with the coweight lattice Λ; and $\mu \in \Lambda$ lies in Gr_G^λ if and only if $\mu \in W\lambda$.

For a coweight $\nu \in \Lambda = \mathrm{Gr}_G^T$, we denote by $S_\nu \subset \mathrm{Gr}_G$ (respectively, $T_\nu \subset \mathrm{Gr}_G$) the orbit of $N(\mathcal{K})$ (respectively, of $N_-(\mathcal{K})$) through ν. The intersections $S_\nu \cap \overline{\mathrm{Gr}}_G^\lambda$ (respectively, $T_\nu \cap \overline{\mathrm{Gr}}_G^\lambda$) are the *attractors* (respectively, *repellents*) of $\mathbb{C}^\times$ acting via its homomorphism 2ρ to the Cartan torus $T \curvearrowright \overline{\mathrm{Gr}}_G^\lambda$: $S_\nu \cap \overline{\mathrm{Gr}}_G^\lambda = \{x \in \overline{\mathrm{Gr}}_G^\lambda : \lim_{c \to 0} 2\rho(c) \cdot x = \nu\}$ and $T_\nu \cap \overline{\mathrm{Gr}}_G^\lambda = \{x \in \overline{\mathrm{Gr}}_G^\lambda : \lim_{c \to \infty} 2\rho(c) \cdot x = \nu\}$. Going to the limit $\mathrm{Gr}_G = \lim_{\lambda \in \Lambda^+} \overline{\mathrm{Gr}}_G^\lambda$, S_ν (respectively, T_ν) is the attractor (respectively, repellent) of ν in Gr_G. The closure $\overline{S}_\nu$ is the union $\bigsqcup_{\mu \leqslant \nu} S_\mu$, while $\overline{T}_\nu = \bigsqcup_{\mu \geqslant \nu} T_\mu$.

Definition 1.1. (a) For $\theta \in \Lambda^{\mathrm{pos}}$ we denote by Sch_θ (respectively, $\overline{\mathrm{Sch}}_\theta$) the intersection $S_\theta \cap T_0$ (respectively, $\overline{S}_\theta \cap \overline{T}_0$)‡. It is equidimensional of dimension $\langle \rho^\vee, \theta \rangle$ (see [**6**, § 6.3])§.

(b) We set $\mathcal{A}_\theta := H_c^{\langle 2\rho^\vee, \theta \rangle}(\mathrm{Sch}_\theta, \mathbb{C}) = H_c^{\langle 2\rho^\vee, \theta \rangle}(\overline{\mathrm{Sch}}_\theta, \mathbb{C})$, and $\mathcal{A} := \bigoplus_{\theta \in \Lambda^{\mathrm{pos}}} \mathcal{A}_\theta$.

Given a smooth curve X and $\theta \in \Lambda^{\mathrm{pos}}$, the open zastava space $\mathring{Z}^\theta$ (see, for example, [**6**]) is equipped with the projection $\pi_\theta \colon \mathring{Z}^\theta \to X^\theta$ to the degree θ configuration space of X. It enjoys the factorization property, and for any $x \in X$, we have a canonical isomorphism $\pi_\theta^{-1}(\theta \cdot x) \xrightarrow{\sim} \mathrm{Sch}_\theta$. Given $\theta_1, \theta_2 \in \Lambda^{\mathrm{pos}}$ such that $\theta_1 + \theta_2 = \theta$, the coproduct $\Delta_{\theta_1,\theta_2} \colon \mathcal{A}_\theta \to \mathcal{A}_{\theta_1} \otimes \mathcal{A}_{\theta_2}$ is defined just like in 1.1 via the cospecialization morphism for the subfamily $\pi_\theta^{-1}(\theta_1 \cdot x + \theta_2 \cdot y)$.

†Terminology of [**22**, 6.2.7].

‡Here Sch stands for Schieder.

§Strictly speaking, only the inequality $\dim(S_\theta \cap T_0) \leqslant \langle \rho^\vee, \theta \rangle$ is verified right after the proof of [**6**, Proposition 6.4]. The opposite inequality follows, for example, from the existence of the factorizable family $\pi_\theta \colon \mathring{Z}^\theta \to X^\theta$ mentioned in the next paragraph.

To construct the product $\mathsf{m}\colon \bigoplus_{\theta_1+\theta_2=\theta} \mathcal{A}_{\theta_1} \otimes \mathcal{A}_{\theta_2} \to \mathcal{A}_\theta$ we need the Drinfeld–Gaitsgory interpolation $\widetilde{\overline{\mathrm{Sch}}_\theta} \to \mathbb{A}^1$ [10, § 2.2] constructed with respect to the $\mathbb{C}^\times$-action on $\overline{\mathrm{Sch}}_\theta$ arising from the cocharacter 2ρ of T. The key property of $\widetilde{\overline{\mathrm{Sch}}_\theta} \to \mathbb{A}^1$ is that the fibers over $a \neq 0$ are all isomorphic to $\overline{\mathrm{Sch}}_\theta$, while the zero fiber $(\widetilde{\overline{\mathrm{Sch}}_\theta})_0$ is isomorphic to the disjoint union $\bigsqcup_\lambda \overline{\mathrm{Sch}}_\theta^{+,\lambda} \times \overline{\mathrm{Sch}}_\theta^{-,\lambda}$. Here λ (a coweight in Λ^{pos} such that $\lambda \leqslant \theta$) runs through the set of $\mathbb{C}^\times$-fixed points of $\overline{\mathrm{Sch}}_\theta$, and $\overline{\mathrm{Sch}}_\theta^{+,\lambda}$ (respectively, $\overline{\mathrm{Sch}}_\theta^{-,\lambda}$) stands for the corresponding attractor (respectively, repellent), equal to $S_\lambda \cap \overline{T}_0$ (respectively, to $\overline{S}_\theta \cap T_\lambda$). It is easy to see that $H_c^{\langle 2\rho^\vee,\theta\rangle}((\widetilde{\overline{\mathrm{Sch}}_\theta})_0, \mathbb{C}) = \bigoplus_{\theta_1+\theta_2=\theta} H_c^{\langle 2\rho^\vee,\theta_1\rangle}(\mathrm{Sch}_{\theta_1}, \mathbb{C}) \otimes H_c^{\langle 2\rho^\vee,\theta_2\rangle}(\mathrm{Sch}_{\theta_2}, \mathbb{C})$, and the desired product m is nothing but the cospecialization morphism for the compactly supported cohomology of the fibers of the Drinfeld–Gaitsgory family.

Schieder conjectured that the bialgebra $\mathcal{A}$ is isomorphic to the universal enveloping algebra $U(\mathfrak{n}^\vee)$ of $\mathrm{Lie}(N^\vee)$ where $N^\vee \subset B^\vee \subset G^\vee$ is the unipotent radical of Borel subgroup of $G^\vee$. The goal of the present work is a proof of Schieder's conjecture.

1.3

In order to produce an isomorphism $U(\mathfrak{n}^\vee) \xrightarrow{\sim} \mathcal{A}$, we construct an action of $\mathcal{A}$ on the geometric Satake fiber functor. More precisely, we denote by $r_{\nu,+}$ (respectively, $r_{\nu,-}$) the locally closed embedding $S_\nu \hookrightarrow \mathrm{Gr}_G$ (respectively, $T_\nu \hookrightarrow \mathrm{Gr}_G$). We also denote by $\iota_{\nu,+}$ (respectively, $\iota_{\nu,-}$) the closed embedding of the point ν into S_ν (respectively, into T_ν).

According to [4, 10], there is a canonical isomorphism of functors $\iota_{\nu,-}^* r_{\nu,-}^! \simeq \iota_{\nu,+}^! r_{\nu,+}^*\colon D^b_{G_\mathcal{O}}(\mathrm{Gr}_G) \to D^b(\mathrm{Vect})$. For a sheaf $\mathcal{P} \in D^b_{G_\mathcal{O}}(\mathrm{Gr}_G)$, its hyperbolic stalk at ν is defined as $\Phi_\nu(\mathcal{P}) := \iota_{\nu,-}^* r_{\nu,-}^! \mathcal{P} \simeq \iota_{\nu,+}^! r_{\nu,+}^* \mathcal{P}$. According to [18], for $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$, the hyperbolic stalk $\Phi_\nu(\mathcal{P})$ is concentrated in degree $\langle 2\rho^\vee, \nu\rangle$, and there is a canonical direct sum decomposition $H^\bullet(\mathrm{Gr}_G, \mathcal{P}) = \bigoplus_{\nu\in\Lambda} \Phi_\nu(\mathcal{P})$. Moreover, the abelian category $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ is monoidal with respect to the convolution operation $\star$, and the functor $H^\bullet(\mathrm{Gr}_G, -)\colon (\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G), \star) \to (\mathrm{Vect}, \otimes)$ is a fiber functor identifying $(\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G), \star)$ with the tensor category $\mathrm{Rep}(G^\vee)$ (geometric Satake equivalence).

We define a morphism of functors $\mathcal{A}_\theta \otimes \Phi_\nu \to \Phi_{\nu+\theta}$ in 4.1.1. To this end (and also in order to check various tensor compatibilities) we consider the *Drinfeld–Gaitsgory–Vinberg interpolation Grassmannian*: a relative compactification $\mathrm{VinGr}_G^{\mathrm{princ}}$ of the Drinfeld–Gaitsgory interpolation $\widetilde{\mathrm{Gr}}_G \to \mathbb{A}^1$. We also consider an extended version $\mathrm{VinGr}_G \to T_{\mathrm{ad}}^+ := \mathrm{Spec}\,\mathbb{C}[\Lambda^{\vee,\mathrm{pos}}]$ and its version VinGr_{G,X^n} for the Beilinson–Drinfeld Grassmannian. It was implicit already in Schieder's work, and it was made explicit by Gaitsgory and Nadler, cf. an earlier work [15]. We believe it is a very interesting object in its own right. For example, let $\omega \in \Lambda^+$ be a minuscule dominant coweight. Then the Schubert variety Gr_G^ω is isomorphic to a parabolic flag variety G/P_ω, and the corresponding subvariety VinGr_G^ω of VinGr_G is isomorphic to Brion's degeneration of Δ_{G/P_ω} in $\mathrm{Hilb}(G/P_\omega \times G/P_\omega) \times T_{\mathrm{ad}}^+$ [9, § 3].

Remark 1.2. (a) By the geometric Satake equivalence, for $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$, the cohomology $H^\bullet(\mathrm{Gr}_G, \mathcal{P})$ is equipped with an action of $U(\mathfrak{g}^\vee)$. For example, the action of the Cartan subalgebra $U(\mathfrak{t}^\vee) \subset U(\mathfrak{g}^\vee)$ comes from the grading $H^\bullet(\mathrm{Gr}_G, \mathcal{P}) = \bigoplus_{\nu\in\Lambda} \Phi_\nu(\mathcal{P})$. The action of $U(\mathfrak{n}^\vee)$ comes from the geometric action of the Schieder bialgebra $\mathcal{A}$ on the geometric Satake fiber functor, and the isomorphism $U(\mathfrak{n}^\vee) \xrightarrow{\sim} \mathcal{A}$. Finally, the action of $U(\mathfrak{n}_-^\vee)$ is conjugate to the action of $U(\mathfrak{n}^\vee)$ with respect to the Lefschetz bilinear form on $H^\bullet(\mathrm{Gr}_G, \mathcal{P})$.

(b) By construction, the Schieder algebra $\mathcal{A}$ comes equipped with a basis (fundamental classes of irreducible components of Sch_θ). The corresponding integral form is denoted $\mathcal{A}_\mathbb{Z} \subset \mathcal{A}$. On the other hand, $U(\mathfrak{n}^\vee)$ is equipped with the *semi-canonical basis* [16][†]. The

[†]Constructed under the assumption that G is simply laced.

corresponding integral form is nothing but the Chevalley–Kostant integral form $U(\mathfrak{n}^\vee)_\mathbb{Z}$. According to Proposition 4.20, the isomorphism $U(\mathfrak{n}^\vee) \xrightarrow{\sim} \mathcal{A}$ gives rise to an isomorphism of their integral forms: $U(\mathfrak{n}^\vee) \supset U(\mathfrak{n}^\vee)_\mathbb{Z} \xrightarrow{\sim} \mathcal{A}_\mathbb{Z} \subset \mathcal{A}$. In the simplest example when $G = SL(2)$, $\mathcal{A}$ is $\mathbb{N}$-graded, and each graded component $\mathcal{A}_n$ is one dimensional with the basis vector e_n; one can check $e_n e_m = \binom{n+m}{n} e_{n+m}$. Hence the two bases match under the isomorphism $U(\mathfrak{n}^\vee)_\mathbb{Z} \xrightarrow{\sim} \mathcal{A}_\mathbb{Z}$. However, for general G the two bases do *not* match, as seen in an example for G of type A_5 in degree (2,4,4,4,2) in [**1**, Appendix A]. Thus the Higgs branch realization [**16**] of $U(\mathfrak{n}^\vee)$ is different from the Coulomb branch realization [**22**] of $U(\mathfrak{n}^\vee)$.

1.4

The paper is organized as follows. In a lengthy § 2 we review the works of Schieder and other related materials. To simplify the exposition somewhat we assume that the derived subgroup $[G, G] \subset G$ is simply connected. The Schieder algebra $\mathcal{A}$ is defined in § 2.6. Note that the construction of multiplication in § 2.6.3 does not use the Drinfeld–Gaitsgory interpolation in contrast to the construction at the end of § 1.2. However, the two constructions are equivalent as a consequence of Proposition 3.17. We define the Drinfeld–Gaitsgory–Vinberg interpolation Grassmannian in § 3. We define the action of the Schieder algebra $\mathcal{A}$ on the geometric Satake fiber functor Φ and check various compatibilities in § 4. We deduce the Schieder conjecture $\mathcal{A} \simeq U(\mathfrak{n}^\vee)$ in Corollary 4.18. In § 4.5, we identify the above action of $\mathcal{A}$ on Φ with another action going back to [**12**]. Finally in § 5, we explain the changes needed in the case of arbitrary reductive G. In § 5.4, for a quiver Q without loop edges, we propose a conjectural geometric construction of $U(\mathfrak{g}_Q^+)$ (positive subalgebra of the corresponding symmetric Kac–Moody Lie algebra) in the framework of Coulomb branch of the corresponding quiver gauge theory. The Appendix written by Gaitsgory contains proofs of Proposition 2.16 stating that the Drinfeld–Lafforgue–Vinberg compactification $\overline{\mathrm{Bun}}_G$ of Bun_G is proper over $\mathrm{Bun}_G \times \mathrm{Bun}_G$, and of Proposition 3.17 stating that the Drinfeld–Gaitsgory–Vinberg interpolation Grassmannian $\mathrm{VinGr}_G^{\mathrm{princ}}$ is a relative compactification of the Drinfeld–Gaitsgory interpolation $\widetilde{\mathrm{Gr}}_G$.

2. *Review of Schieder's work*

Till § 5 we assume that the derived subgroup $[G, G] \subset G$ is simply connected. It implies in particular that $\Lambda^{\mathrm{pos}} := \bigoplus_{i \in I} \mathbb{N}\alpha_i$ is equal to $\Lambda_{\geqslant 0} := \{\mu \in \Lambda : \langle \lambda^\vee, \mu \rangle \geqslant 0 \ \forall \lambda^\vee \in \Lambda^{\vee +}\}$.

2.1. *General recollections*

2.1.1. *The affine Grassmannian* [**3**, § 4.5*;* **27**, § 1,2 *and* 1.4]. Let X be a smooth projective curve over $\mathbb{C}$. Let us fix a point $x \in X$. The functor of points

$$\mathrm{Gr}_G \colon \mathbf{Sch} \to \mathbf{Set}, \ S \mapsto \mathrm{Gr}_G(S)$$

can be described as follows. For a scheme S the set $\mathrm{Gr}_G(S)$ consists of the following data.

(1) A G-bundle $\mathcal{F}$ on $S \times X$.
(2) A trivialization σ of $\mathcal{F}$ on $S \times (X \setminus \{x\})$.

Remark 2.1. For a space $\mathcal{Y}$, one defines a space $\mathrm{Maps}(X, \mathcal{Y})$ parametrizing morphisms from the curve X to $\mathcal{Y}$ as $\mathrm{Maps}(X, \mathcal{Y})(S) := \mathcal{Y}(X \times S)$. Note that the ind-scheme Gr_G is isomorphic to the fiber product

$$\mathrm{Maps}(X, \mathrm{pt}/G) \underset{\mathrm{Maps}(X \setminus \{x\}, \mathrm{pt}/G)}{\times} \mathrm{pt},$$

where the morphism $\mathrm{pt} \to \mathrm{Maps}(X\backslash\{x\}, \mathrm{pt}/G)$ is the composition of the isomorphism $\mathrm{pt} \xrightarrow{\sim} \mathrm{Maps}(X\backslash\{x\}, \mathrm{pt})$ with the morphism $\mathrm{Maps}(X\backslash\{x\}, \mathrm{pt}) \to \mathrm{Maps}(X\backslash\{x\}, \mathrm{pt}/G)$ induced by the morphism $\mathrm{pt} \to \mathrm{pt}/G$.

2.1.2. *The Beilinson–Drinfeld Grassmannian* [**3**, § 5.3.10 *and* 5.3.11; **18**, § 5; **27**, § 3.1]. For $n \in \mathbb{N}$, Gr_{G,X^n} is the moduli space of the following data: it associates to a scheme S

(1) a collection of S-points $\underline{x} = (x_1, \ldots, x_n) \in X^n(S)$ of the curve X;
(2) a G-bundle $\mathcal{F}$ on $S \times X$;
(3) a trivialization σ of $\mathcal{F}$ on $(S \times X) \setminus \{\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_n}\}$, where $\Gamma_{x_k} \subset S \times X$ is the graph of x_k.

We have a projection $\pi_n \colon \mathrm{Gr}_{G,X^n} \to X^n$ that forgets the data of $\mathcal{F}$ and σ.

Let us denote by $\Delta_X \hookrightarrow X^n$ the diagonal embedding. Take a point $x \in X$. Note that the fiber of the morphism π_n over the point $(x, \ldots, x) \in \Delta_X$ is isomorphic to Gr_G. We will denote this fiber by the same symbol.

Remark 2.2. Let us fix $n \in \mathbb{N}$. Let us denote by $\mathrm{Bun}_G(\mathfrak{U}_n)$ the following stack over X^n: it associates to a scheme S

(1) a collection of points $\underline{x} = (x_1, \ldots, x_n) \in X^n(S)$ of the curve X;
(2) a G-bundle $\mathcal{F}$ on $(S \times X) \setminus \{\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_n}\}$.

We have a restriction morphism $X^n \times \mathrm{Maps}(X, \mathrm{pt}/G) \to \mathrm{Bun}_G(\mathfrak{U}_n)$. We also have a morphism $X^n \to \mathrm{Bun}_G(\mathfrak{U}_n)$ that sends a collection of S-points $\underline{x} \in X^n(S)$ to the data $(\underline{x}, \mathcal{F}^{\mathrm{triv}}_{(S\times X)\backslash\{\Gamma_{x_1}\cup\cdots\cup\Gamma_{x_n}\}}) \in \mathrm{Bun}_G(\mathfrak{U}_n)(S)$. The Beilinson–Drinfeld Grassmannian Gr_{G,X^n} is nothing but

$$(X^n \times \mathrm{Maps}(X, \mathrm{pt}/G)) \underset{\mathrm{Bun}_G(\mathfrak{U}_n)}{\times} X^n.$$

2.1.3. Let $G_{X,\mathcal{O}}$ be the group-scheme (over X) that represents the following functor. It associates to a scheme S

(1) an S-point $x \in X(S)$ of the curve X;
(2) a point $\sigma \in G(\hat{\Gamma}_x)$, where $\hat{\Gamma}_x$ is the completion of Γ_x in $S \times X$.

We have a projection $G_{X,\mathcal{O}} \to X$ that forgets the data of σ. Recall the projection $\pi_1 \colon \mathrm{Gr}_{G,X} \to X$ in § 2.1.2. Thus schemes $\mathrm{Gr}_{G,X}, G_{X,\mathcal{O}}$ are endowed with structures of schemes over the curve X. We have an action $G_{X,\mathcal{O}} \curvearrowright \mathrm{Gr}_{G,X}$ by changing the trivialization. Hence we can define a category $\mathrm{Perv}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X})$ as the category of $G_{X,\mathcal{O}}$-equivariant perverse sheaves on $\mathrm{Gr}_{G,X}$. Let us fix a point $x \in X$. Consider the closed embedding $\iota_x \colon \mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{G,X}$. It follows from [**18**, Remark 5.1; **27**, § 5.4] that there exists a functor $\mathfrak{p}^0 \colon \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G) \to \mathrm{Perv}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X})$ such that the composition $\iota_x^*[-1] \circ \mathfrak{p}^0$ is isomorphic to Id.

2.1.4. *Tensor structure on* $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ *via Beilinson–Drinfeld Grassmanian.* Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$. Let us describe the convolution $\mathcal{P}_1 \star \mathcal{P}_2$. Recall the projection $\pi_2 \colon \mathrm{Gr}_{G,X^2} \to X^2$ in § 2.1.2. Let $\Delta_X \hookrightarrow X^2$ be the closed embedding of the diagonal. Let $U \hookrightarrow X^2$ be the open embedding of the complement to the diagonal. It follows from [**3**, § 5.3.12; **27**, Proposition 3.1.13] that the restriction of the family $\pi_2 \colon \mathrm{Gr}_{G,X^2} \to X^2$ to the open subvariety $U \hookrightarrow X^2$ is isomorphic to $(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U$ and the restriction of the family $\pi_2 \colon \mathrm{Gr}_{G,X^2} \to X^2$ to the closed subvariety $\Delta_X \hookrightarrow X^2$ is isomorphic to $\mathrm{Gr}_{G,X}$.

Let us denote by $\mathfrak{j}$ the open embedding

$$(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U \simeq \pi_2^{-1}(U) \hookrightarrow \mathrm{Gr}_{G,X^2}.$$

Set

$$\mathcal{P}_1 \underset{X}{\circ} \mathcal{P}_2 := (\mathfrak{p}^0\mathcal{P}_1 \boxtimes \mathfrak{p}^0\mathcal{P}_2)|_U,\ \mathcal{P}_1 \underset{X}{\star} \mathcal{P}_2 := \mathrm{j}_{!*}(\mathcal{P}_1 \underset{X}{\circ} \mathcal{P}_2).$$

Then according to [**18**, § 5], we have $\mathfrak{p}^0(\mathcal{P}_1 \star \mathcal{P}_2) \simeq \mathcal{P}_1 \underset{X}{\star} \mathcal{P}_2$, $\mathcal{P}_1 \star \mathcal{P}_2 \simeq \iota_x^*[-1](\mathcal{P}_1 \underset{X}{\star} \mathcal{P}_2)$.

2.1.5. *Rational morphisms.* Let $\mathcal{F}$ be a coherent sheaf on $S \times X$. Let us fix two numbers $n, m \in \mathbb{N}$. Let us also fix a collection of S-points $\underline{x} = (x_1, \dots, x_n) \in X^n(S)$ of the curve X. Let us denote by $\mathrm{QM}^m_{\underline{x}}(\mathcal{O}_{S\times X}, \mathcal{F})$ the set of morphisms $\mathcal{O}_{S\times X}(-m \cdot (\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n})) \to \mathcal{F}$. For $m_1, m_2 \in \mathbb{N}$, $m_1 \leqslant m_2$, we have the natural embeddings $\mathrm{QM}^{m_1}_{\underline{x}}(\mathcal{O}_{S\times X}, \mathcal{F}) \hookrightarrow \mathrm{QM}^{m_2}_{\underline{x}}(\mathcal{O}_{S\times X}, \mathcal{F})$ given by the composition with the morphism

$$\mathcal{O}_{S\times X}(-m_2 \cdot (\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n})) \hookrightarrow \mathcal{O}_{S\times X}(-m_1 \cdot (\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n})).$$

Let us denote by $\mathrm{RM}_{\underline{x}}(\mathcal{O}_{S\times X}, \mathcal{F})$ the inductive limit $\lim_{m\in\mathbb{N}} \mathrm{QM}^m_{\underline{x}}(\mathcal{O}_{S\times X}, \mathcal{F})$. The elements of the set $\mathrm{RM}_{\underline{x}}(\mathcal{O}_{S\times X}, \mathcal{F})$ will be called rational morphisms from $\mathcal{O}_{S\times X}$ to $\mathcal{F}$ regular on $U := (S \times X) \setminus \{\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n}\}$.

The set of rational morphisms from $\mathcal{F}$ to $\mathcal{O}_{S\times X}$ regular on U is defined analogously as the limit of the sets of morphisms $\mathcal{F} \to \mathcal{O}_{S\times X}(m \cdot (\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n}))$.

2.1.6. *The Tannakian approach.* For an algebraic group H, a right H-torsor $\mathcal{E}$ over a scheme X and $V \in \mathrm{Rep}(H)$ we denote by $\mathcal{V}_{\mathcal{E}}$ the associated vector bundle over X: $\mathcal{V}_{\mathcal{E}} := \mathcal{E} \overset{H}{\times} V$. For every $\lambda^\vee \in \Lambda^{\vee+}$ let us fix a highest weight vector $v_{\lambda^\vee} \in V^{\lambda^\vee}$ with respect to the Borel subgroup $B \subset G$. For $\lambda^\vee, \mu^\vee \in \Lambda^{\vee+}$ we get the G-morphism $\mathrm{pr}_{\lambda^\vee,\mu^\vee}\colon V^{\lambda^\vee} \otimes V^{\mu^\vee} \twoheadrightarrow V^{\lambda^\vee+\mu^\vee}$ that is uniquely determined by the following property: $\mathrm{pr}_{\lambda^\vee,\mu^\vee}(v_{\lambda^\vee} \otimes v_{\mu^\vee}) = v_{\lambda^\vee+\mu^\vee}$. Let Gr'_{G,X^n} be the following moduli space. It associates to a scheme S

(1) a collection of S-points $\underline{x} = (x_1, \dots, x_n) \in X^n(S)$ of the curve X;
(2) a G-bundle $\mathcal{F}$ on $S \times X$;
(3) rational N_- and N-structures on $\mathcal{F}$, that is, (cf. [**8**, Theorem 1.1.2]) for every $\lambda^\vee \in \Lambda^{\vee+}$, rational morphisms

$$\mathcal{O}_{S\times X} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}$$

regular on $U := (S \times X) \setminus \{\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n}\}$, satisfying the following conditions.
(a) For every $\lambda^\vee \in \Lambda^{\vee+}$ the composition

$$(\zeta_{\lambda^\vee} \circ \eta_{\lambda^\vee})|_{(S\times X)\setminus\{\Gamma_{x_1}\cup\dots\cup\Gamma_{x_n}\}}$$

is the identity morphism.
(b) For every $\lambda^\vee, \mu^\vee \in \Lambda^{\vee+}$ let $\mathrm{pr}_{\lambda^\vee,\mu^\vee}\colon V^{\lambda^\vee} \otimes V^{\mu^\vee} \twoheadrightarrow V^{\lambda^\vee+\mu^\vee}$ be the projection morphism. We have the corresponding morphisms

$$\mathrm{pr}^{\mathcal{F}}_{\lambda^\vee,\mu^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}} \to \mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}}.$$

Then the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{O}_U \otimes \mathcal{O}_U & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathcal{O}_U \\ \Big\downarrow{\scriptstyle \eta_{\lambda^\vee}\otimes\eta_{\mu^\vee}} & & \Big\downarrow{\scriptstyle \eta_{\lambda^\vee+\mu^\vee}} \\ (\mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}})|_U & \xrightarrow{\mathrm{pr}^{\mathcal{F}}_{\lambda^\vee,\mu^\vee}|_U} & (\mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}})|_U, \end{array}$$

$$\begin{array}{ccc} (\mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}})|_U & \xrightarrow{\mathrm{pr}^{\mathcal{F}}_{\lambda^\vee,\mu^\vee}|_U} & (\mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}})|_U \\ \downarrow{\scriptstyle \zeta_{\lambda^\vee}\otimes\zeta_{\mu^\vee}} & & \downarrow{\scriptstyle \zeta_{\lambda^\vee+\mu^\vee}} \\ \mathcal{O}_U \otimes \mathcal{O}_U & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathcal{O}_U. \end{array}$$

(c) Given a morphism $\mathrm{pr}\colon V^{\lambda^\vee} \otimes V^{\mu^\vee} \to V^{\nu^\vee}$ for $\lambda^\vee, \mu^\vee, \nu^\vee \in \Lambda^{\vee+}$, $\nu^\vee < \lambda^\vee + \mu^\vee$, we have

$$\mathrm{pr}^{\mathcal{F}} \circ (\eta_{\lambda^\vee} \otimes \eta_{\mu^\vee}) = 0,\ (\zeta_{\lambda^\vee} \otimes \zeta_{\mu^\vee}) \circ \mathrm{pr}^{\mathcal{F}} = 0.$$

(d) For $\lambda^\vee = 0$ we have $\zeta_{\lambda^\vee} = \mathrm{Id}$ and $\eta_{\lambda^\vee} = \mathrm{Id}$.

Proposition 2.3. *For $n \in \mathbb{N}$, the functors Gr_{G,X^n} and Gr'_{G,X^n} are isomorphic.*

Proof. Let us construct a morphism of functors $\Xi\colon \mathrm{Gr}'_{G,X^n} \to \mathrm{Gr}_{G,X^n}$. Take an S-point $(\underline{x}, \mathcal{F}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \in \mathrm{Gr}'_{G,X^n}(S)$. Restrictions of the morphisms $\eta_{\lambda^\vee}, \zeta_{\lambda^\vee}$ to the open subvariety $U \subset S \times X$ define the transversal N and N_--structures in the G-bundle $\mathcal{F}|_U$. They define a trivialization σ of $\mathcal{F}|_U$. Set $\Xi(\underline{x}, \mathcal{F}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) := (\underline{x}, \mathcal{F}, \sigma) \in \mathrm{Gr}_{G,X^n}$.

Let us construct the inverse morphism $\Xi^{-1}\colon \mathrm{Gr}_{G,X^n} \to \mathrm{Gr}'_{G,X^n}$. Take an S-point $(\underline{x}, \mathcal{F}, \sigma) \in \mathrm{Gr}_{G,X^n}$. Note that the standard N, N_--structures in the trivial G-bundle on $S \times X$ define via σ the N and N_--structures in $\mathcal{F}|_U$. Thus for every $\lambda^\vee \in \Lambda^{\vee+}$ we have morphisms of vector bundles $\mathcal{O}_U \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}}|_U \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_U$.

The morphisms $\eta'_{\lambda^\vee}$, $\zeta'_{\lambda^\vee}$ come from morphisms

$$\mathcal{O}_{S\times X}(l_{\lambda^\vee} \cdot (\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_n})) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}(k_{\lambda^\vee} \cdot (\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_n}))$$

for some integers $l_{\lambda^\vee}, k_{\lambda^\vee}$.

Set $\Xi^{-1}(\underline{x}, \mathcal{F}, \sigma) := (\underline{x}, \mathcal{F}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee})$. It is easy to see that the morphisms Ξ, Ξ^{-1} are mutually inverse. □

2.1.7. *Definition of $\overline{S}_\nu$ via Tannakian approach.* We fix a point $x \in X$. Let us give the Tannakian definition of the ind-scheme $\overline{S}_\nu \subset \mathrm{Gr}_G$. The corresponding functor of points associates to a scheme S

(1) a G-bundle $\mathcal{F}$ on $S \times X$;
(2) for every $\lambda^\vee \in \Lambda^{\vee+}$, morphisms of sheaves $\eta_{\lambda^\vee}\colon \mathcal{O}_{S\times X}(-\langle\lambda^\vee, \nu\rangle \cdot (S \times x)) \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}}$ and rational morphisms $\zeta_{\lambda^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \to \mathcal{O}_{S\times X}$ regular on $S \times (X \setminus \{x\})$, satisfying the same conditions as in the definition of Gr'_{G,X^n} in § 2.1.6.

Allowing $x \in X$ to vary, we obtain an ind-scheme $\overline{S}_{\nu,X} \to X$.

Remark 2.4. Note that the open ind-subscheme $S_\nu \subset \overline{S}_\nu$ consists of such $(\mathcal{F}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \in \overline{S}_\nu$ that $\eta_{\lambda^\vee}$ are injective morphisms of vector bundles.

2.1.8. *Definition of $\overline{T}_\nu$ via Tannakian approach.* We fix a point $x \in X$. Let us give the Tannakian definition of the ind-scheme $\overline{T}_\nu$. The corresponding functor of points associates to a scheme S

(1) a G-bundle $\mathcal{F}$ on $X \times S$;
(2) for every $\lambda^\vee \in \Lambda^{\vee+}$, rational morphisms $\eta_{\lambda^\vee}\colon \mathcal{O}_{S\times X} \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}}$ regular on $S \times (X \setminus \{x\})$, and morphisms of sheaves $\zeta_{\lambda^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \to \mathcal{O}_{S\times X}(-\langle\lambda^\vee, \nu\rangle \cdot (S \times x))$, satisfying the same conditions as in the definition of Gr'_{G,X^n} in § 2.1.6.

Allowing $x \in X$ to vary, we obtain an ind-scheme $\overline{T}_{\nu,X} \to X$.

Remark 2.5. Note that the open ind-subscheme $T_\nu \subset \overline{T}_\nu$ consists of such $(\mathcal{F}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \in \overline{T}_\nu$ that $\zeta_{\lambda^\vee}$ are surjective morphisms of vector bundles.

2.1.9. *Definition of* $\overline{S}_{\theta_1,\theta_2}$. Fix two cocharacters $\theta_1, \theta_2 \in \Lambda$. Let $\overline{S}_{\theta_1,\theta_2}$ be the following moduli space: it associates to a scheme S

(1) a pair of S-points $(x_1, x_2) \in X^2(S)$ of the curve X;
(2) a G-bundle $\mathcal{F}$ on $S \times X$;
(3) for every $\lambda^\vee \in \Lambda^{\vee+}$, morphisms of sheaves $\eta_{\lambda^\vee}$,

$$\eta_{\lambda^\vee} : \mathcal{O}_{S\times X}(-\langle\lambda^\vee, \theta_1\rangle \cdot \Gamma_{x_1} - \langle\lambda^\vee, \theta_2\rangle \cdot \Gamma_{x_2}) \to \mathcal{V}_{\mathcal{F}}^{\lambda^\vee},$$

and rational morphisms $\zeta_{\lambda^\vee} : \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \to \mathcal{O}_{S\times X}$ regular on $(S \times X) \setminus \{\Gamma_{x_1} \cup \Gamma_{x_2}\}$, satisfying the same conditions as in the definition of Gr'_{G,X^n} in § 2.1.6.

2.1.10. *Definition of* $\overline{T}_{\theta_1,\theta_2}$. Fix two cocharacters $\theta_1, \theta_2 \in \Lambda$. Let $\overline{T}_{\theta_1,\theta_2}$ be the following moduli space: it associates to a scheme S

(1) a pair of S-points $(x_1, x_2) \in X^2(S)$ of the curve X;
(2) a G-bundle $\mathcal{F}$ on $S \times X$;
(3) for every $\lambda^\vee \in \Lambda^{\vee+}$, morphisms of sheaves,

$$\zeta_{\lambda^\vee} : \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \to \mathcal{O}_{S\times X}(-\langle\lambda^\vee, \theta_1\rangle \cdot \Gamma_{x_1} - \langle\lambda^\vee, \theta_2\rangle \cdot \Gamma_{x_2})$$

and rational morphisms $\eta_{\lambda^\vee} : \mathcal{O}_{S\times X} \to \mathcal{V}_{\mathcal{F}}^{\lambda^\vee}$ regular on $(S \times X) \setminus \{\Gamma_{x_1} \cup \Gamma_{x_2}\}$, satisfying the same conditions as in the definition of Gr'_{G,X^n} in § 2.1.6.

2.2. *The Vinberg semi-group*

In [**24**, **25**], Vinberg has defined a multiparameter degeneration $\mathrm{Vin}_G \to \mathbb{A}^r$ of a reductive group G of semi-simple rank r. Let us recall two equivalent constructions of Vin_G.

2.2.1. *Rees construction of the Vinberg semi-group* [**25**]. Consider the regular action of $G \times G$ on G: $(g_1, g_2) \cdot g := g_1 \cdot g \cdot g_2^{-1}$.

Proposition 2.6 (Peter–Weyl theorem). *The morphism*

$$\psi\colon \bigoplus_{\lambda^\vee\in\Lambda^{\vee+}} V^{\lambda^\vee} \otimes (V^{\lambda^\vee})^* \xrightarrow{\sim} \mathbb{C}[G],\ v \otimes v^\vee \mapsto [g \mapsto \langle gv, v^\vee\rangle]$$

is an isomorphism of $G \times G$*-modules.*

The isomorphism ψ induces a $\Lambda^{\vee+}$-grading on $\mathbb{C}[G]$. This grading is not compatible with the algebra structure on $\mathbb{C}[G]$. It is easy to see that the algebra structure on $\mathbb{C}[G]$ is compatible with the corresponding $\Lambda^\vee$-filtration (the Peter–Weyl filtration):

$$\mathbb{C}[G]_{\leqslant\lambda^\vee} := \psi\left(\bigoplus_{\mu^\vee\leqslant\lambda^\vee} V^{\mu^\vee} \otimes (V^{\mu^\vee})^*\right).$$

Let $\mathbb{C}[\Lambda^\vee]$ denote the group algebra of $\Lambda^\vee$: it is generated by the formal variables $t^{\lambda^\vee}$ with relations $t^{\lambda^\vee} \cdot t^{\mu^\vee} = t^{\lambda^\vee+\mu^\vee}$ for $\lambda^\vee, \mu^\vee \in \Lambda^\vee$.

DEFINITION 2.7. The Vinberg semi-group Vin_G for G is defined as the spectrum of the Rees algebra for $\mathbb{C}[G]$ with the Peter–Weyl filtration:

$$\mathrm{Vin}_G := \mathrm{Spec}\left(\bigoplus_{\lambda^\vee\in\Lambda^\vee}\mathbb{C}[G]_{\leqslant\lambda^\vee}t^{\lambda^\vee}\right).$$

Note that Vin_G is equipped with a natural $G\times G$-action. Let us also note that the algebra $\mathbb{C}[\mathrm{Vin}_G]$ can be equipped with the comultiplication morphism

$$\Delta\colon \mathbb{C}[\mathrm{Vin}_G]\to\mathbb{C}[\mathrm{Vin}_G]\otimes\mathbb{C}[\mathrm{Vin}_G]$$

that is induced from the comultiplication morphisms for Hopf algebras $\mathbb{C}[G]$, $\mathbb{C}[\Lambda^\vee]$. It follows that $\mathbb{C}[\mathrm{Vin}_G]$ carries a bialgebra structure. So Vin_G is an algebraic monoid (semi-group).

2.2.2. *Construction of the morphism* $\Upsilon\colon \mathrm{Vin}_G\to T^+_{\mathrm{ad}}$. We consider the polynomial subalgebra $\mathbb{C}[t^{\alpha_i^\vee},\ i\in I]$ of $\mathbb{C}[\Lambda^\vee]$ generated by the elements $t^{\alpha_i^\vee},\ i\in I$. Set $T^+_{\mathrm{ad}} := \mathrm{Spec}(\mathbb{C}[t^{\alpha_i^\vee},\ i\in I])\simeq\mathbb{A}^r$. Let us denote by $Z_G\subset T$ the center of the group G. Set $T_{\mathrm{ad}} := T/Z_G$. We have a natural open embedding $T_{\mathrm{ad}} = \mathrm{Spec}(\mathbb{C}[t^{\pm\alpha_i^\vee},\ i\in I])\hookrightarrow T^+_{\mathrm{ad}}$. Thus T^+_{ad} is the toric variety for T_{ad}, and the corresponding cone is $\Lambda^{\vee,\mathrm{pos}}$. Note that the variety T^+_{ad} has a unique monoid structure extending the multiplication in T_{ad}.

Observe that $t^{\alpha_i^\vee}\in\mathbb{C}[\mathrm{Vin}_G]$ for any simple root $\alpha_i^\vee$, so there is a homomorphism $\mathbb{C}[t^{\alpha_i^\vee},\ i\in I]\hookrightarrow\mathbb{C}[\mathrm{Vin}_G]$. Thus we get a homomorphism of monoids $\Upsilon\colon \mathrm{Vin}_G\to T^+_{\mathrm{ad}}$. Note that the algebra of functions $\mathbb{C}[\mathrm{Vin}_G]$ is $\Lambda^\vee$-graded. It follows that the torus $\mathrm{Spec}(\mathbb{C}[\Lambda^\vee]) = T$ acts on the scheme Vin_G. It also acts on the affine space T^+_{ad} in the following way:

$$t\cdot(a_i) := (\alpha_i^\vee(t)\cdot a_i)\ \text{ in coordinates }\ \alpha_i^\vee,\ i\in I.$$

The morphism Υ is T-equivariant. Recall that $Z_G\subset T$ is the center of the group G. It acts trivially on T^+_{ad}, thus the action of T on T^+_{ad} factors through the action of $T_{\mathrm{ad}} := T/Z_G$.

The fiber $\Upsilon^{-1}(1)$ over $1\in T_{\mathrm{ad}}\subset T^+_{\mathrm{ad}}$ naturally identifies with the group G. Recall that the morphism Υ is T-equivariant. It follows that the preimage $\Upsilon^{-1}(T_{\mathrm{ad}})$ is isomorphic to

$$(G\times T)/Z_G =: G_{\mathrm{enh}},$$

where the action of $Z_G\curvearrowright G\times T$ is given by the formula $(g,t)\cdot z := (z\cdot g, z^{-1}\cdot t)$.

We denote by j the corresponding open embedding $G_{\mathrm{enh}}\hookrightarrow\mathrm{Vin}_G$.

2.2.3. *The section of the morphism* Υ. Let us construct a surjective morphism $\mathfrak{s}^*\colon\mathbb{C}[\mathrm{Vin}_G]\twoheadrightarrow\mathbb{C}[T^+_{\mathrm{ad}}]$. Decompose

$$\mathbb{C}[\mathrm{Vin}_G] = \bigoplus_{\mu^\vee\leqslant\lambda^\vee\in\Lambda^\vee}\psi(V^{\mu^\vee}\otimes(V^{\mu^\vee})^*)t^{\lambda^\vee} = \bigoplus_{\mu^\vee\leqslant\lambda^\vee\in\Lambda^\vee,\ \nu_1^\vee,\nu_2^\vee\in\Lambda^\vee}\psi\left(V^{\mu^\vee}_{\nu_1^\vee}\otimes(V^{\mu^\vee}_{\nu_2^\vee})^*\right)t^{\lambda^\vee}.$$

Set $\mathfrak{s}^*(\psi(w_{\nu_1^\vee}\otimes f_{\nu_2^\vee})t^{\lambda^\vee}) := \langle w_{\nu_1^\vee}, f_{\nu_2^\vee}\rangle t^{\lambda^\vee-\nu_1^\vee}$ for $w_{\nu_1^\vee}\in V^{\mu^\vee}_{\nu_1^\vee}, f_{\nu_2^\vee}\in(V^{\mu^\vee}_{\nu_2^\vee})^*$. The morphism $\mathfrak{s}^*$ corresponds to the section $\mathfrak{s}\colon T^+_{\mathrm{ad}}\hookrightarrow\mathrm{Vin}_G$ of the morphism Υ.

2.2.4. *The nondegenerate locus of the Vinberg semi-group* [**11**, *Section* D.4*;* **25**, § 0.8]. The Vinberg semi-group contains a dense open subvariety ${}_0\mathrm{Vin}_G\subset\mathrm{Vin}_G$, the *nondegenerate locus* of Vin_G. It is uniquely characterized by the fact that it meets each fiber of the morphism $\Upsilon\colon \mathrm{Vin}_G\to T^+_{\mathrm{ad}}$ in the open $G\times G$-orbit of that fiber; that is, for any $t\in T^+_{\mathrm{ad}}$, we have

$$\mathrm{Vin}_G|_t\cap{}_0\mathrm{Vin}_G = G\cdot\mathfrak{s}(t)\cdot G.$$

The Tannakian characterization of ${}_0\mathrm{Vin}_G$ will be given in § 2.2.10.

2.2.5. *The zero fiber of* Υ. Let us describe the zero fiber of the morphism $\Upsilon\colon \mathrm{Vin}_G \to T_{\mathrm{ad}}^+$. Let us denote $(\mathrm{Vin}_G)_0 := \Upsilon^{-1}(0)$.

LEMMA 2.8 (cf. [21, Lemma 2.1.11]). *The scheme* $(\mathrm{Vin}_G)_0$ *is isomorphic to*

$$(\overline{G/N}\times\overline{G/N_-})/\!\!/T := \mathrm{Spec}((\mathbb{C}[G]^N\otimes\mathbb{C}[G]^{N_-})^T),$$

where the action of $T \curvearrowright G/N,\ G/N_-$ *is given by the right multiplication. The* $G\times G$*-action on* $(\mathrm{Vin}_G)_0$ *corresponds to the action via the left multiplication* $G\times G \curvearrowright (\overline{G/N}\times\overline{G/N_-})/\!\!/T$.

Proof. It follows from Definition 2.7 that the $G\times G$-module $\mathbb{C}[(\mathrm{Vin}_G)_0]$ is isomorphic to $\bigoplus_{\lambda^\vee\in\Lambda^{\vee+}} V^{\lambda^\vee}\otimes(V^{\lambda^\vee})^*$, and the algebra structure on $\mathbb{C}[(\mathrm{Vin}_G)_0]$ is given by the projection morphisms

$$(V^{\lambda_1^\vee}\otimes(V^{\lambda_1^\vee})^*)\otimes(V^{\lambda_2^\vee}\otimes(V^{\lambda_2^\vee})^*)\twoheadrightarrow V^{\lambda_1^\vee+\lambda_2^\vee}\otimes(V^{\lambda_1^\vee+\lambda_2^\vee})^*.$$

So there is an embedding of algebras

$$\mathbb{C}[(\mathrm{Vin}_G)_0] = \bigoplus_{\lambda^\vee\in\Lambda^{\vee+}} V^{\lambda^\vee}\otimes(V^{\lambda^\vee})^* \hookrightarrow \bigoplus_{\lambda^\vee,\mu^\vee\in\Lambda^{\vee+}} V^{\lambda^\vee}\otimes(V^{\mu^\vee})^* = \mathbb{C}[G]^N\otimes\mathbb{C}[G]^{N_-}.$$

It is easy to see that the image of $\mathbb{C}[(\mathrm{Vin}_G)_0]$ coincides with $(\mathbb{C}[G]^N\otimes\mathbb{C}[G]^{N_-})^T$. □

REMARK 2.9. It follows from Lemma 2.8 that the open subscheme ${}_0\mathrm{Vin}_G\cap(\mathrm{Vin}_G)_0 \subset (\mathrm{Vin}_G)_0$ is isomorphic to $(G/N\times G/N_-)/T \subset (\overline{G/N}\times\overline{G/N_-})/\!\!/T$.

2.2.6. *The scheme* $\mathrm{Vin}_G/\!\!/(N\times N_-)$. Let us denote by $\overline{T}$ the monoid

$$\overline{T} := (\mathrm{Vin}_G)_0/\!\!/(N\times N_-) = \mathrm{Spec}((\mathbb{C}[G]^{N\times N}\otimes\mathbb{C}[G]^{N_-\times N_-})^T) = \mathrm{Spec}\,\mathbb{C}[\Lambda^{\vee+}].$$

The monoid $\overline{T}$ is the toric variety for the torus T.

REMARK 2.10. Let us point out the difference between the monoids T_{ad}^+, $\overline{T}$. The monoid T_{ad}^+ is the toric variety for the torus T_{ad} and the corresponding cone is $\Lambda^{\vee,\mathrm{pos}}$. The monoid $\overline{T}$ is the toric variety for the torus T, the corresponding cone is $\Lambda^{\vee+}$.

LEMMA 2.11 (cf. [22, Lemma 4.1.3]). *The scheme* $\mathrm{Vin}_G/\!\!/(N\times N_-)$ *is isomorphic to*

$$((\mathrm{Vin}_G)_0/\!\!/(N\times N_-))\times T_{\mathrm{ad}}^+ = \overline{T}\times T_{\mathrm{ad}}^+.$$

Proof. Note that

$$\mathbb{C}[\mathrm{Vin}_G/\!\!/(N\times N_-)] := \mathbb{C}[\mathrm{Vin}_G]^{N\times N_-} = \bigoplus_{\lambda^\vee\in\Lambda^\vee}\mathbb{C}[G]^{N\times N_-}_{\leqslant\lambda^\vee}t^{\lambda^\vee}.$$

The algebra $\mathbb{C}[G]^{N\times N_-}$ is isomorphic to $\bigoplus_{\lambda^\vee\in\Lambda^{\vee+}}\mathbb{C}\cdot(v_{\lambda^\vee}\otimes v^\vee_{\lambda^\vee})\simeq\mathbb{C}[\overline{T}]$, where $v_{\lambda^\vee}\in V^{\lambda^\vee}$ (respectively, $v^\vee_{\lambda^\vee}\in(V^{\lambda^\vee})^*$) is the highest (respectively, lowest) vector. Thus we have

$$\mathbb{C}[G]^{N\times N_-}_{\leqslant\lambda^\vee} = \bigoplus_{\mu^\vee\leqslant\lambda^\vee\in\Lambda^\vee}\mathbb{C}\cdot(v_{\mu^\vee}\otimes v^\vee_{\mu^\vee}).$$

Now the isomorphism

$$\mathbb{C}[\overline{T}\times T_{\mathrm{ad}}^+]\simeq\bigoplus_{\mu^\vee\in\Lambda^{\vee+},\ \lambda^\vee\in\Lambda^{\vee,\mathrm{pos}}}\mathbb{C}\cdot(v_{\mu^\vee}\otimes v^\vee_{\mu^\vee}\otimes t^{\lambda^\vee})\xrightarrow{\sim}\bigoplus_{\lambda^\vee\in\Lambda^\vee}\mathbb{C}[G]^{N\times N_-}_{\leqslant\lambda^\vee}t^{\lambda^\vee}$$
$$=\mathbb{C}[\mathrm{Vin}_G/\!\!/(N\times N_-)]$$

is given by $v_{\mu^\vee}\otimes v^\vee_{\mu^\vee}\otimes t^{\lambda^\vee}\mapsto(v_{\mu^\vee}\otimes v^\vee_{\mu^\vee})\cdot t^{\lambda^\vee+\mu^\vee}$. □

2.2.7. *The example* $G = \mathrm{SL}_2$. For $G = \mathrm{SL}_2$ the semi-group Vin_G is equal to the semi-group of 2×2 matrices $\mathrm{Mat}_{2\times 2}$. The $\mathrm{SL}_2 \times \mathrm{SL}_2$-action is given by the left and right multiplication and the action of $T \simeq \mathbb{C}^\times$ is given by the scalar multiplication. The morphism Υ is equal to the determinant map:

$$\Upsilon\colon\ \mathrm{Vin}_G = \mathrm{Mat}_{2\times 2} \xrightarrow{\det} \mathbb{A}^1 = T_{\mathrm{ad}}^+.$$

It follows that the preimage $\Upsilon^{-1}(T_{\mathrm{ad}})$ is equal to GL_2. The nondegenerate locus ${}_0\mathrm{Vin}_G$ is equal to $\mathrm{Mat}_{2\times 2} \setminus \{0\}$. The section $\mathfrak{s}$ in § 2.2.3 is given by $a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$.

2.2.8. *Tannakian definition of the Vinberg semi-group.* Recall the open dense embedding $j\colon G_{\mathrm{enh}} \hookrightarrow \mathrm{Vin}_G$ of algebraic monoids (§ 2.2.2). Its image coincides with the group of units in Vin_G. The group G_{enh} is reductive, and Vin_G is irreducible and affine. The tensor category $\mathrm{Rep}(\mathrm{Vin}_G)$ of finite-dimensional representations of Vin_G is the full tensor subcategory in the category $\mathrm{Rep}(G_{\mathrm{enh}})$. Let us describe this subcategory $\mathrm{Rep}(\mathrm{Vin}_G) \subset \mathrm{Rep}(G_{\mathrm{enh}})$.

To do so, we first introduce the following notation. Any representation V of G_{enh} admits a canonical decomposition as G_{enh}-representations $V = \bigoplus_{\lambda^\vee \in \Lambda^\vee} V_{\lambda^\vee}$ according to the action of the center $Z_{G_{\mathrm{enh}}} = (Z_G \times T)/Z_G \simeq T$: the center $Z_{G_{\mathrm{enh}}}$ acts on the summand $V_{\lambda^\vee}$ via the character $\lambda^\vee$. Each summand $V_{\lambda^\vee}$ in this decomposition is a G-representation via the inclusion $G \hookrightarrow G_{\mathrm{enh}},\ g \mapsto (g, 1)$.

The subcategory $\mathrm{Rep}(\mathrm{Vin}_G) \subset \mathrm{Rep}(G_{\mathrm{enh}})$ consists of $V \in \mathrm{Rep}(G_{\mathrm{enh}})$ such that for each $\lambda^\vee \in \Lambda^\vee$ the weights of the summand $V_{\lambda^\vee}$ considered as a G-representation are all $\leqslant \lambda^\vee$.

According to the Tannakian formalism, the functor of points

$$\mathrm{Vin}_G\colon \mathbf{Sch} \to \mathbf{Set},\ \ S \mapsto \mathrm{Vin}_G(S)$$

can be described as follows. The set $\mathrm{Vin}_G(S)$ consists of the following data.

(1) For every $\lambda^\vee \in \Lambda^{\vee+}$, a morphism $g_{\lambda^\vee}: S \to \mathrm{End}(V^{\lambda^\vee})$.

(2) For every $\mu^\vee \in \Lambda^{\vee,\mathrm{pos}}$, a regular function $\tau_{\mu^\vee}$ on S, such that the following conditions hold.

- (a) For any $\mu_1^\vee, \mu_2^\vee \in \Lambda^{\vee,\mathrm{pos}}$, we have $\tau_{\mu_1^\vee} \cdot \tau_{\mu_2^\vee} = \tau_{\mu_1^\vee + \mu_2^\vee}$.
- (b) For any $\lambda_1^\vee, \lambda_2^\vee, \nu^\vee \in \Lambda^{\vee+}$ such that $V^{\nu^\vee}$ enters $V^{\lambda_1^\vee} \otimes V^{\lambda_2^\vee}$ with nonzero multiplicity, we denote by $\iota\colon W^{\nu^\vee} \hookrightarrow V^{\lambda_1^\vee} \otimes V^{\lambda_2^\vee}$ the embedding of the corresponding isotypical component, and by $\mathrm{pr}\colon V^{\lambda_1^\vee} \otimes V^{\lambda_2^\vee} \twoheadrightarrow W^{\nu^\vee}$ the corresponding projection. Then we have

$$\mathrm{pr} \circ (g_{\lambda_1^\vee} \otimes g_{\lambda_2^\vee}) \circ \iota = \tau_{\lambda_1^\vee + \lambda_2^\vee - \nu^\vee} \cdot g_{\nu^\vee}.$$

- (c) g_0 sends S to $\mathrm{Id} \in \mathrm{End}(\mathbb{C})$, τ_0 sends S to $1 \in \mathbb{C}$.

2.2.9. *The morphism* Υ *and the section* $\mathfrak{s}$ *via Tannakian approach.* The functor of points

$$T_{\mathrm{ad}}^+\colon \mathbf{Sch} \to \mathbf{Set},\ \ S \mapsto T_{\mathrm{ad}}^+(S)$$

can be described as follows. The set $T_{\mathrm{ad}}^+(S)$ consists of the following data.

For every $\mu^\vee \in \Lambda^{\vee,\mathrm{pos}}$, a regular function $\tau_{\mu^\vee}$ on S, such that for any $\mu_1^\vee, \mu_2^\vee \in \Lambda^{\vee,\mathrm{pos}}$ we have $\tau_{\mu_1^\vee} \cdot \tau_{\mu_2^\vee} = \tau_{\mu_1^\vee + \mu_2^\vee}$. and τ_0 sends S to $1 \in \mathbb{C}$.

The morphism $\Upsilon\colon\ \mathrm{Vin}_G \to T_{\mathrm{ad}}^+$ in § 2.2.2 forgets the data of $g_{\lambda^\vee}$. The morphism $\mathfrak{s}$ in § 2.2.3 sends an S-point $(\tau_{\mu^\vee}) \in T_{\mathrm{ad}}^+(S)$ to an S-point $(g_{\lambda^\vee}, \tau_{\mu^\vee}) \in \mathrm{Vin}_G(S)$ where $g_{\lambda^\vee}: S \to \mathrm{End}(V^{\lambda^\vee})$ acts on the weight component $(V^{\lambda^\vee})_{\lambda^\vee - \mu^\vee}$ via the multiplication by $\tau_{\mu^\vee}$.

2.2.10. *Nondegenerate locus of the Vinberg semi-group via Tannakian aproach* [**11**, § *D*.4]. Recall the nondegenerate locus ${}_0\mathrm{Vin}_G \subset \mathrm{Vin}_G$ in § 2.2.4.

In Tannakian terms it corresponds to $(g_{\lambda^\vee}, \tau_{\mu^\vee}) \in \mathrm{Vin}_G(S)$ such that $g_{\lambda^\vee} \neq 0$ for any $\lambda^\vee \in \Lambda^{\vee+}$.

REMARK 2.12. The scheme ${}_0\mathrm{Vin}_G/T$ is nothing but the De Concini–Procesi wonderful compactification of $G_{\mathrm{ad}} := G/Z_G$ [**25**, 8.6].

2.2.11. *The principal degeneration.* Let us denote by $\mathrm{Vin}_G^{\mathrm{princ}}$ the restriction of the multiparameter family $\Upsilon\colon \mathrm{Vin}_G \to T_{\mathrm{ad}}^+$ in § 2.2.2 to the 'principal' line

$$\mathbb{A}^1 \hookrightarrow T_{\mathrm{ad}}^+,\ a \mapsto (a, \dots, a) \text{ in coordinates } \alpha_i^\vee,\ i \in I.$$

Let us denote the corresponding morphism $\mathrm{Vin}_G^{\mathrm{princ}} \to \mathbb{A}^1$ by $\Upsilon^{\mathrm{princ}}$. The morphism $\Upsilon^{\mathrm{princ}}$ is $\mathbb{C}^\times$-equivariant with respect to the 'diagonal' $\mathbb{C}^\times$-action via

$$\mathbb{C}^\times \hookrightarrow T,\ c \mapsto 2\rho(c).$$

It follows from § 2.2.2 that the preimage $(\Upsilon^{\mathrm{princ}})^{-1}(\mathbb{G}_m)$ is isomorphic to $G \times \mathbb{G}_m$.

Let us denote by ${}_0\mathrm{Vin}_G^{\mathrm{princ}}$ the intersection

$${}_0\mathrm{Vin}_G^{\mathrm{princ}} := {}_0\mathrm{Vin}_G \cap \mathrm{Vin}_G^{\mathrm{princ}}.$$

2.3. *Drinfeld–Lafforgue–Vinberg degeneration of* $\mathrm{Bun}_G(X)$ [**21**, § 2.2]

Let X be a smooth projective curve over $\mathbb{C}$. Let us now recall the definition of Drinfeld–Lafforgue–Vinberg multiparameter degeneration $\mathrm{VinBun}_G(X)$ of the moduli stack $\mathrm{Bun}_G(X)$. Since the curve X is fixed throughout the paper, we will use the simplified notations $\mathrm{VinBun}_G, \mathrm{Bun}_G$ instead of $\mathrm{VinBun}_G(X), \mathrm{Bun}_G(X)$.

Recall that for a stack $\mathcal{Y}$, one defines a stack $\mathrm{Maps}(X, \mathcal{Y})$ parametrizing morphisms from the curve X to $\mathcal{Y}$ as $\mathrm{Maps}(X, \mathcal{Y})(S) := \mathcal{Y}(X \times S)$.

Thus, for example, we have $\mathrm{Bun}_G = \mathrm{Maps}(X, \mathrm{pt}/G)$. Similarly, given an open substack ${}_0\mathcal{Y} \subset \mathcal{Y}$, a stack $\mathrm{Maps}_{\mathrm{gen}}(X, \mathcal{Y} \supset {}_0\mathcal{Y})$ associates to a scheme S the following data: morphisms $f\colon S \times X \to \mathcal{Y}$ such that for every geometric point $\bar{s} \to S$ there exists an open dense subset $U \subset \bar{s} \times X$ on which the restricted morphism $f|_U\colon U \to \mathcal{Y}$ factors through the open substack ${}_0\mathcal{Y} \subset \mathcal{Y}$.

DEFINITION 2.13 [**21**, § 2.2.2]. Consider an algebraic stack $\mathrm{Vin}_G/(G \times G)$ and an open substack ${}_0\mathrm{Vin}_G/(G \times G) \subset \mathrm{Vin}_G/(G \times G)$. We define

$$\mathrm{VinBun}_G := \mathrm{Maps}_{\mathrm{gen}}(X, \mathrm{Vin}_G/(G \times G) \supset {}_0\mathrm{Vin}_G/(G \times G)).$$

We denote by $\wp_{1,2}\colon \mathrm{VinBun}_G \to \mathrm{Bun}_G = \mathrm{Maps}(X, \mathrm{pt}/G)$ the two natural projections arising from $\mathrm{Vin}_G/(G \times G) \to \mathrm{pt}/(G \times G)$.

Note that the action of $T \curvearrowright \mathrm{Vin}_G$ in § 2.2.2 commutes with the $G \times G$-action on Vin_G. It follows that the T-action on Vin_G induces the T-action $T \curvearrowright \mathrm{VinBun}_G$.

2.3.1. *The degeneration morphism* $\mathrm{VinBun}_G \to T_{\mathrm{ad}}^+$. The morphism $\Upsilon\colon \mathrm{Vin}_G \to T_{\mathrm{ad}}^+$ is $G \times G$-equivariant, so we obtain the morphism

$$\Upsilon_{/(G\times G)}\colon\ \mathrm{Vin}_G/(G \times G) \to T_{\mathrm{ad}}^+/(G \times G) = T_{\mathrm{ad}}^+ \times (\mathrm{pt}/(G \times G)).$$

Let $\mathrm{pr}_1\colon T_{\mathrm{ad}}^+ \times (\mathrm{pt}/(G \times G)) \to T_{\mathrm{ad}}^+$ denote the projection morphism to the first factor. The curve X is proper while the variety T_{ad}^+ is affine, so it follows that the morphism

$$T_{\mathrm{ad}}^+ = \mathrm{Maps}(\mathrm{pt}, T_{\mathrm{ad}}^+) \xrightarrow{\sim} \mathrm{Maps}(X, T_{\mathrm{ad}}^+)$$

induced by the morphism $X \to \mathrm{pt}$ is an isomorphism.

Define

$$\Upsilon\colon\ \mathrm{VinBun}_G \to T_{\mathrm{ad}}^+,\ \Upsilon(f) := \mathrm{pr}_1 \circ (\Upsilon_{/(G\times G)}) \circ f.$$

The morphism Υ is T-equivariant. From § 2.2.2 it follows that the preimage $\Upsilon^{-1}(T_{\mathrm{ad}})$ is isomorphic to $\mathrm{Bun}_G \times T_{\mathrm{ad}}$.

2.3.2. *Defect free locus of* VinBun_G. Set ${}_0\mathrm{VinBun}_G := \mathrm{Maps}(X,\ {}_0\mathrm{Vin}_G/(G\times G))$. The stack ${}_0\mathrm{VinBun}_G$ is a smooth open substack of VinBun_G. Let us denote by ${}_0\Upsilon$ the restriction of the morphism Υ to ${}_0\mathrm{VinBun}_G$. Denote $({}_0\mathrm{VinBun}_G)_0 :=\ {}_0\Upsilon^{-1}(0)$.

Proposition 2.14 [**21**, § 2.4]. *The stack* $({}_0\mathrm{VinBun}_G)_0$ *is isomorphic to the fiber product* $\mathrm{Bun}_{B_-} \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B$.

Proof. Follows from Remark 2.9. □

2.3.3. *The relative compactification* $\overline{\mathrm{Bun}}_G$. Define $\overline{\mathrm{Bun}}_G := \mathrm{VinBun}_G/T$.

Remark 2.15. The stack $\overline{\mathrm{Bun}}_G$ contains the stack $\mathrm{Bun}_G \times (\mathrm{pt}/Z_G)$ as a smooth open substack.

Proposition 2.16 (Gaitsgory). *The natural morphism* $\kappa\colon \overline{\mathrm{Bun}}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ *is proper.*

Proof. Will be given in Appendix A.1. □

2.3.4. *Tannakian approach to the definition of* VinBun_G. Let us explain how to deduce the Tannakian description of VinBun_G from the Tannakian description of Vin_G in § 2.2.8. An S-point of VinBun_G is a morphism f from $S\times X$ to $\mathrm{Vin}_G/(G\times G)$. Such a morphism is the same as a pair of G-torsors $\mathcal{F}_1, \mathcal{F}_2$ on $S\times X$ and a $G\times G$-equivariant morphism from $\mathcal{F}_1\times\mathcal{F}_2$ to Vin_G. It follows from § 2.2.8 that this morphism is the same as the family of $G\times G$-equivariant morphisms $g_{\lambda^\vee}$ $(\lambda^\vee\in\Lambda^{\vee+})$ from $\mathcal{F}_1\times\mathcal{F}_2$ to $\mathrm{End}(V^{\lambda^\vee})$ and a $G\times G$-equivariant regular functions $\tau_{\mu^\vee}$, $\mu^\vee\in\Lambda^{\vee,\mathrm{pos}}$, on $\mathcal{F}_1\times\mathcal{F}_2$, satisfying certain conditions. Note that a $G\times G$-equivariant morphism from $\mathcal{F}_1\times\mathcal{F}_2$ to $\mathrm{End}(V^{\lambda^\vee})$ is the same as the morphism from $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}$ to $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}$ and a $G\times G$-equivariant regular function on $\mathcal{F}_1\times\mathcal{F}_2$ is the same as the regular function on $S\times X$.

It follows that the stack $\mathrm{VinBun}_G\colon \mathbf{Sch}\to\mathbf{Gpd}$ can be described in the following way. It assigns to a scheme S the following data.

(1) Two right G-torsors $\mathcal{F}_1, \mathcal{F}_2$ on X.
(2) For every $\lambda^\vee\in\Lambda^{\vee+}$, a morphism $\varphi_{\lambda^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}\to\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}$.
(3) For every $\mu^\vee\in\Lambda^{\vee,\mathrm{pos}}$, a morphism $\tau_{\mu^\vee}\colon \mathcal{O}_{S\times X}\to\mathcal{O}_{S\times X}$, such that the following conditions hold.

(a) For any $\mu_1^\vee,\mu_2^\vee\in\Lambda^{\vee,\mathrm{pos}}$, we have $\tau_{\mu_1^\vee}\otimes\tau_{\mu_2^\vee} = \tau_{\mu_1^\vee+\mu_2^\vee}$.

(b) For every geometric point $\overline{s}\to S$ and any dominant character $\lambda^\vee\in\Lambda^\vee$, the morphism $\varphi_{\lambda^\vee}|_{\overline{s}\times X}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}|_{\overline{s}\times X}\to\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}|_{\overline{s}\times X}$ is nonzero.

(c) For any $\lambda_1^\vee,\lambda_2^\vee,\nu^\vee\in\Lambda^{\vee+}$ such that $V^{\nu^\vee}$ enters $V^{\lambda_1^\vee}\otimes V^{\lambda_2^\vee}$ with nonzero multiplicity, we denote by $\iota\colon W^{\nu^\vee}\hookrightarrow V^{\lambda_1^\vee}\otimes V^{\lambda_2^\vee}$ the embedding of the corresponding isotypical component and by $\mathrm{pr}\colon V^{\lambda_1^\vee}\otimes V^{\lambda_2^\vee}\twoheadrightarrow W^{\nu^\vee}$ the corresponding projection. We denote by $\iota^{\mathcal{F}_1}, \mathrm{pr}^{\mathcal{F}_2}$ the corresponding morphisms between the induced vector bundles. Then we have

$$\mathrm{pr}^{\mathcal{F}_2}\circ(\varphi_{\lambda_1^\vee}\otimes\varphi_{\lambda_2^\vee})\circ\iota^{\mathcal{F}_1} = (\tau_{\lambda_1^\vee+\lambda_2^\vee-\nu^\vee})\otimes\varphi_{\nu^\vee}.$$

(d) The morphism φ_0 coincides with the identity morphism, and the morphism τ_0 coincides with the identity morphism.

Condition (b) is a consequence of the fact that the morphism f generically lands into ${}_0\mathrm{Vin}_G/(G\times G)$. Conditions (a), (c) and (d) follow from the conditions (a), (b) and (c) of in 2.2.8, respectively.

In Tannakian terms, the morphism $\Upsilon\colon \mathrm{VinBun}_G\to T^+_{\mathrm{ad}}$ sends an S-point $(\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee})\in \mathrm{VinBun}_G(S)$ to the point $(\tau_{\mu^\vee})\in T^+_{\mathrm{ad}}(S)$.

2.3.5. *Description of* VinBun_G *for* $G=SL_2$. Following [**20**, Section 2.1.1] we can describe the stack VinBun_{SL_2} in the following way. It assigns to a scheme S a pair of vector bundles $\mathcal{F}_1,\mathcal{F}_2$ of rank 2 on $X\times S$ together with trivializations of their determinant line bundles $\det\mathcal{F}_1,\det\mathcal{F}_2$ and a morphism of coherent sheaves $\varphi\colon\mathcal{F}_1\to\mathcal{F}_2$ satisfying the following condition: For each geometric point $\overline{s}\to S$ the morphism $\varphi|_{X\times\overline{s}}$ is nonzero. In these terms the morphism $\Upsilon\colon \mathrm{VinBun}_{SL_2}\to T^+_{\mathrm{ad}}=\mathbb{A}^1$ sends an S-point to the determinant $\det\varphi\in\Gamma(\mathcal{O}_{X\times S})=\mathbb{A}^1(S)$.

2.4. *Local model for* Bun_B

We now recall the construction of a certain *local model* for Bun_B from [**6**]. This local model is called *open zastava*. We need some notations first.

2.4.1. *The open Bruhat locus of* G. Let us define *the open Bruhat locus*$G^{\mathrm{Bruhat}}\subset G$ as the $B\times N_-$-orbit of $1\in G$.

DEFINITION 2.17. The open zastava is defined as

$$\overset{\circ}{Z}:=\mathrm{Maps}_{\mathrm{gen}}(X,G/(B\times N_-)\supset G^{\mathrm{Bruhat}}/(B\times N_-)=\mathrm{pt}).$$

2.4.2. *Connected components of the open zastava.* For any positive integer $n\in\mathbb{N}$ we denote the nth symmetric power of the curve X by $X^{(n)}$. Let $\theta\in\Lambda^{\mathrm{pos}}$. A point $D\in X^\theta$ is a collection of effective divisors $D_{\lambda^\vee}\in X^{(\langle\lambda^\vee,\theta\rangle)}$ for $\lambda\in\Lambda^{\vee+}$ such that for every $\lambda_1^\vee,\lambda_2^\vee\in\Lambda^{\vee+}$ we have $D_{\lambda_1^\vee}+D_{\lambda_2^\vee}=D_{\lambda_1^\vee+\lambda_2^\vee}$. Since the derived subgroup of G is assumed to be simply connected, for $\theta=\sum_{i\in I}n_i\alpha_i\in\Lambda^{\mathrm{pos}}$ we have $X^\theta=\prod_{i\in I}X^{(n_i)}$.

Let us describe the connected components of the scheme $\overset{\circ}{Z}$. Note that the natural morphism $G/(N\times N_-)\to G/\!\!/(N\times N_-)\simeq\overline{T}$ induces the morphism $\pi\colon\overset{\circ}{Z}\to\mathrm{Maps}_{\mathrm{gen}}(X,\overline{T}/T\supset\mathrm{pt})$. The scheme $\mathrm{Maps}_{\mathrm{gen}}(X,\overline{T}/T\supset T/T=\mathrm{pt})$ is isomorphic to $\coprod_{\theta\in\Lambda^{\mathrm{pos}}}X^\theta$. For $\theta\in\Lambda^{\mathrm{pos}}$, set $\overset{\circ}{Z}{}^\theta:=\pi^{-1}(X^\theta)$.

PROPOSITION 2.18 [**5**, Proposition 2.25]. *The decomposition* $\overset{\circ}{Z}=\coprod_{\theta\in\Lambda^{\mathrm{pos}}}\overset{\circ}{Z}{}^\theta$ *coincides with the decomposition of* $\overset{\circ}{Z}$ *into connected components.*

2.4.3. *Factorization of zastava* [**6**, § 2.3]. Let us fix $\theta_1,\theta_2,\theta\in\Lambda^{\mathrm{pos}}$ such that $\theta=\theta_1+\theta_2$. We have an addition morphism $\mathrm{add}\colon X^{\theta_1}\times X^{\theta_2}\to X^\theta$. An open subset $(X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}}\subset X^{\theta_1}\times X^{\theta_2}$ is formed by the pairs $(D_{\theta_1},D_{\theta_2})$ of disjoint divisors. The *factorization* is a canonical isomorphism

$$\mathfrak{f}_{\theta_1,\theta_2}\colon (X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}}\underset{X^\theta}{\times}\overset{\circ}{Z}{}^\theta\xrightarrow{\sim}(X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}}\underset{X^{\theta_1}\times X^{\theta_2}}{\times}(\overset{\circ}{Z}{}^{\theta_1}\times\overset{\circ}{Z}{}^{\theta_2}).$$

For a point $x\in X$ and a cocharacter $\theta\in\Lambda^{\mathrm{pos}}$, let us denote by $\overset{\circ}{\mathfrak{Z}}{}^\theta$ the fiber $\pi^{-1}(\theta\cdot x)$. For two points $x,y\in X$, let us denote by $\theta_1\cdot x+\theta_2\cdot y\in X^\theta$ the point $\mathrm{add}(\theta_1\cdot x,\theta_2\cdot y)$. It follows from factorization that for $x \neq y$ the fiber $\pi^{-1}(\theta_1 \cdot x + \theta_2 \cdot y)$ can be canonically identified with $\mathring{\mathfrak{Z}}^{\theta_1}_x \times \mathring{\mathfrak{Z}}^{\theta_2}_y$.

2.4.4. *Tannakian approach to the definition of Z^θ.* For $\lambda^\vee \in \Lambda^\vee$, let $\mathbb{C}^{\lambda^\vee}$ be the one-dimensional representation of T via character $\lambda^\vee : T \to \mathbb{C}^\times$. Fix a positive cocharacter $\theta \in \Lambda^{\mathrm{pos}}$. We recall the Tannakian definition of the functor $Z^\theta \colon \mathbf{Sch} \to \mathbf{Set},\ S \mapsto Z^\theta(S)$. It associates to a scheme S

(1) a G-bundle $\mathcal{F}$ on $S \times X$;
(2) a T-bundle $\mathcal{T}$ on $S \times X$ of degree $-\theta$;
(3) for every $\lambda^\vee \in \Lambda^{\vee+}$, a morphism of coherent sheaves $\mathbb{C}^{\lambda^\vee}_{\mathcal{T}} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}}$ and a surjective morphism of vector bundles $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}$ satisfying the following conditions.
    (a) For every $\lambda^\vee \in \Lambda^{\vee+}$ the composition $(\zeta_{\lambda^\vee} \circ \eta_{\lambda^\vee})$ is an isomorphism generically.
    (b) The Plücker relations hold.

Remark 2.19. The open subscheme $\mathring{Z}^\theta \subset Z^\theta$ consists of $(\mathcal{F}, \mathcal{T}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \in Z^\theta$ such that $\eta_{\lambda^\vee}$ are injective morphisms of vector bundles.

2.4.5. *Matrix description of open zastava for $G = SL_2$.* Let $G = SL_2$, $T = \{\mathrm{diag}(t, t^{-1}) \,|\, t \in \mathbb{C}^\times\} \subset SL_2$ and identify $\mathbb{Z} \xrightarrow{\sim} \Lambda$ via $n \mapsto (t \mapsto \mathrm{diag}(t^n, t^{-n}))$. We assume that $X = \mathbb{A}^1$ (note that zastava spaces are defined even if X is not projective). Fix $n \in \mathbb{Z}_{\geqslant 0} = \Lambda^{\mathrm{pos}}$. The variety $\mathring{Z}^n$ can be identified with the space of matrices $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{2\times 2}[z]$ such that A is a monic polynomial of degree n, while the degrees of B and C are strictly less than n, and $\det \mathbf{M} = 1$ (see, for example, [7, § 2(xii)]). The morphism $\pi \colon \mathring{Z}^n \to \mathbb{A}^{(n)}$ in these terms sends matrix $\mathbf{M}$ to the set of roots of A with multiplicities.

2.4.6. Let us fix two cocharacters $\theta_1, \theta_2 \in \Lambda, \theta_1 \geqslant \theta_2$ and a point $x \in X$.

Definition 2.20. $\overline{\mathfrak{Z}}^{\theta_1,\theta_2}$ is the moduli space of the following data: it associates to a scheme S

(1) a G-bundle $\mathcal{F}$ on $S \times X$;
(2) for every $\lambda^\vee \in \Lambda^{\vee+}$, a morphism of *sheaves* $\mathcal{O}_{S\times X}(-\langle\lambda^\vee, \theta_1\rangle \cdot (S \times x)) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}}$ and a morphism of *sheaves* $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}(-\langle\lambda^\vee, \theta_2\rangle \cdot (S \times x))$ satisfying the following conditions.
    (a) For every $\lambda^\vee \in \Lambda^{\vee+}$ the composition $(\zeta_{\lambda^\vee} \circ \eta_{\lambda^\vee})$ coincides with the canonical embedding $\mathcal{O}_{S\times X}(-\langle\lambda^\vee, \theta_1\rangle \cdot (S \times x)) \hookrightarrow \mathcal{O}_{S\times X}(-\langle\lambda^\vee, \theta_2\rangle \cdot (S \times x))$,
    (b) The Plücker relations hold.

Remark 2.21. Note that the scheme $\overline{\mathfrak{Z}}^{\theta_1,\theta_2}$ naturally identifies with $\overline{T}_{\theta_2} \cap \overline{S}_{\theta_1}$.

2.5. *Local model for* VinBun_G

We now recall the construction of certain *local model* for VinBun_G in [21, § 6.1.6]. First we need some notations.

2.5.1. *The open Bruhat locus of the Vinberg semi-group.* Let us define *the open Bruhat locus* $\mathrm{Vin}^{\mathrm{Bruhat}}_G \subset \mathrm{Vin}_G$ as $B \times N_-$-orbit of the image of the section $\mathfrak{s} \colon T^+_{\mathrm{ad}} \hookrightarrow \mathrm{Vin}_G$.

Definition 2.22 [21, § 6.1.6]. The local model is defined as

$$Y := \mathrm{Maps}_{\mathrm{gen}}(X, \mathrm{Vin}_G/(B \times N_-) \supset \mathrm{Vin}_G^{\mathrm{Bruhat}}/(B \times N_-)).$$

2.5.2. *Defect free locus of Y.* Set

$${}_0Y := \mathrm{Maps}_{\mathrm{gen}}(X, {}_0\mathrm{Vin}_G/(B \times N_-) \supset \mathrm{Vin}_G^{\mathrm{Bruhat}}/(B \times N_-)).$$

The scheme ${}_0Y$ is a smooth open subscheme of Y.

2.5.3. *Local models Y^θ.* Using Lemma 2.11, we get the natural morphism

$$\mathrm{Vin}_G/(N \times N_-) \to \mathrm{Vin}_G /\!\!/ (N \times N_-) \simeq \overline{T} \times T_{\mathrm{ad}}^+$$

that gives us the morphism $\mathrm{Vin}_G/(B \times N_-) \to (\overline{T}/T) \times T_{\mathrm{ad}}^+$, which, in turn, induces the morphism

$$\pi\colon Y \to \mathrm{Maps}_{\mathrm{gen}}(X, (\overline{T}/T) \times T_{ad}^+ \supset \mathrm{pt} \times T_{\mathrm{ad}}^+) \simeq \mathrm{Maps}_{\mathrm{gen}}(X, \overline{T}/T \supset \mathrm{pt}).$$

The ind-scheme $\mathrm{Maps}_{\mathrm{gen}}(X, \overline{T}/T \supset T/T = \mathrm{pt})$ is isomorphic to $\coprod_{\theta \in \Lambda^{\mathrm{pos}}} X^\theta$. For $\theta \in \Lambda^{\mathrm{pos}}$ set $Y^\theta := \pi^{-1}(X^\theta)$.

For a positive cocharacter $\theta \in \Lambda^{\mathrm{pos}}$ set ${}_0Y^\theta := {}_0Y \cap Y^\theta$: an open dense smooth subscheme of Y^θ.

2.5.4. *Tannakian approach to the definition of Y^θ.* The scheme Y^θ can be described in the following way. The corresponding functor of points assigns to a scheme S the following data.

(1) An S-point $(\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}) \in \mathrm{VinBun}_G(S)$ of VinBun_G.
(2) A T-bundle $\mathcal{T}$ on $S \times X$ of degree $-\theta$.
(3) For every $\lambda^\vee \in \Lambda^{\vee +}$, morphisms of vector bundles

$$\eta_{\lambda^\vee} : \mathbb{C}_{\mathcal{T}}^{\lambda^\vee} \hookrightarrow \mathcal{V}_{\mathcal{F}_1}^{\lambda^\vee},\ \zeta_{\lambda^\vee} : \mathcal{V}_{\mathcal{F}_2}^{\lambda^\vee} \twoheadrightarrow \mathcal{O}_{S \times X},$$

satisfying the following conditions.

(a) For every $\lambda^\vee \in \Lambda^{\vee +}$, the composition

$$\zeta_{\lambda^\vee} \circ \eta_{\lambda^\vee} : \mathbb{C}_{\mathcal{T}}^{\lambda^\vee} \to \mathcal{O}_{S \times X}$$

is an isomorphism generically.
(b) The Plücker relations hold.

We have the morphism $\Upsilon^\theta\colon Y^\theta \to T_{\mathrm{ad}}^+$ that forgets the data of $\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}$.

2.5.5. *Matrix description of Y^θ for $G = SL_2$.* Let $G = SL_2, X = \mathbb{A}^1$. We keep the notations of Section 2.4.5. Fix $n \in \mathbb{Z}_{\geqslant 0} = \Lambda^{\mathrm{pos}}$. It can be deduced from [20, Section 5.3] that the variety Y^n is isomorphic to the space of matrices $\mathbf{M} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Mat}_{2\times 2}[z]$ such that A is a monic polynomial of degree n, while the degrees of B and C are strictly less than n, and $\det \mathbf{M} \in \mathbb{C} \subset \mathbb{C}[z]$. The morphism Υ^n in these terms sends $\mathbf{M}$ to $\det \mathbf{M}$. The morphism π sends $\mathbf{M}$ to the set of roots of A with multiplicities.

2.5.6. *Compactified local model $\overline{Y}^\theta$.* For a positive cocharacter $\theta \in \Lambda^{\mathrm{pos}}$, we define a certain compactification $\overline{Y}^\theta$ of the local model Y^θ. It associates to a scheme S

(1) an S-point $(\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}) \in \mathrm{VinBun}_G(S)$ of VinBun_G;
(2) a T-bundle $\mathcal{T}$ on $S \times X$ of degree $-\theta$;

(3) for every $\lambda^\vee \in \Lambda^{\vee+}$, morphisms of coherent sheaves

$$\eta_{\lambda^\vee}\colon \mathbb{C}_{\mathcal{T}}^{\lambda^\vee} \hookrightarrow \mathcal{V}_{\mathcal{F}_1}^{\lambda^\vee},\ \zeta_{\lambda^\vee}\colon \mathcal{V}_{\mathcal{F}_2}^{\lambda^\vee} \to \mathcal{O}_{S\times X},$$

satisfying the following conditions.

(a) For every $\lambda^\vee \in \Lambda^{\vee+}$, the composition

$$\zeta_{\lambda^\vee}\circ\eta_{\lambda^\vee}\colon \mathbb{C}_{\mathcal{T}}^{\lambda^\vee} \to \mathcal{O}_{S\times X}$$

is an isomorphism generically.

(b) The Plücker relations hold.

We have the morphism $\Upsilon^\theta\colon \overline{Y}^\theta \to T_{\mathrm{ad}}^+$ that forgets the data of $\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}$. Let us denote by $\overline{Y}^{\theta,\mathrm{princ}}$ the restriction of the multiparameter family $\overline{Y}^\theta \to T_{\mathrm{ad}}^+$ to the 'principal' line

$$\mathbb{A}^1 \hookrightarrow T_{\mathrm{ad}}^+,\ a \mapsto (a,\dots,a) \text{ in coordinates } \alpha_i^\vee,\ i\in I.$$

Let us denote by ${}_0\Upsilon^{\theta,\mathrm{princ}}$ the restriction of the morphism Υ^θ to ${}_0Y^\theta \subset \overline{Y}^\theta$.

Recall the definition of X^θ in § 2.4.2. Let $\overline{\pi}_\theta\colon \overline{Y}^\theta \to X^\theta$ be the morphism given by

$$(\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \mapsto (\mathrm{Div}(\zeta_{\lambda^\vee}\circ\eta_{\lambda^\vee})).$$

Here by $\mathrm{Div}(\zeta_{\lambda^\vee}\circ\eta_{\lambda^\vee})$ we mean the divisor of zeros of $\zeta_{\lambda^\vee}\circ\eta_{\lambda^\vee}$ considered as a global section of $(\mathbb{C}_{\mathcal{T}}^{\lambda^\vee})^*$. Note that the fiber of $\overline{\pi}_\theta$ over $1 \in T_{\mathrm{ad}} \subset T_{\mathrm{ad}}^+$ is nothing but the compactified zastava space $\overline{Z}^\theta$ of [**14**, § 7.2].

For $x \in X$, set $\overline{\mathfrak{Y}}^\theta := \overline{\pi}_\theta^{-1}(\theta\cdot x)$, $\overline{\mathfrak{Y}}^{\theta,\mathrm{princ}} := \overline{\mathfrak{Y}}^\theta \cap \overline{Y}^{\theta,\mathrm{princ}}$.

REMARK 2.23 (Gaitsgory). We have

$$\bigsqcup_{\theta\in\Lambda^{\mathrm{pos}}} \overline{Y}^\theta$$
$$= \mathrm{Maps}_{\mathrm{gen}}\Big(X, G\backslash\Big(\overline{G/N_-}\times \mathrm{Vin}_G \times \overline{N\backslash G}/T\Big)/G \supset G\backslash\Big((G/N_-)\times \mathrm{Vin}_G^{\mathrm{Bruhat}} \times (B\backslash G)\Big)/G\Big).$$

2.5.7. *Factorization of compactified local models.* The compactified local models $\overline{Y}^\theta$ factorize in families over T_{ad}^+ in the sense of the following lemma:

LEMMA 2.24. *Let $\theta_1, \theta_2 \in \Lambda^{\mathrm{pos}}$ and let $\theta := \theta_1+\theta_2$. Then the addition morphism $(X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}} \xrightarrow{\mathrm{add}} X^\theta$ induces a Cartesian square*

$$\begin{array}{ccc} (X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}} \underset{X^{\theta_1}\times X^{\theta_2}}{\times} (\overline{Y}^{\theta_1} \underset{T_{\mathrm{ad}}^+}{\times} \overline{Y}^{\theta_2}) & \longrightarrow & \overline{Y}^\theta \\ \downarrow & & \downarrow \overline{\pi}_\theta \\ (X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}} & \xrightarrow{\mathrm{add}} & X^\theta. \end{array}$$

Proof. The proof is analogous to the one of [**20**, Proposition 5.1.2]. □

The above Lemma implies that the fibers of the morphism $\overline{Y}^\theta \to T_{\mathrm{ad}}^+$ are factorizable. That is for each $t \in T_{\mathrm{ad}}^+$ we have an isomorphism

$$(X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}} \underset{X^\theta}{\times} (\overline{Y}^\theta|_t) \xrightarrow{\sim} (X^{\theta_1}\times X^{\theta_2})_{\mathrm{disj}} \underset{X^{\theta_1}\times X^{\theta_2}}{\times} (\overline{Y}^{\theta_1}|_t \times \overline{Y}^{\theta_2}|_t).$$

2.5.8. Let us fix two cocharacters $\theta_1, \theta_2 \in \Lambda$, $\theta_1 \geqslant \theta_2$, and a point $x \in X$.

Definition 2.25. The space $\overline{\mathfrak{Y}}^{\theta_1,\theta_2}$ associates to a scheme S

(1) an S-point $(\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee})\in \mathrm{VinBun}_G(S)$ of VinBun_G;
(2) for every $\lambda^\vee\in\Lambda^{\vee+}$, a morphism of vector bundles $\mathcal{O}_{S\times X}(-\langle\lambda^\vee,\theta_1\rangle\cdot(S\times x))\xrightarrow{\eta_{\lambda^\vee}}\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}$ and a morphism of vector bundles $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}\xrightarrow{\zeta_{\lambda^\vee}}\mathcal{O}_{S\times X}(-\langle\lambda^\vee,\theta_2\rangle\cdot(S\times x))$ satisfying the following conditions.
   (a) For every $\lambda^\vee\in\Lambda^{\vee+}$ the composition $(\zeta_{\lambda^\vee}\circ\varphi_{\lambda^\vee}\circ\eta_{\lambda^\vee})$ coincides with the canonical morphism $\mathcal{O}_{S\times X}(-\langle\lambda^\vee,\theta_1\rangle\cdot(S\times x))\hookrightarrow\mathcal{O}_{S\times X}(-\langle\lambda^\vee,\theta_2\rangle\cdot(S\times x))$.
   (b) The Plücker relations hold.

Let $\Upsilon^{\theta_1}_{\theta_2}\colon\overline{\mathfrak{Y}}^{\theta_1,\theta_2}\to T^+_{\mathrm{ad}}$ be the morphism given by

$$(\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee},\eta_{\lambda^\vee},\zeta_{\lambda^\vee})\mapsto(\tau_{\mu^\vee}).$$

We denote by $\overline{\mathfrak{Y}}^{\theta_1,\theta_2,\mathrm{princ}}$ the restriction of the multiparameter family $\Upsilon^{\theta_1}_{\theta_2}\colon\overline{\mathfrak{Y}}^{\theta_1,\theta_2}\to T^+_{\mathrm{ad}}$ to the 'principal' line

$$\mathbb{A}^1\hookrightarrow T^+_{\mathrm{ad}},\ a\mapsto(a,\dots,a)\ \text{in coordinates}\ \alpha_i^\vee,\ i\in I.$$

We will prove in Lemma 3.16 that the family $\overline{\mathfrak{Y}}^{\theta_1,\theta_2}\to T^+_{\mathrm{ad}}$ is isomorphic to $\overline{\mathfrak{Y}}^{\theta_1-\theta_2}\to T^+_{\mathrm{ad}}$.

2.6. *Schieder bialgebra*

Let us recall the construction of certain bialgebra from [22]. We set

$$\mathcal{A}:=\bigoplus_{\theta\in\Lambda^{\mathrm{pos}}}H_c^{\langle 2\rho^\vee,\theta\rangle}(\mathring{\mathfrak{Z}}^\theta).$$

Remark 2.26. It follows from [6, § 6] that for $\theta\in\Lambda^{\mathrm{pos}}$ the vector space $H_c^{\langle 2\rho^\vee,\theta\rangle}(\mathring{\mathfrak{Z}}^\theta)$ can be identified with the θ-weight component $U(\mathfrak{n}^\vee)_\theta$. It follows that the algebra $\mathcal{A}$ is isomorphic to $U(\mathfrak{n}^\vee)$ as a graded vector space.

2.6.1. *The two-parameter degeneration* [22, § 6.2.2]. Take $\theta\in\Lambda^{\mathrm{pos}}$. Let us consider the morphism

$$\pi_\theta\times{}_0\Upsilon^{\theta,\mathrm{princ}}\colon{}_0Y^{\theta,\mathrm{princ}}\to X^\theta\times\mathbb{A}^1.$$

Fix $x\in X$ and positive cocharacters $\theta_1,\theta_2\in\Lambda^{\mathrm{pos}}$ such that $\theta_1+\theta_2=\theta$. We define the family $\Pi\colon Q\to X\times\mathbb{A}^1$ as the pullback of the family ${}_0Y^{\theta,\mathrm{princ}}\to X^\theta\times\mathbb{A}^1$ along

$$X\times\mathbb{A}^1\hookrightarrow X^\theta\times\mathbb{A}^1,\ (y,a)\mapsto(\theta_1\cdot x+\theta_2\cdot y,\ a).$$

Let Π_{comult} denote the one-parameter family over X obtained by restricting the family Π above along the inclusion $X\times\{1\}\hookrightarrow X\times\mathbb{A}^1$. Similarly, let Π_{mult} denote the one-parameter family over $\mathbb{A}^1$ obtained by restricting the family Π above along the inclusion $\{x\}\times\mathbb{A}^1\hookrightarrow X\times\mathbb{A}^1$.

Proposition 2.27 [22, Corollary 6.2.5].

(a) *The one-parameter family Π_{comult} is trivial over $X\setminus\{x\}$. The special fiber is*

$$Q|_{(\theta\cdot x,1)}=\mathring{\mathfrak{Z}}^\theta.$$

A general fiber is

$$Q|_{(\theta_1\cdot x+\theta_2\cdot y,1)}=\mathring{\mathfrak{Z}}^{\theta_1}\times\mathring{\mathfrak{Z}}^{\theta_2}.$$

(b) *The one-parameter family* Π_{mult} *is trivial over* $\mathbb{A}^1 \setminus \{0\}$. *The special fiber is*

$$Q|_{(\theta\cdot x,0)} = \bigsqcup_{\theta_1+\theta_2=\theta} \overset{\circ}{\mathfrak{Z}}{}^{\theta_1} \times \overset{\circ}{\mathfrak{Z}}{}^{\theta_2}.$$

A general fiber is $Q|_{(\theta\cdot x,1)} = \overset{\circ}{\mathfrak{Z}}{}^{\theta}$.

2.6.2. *Comultiplication* [**12**, § 2.11.1; **22**, § 6.3.2]. Given positive coweights $\theta, \theta_1, \theta_2 \in \Lambda^{\mathrm{pos}}$ with $\theta_1 + \theta_2 = \theta$ we define a morphism of vector spaces

$$\Delta_{\theta_1,\theta_2} \colon \mathcal{A}_\theta \to \mathcal{A}_{\theta_1} \otimes \mathcal{A}_{\theta_2}$$

as the cospecialization morphism

$$H_c^{\langle 2\rho^\vee,\theta\rangle}(\overset{\circ}{\mathfrak{Z}}{}^{\theta}) = H_c^{\langle 2\rho^\vee,\theta\rangle}(Q|_{(\theta\cdot x,1)}) \to H_c^{\langle 2\rho^\vee,\theta\rangle}(Q|_{(\theta_1\cdot x+\theta_2\cdot y,1)}) = H_c^{\langle 2\rho^\vee,\theta_1\rangle}(\overset{\circ}{\mathfrak{Z}}{}^{\theta}_1) \otimes H_c^{\langle 2\rho^\vee,\theta_2\rangle}(\overset{\circ}{\mathfrak{Z}}{}^{\theta}_2)$$

corresponding to the one-parameter degeneration Π_{comult}. Summing over all such triples $(\theta, \theta_1, \theta_2)$ we obtain a comultiplication morphism

$$\Delta \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}.$$

2.6.3. *Multiplication* [**22**, § 6.3.3]. Given a positive coweight $\theta \in \Lambda^{\mathrm{pos}}$ we define a morphism of vector spaces

$$\mathbf{m}_\theta := \bigoplus_{\theta_1+\theta_2=\theta} \mathbf{m}_{\theta_1,\theta_2} \colon \bigoplus_{\theta_1+\theta_2=\theta} \mathcal{A}_{\theta_1} \otimes \mathcal{A}_{\theta_2} \to \mathcal{A}_\theta$$

as the cospecialization morphism

$$\bigoplus_{\theta_1+\theta_2=\theta} H_c^{\langle 2\rho^\vee,\theta_1\rangle}(\overset{\circ}{\mathfrak{Z}}{}^{\theta}_1) \otimes H_c^{\langle 2\rho^\vee,\theta_2\rangle}(\overset{\circ}{\mathfrak{Z}}{}^{\theta}_2) = H_c^{\langle 2\rho^\vee,\theta\rangle}(Q|_{(\theta x,0)}) \to H_c^{\langle 2\rho^\vee,\theta\rangle}(Q|_{(\theta x,1)}) = H_c^{\langle 2\rho^\vee,\theta\rangle}(\overset{\circ}{\mathfrak{Z}}{}^{\theta})$$

corresponding to the one-parameter degeneration Π_{mult}. Summing over all $\theta \in \Lambda^{\mathrm{pos}}$ we obtain a multiplication morphism

$$\mathbf{m} \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

3. *Drinfeld–Gaitsgory–Vinberg interpolation Grassmannian*

3.1. *A degeneration of the Beilinson–Drinfeld Grassmannian*

3.1.1. *The example* $G = \mathrm{SL}_2$. Let us start from the case $G = \mathrm{SL}_2$. Let us fix a point $x \in X$.

Definition 3.1. For $G = \mathrm{SL}_2$, an S-point of $\mathrm{VinGr}_{G,x}$ consists of the following data.

(1) Two vector bundles $\mathcal{V}_1, \mathcal{V}_2$ of rank 2 on $S \times X$, together with trivializations of their determinant line bundles $\det \mathcal{V}_1, \det \mathcal{V}_2$ and a morphism of coherent sheaves $\varphi \colon \mathcal{V}_1 \to \mathcal{V}_2$.

(2) The rational morphisms $\eta \colon \mathcal{O} \to \mathcal{V}_1$, $\zeta \colon \mathcal{V}_2 \to \mathcal{O}$ regular on $S \times (X \setminus \{x\})$ such that the composition

$$(\zeta \circ \varphi \circ \eta)|_{S\times(X\setminus\{x\})} \colon \mathcal{O}|_{S\times(X\setminus\{x\})} \to \mathcal{O}|_{S\times(X\setminus\{x\})}$$

is the identity morphism.

We have a morphism $\Upsilon \colon \mathrm{VinGr}_{G,x} \to \mathbb{A}^1$ which sends an S-point above to the determinant

$$\det \varphi \in \Gamma(\mathcal{O}_{X\times S}) = \Gamma(\mathcal{O}_S) = \mathbb{A}^1(S).$$

It follows from Proposition 3.12 that the functor $\mathrm{VinGr}_{G,x}$ is represented by an ind-scheme (ind-projective over $\mathbb{A}^1$) denoted by the same symbol.

Let $(\mathrm{VinGr}_{G,x})_a$ denote the fiber of the morphism Υ over $a \in \mathbb{A}^1$.

Proposition 3.2. *The fiber of the morphism Υ over the point $1 \in \mathbb{A}^1$ is isomorphic to the affine Grassmannian Gr_G.*

Proof. Let us construct a morphism of functors $\Theta\colon (\mathrm{VinGr}_{G,x})_1 \to \mathrm{Gr}_G$. Note that the vector bundles $\mathcal{V}_1,\ \mathcal{V}_2$ are identified via φ. From condition 3.1(2) it follows that the rational morphisms η, ζ define transversal N and N_- structures in the vector bundle

$$\mathcal{V} := \mathcal{V}_1|_{S\times(X\setminus\{x\})} \simeq \mathcal{V}_2|_{S\times(X\setminus\{x\})}.$$

Thus we get a trivialization σ of the vector bundle $\mathcal{V}|_{S\times(X\setminus\{x\})}$. Set

$$\Theta(\mathcal{V}_1, \mathcal{V}_2, \varphi, \zeta, \eta) := (\mathcal{V}, \sigma) \in \mathrm{Gr}_G(S).$$

Let us construct the inverse morphism $\Theta^{-1}\colon \mathrm{Gr}_G \to (\mathrm{VinGr}_{G,x})_1$. Consider a point $(\mathcal{V}, \sigma) \in \mathrm{Gr}_G(S)$. Set $\mathcal{V}_1 := \mathcal{V} =: \mathcal{V}_2$, $\varphi := \mathrm{Id}\colon \mathcal{V}_1 = \mathcal{V} \to \mathcal{V} = \mathcal{V}_2$. Let us define $\eta\colon \mathcal{O} \to \mathcal{V} = \mathcal{V}_1$ as the composition of $\eta_0\colon \mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathcal{O}$, $s \mapsto (s, 0)$ and σ. Let us define $\zeta\colon \mathcal{V}_2 = \mathcal{V} \to \mathcal{O}$ as the composition of σ^{-1} and $\zeta_0\colon \mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$, $(s_1, s_2) \mapsto s_1$. Finally, we set $\Theta^{-1}(\mathcal{V}, \sigma) := (\mathcal{V}_1, \mathcal{V}_2, \varphi, \zeta, \eta)$. □

3.1.2. *The degeneration of Gr_G for arbitrary G via Tannakian approach.* We now define a degeneration $\mathrm{VinGr}_{G,x}$ of the affine Grassmannian Gr_G for arbitrary reductive group G. An S-point of $\mathrm{VinGr}_{G,x}$ consists of the following data.

(1) An S-point $(\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}) \in \mathrm{VinBun}_G(S)$.
(2) For every $\lambda^\vee \in \Lambda^{\vee+}$, the rational morphisms

$$\eta_{\lambda^\vee} : \mathcal{O}_{S\times X} \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1},\ \zeta_{\lambda^\vee} : \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \to \mathcal{O}_{S\times X}$$

regular on $U := S \times (X \setminus \{x\})$, satisfying the following conditions.

(a) For every $\lambda^\vee \in \Lambda^{\vee+}$, the composition

$$(\zeta_{\lambda^\vee} \circ \varphi_{\lambda^\vee} \circ \eta_{\lambda^\vee})|_U$$

is the identity morphism.

(b) For every $\lambda^\vee, \mu^\vee \in \Lambda^{\vee+}$ let $\mathrm{pr}_{\lambda^\vee,\mu^\vee} : V^{\lambda^\vee} \otimes V^{\mu^\vee} \twoheadrightarrow V^{\lambda^\vee+\mu^\vee}$ be the projection morphism in § 2.1.6. We have the corresponding morphisms

$$\mathrm{pr}^{\mathcal{F}_1}_{\lambda^\vee,\mu^\vee} : \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}_1} \to \mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}_1},\ \mathrm{pr}^{\mathcal{F}_2}_{\lambda^\vee,\mu^\vee} : \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}_2} \to \mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}_2}.$$

Then the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{O}_U \otimes \mathcal{O}_U & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathcal{O}_U \\ \big\downarrow{\scriptstyle \eta_{\lambda^\vee}\otimes\eta_{\mu^\vee}} & & \big\downarrow{\scriptstyle \eta_{\lambda^\vee+\mu^\vee}} \\ (\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}_1})|_U & \xrightarrow{\mathrm{pr}^{\mathcal{F}_1}_{\lambda^\vee,\mu^\vee}} & (\mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}_1})|_U, \end{array}$$

$$\begin{array}{ccc} (\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}_2})|_U & \xrightarrow{\mathrm{pr}^{\mathcal{F}_2}_{\lambda^\vee,\mu^\vee}} & (\mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}_2})|_U \\ \big\downarrow{\scriptstyle \zeta_{\lambda^\vee}\otimes\zeta_{\mu^\vee}} & & \big\downarrow{\scriptstyle \zeta_{\lambda^\vee+\mu^\vee}} \\ \mathcal{O}_U \otimes \mathcal{O}_U & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathcal{O}_U. \end{array}$$

(c) Given a morphism $\mathrm{pr}\colon V^{\lambda^\vee}\otimes V^{\mu^\vee}\to V^{\nu^\vee}$ for $\lambda^\vee,\mu^\vee,\nu^\vee\in\Lambda^+$, $\nu^\vee<\lambda^\vee+\mu^\vee$, we have

$$\mathrm{pr}^{\mathcal{F}_1}\circ(\eta_{\lambda^\vee}\otimes\eta_{\mu^\vee})=0,\ \ (\zeta_{\lambda^\vee}\otimes\zeta_{\mu^\vee})\circ\mathrm{pr}^{\mathcal{F}_2}=0.$$

(d) For $\lambda^\vee=0$, we have $\zeta_{\lambda^\vee}=\mathrm{Id}$ and $\eta_{\lambda^\vee}=\mathrm{Id}$.

We have a projection $\mathrm{VinGr}_{G,x}\to\mathrm{VinBun}_G$ that forgets the data $\eta_{\lambda^\vee},\zeta_{\lambda^\vee}$.

3.1.3. *The degeneration* $\mathrm{VinGr}_{G,x}$ *via mapping stacks.* Recall the family $\Upsilon\colon \mathrm{Vin}_G\to T^+_{\mathrm{ad}}$ in § 2.2.2 considered as the scheme over T^+_{ad}. Fix a point $x\in X$. Denote by $\mathrm{VinGr}'_{G,x}$ the fiber product

$$\mathrm{Maps}_{T^+_{\mathrm{ad}}}(X\times T^+_{\mathrm{ad}},\mathrm{Vin}_G/(G\times G))\underset{\mathrm{Maps}_{T^+_{\mathrm{ad}}}((X\setminus\{x\})\times T^+_{\mathrm{ad}},\mathrm{Vin}_G/(G\times G))}{\times}T^+_{\mathrm{ad}},$$

where the morphism $T^+_{\mathrm{ad}}\to\mathrm{Maps}_{T^+_{\mathrm{ad}}}((X\setminus\{x\})\times T^+_{\mathrm{ad}},\mathrm{Vin}_G/(G\times G))$ is the composition of the isomorphism $T^+_{\mathrm{ad}}\xrightarrow{\sim}\mathrm{Maps}_{T^+_{\mathrm{ad}}}((X\setminus\{x\})\times T^+_{\mathrm{ad}},T^+_{\mathrm{ad}})$ with the morphism

$$\mathrm{Maps}_{T^+_{\mathrm{ad}}}((X\setminus\{x\})\times T^+_{\mathrm{ad}},T^+_{\mathrm{ad}})\to\mathrm{Maps}_{T^+_{\mathrm{ad}}}((X\setminus\{x\})\times T^+_{\mathrm{ad}},\mathrm{Vin}_G/(G\times G))$$

induced by the morphism $T^+_{\mathrm{ad}}\xrightarrow{\mathfrak{s}}\mathrm{Vin}_G\to\mathrm{Vin}_G/(G\times G)$.

Let us denote by $\Upsilon'\colon\ \mathrm{VinGr}'_{G,x}\to T^+_{\mathrm{ad}}$ the projection to the second factor.

PROPOSITION 3.3. *The families* $\Upsilon\colon\ \mathrm{VinGr}_{G,x}\to T^+_{\mathrm{ad}}$ *and* $\Upsilon'\colon\ \mathrm{VinGr}'_{G,x}\to T^+_{\mathrm{ad}}$ *are isomorphic.*

Proof. Follows from Proposition 3.4. □

3.1.4. *Definition of* VinGr_{G,X^n} *via Tannakian approach.* We now define a degeneration VinGr_{G,X^n} of the Beilinson–Drinfeld Grassmannian Gr_{G,X^n} for arbitrary reductive group G. An S-point of VinGr_{G,X^n} consists of the following data.

(1) A collection of S-points $\underline{x}=(x_1,\dots,x_n)\in X^n(S)$ of the curve X.
(2) An S-point $(\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee})\in\mathrm{VinBun}_G(S)$.
(3) For every $\lambda^\vee\in\Lambda^{\vee+}$, the rational morphisms

$$\eta_{\lambda^\vee}:\mathcal{O}_{S\times X}\to\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1},\ \ \zeta_{\lambda^\vee}:\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}\to\mathcal{O}_{S\times X}$$

regular on $(S\times X)\setminus\{\Gamma_{x_1}\cup\dots\cup\Gamma_{x_n}\}$, satisfying the same conditions as in the definition of $\mathrm{VinGr}_{G,x}$ in § 3.1.2 with U replaced by $(S\times X)\setminus\{\Gamma_{x_1}\cup\dots\cup\Gamma_{x_n}\}$.

We have a projection $\mathrm{VinGr}_{G,X^n}\to\mathrm{VinBun}_G$ that forgets the data of $\underline{x},\eta_{\lambda^\vee},\zeta_{\lambda^\vee}$. We have a projection $\pi^{\mathrm{Vin}}_n\colon\ \mathrm{VinGr}_{G,X^n}\to X^n$ that forgets the data of $\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\lambda^\vee},\zeta_{\lambda^\vee},\eta_{\lambda^\vee}$.

3.1.5. *The degeneration morphism* $\Upsilon\colon\ \mathrm{VinGr}_{G,X^n}\to T^+_{\mathrm{ad}}$. The morphism $\Upsilon\colon\ \mathrm{VinGr}_{G,X^n}\to T^+_{\mathrm{ad}}$ is defined as the composition of the morphisms

$$\mathrm{VinGr}_{G,X^n}\to\mathrm{VinBun}_G\to T^+_{\mathrm{ad}}.$$

3.1.6. *Second definition of* VinGr_{G,X^n}. (cf. Remark 2.2)
Let us fix $n\in\mathbb{N}$. We denote by $\mathrm{VinBun}_G(\mathfrak{U}_n)$ the following stack over $T^+_{\mathrm{ad}}\times X^n$: it associates to a scheme S

(1) a collection of S-points $\underline{x}=(x_1,\dots,x_n)\in X^n(S)$ of the curve X;
(2) an element of $\mathit{Maps}_{T^+_{\mathrm{ad}}}(T^+_{\mathrm{ad}}\times((S\times X)\setminus\{\Gamma_{x_1}\cup\dots\cup\Gamma_{x_n}\}),\mathrm{Vin}_G/(G\times G))$.

We have a restriction morphism $X^n \times \mathrm{Maps}_{T^+_{\mathrm{ad}}}(T^+_{\mathrm{ad}} \times X, \mathrm{Vin}_G/(G \times G)) \to \mathrm{VinBun}_G(\mathfrak{U}_n)$. We also have a morphism $T^+_{\mathrm{ad}} \times X^n \to \mathrm{VinBun}_G(\mathfrak{U}_n)$ induced by the morphism $T^+_{\mathrm{ad}} \xrightarrow{\mathfrak{s}} \mathrm{Vin}_G \to \mathrm{Vin}_G/(G \times G)$. We define VinGr'_{G,X^n} as the following family over the scheme T^+_{ad}:

$$(X^n \times \mathrm{Maps}_{T^+_{\mathrm{ad}}}(T^+_{\mathrm{ad}} \times X, \mathrm{Vin}_G/(G \times G)) \underset{\mathrm{VinBun}_G(\mathfrak{U}_n)}{\times} (T^+_{\mathrm{ad}} \times X^n).$$

The morphism $\mathrm{VinGr}'_{G,X^n} \to T^+_{\mathrm{ad}}$ is denoted by Υ'.

Proposition 3.4. *The families* $\Upsilon\colon \mathrm{VinGr}_{G,X^n} \to T^+_{\mathrm{ad}}$ *and* $\Upsilon'\colon \mathrm{VinGr}'_{G,X^n} \to T^+_{\mathrm{ad}}$ *are isomorphic.*

Proof. Let us construct a morphism $\mho\colon \mathrm{VinGr}'_{G,X^n} \to \mathrm{VinGr}_{G,X^n}$. Let $f\colon S \to T^+_{\mathrm{ad}}$ be a scheme over T^+_{ad}. An S-point of the family VinGr'_{G,X^n} over T^+_{ad} is the following data.

(1) A collection of S-points $\underline{x} = (x_1, \ldots, x_n) \in X^n(S)$ of the curve X.
(2) Two G-bundles $\mathcal{F}_1, \mathcal{F}_2$ on $S \times X$.
(3) A $G \times G$-equivariant morphism $\digamma\colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathrm{Vin}_G$ of schemes over T^+_{ad}.
(4) Trivializations σ_1, σ_2 of $\mathcal{F}_1, \mathcal{F}_2$ on $U := (S \times X) \setminus \{\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_n}\}$ such that the morphism $\digamma$ is identified with the morphism $\digamma^{\mathrm{triv}}\colon G \times U \times G \to \mathrm{Vin}_G$ given by $(g_1, p, x, g_2) \mapsto g_1 \cdot \mathfrak{s}(f(p)) \cdot {g_2}^{-1}$, where $\mathfrak{s}$ is the section of the morphism $\Upsilon\colon \mathrm{Vin}_G \to T^+_{\mathrm{ad}}$ defined in § 2.2.3 and § 2.2.9.

From the Tannakian description of Vin_G in § 2.2.8 and the description of T^+_{ad} in § 2.2.9 it follows that the $G \times G$-equivariant morphism $\digamma\colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathrm{Vin}_G$ of schemes over T^+_{ad} is the same as the collection of $G \times G$-equivariant morphisms $\digamma_{\lambda^\vee}\colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathrm{End}(V^{\lambda^\vee})$ for each $\lambda^\vee \in \Lambda^{\vee +}$ and (fixed) $G \times G$-equivariant morphisms $\widetilde{\tau}_{\mu^\vee}\colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathbb{C}$ satisfying the same conditions as in the definition of Vin_G in § 2.2.8. Each $G \times G$-equivariant morphism $\digamma_{\lambda^\vee}\colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathrm{End}(V^{\lambda^\vee})$ induces the morphism $\varphi_{\lambda^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}$. Note also that the morphisms $\widetilde{\tau}_{\mu^\vee}$ induce the morphisms $\tau_{\mu^\vee}\colon \mathcal{O}_{S\times X} = \mathbb{C}_{\mathcal{F}_1} \to \mathbb{C}_{\mathcal{F}_2} = \mathcal{O}_{S\times X}$. The trivializations σ_1, σ_2 induce the trivializations of the vector bundles $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}, \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}$ such that the morphisms $\varphi_{\lambda^\vee}|_U\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}|_U \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}|_U$ get identified with $g_{\lambda^\vee}\colon V^{\lambda^\vee} \otimes \mathcal{O}_U \to V^{\lambda^\vee} \otimes \mathcal{O}_U$, where $g_{\lambda^\vee}$ is the tensor product of the morphisms

$$\mathrm{Id} \otimes (\tau_{\mu^\vee}|_U)\colon (V^{\lambda^\vee})_{\lambda^\vee - \mu^\vee} \otimes \mathcal{O}_U \to (V^{\lambda^\vee})_{\lambda^\vee - \mu^\vee} \otimes \mathcal{O}_U$$

(cf. § 2.2.9). The standard N, N_--structures in the trivial G-bundle on $S \times X$ define via σ_1, σ_2 the N and N_--structures in $\mathcal{F}_1|_U,\ \mathcal{F}_2|_U$, respectively. They give us the collection of rational morphisms $\mathcal{O}_{S\times X} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1},\ \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}$ (cf. proof of Proposition 2.3). Set

$$\mho(\underline{x}, \mathcal{F}_1, \mathcal{F}_2, \digamma, \sigma_1, \sigma_2) := (\underline{x}, \mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}, \zeta_{\lambda^\vee}, \eta_{\lambda^\vee}).$$

The inverse morphism $\mho^{-1}$ can be constructed using the same arguments. □

Recall the action $T \curvearrowright T^+_{\mathrm{ad}}$ in § 2.2.2. Note that the torus T acts on the space $\mathrm{VinGr}'_{G,X^n} = \mathrm{VinGr}_{G,X^n}$ via the actions $T \curvearrowright T^+_{\mathrm{ad}}, \mathrm{Vin}_G$ in § 2.2.2. It is easy to see that the morphism Υ is T-equivariant.

Lemma 3.5. *The family* $\Upsilon^{-1}(T_{\mathrm{ad}})$ *is isomorphic to* $\mathrm{Gr}_{G,X^n} \times T_{\mathrm{ad}}$.

Proof. The morphism Υ is T-equivariant so it is enough to identify the fiber $\Upsilon^{-1}(1)$ with Gr_{G,X^n}. The argument is analogous to the proof of Proposition 3.2. □

Let $\Delta_X \hookrightarrow X^2$ be the closed embedding of the diagonal. Let $U \hookrightarrow X^2$ be the open embedding of the complement to the diagonal.

PROPOSITION 3.6. (a) *The restriction of the family* $\pi_2^{\mathrm{Vin}}\colon \mathrm{VinGr}_{G,X^2} \to X^2$ *to the closed subvariety* $\Delta_X \hookrightarrow X^2$ *is isomorphic to* $\mathrm{VinGr}_{G,X}$.

(b) *The restriction of the family* $\pi_2^{\mathrm{Vin}}\colon \mathrm{VinGr}_{G,X^2} \to X^2$ *to the open subvariety* $U \hookrightarrow X^2$ *is isomorphic to* $(\mathrm{VinGr}_{G,X} \underset{T^+_{\mathrm{ad}}}{\times} \mathrm{VinGr}_{G,X})|_U$.

Proof. Part (a) is clear, let us prove (b). The proof is the same as the one of [**27**, Proposition 3.1.13]. Take $\underline{x} = (x_1, x_2) \in U(S) \subset X^2(S)$. We define a morphism $(\mathrm{VinGr}_{G,X^2})|_U(S) \to \mathrm{VinGr}_{G,X}$ by sending $(\underline{x}, \mathcal{F}_1, \mathcal{F}_2, F, \sigma_1, \sigma_2)$ to $(x_1, (\mathcal{F}_1)_{x_1}, (\mathcal{F}_2)_{x_1}, F_{x_1}, (\sigma_1)_{x_1}, (\sigma_2)_{x_1})$, where $(\mathcal{F}_k)_{x_1}$ $(k \in \{1,2\})$ is obtained by gluing $(\mathcal{F}_k)|_{((S\times X)\setminus\{\Gamma_{x_2}\})}$ and $(\mathcal{F}^{\mathrm{triv}})|_{((S\times X)\setminus\{\Gamma_{x_1}\})}$ via σ_k and therefore is equipped with a trivialization $(\sigma_k)_{x_1}$. The morphism F_{x_1} is obtained by gluing $F|_{((S\times X)\setminus\{\Gamma_{x_2}\})}$ and $F^{\mathrm{triv}}|_{((S\times X)\setminus\{\Gamma_{x_1}\})}$. Similarly, we have another morphism $(\mathrm{VinGr}_{G,X^2})|_U(S) \to \mathrm{VinGr}_{G,X}$. Together, they define a morphism $(\mathrm{VinGr}_{G,X^2})|_U \to (\mathrm{VinGr}_{G,X} \underset{T^+_{\mathrm{ad}}}{\times} \mathrm{VinGr}_{G,X})|_U$.

Conversely, if we have $(x_1, (\mathcal{F}_1)_{x_1}, (\mathcal{F}_2)_{x_1}, F_{x_1}, (\sigma_1)_{x_1}, (\sigma_2)_{x_1}) \in \mathrm{VinGr}_{G,X}(S)$ and $(x_2, (\mathcal{F}_1)_{x_2}, (\mathcal{F}_2)_{x_2}, F_{x_2}, (\sigma_1)_{x_2}, (\sigma_2)_{x_2}) \in \mathrm{VinGr}_{G,X}(S)$ such that $(x_1, x_2) \in U(S)$, we can construct $\mathcal{F}_k$ $(k \in \{1,2\})$ by gluing $(\mathcal{F}_k)_{x_1}|_{((S\times X)\setminus\{\Gamma_{x_2}\})}$ and $(\mathcal{F}_k)_{x_2}|_{((S\times X)\setminus\{\Gamma_{x_1}\})}$ by $(\sigma_k)^{-1}_{x_2}(\sigma_k)_{x_1}$, which by definition is equipped with a trivialization σ_k on $(S\times X)\setminus\{\Gamma_{x_1}\cup\Gamma_{x_2}\}$. The morphism F is obtained by gluing F_{x_1} and F_{x_2}. □

3.1.7. *Embedding of* VinGr_{G,X^n} *into the product* $(\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}) \times T^+_{\mathrm{ad}}$. Let us construct a closed embedding

$$\vartheta\colon \mathrm{VinGr}_{G,X^n} \hookrightarrow \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T^+_{\mathrm{ad}}.$$

It sends an S-point

$$(\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}, \zeta_{\lambda^\vee}, \eta_{\lambda^\vee}) \in \mathrm{VinGr}_{G,X^n}(S)\colon \mathcal{O}_{S\times X} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} \xrightarrow{\varphi_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}$$

to the point

$$(\mathcal{F}_1, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}\circ\varphi_{\lambda^\vee}) \times (\mathcal{F}_2, \varphi_{\lambda^\vee}\circ\eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \times \tau_{\mu^\vee}.$$

LEMMA 3.7. *The morphism* ϑ *is a closed embedding* (*cf.* [**20**, *Lemma* 5.2.7]).

Proof. Let us show that ϑ induces an injective map on the level of S-points. Let us identify T^+_{ad} with $\mathrm{Maps}(X, T^+_{\mathrm{ad}})$. We already used this identification in the definition of the morphism ϑ because the morphisms $\tau_{\mu^\vee}: \mathcal{O}_{S\times X} \to \mathcal{O}_{S\times X}$ define an S-point of $\mathrm{Maps}(X, T^+_{\mathrm{ad}}) \simeq T^+_{\mathrm{ad}}$. Take an S-point

$$(P, \tau_{\mu^\vee}) \in \left(\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}\right) \times \mathrm{Maps}(X, T^+_{\mathrm{ad}}).$$

The point $P \in (\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n})$ is represented by a collection of the following outer diamonds:

$$
\begin{array}{ccccc}
 & & \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} & & \\
 & \overset{\eta_{\lambda^\vee}}{\nearrow} & \vdots & \overset{\zeta_{\lambda^\vee}}{\searrow} & \\
\mathcal{O}_{X\times S} & & \vdots\, \varphi_{\lambda^\vee} & & \mathcal{O}_{X\times S}. \\
 & \underset{\eta'_{\lambda^\vee}}{\searrow} & \downarrow & \underset{\zeta'_{\lambda^\vee}}{\nearrow} & \\
 & & \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} & &
\end{array}
\tag{3.8}
$$

We must show that there is at most one collection of dotted arrows $\varphi_{\lambda^\vee}$ making both triangles commutative. Note that the collection of morphisms $(\eta_{\lambda^\vee}, \zeta_{\lambda^\vee}, \eta'_{\lambda^\vee}, \zeta'_{\lambda^\vee})$ defines the trivializations of G-bundles

$$\mathcal{F}_1|_{((S\times X)\setminus\{\Gamma_{x_1}\cup\cdots\cup\Gamma_{x_n}\})},\ \mathcal{F}_2|_{((S\times X)\setminus\{\Gamma_{x_1}\cup\cdots\cup\Gamma_{x_n}\})}.$$

After these trivializations and restrictions to

$$U := (S\times X)\setminus\{\Gamma_{x_1}\cup\cdots\cup\Gamma_{x_n}\},$$

the outer diamond (3.8) takes the following form:

$$
\begin{array}{ccccc}
 & & V^{\lambda^\vee}\otimes\mathcal{O}_U & & \\
 & \overset{\iota_{\lambda^\vee}}{\nearrow} & \vdots & \overset{\zeta_{\lambda^\vee}}{\searrow} & \\
\mathcal{O}_U & & \vdots\, \varphi_{\lambda^\vee}|_U & & \mathcal{O}_U, \\
 & \underset{\iota_{\lambda^\vee}}{\searrow} & \downarrow & \underset{\zeta_{\lambda^\vee}}{\nearrow} & \\
 & & V^{\lambda^\vee}\otimes\mathcal{O}_U & &
\end{array}
$$

where the morphisms $\iota_{\lambda^\vee}$ correspond to the highest vector embeddings $\mathbb{C}\hookrightarrow V^{\lambda^\vee}$ and the morphisms $(\zeta_{\lambda^\vee})^*$ correspond to the lowest vector embeddings $\mathbb{C}\hookrightarrow (V^{\lambda^\vee})^*$. It follows that $(\varphi_{\lambda^\vee}|_U,\ \tau_{\mu^\vee})$ actually corresponds to the unique point in $\mathrm{Vin}_G(U)$ that can be described as $\mathfrak{s}(\tau_{\mu^\vee}|_U)$ (see § 2.2.3 for the definition of $\mathfrak{s}$). Indeed, the submonoid of Vin_G consisting of $g\in\mathrm{Vin}_G$ such that $g_\lambda|_{(V^{\lambda^\vee})_{\lambda^\vee}} = \mathrm{Id}$ for any $\lambda\in\Lambda^{\vee+}$ is $N\cdot\mathfrak{s}(T^+_{\mathrm{ad}})$ and the submonoid of Vin_G consisting of $g\in\mathrm{Vin}_G$ such that $g^*_\lambda|_{(V^{\lambda^\vee})^*_{-\lambda^\vee}} = \mathrm{Id}$ is $N_-\cdot\mathfrak{s}(T^+_{\mathrm{ad}})$, so their intersection is $\mathfrak{s}(T^+_{\mathrm{ad}})$. The claim follows.

Let us now show that the morphism ϑ is proper. Consider the following fiber product:

$$\mathcal{X} := \left(\left(\mathrm{Gr}_{G,X^n}\underset{X^n}{\times}\mathrm{Gr}_{G,X^n}\right)\times T^+_{\mathrm{ad}}\right)\underset{\mathrm{Bun}_G\times\mathrm{Bun}_G}{\times}\mathrm{VinBun}_G\,.$$

We have a projection

$$\mathcal{X}\to\left(\mathrm{Gr}_{G,X^n}\underset{X^n}{\times}\mathrm{Gr}_{G,X^n}\right)\times T^+_{\mathrm{ad}}.$$

Recall the projection $\mathrm{VinGr}_{G,X^n}\to\mathrm{VinBun}_G$ in § 3.1.4. Together with the morphism ϑ, it induces a morphism $\varsigma\colon \mathrm{VinGr}_{G,X^n}\to\mathcal{X}$. It is easy to see that the morphism ς is a closed embedding.

Recall the action $T\curvearrowright\mathrm{VinBun}_G$ in § 2.3. It follows from Proposition 2.16 that the morphism

$$\kappa\colon\ \mathrm{VinBun}_G/T\to\mathrm{Bun}_G\times\mathrm{Bun}_G$$

is proper. Then it follows that the morphism

$$\widetilde{\kappa}\colon \left(\left(\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}\right) \underset{\mathrm{Bun}_G \times \mathrm{Bun}_G}{\times} \mathrm{VinBun}_G /T\right) \times T_{\mathrm{ad}}^{+} \to \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T_{\mathrm{ad}}^{+} \tag{3.9}$$

obtained by the base change of the morphism κ is proper. Let us denote the left-hand side of (3.9) by $\mathbb{P}\mathcal{X}$. We consider the natural morphism $\varrho\colon \mathcal{X} \to \mathbb{P}\mathcal{X}$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{VinGr}_{G,X^n} & \xrightarrow{\quad\varsigma\quad} & \mathcal{X} \\ \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\varrho} \\ (\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}) \times T_{\mathrm{ad}}^{+} & \xleftarrow{\quad\widetilde{\kappa}\quad} & \mathbb{P}\mathcal{X} \end{array} \tag{3.10}$$

We claim that the composition $\varrho \circ \varsigma$ is a closed embedding. To prove this let us consider the image of the morphism ς. It is a closed subspace of the T-torsor $\mathcal{X} \to \mathbb{P}\mathcal{X}$ that intersects each T-orbit in a unique point. It follows that the image of $\varrho \circ \varsigma$ is closed. The claim follows. Now from the commutativity of the diagram (3.10) we see that the morphism ϑ is the composition of two proper morphisms, $\vartheta = \widetilde{\kappa} \circ (\varrho \circ \varsigma)$. So ϑ is proper. □

LEMMA 3.11. *The morphism*

$$\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \to \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}$$

that sends

$$(\underline{x}, \mathcal{O}_{X\times S} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{X\times S},\ \mathcal{O}_{X\times S} \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{X\times S}) \in \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}$$

to the point

$$\left(\underline{x}, \mathcal{O}_{X\times S} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}\circ\eta'_{\lambda^\vee}\circ\zeta_{\lambda^\vee}} \mathcal{O}_{X\times S},\ \mathcal{O}_{X\times S} \xrightarrow{\eta'_{\lambda^\vee}\circ\zeta_{\lambda^\vee}\circ\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{X\times S}\right)$$
$$\in \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}$$

is the identity morphism.

Proof. Obvious. □

PROPOSITION 3.12. *The functor* VinGr_{G,X^n} *is represented by an ind-scheme ind-projective over* T_{ad}^{+}.

Proof. In Lemma 3.7 it is proved that VinGr_{G,X^n} is a closed subfunctor of

$$\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T_{\mathrm{ad}}^{+}$$

and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{VinGr}_{G,X^n} & \xrightarrow{\quad\vartheta\quad} & \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T_{\mathrm{ad}}^{+} \\ & {\scriptstyle\Upsilon}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_3} & \\ & T_{\mathrm{ad}}^{+} & \end{array}$$

where the morphism $\mathrm{pr}_3 \colon \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T_{\mathrm{ad}}^{+} \to T_{\mathrm{ad}}^{+}$ is the projection to the third factor. It is known [**3**, § 5.3.10; **27**, Theorem 3.1.3] that the functor Gr_{G,X^n} is represented by an ind-projective scheme. It follows that the functor VinGr_{G,X^n} is represented by a closed ind-subscheme of $\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T_{\mathrm{ad}}^{+}$ ind-projective over T_{ad}^{+}. □

3.1.8. *Defect free locus of* $\mathrm{VinGr}_{G,x}$. Set

$$ {}_0\mathrm{VinGr}_{G,x} := \mathrm{Maps}_{T_{\mathrm{ad}}^{+}}(X \times T_{\mathrm{ad}}^{+},\ {}_0\mathrm{Vin}_G/(G\times G)) \underset{\mathrm{Maps}_{T_{\mathrm{ad}}^{+}}((X\setminus\{x\})\times T_{\mathrm{ad}}^{+},\ {}_0\mathrm{Vin}_G/(G\times G))}{\times} T_{\mathrm{ad}}^{+}. $$

We denote by ${}_0\Upsilon\colon {}_0\mathrm{VinGr}_{G,x} \to T_{\mathrm{ad}}^{+}$ the projection to the second factor.

3.1.9. *The principal degeneration* $\mathrm{VinGr}_{G,X^n}^{\mathrm{princ}}$. We denote by $\mathrm{VinGr}_{G,X^n}^{\mathrm{princ}}$ (respectively, ${}_0\mathrm{VinGr}_{G,x}^{\mathrm{princ}}$) the restriction of the degeneration $\Upsilon\colon \mathrm{VinGr}_{G,X^n} \to T_{\mathrm{ad}}^{+}$ (respectively, ${}_0\Upsilon$) to the 'principal' line

$$ \mathbb{A}^1 \hookrightarrow T_{\mathrm{ad}}^{+},\ a \mapsto (a,\dots,a) \text{ in coordinates } \alpha_i^{\vee},\ i\in I. $$

We denote the corresponding morphism $\mathrm{VinGr}_{G,X^n}^{\mathrm{princ}} \to \mathbb{A}^1$ by $\Upsilon^{\mathrm{princ}}$ (respectively, ${}_0\mathrm{VinGr}_{G,x}^{\mathrm{princ}} \to \mathbb{A}^1$ by ${}_0\Upsilon^{\mathrm{princ}}$).

Let us denote by $\vartheta^{\mathrm{princ}}\colon \mathrm{VinGr}_{G,X^n}^{\mathrm{princ}} \to (\mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n}) \times \mathbb{A}^1$ the restriction of the morphism ϑ oin § 3.1.7 to $\mathrm{VinGr}_{G,X^n}^{\mathrm{princ}}$.

3.1.10. *The special fiber of the degeneration* $\mathrm{VinGr}_{G,x}$. Let us describe the fiber over $0 \in T_{\mathrm{ad}}^{+}$ of the morphism Υ. Set $(\mathrm{VinGr}_{G,x})_0 := \Upsilon^{-1}(0)$. Note that the morphism

$$ \vartheta\colon \mathrm{VinGr}_{G,x} \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times T_{\mathrm{ad}}^{+} $$

in § 3.1.7 induces the closed embedding

$$ \vartheta_0\colon (\mathrm{VinGr}_{G,x})_0 \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G. $$

LEMMA 3.13. *The morphism ϑ_0 induces an isomorphism between $(\mathrm{VinGr}_{G,x})_0$ and*

$$ \bigcup_{\mu\in\Lambda} \overline{T}_\mu \times \overline{S}_\mu \subset \mathrm{Gr}_G \times \mathrm{Gr}_G, $$

considered as the reduced ind-schemes.

Proof. We define a morphism

$$ \varsigma\colon \bigcup_{\mu\in\Lambda} \overline{T}_\mu \times \overline{S}_\mu \hookrightarrow (\mathrm{VinGr}_{G,x})_0. $$

It sends a pair of points

$$ \mathcal{O}_{S\times X} \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}'}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{S\times X}(\langle -\lambda^\vee, \mu\rangle \cdot (S\times x)) \in \overline{T}_\mu(S), $$
$$ \mathcal{O}_{S\times X}(\langle -\lambda^\vee, \mu\rangle \cdot (S\times x)) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X} \in \overline{S}_\mu(S) $$

to the point

$$ \mathcal{O}_{S\times X} \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}'}^{\lambda^\vee} \xrightarrow{\eta_{\lambda^\vee}\circ\zeta'_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}}^{\lambda^\vee} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X} \in (\mathrm{VinGr}_{\mathrm{G,x}})_0. $$

It follows from Lemma 3.11 that the composition $\vartheta_0 \circ \varsigma$ coincides with the natural closed embedding

$$ \bigcup_{\mu\in\Lambda} \overline{T}_\mu \times \overline{S}_\mu \subset \mathrm{Gr}_G \times \mathrm{Gr}_G. $$

It follows that the morphism ς is a closed embedding. It suffices to show now that the morphism ς is surjective on the level of $\mathbb{C}$-points. Take a $\mathbb{C}$-point

$$\mathrm{P} := (\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \zeta_{\lambda^\vee}, \eta_{\lambda^\vee}) \in (\mathrm{VinGr}_{G,x})_0.$$

The morphisms $\varphi_{\lambda^\vee}$ admit a unique factorization as

$$\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{L}^{\lambda^\vee} \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2},$$

where $\mathcal{L}^{\lambda^\vee}$ is a line bundle on the curve X and the first morphism is a surjection of vector bundles. Note that for any $\lambda^\vee_1, \lambda^\vee_2 \in \Lambda^{\vee+}$ we have the identification $\mathcal{L}^{\lambda^\vee_1} \otimes \mathcal{L}^{\lambda^\vee_2} = \mathcal{L}^{\lambda^\vee_1+\lambda^\vee_2}$ and $\mathcal{L}^0 = \mathcal{O}_{S\times X}$. It follows that there exists $\mu \in \Lambda$ such that

$$\mathcal{L}^{\lambda^\vee} \simeq \mathcal{O}_{S\times X}(\langle -\lambda^\vee, \mu\rangle \cdot (S \times x)).$$

So the $\mathbb{C}$-point P is the image of the $\mathbb{C}$-point

$$(\mathcal{F}_1, \eta_{\lambda^\vee}, \zeta'_{\lambda^\vee}) \times (\mathcal{F}_2, \eta'_{\lambda^\vee}, \zeta_{\lambda^\vee}) \in \overline{T}_\mu \times \overline{S}_\mu$$

under the morphism ς. □

REMARK 3.14. It follows from Proposition 2.14 that the morphism ϑ_0 restricts to the isomorphism between the ind-schemes $(\mathrm{VinGr}_{G,x})_0 \cap {}_0\mathrm{VinGr}_{G,x}$ and $\bigsqcup_{\mu\in\Lambda} T_\mu \times S_\mu$.

3.1.11. *Ind-scheme* $\mathrm{Vin}\,\overline{S}_\nu$. Fix a point $x \in X$ and a cocharacter $\nu \in \Lambda$. We consider a functor

$$\mathrm{Vin}\,\overline{S}_\nu \colon \mathbf{Sch} \to \mathbf{Sets},$$

associating to a scheme S

(1) an S-point $(\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}) \in \mathrm{VinBun}_G(S)$ of VinBun_G;
(2) for every $\lambda^\vee \in \Lambda^{\vee+}$, morphisms of sheaves

$$\eta_{\lambda^\vee} : \mathcal{O}_{S\times X}(\langle -\lambda^\vee, \nu\rangle \cdot (S \times x)) \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1},\ \zeta_{\lambda^\vee} : \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \to \mathcal{O}_{S\times X},$$

satisfying the same conditions as in § 3.1.2.

Let $\widetilde{r}_{\nu,+} \colon \mathrm{Vin}\,\overline{S}_\nu \hookrightarrow \mathrm{VinGr}_{G,x}$ be the natural embedding.

LEMMA 3.15. *The morphism* $\widetilde{r}_{\nu,+}$ *is a closed embedding.*

Proof. Recall the closed embedding $\vartheta\colon \mathrm{VinGr}_{G,x} \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}}$ in § 3.1.7. It is enough to show that the composition $\vartheta \circ \widetilde{r}_\nu$ is a closed embedding. The proof is the same as the proof of Lemma 3.7. □

The morphism $\Upsilon_\nu\colon \mathrm{Vin}\,\overline{S}_\nu \to T^+_{\mathrm{ad}}$ is defined as the composition of the morphisms

$$\mathrm{Vin}\,\overline{S}_\nu \to \mathrm{VinBun}_G \to T^+_{\mathrm{ad}}.$$

3.1.12. *The principal degeneration* $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu$. Let us denote by $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu$ the restriction of the degeneration $\Upsilon_\nu\colon \mathrm{Vin}\,\overline{S}_\nu \to T^+_{\mathrm{ad}}$ to the 'principal' line

$$\mathbb{A}^1 \hookrightarrow T^+_{\mathrm{ad}},\ a \mapsto (a, \ldots, a) \text{ in coordinates } \alpha^\vee_i,\ i \in I.$$

Let us denote the corresponding morphism $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu \to \mathbb{A}^1$ by $\Upsilon^{\mathrm{princ}}_\nu$.

It follows from Lemma 3.5 that the fiber of the morphism $\Upsilon_\nu^{\mathrm{princ}}$ over the point $1 \in \mathbb{A}^1$ is isomorphic to $\overline{S}_\nu$. It follows from Lemma 3.13 that the morphism $\widetilde{r}_{\nu,+}$ induces an isomorphism from $(\mathrm{Vin}\,\overline{S}_\nu)_0$ to

$$\bigcup_{\mu\in\Lambda,\mu\leqslant\nu} (\overline{T}_\mu \cap \overline{S}_\nu) \times \overline{S}_\mu.$$

3.1.13. Let us fix $\theta_1, \theta_2 \in \Lambda$, $\theta_1 \geqslant \theta_2$, and set $\theta := \theta_1 - \theta_2$. Recall the families $\overline{\mathfrak{Y}}^\theta, \overline{\mathfrak{Y}}^{\theta_1,\theta_2}$ over T^+_{ad} in § 2.5.6 and 2.5.8. Let us denote by $\imath_{\theta_1,\theta_2}$, $\imath_\theta$ the closed embeddings of the families $\overline{\mathfrak{Y}}^\theta, \overline{\mathfrak{Y}}^{\theta_1,\theta_2}$ into $\mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}}$ that are constructed in the same way as the embedding ϑ in § 3.1.7.

LEMMA 3.16 (cf. [6, § 6.3]). *The family* $\overline{\mathfrak{Y}}^{\theta_1,\theta_2} \to T^+_{\mathrm{ad}}$ *is isomorphic to the family* $\overline{\mathfrak{Y}}^\theta \to T^+_{\mathrm{ad}}$.

Proof. Recall the identification $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$ in § 1.2. Recall that $\theta_2 \in \Lambda$ defines an element $z^{\theta_2} \in T(\mathcal{K})$. We get the isomorphism

$$\aleph_{\theta_2} \colon \mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}} \xrightarrow{\sim} \mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}}$$

given by $([g_1],[g_2],t) \mapsto ([z^{\theta_2}g_1],[z^{\theta_2}g_2],t)$. It is easy to see that the isomorphism $\aleph_{\theta_2}$ identifies the subscheme $\imath_\theta(\overline{\mathfrak{Y}}^\theta) \simeq \overline{\mathfrak{Y}}^\theta$ of $\mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}}$ with the subscheme $\imath_{\theta_1,\theta_2}(\overline{\mathfrak{Y}}^{\theta_1,\theta_2}) \simeq \overline{\mathfrak{Y}}^{\theta_1,\theta_2}$ of $\mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}}$. □

3.2. *Drinfeld–Gaitsgory interpolation*

3.2.1. *The space of* $\mathbb{C}^\times$*-equivariant morphisms.* Let Z_1 and Z_2 be schemes equipped with an action of $\mathbb{C}^\times$. Then we define the space $\mathrm{Maps}^{\mathbb{C}^\times}(Z_1,Z_2)$ as follows: for any scheme S,

$$\mathrm{Maps}^{\mathbb{C}^\times}(Z_1,Z_2)(S) := \mathrm{Mor}(S\times Z_1, Z_2)^{\mathbb{C}^\times}.$$

3.2.2. *Attractors and repellents.* (c.f. [10, Sections 1.4, 1.5]). Let Z be a scheme equipped with an action of $\mathbb{C}^\times$. Then we set $Z^{\mathrm{attr}} := \mathrm{Maps}^{\mathbb{C}^\times}(\mathbb{A}^1, Z)$, where $\mathbb{C}^\times$ acts on $\mathbb{A}^1$ by dilations. It follows from [10, Corollary 1.5.3(ii)] that the functor Z^{attr} is represented by a scheme.

Let $\mathbb{A}^1_-$ be an affine line equipped with the following action of $\mathbb{C}^\times$:

$$\mathbb{C}^\times \times \mathbb{A}^1_- \to \mathbb{A}^1_-,\ (c,a)\mapsto c^{-1}a.$$

We set $Z^{\mathrm{rep}} := \mathrm{Maps}^{\mathbb{C}^\times}(\mathbb{A}^1_-, Z)$. The scheme Z^{attr} is called the *attractor* of Z, and Z^{rep} is called the *repellent* of Z.

Recall that $Z^{\mathbb{C}^\times} := \mathrm{Maps}^{\mathbb{C}^\times}(\mathrm{pt}, Z)$. The $\mathbb{C}^\times$-equivariant morphisms $0\colon \mathrm{pt}\to\mathbb{A}^1$, $0_-\colon \mathrm{pt}\to \mathbb{A}^1_-$ induce the morphisms $q^+\colon Z^{\mathrm{attr}} \to Z^{\mathbb{C}^\times}$, $q^-\colon Z^{\mathrm{rep}}\to Z^{\mathbb{C}^\times}$.

3.2.3. *Definition of the interpolation.* Let Z be a scheme equipped with a $\mathbb{C}^\times$-action. In [10, § 2], certain interpolation $\widetilde{Z}$ over $\mathbb{A}^1$ was defined. Let us recall the construction of $\widetilde{Z}$.

Set $\mathbb{X} := \mathbb{A}^2$ and consider the morphism $\mathbb{X}\to\mathbb{A}^1$, $(\tau_1,\tau_2)\mapsto\tau_1\tau_2$. For any scheme S over $\mathbb{A}^1$ set $\mathbb{X}_S := \mathbb{X}\underset{\mathbb{A}^1}{\times} S$. Let us consider the following $\mathbb{C}^\times$-action on $\mathbb{X}$: $c\cdot(\tau_1,\tau_2) := (c\cdot\tau_1, c^{-1}\cdot\tau_2)$. This action preserves the morphism $\mathbb{X}\to\mathbb{A}^1$, so for any scheme S one obtains a $\mathbb{C}^\times$-action on $\mathbb{X}_S$.

Define $\widetilde{Z}$ to be the following space over $\mathbb{A}^1$:

$$\mathrm{Mor}_{\mathbb{A}^1}(S,\widetilde{Z}) := \mathrm{Mor}(\mathbb{X}_S, Z)^{\mathbb{C}^\times}.$$

3.2.4. *Properties of the interpolation.* Let us recall the main properties of $\widetilde{Z}$ from [10]. The projection $\mathbb{X} \to \mathbb{A}^1$ admits two sections:

$$s_1(a) := (1, a),\ \ s_2(a) := (a, 1).$$

The sections s_1, s_2 define morphisms $\gamma_1 \colon \widetilde{Z} \to Z,\ \gamma_2 \colon \widetilde{Z} \to Z$. Let $\gamma \colon \widetilde{Z} \to Z \times Z \times \mathbb{A}^1$ denote the morphism whose third component is the tautological projection $\widetilde{Z} \to \mathbb{A}^1$, and the first and the second components are γ_1 and γ_2, respectively. It follows from [10, Proposition 2.2.6] that the morphism γ induces an isomorphism between $\widetilde{Z}|_{\mathbb{G}_m}$ and the graph of the action morphism $\mathbb{C}^\times \times Z \to Z$.

3.2.5. *The special fiber of the interpolation $\widetilde{Z}$.* For $a \in \mathbb{A}^1$ let us denote by $\widetilde{Z}_a$ the fiber of $\widetilde{Z}$ over a. It follows from [10, Proposition 2.2.9] that the following diagram is cartesian:

$$\begin{array}{ccc} \widetilde{Z}_0 & \xrightarrow{\gamma_1} & Z^{\mathrm{attr}} \\ \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle q^+} \\ Z^{\mathrm{rep}} & \xrightarrow{q^-} & Z^{\mathbb{C}^\times} \end{array}$$

that is, the fiber $\widetilde{Z}_0$ is canonically isomorphic to $Z^{\mathrm{attr}} \underset{Z^{\mathbb{C}^\times}}{\times} Z^{\mathrm{rep}}$.

3.2.6. *Drinfeld–Gaitsgory interpolation of* Gr_G. Recall that Gr_G is the union of the projective schemes $\overline{\mathrm{Gr}}{}^\lambda_G,\ \lambda \in \Lambda^+$. We consider the $\mathbb{C}^\times$-action on Gr_G arising from the coweight $2\rho \colon \mathbb{C}^\times \to T \curvearrowright \mathrm{Gr}_G$. It follows from [10] that the closed embeddings of the Schubert varieties induce the closed embeddings of the corresponding Drinfeld–Gaitsgory interpolations $\widetilde{\overline{\mathrm{Gr}}}{}^\mu_G \subset \widetilde{\overline{\mathrm{Gr}}}{}^\lambda_G,\ \mu \leqslant \lambda \in \Lambda^+$, and we define $\widetilde{\mathrm{Gr}}_G := \lim_\lambda \widetilde{\overline{\mathrm{Gr}}}{}^\lambda_G$ (the inductive limit).

3.2.7. *The special fiber of the interpolation $\widetilde{\mathrm{Gr}}_G$.* From § 3.2.5 it follows that

$$(\widetilde{\mathrm{Gr}}_G)_0 \simeq \mathrm{Gr}^{\mathrm{attr}}_G \times_{(\mathrm{Gr}_G)^{\mathbb{C}^\times}} \mathrm{Gr}^{\mathrm{rep}}_G \simeq \bigsqcup_{\mu \in \Lambda} S_\mu \times T_\mu.$$

3.2.8. *The open embedding $\jmath \colon \widetilde{\mathrm{Gr}}_G \hookrightarrow \mathrm{VinGr}^{\mathrm{princ}}_{G,x}$.* Let us construct an open embedding of the interpolation $\widetilde{\mathrm{Gr}}_G \to \mathbb{A}^1$ into the degeneration $\mathrm{VinGr}^{\mathrm{princ}}_{G,x} \to \mathbb{A}^1$ (considered as the schemes over $\mathbb{A}^1$) such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}_G & \xrightarrow{\jmath} & \mathrm{VinGr}^{\mathrm{princ}}_{G,x} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \vartheta} \\ \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 & \xrightarrow{\mathrm{Id}} & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1. \end{array}$$

Recall the open ind-subscheme ${}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x} \subset \mathrm{VinGr}^{\mathrm{princ}}_{G,x}$ in § 3.1.9.

Proposition 3.17 (Gaitsgory). *There exists an isomorphism $\eta \colon {}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x} \xrightarrow{\sim} \widetilde{\mathrm{Gr}}_G$ of the families over $\mathbb{A}^1$ such that the following diagram is commutative:*

$$\begin{array}{ccc} {}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x} & \xrightarrow{\eta} & \widetilde{\mathrm{Gr}}_G \\ \downarrow{\scriptstyle \vartheta|_{{}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x}}} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 & \xrightarrow{\mathrm{Id}} & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1. \end{array} \tag{3.18}$$

Proof. Will be given in Appendix A.2. □

4. *The action of Schieder bialgebra on the fiber functor*

4.1. *Construction of the action*

Fix $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$. Set $V := H^\bullet(\mathrm{Gr}_G, \mathcal{P})$. Recall the closed embedding

$$\vartheta^{\mathrm{princ}}\colon\ \mathrm{VinGr}^{\mathrm{princ}}_{G,x} \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$$

in § 3.1.9. Let us fix a cocharacter $\nu \in \Lambda$. Recall the closed embedding $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu \hookrightarrow \mathrm{VinGr}^{\mathrm{princ}}_{G,x}$ in § 3.1.11. Let $\vartheta^{\mathrm{princ}}_\nu\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$ be the composition

$$\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu \hookrightarrow \mathrm{VinGr}^{\mathrm{princ}}_{G,x} \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1.$$

For $\mu \in \Lambda$ set $\mathcal{P}_\mu := \mathcal{P}|_{\overline{S}_\mu}$, $\widetilde{\mathcal{P}}_\nu := (\underline{\mathbb{C}} \boxtimes \mathcal{P} \boxtimes \underline{\mathbb{C}})|_{\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu}$ (the $*$-restrictions to the corresponding closed subvarieties). We will see in Remark 4.2 that the support of the complex $\widetilde{\mathcal{P}}_\nu$ is finite dimensional.

Recall the one-parameter deformation $\Upsilon^{\mathrm{princ}}_\nu\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_\nu \to \mathbb{A}^1$ in § 3.1.12.

PROPOSITION 4.1. *The one-parametric family $\Upsilon^{\mathrm{princ}}_\nu$ is trivial over $\mathbb{G}_m$. The special fiber ${\Upsilon^{\mathrm{princ}}_\nu}^{-1}(0)$ is*

$$\bigcup_{\mu\leqslant\nu,\mu\in\Lambda} (\overline{S}_\nu \cap \overline{T}_\mu) \times \overline{S}_\mu.$$

A general fiber is $\overline{S}_\nu$. The restriction $\widetilde{\mathcal{P}}_\nu|_{{\Upsilon^{\mathrm{princ}}_\nu}^{-1}(\mathbb{G}_m)}$ is isomorphic to $\mathcal{P}|_{\overline{S}_\nu} \boxtimes \underline{\mathbb{C}}_{\mathbb{G}_m}$. The restriction of $\widetilde{\mathcal{P}}_\nu$ to $(\overline{S}_\nu \cap \overline{T}_\mu) \times \overline{S}_\mu$ is isomorphic to $\underline{\mathbb{C}}_{(\overline{S}_\nu\cap\overline{T}_\mu)} \boxtimes (\mathcal{P}|_{\overline{S}_\mu})$.

Proof. Follows from § 3.1.12. □

REMARK 4.2. Let us show that the support of the complex $\widetilde{\mathcal{P}}_\nu$ is finite dimensional. It is enough to show that the supports of $(\widetilde{\mathcal{P}}_\nu)|_{{\Upsilon^{\mathrm{princ}}_\nu}^{-1}(\mathbb{G}_m)}$ and $(\widetilde{\mathcal{P}}_\nu)|_{{\Upsilon^{\mathrm{princ}}_\nu}^{-1}(0)}$ are finite dimensional. It follows from Proposition 4.1 using the fact that there are finitely many $\mu \leqslant \nu$ such that $\mathcal{P}_\mu$ is nonzero.

4.1.1. *The action.* Given a positive coweight $\nu \in \Lambda^{\mathrm{pos}}$ we define a morphism of vector spaces

$$\bigoplus_{\mu\leqslant\nu}(\mathcal{A}_{\nu-\mu} \otimes V_\mu) = H_c^{\langle 2\rho^\vee,\nu\rangle}({\Upsilon^{\mathrm{princ}}_\nu}^{-1}(0), (\widetilde{\mathcal{P}}_\nu)_0) \to H_c^{\langle 2\rho^\vee,\nu\rangle}({\Upsilon^{\mathrm{princ}}_\nu}^{-1}(1), (\widetilde{\mathcal{P}}_\nu)_1) = V_\nu,$$

as the cospecialization morphism (with coefficients in the sheaf $\widetilde{\mathcal{P}}_\nu$) corresponding to the one-parameter degeneration $\Upsilon^{\mathrm{princ}}_\nu$. Here $(\widetilde{\mathcal{P}}_\nu)_0$ (respectively, $(\widetilde{\mathcal{P}}_\nu)_1$) stands for the restriction of $\widetilde{\mathcal{P}}_\nu$ to the fiber of $\Upsilon^{\mathrm{princ}}_\nu$ over $0 \in \mathbb{A}^1$ (respectively, over $1 \in \mathbb{A}^1$). Summing over all $\nu \in \Lambda^{\mathrm{pos}}$ we obtain the desired morphism

$$\mathrm{act}_V\colon \mathcal{A} \otimes V \to V.$$

4.2. *Associativity*

4.2.1. *The two-parameter deformation of Grassmannian.* Recall two projections

$$\mathrm{Bun}_G \xleftarrow{\wp_1} \mathrm{VinBun}_G \xrightarrow{\wp_2} \mathrm{Bun}_G$$

of Definition 2.13. Set $\mathcal{W} := \mathrm{VinBun}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_G$. For $n \in \mathbb{N}$ let $\mathrm{Vin}^2 \mathrm{Gr}_{G,X^n}$ be the moduli space of the following data: it associates to a scheme S

(1) a collection of S-points $(x_1, \dots, x_n) \in X^n(S)$ of the curve X;
(2) an S-point $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \varphi_{1,\lambda^\vee}, \varphi_{2,\lambda^\vee}, \tau_{1,\mu^\vee}, \tau_{2,\mu^\vee}) \in \mathcal{W}(S)$;
(3) for every $\lambda \in \Lambda^{\vee+}$, the rational morphisms

$$\eta_{\lambda^\vee} : \mathcal{O}_{S\times X} \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}, \ \zeta_{\lambda^\vee} : \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_3} \to \mathcal{O}_{S\times X},$$

regular on $(S \times X) \setminus \{\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_n}\}$, such that the data

$$(\mathcal{F}_1, \mathcal{F}_3, \varphi_{2,\lambda^\vee} \circ \varphi_{1,\lambda^\vee}, \tau_{2,\mu^\vee} \circ \tau_{1,\mu^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee})$$

define an S-point of VinGr_{G,X^n}.

Let us denote by ${}_0\mathcal{W}$ the substack of $\mathcal{W}$ consisting of points

$$(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \varphi_{1,\lambda^\vee}, \varphi_{2,\lambda^\vee}, \tau_{1,\mu^\vee}, \tau_{2,\mu^\vee}) \in \mathcal{W}$$

such that the data

$$(\mathcal{F}_1, \mathcal{F}_3, \varphi_{2,\lambda^\vee} \circ \varphi_{1,\lambda^\vee}, \tau_{2,\mu^\vee} \circ \tau_{1,\mu^\vee})$$

define a point of VinBun_G.

REMARK 4.3. The family $\mathrm{Vin}^2 \mathrm{Gr}_{G,X^n}$ can be obtained as the fiber product:

$$\begin{array}{ccc} \mathrm{Vin}^2\mathrm{Gr}_{G,X^n} = (\mathrm{VinGr}_{G,X^n} \underset{\mathrm{VinBun}_G}{\times} {}_0\mathcal{W}) & \longrightarrow & {}_0\mathcal{W} \\ \downarrow & & \downarrow \mathrm{m} \\ \mathrm{VinGr}_{G,X^n} & \longrightarrow & \mathrm{VinBun}_G \end{array}$$

where the morphism $\mathrm{m}\colon {}_0\mathcal{W} \to \mathrm{VinBun}_G$ corresponds to the multiplication in Vin_G.

The degeneration morphism $\Upsilon^2\colon \mathrm{Vin}^2 \mathrm{Gr}_{G,X^n} \to T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}$ is defined as the composition of the morphisms

$$\mathrm{Vin}^2 \mathrm{Gr}_{G,X^n} \to \mathcal{W} \to T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}.$$

Let us denote by $\mathrm{Vin}^2 \mathrm{Gr}^{\mathrm{princ}}_{G,X^n}$ the restriction of the degeneration $\Upsilon^2\colon \mathrm{Vin}^2 \mathrm{Gr}_{G,X^n} \to T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}$ to the product of 'principal' lines

$$\mathbb{A}^1 \times \mathbb{A}^1 \hookrightarrow T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}, \ (a_1, a_2) \mapsto ((a_1, \dots, a_1), (a_2, \dots, a_2)).$$

Let us denote the corresponding morphism $\mathrm{Vin}^2 \mathrm{Gr}^{\mathrm{princ}}_{G,X^n} \to \mathbb{A}^2$ by $\Upsilon^{2,\mathrm{princ}}$.

LEMMA 4.4. *The restrictions of the two-parameter family* $\Upsilon^{2,\mathrm{princ}}\colon \mathrm{Vin}^2\,\mathrm{Gr}^{\mathrm{princ}}_{G,X^n} \to \mathbb{A}^2$ *to the lines*

$$\mathbb{A}^1 \times \{1\} \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1,\ \{1\} \times \mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1$$

are both isomorphic to the one-parameter family $\Upsilon^{\mathrm{princ}}\colon \mathrm{VinGr}^{\mathrm{princ}}_{G,X^n} \to \mathbb{A}^1$.

Proof. Follows from Remark 4.3. □

Let us construct a closed embedding

$$\vartheta^2\colon\ \mathrm{Vin}^2\,\mathrm{Gr}_{G,X^n} \hookrightarrow \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}.$$

It sends a point

$$(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \varphi_{1,\lambda^\vee}, \varphi_{2,\lambda^\vee}, \tau_{1,\mu^\vee}, \tau_{2,\mu^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}) \in \mathrm{Vin}^2\,\mathrm{Gr}_{G,X^n}:$$

$$\mathcal{O}_{S\times X} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1} \xrightarrow{\varphi_{1,\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2} \xrightarrow{\varphi_{2,\lambda^\vee}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_3} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}$$

to the point

$$(\mathcal{F}_1, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee} \circ \varphi_{2,\lambda^\vee} \circ \varphi_{1,\lambda^\vee}) \times (\mathcal{F}_2, \varphi_{1,\lambda^\vee} \circ \eta_{\lambda^\vee}, \zeta_{\lambda^\vee} \circ \varphi_{2,\lambda^\vee}) \times (\mathcal{F}_3, \varphi_{2,\lambda^\vee} \circ \varphi_{1,\lambda^\vee} \circ \eta_{\lambda^\vee}, \zeta_{\lambda^\vee})$$
$$\times \tau_{1,\mu^\vee} \times \tau_{2,\mu^\vee}.$$

LEMMA 4.5. *The morphism* ϑ^2 *is a closed embedding.*

Proof. The proof is the same as the one of Lemma 3.7. □

Let us denote by

$$(\vartheta^2)^{\mathrm{princ}}\colon\ \mathrm{Vin}^2\,\mathrm{Gr}^{\mathrm{princ}}_{G,X^n} \hookrightarrow \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \underset{X^n}{\times} \mathrm{Gr}_{G,X^n} \times \mathbb{A}^1 \times \mathbb{A}^1$$

the restriction of the morphism ϑ^2 to $\mathrm{Vin}^2\,\mathrm{Gr}^{\mathrm{princ}}_{G,X^n} \subset \mathrm{Vin}^2\,\mathrm{Gr}_{G,X^n}$.

4.2.2. *The two-parameter deformations of closures of semi-infinite orbits.* We fix $x \in X$. Fix a cocharacter $\nu \in \Lambda$. Let $\mathrm{Vin}^2\,\overline{S}_\nu$ be the moduli space of the following data: it associates to a scheme S

(1) an S-point $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \varphi_{1,\lambda^\vee}, \varphi_{2,\lambda^\vee}, \tau_{1,\mu^\vee}, \tau_{2,\mu^\vee}) \in \mathcal{W}$;
(2) for every $\lambda \in \Lambda^{+\vee}$, a morphism of sheaves

$$\eta_{\lambda^\vee}: \mathcal{O}_{S\times X}(\langle -\lambda^\vee, \nu\rangle \cdot (S \times x)) \to \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}$$

and a rational morphism $\zeta_{\lambda^\vee}: \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_3} \to \mathcal{O}_{S\times X}$ regular on $S \times (X \setminus \{x\})$, such that the data

$$(\mathcal{F}_1, \mathcal{F}_3, \varphi_{2,\lambda^\vee} \circ \varphi_{1,\lambda^\vee}, \tau_{2,\mu^\vee} \circ \tau_{1,\mu^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee})$$

define an S-point of $\mathrm{Vin}\,\overline{S}_\nu$.

REMARK 4.6. The family $\mathrm{Vin}^2\,\overline{S}_\nu$ can be obtained as the fiber product:

$$\begin{array}{ccc}
\mathrm{Vin}^2\,\overline{S}_\nu = \mathrm{Vin}\,\overline{S}_\nu \underset{\mathrm{VinBun}_G}{\times} {}_0\mathcal{W} & \longrightarrow & {}_0\mathcal{W} \\
\downarrow & & \downarrow \mathrm{m} \\
\mathrm{Vin}\,\overline{S}_\nu & \longrightarrow & \mathrm{VinBun}_G\,.
\end{array}$$

The degeneration morphism $\Upsilon^2_\nu\colon \operatorname{Vin}^2 \overline{S}_\nu \to T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}$ is defined as the composition of the morphisms

$$\operatorname{Vin}^2 \overline{S}_\nu \to \mathcal{W} \to T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}.$$

Let us denote by $\operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu$ the restriction of the degeneration $\Upsilon^2_\nu\colon \operatorname{Vin}^2 \overline{S}_\nu \to T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}$ to the product of 'principal' lines

$$\mathbb{A}^1 \times \mathbb{A}^1 \hookrightarrow T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}},\ (a_1, a_2) \mapsto ((a_1, \dots, a_1), (a_2, \dots, a_2)).$$

Let us denote the corresponding morphism $\operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu \to \mathbb{A}^2$ by $\Upsilon^{2,\mathrm{princ}}_\nu$.

Let $r^2_{\nu,+}\colon \operatorname{Vin}^2 \overline{S}_\nu \hookrightarrow \operatorname{Vin}^2 \mathrm{Gr}_G$ be the natural closed embedding. Recall the morphism ϑ^2 in § 4.2.1. Let

$$\vartheta^2_\nu\colon \operatorname{Vin}^2 \overline{S}_\nu \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times T^+_{\mathrm{ad}} \times T^+_{\mathrm{ad}}$$

be the composition $\vartheta^2_\nu := r^2_{\nu,+} \circ\ \vartheta^2$. Let

$$(\vartheta^2_\nu)^{\mathrm{princ}}\colon \operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 \times \mathbb{A}^1$$

be the restriction of the morphism ϑ^2_ν to $\operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu \subset \operatorname{Vin}^2 \overline{S}_\nu$.

PROPOSITION 4.7. *Let $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$, and $V = H^\bullet(\mathrm{Gr}_G, \mathcal{P})$. Given $\nu \geqslant \mu_1 \geqslant \mu_2$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}_{\nu-\mu_1} \otimes \mathcal{A}_{\mu_1-\mu_2} \otimes V_{\mu_2} & \xrightarrow{\mathbf{m}_{\nu-\mu_1,\mu_1-\mu_2} \otimes \mathrm{Id}} & \mathcal{A}_{\nu-\mu_2} \otimes V_{\mu_2} \\ \big\downarrow {\scriptstyle \mathrm{Id} \otimes \mathbf{act}_V} & & \big\downarrow {\scriptstyle \mathbf{act}_V} \\ \mathcal{A}_{\nu-\mu_1} \otimes V_{\mu_1} & \xrightarrow{\mathbf{act}_V} & V_\nu. \end{array} \tag{4.8}$$

Proof. Set

$$\widetilde{\mathcal{P}}^2_\nu := (\underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} \boxtimes \mathcal{P} \boxtimes \underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}})|_{\operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu}.$$

It follows from Corollary 4.10 that the support of the complex $\widetilde{\mathcal{P}}^2_\nu$ is finite dimensional. For a point $(a_1, a_2) \in \mathbb{A}^1 \times \mathbb{A}^1$ let us denote by $(\widetilde{\mathcal{P}}^2_\nu)_{(a_1,a_2)}$ the restriction of $\widetilde{\mathcal{P}}^2_\nu$ to the fiber $(\Upsilon^{2,\mathrm{princ}}_\nu)^{-1}(a_1, a_2)$. Let us consider the tautological action $\mathbb{C}^\times \times \mathbb{C}^\times \curvearrowright \mathbb{A}^1 \times \mathbb{A}^1$. Let us consider the stratification

$$\mathbb{A}^1 \times \mathbb{A}^1 = (\mathbb{G}_m \times \mathbb{G}_m) \sqcup (\mathbb{G}_m \times \{0\}) \sqcup (\{0\} \times \mathbb{G}_m) \sqcup (\{0\} \times \{0\})$$

by $\mathbb{C}^\times \times \mathbb{C}^\times$-orbits. Recall the closed embedding

$$(\vartheta^2_\nu)^{\mathrm{princ}}\colon \operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 \times \mathbb{A}^1.$$

LEMMA 4.9. (a) *The restriction of the morphism $(\vartheta^2_\nu)^{\mathrm{princ}}$ of the families over $\mathbb{A}^1 \times \mathbb{A}^1$*

$$\begin{array}{ccc} \operatorname{Vin}^2 \overline{S}^{\mathrm{princ}}_\nu & \xrightarrow{(\vartheta^2_\nu)^{\mathrm{princ}}} & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 \times \mathbb{A}^1 \\ & \searrow \quad \swarrow & \\ & \mathbb{A}^1 \times \mathbb{A}^1 & \end{array}$$

to the stratum $\mathbb{G}_m \times \mathbb{G}_m \subset \mathbb{A}^1 \times \mathbb{A}^1$ *is isomorphic to*

$$\begin{array}{ccc} \overline{S}_\nu \times \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{(\vartheta^2_\nu)^{\mathrm{princ}}|_{(\mathbb{G}_m \times \mathbb{G}_m)}} & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{G}_m \times \mathbb{G}_m \\ & \searrow \quad \swarrow & \\ & \mathbb{G}_m \times \mathbb{G}_m & \end{array}$$

where the morphism $(\vartheta^2_\nu)^{\mathrm{princ}}|_{(\mathbb{G}_m \times \mathbb{G}_m)}$ *is given by* $(g, c_1, c_2) \mapsto (g, 2\rho(c_1) \cdot g, 2\rho(c_1c_2) \cdot g, c_1, c_2)$.

(b) *The restriction of the morphism* $(\vartheta^2_\nu)^{\mathrm{princ}}$ *of the families over* $\mathbb{A}^1 \times \mathbb{A}^1$ *to the stratum* $\mathbb{G}_m \times \{0\} \subset \mathbb{A}^1 \times \mathbb{A}^1$ *is isomorphic to*

$$\begin{array}{ccc} \bigcup\limits_{\mu \in \Lambda, \nu \geq \mu} (\overline{T}_\mu \cap \overline{S}_\nu) \times \overline{S}_\mu \times \mathbb{G}_m & \xrightarrow{(\vartheta^2_\nu)^{\mathrm{princ}}|_{(\mathbb{G}_m \times \{0\})}} & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{G}_m \times \{0\} \\ & \searrow \quad \swarrow & \\ & \mathbb{G}_m \times \{0\} & \end{array}$$

where the map $(\vartheta^2_\nu)^{\mathrm{princ}}|_{(\mathbb{G}_m \times \{0\})}$ *is given by* $(g_1, g_2, c) \mapsto (g_1, 2\rho(c) \cdot g_1, g_2, c, 0)$.

(c) *The restriction of the morphism* $(\vartheta^2_\nu)^{\mathrm{princ}}$ *of the families over* $\mathbb{A}^1 \times \mathbb{A}^1$ *to the stratum* $\{0\} \times \mathbb{G}_m \subset \mathbb{A}^1 \times \mathbb{A}^1$ *is isomorphic to*

$$\begin{array}{ccc} \bigcup\limits_{\mu \in \Lambda, \nu \geq \mu} (\overline{T}_\mu \cap \overline{S}_\nu) \times \overline{S}_\mu \times \mathbb{G}_m & \xrightarrow{(\vartheta^2_\nu)^{\mathrm{princ}}|_{(\{0\} \times \mathbb{G}_m)}} & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \{0\} \times \mathbb{G}_m \\ & \searrow \quad \swarrow & \\ & \{0\} \times \mathbb{G}_m & \end{array}$$

where the map $(\vartheta^2_\nu)^{\mathrm{princ}}|_{(\{0\} \times \mathbb{G}_m)}$ *is given by* $(g_1, g_2, c) \mapsto (g_1, g_2, 2\rho(c) \cdot g_2, 0, c)$.

(d) *The restriction of the morphism* $(\vartheta^2_\nu)^{\mathrm{princ}}$ *of the families over* $\mathbb{A}^1 \times \mathbb{A}^1$ *to the point* $(\{0\} \times \{0\})$ *is isomorphic to*

$$\begin{array}{ccc} \bigcup\limits_{\mu_1, \mu_2 \in \Lambda, \nu \geq \mu_1 \geq \mu_2} (\overline{S}_\nu \cap \overline{T}_{\mu_1}) \times (\overline{S}_{\mu_1} \cap \overline{T}_{\mu_2}) \times \overline{S}_{\mu_2} & \longrightarrow & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G\,. \\ & \searrow \quad \swarrow & \\ & (\{0\} \times \{0\}) & \end{array}$$

Proof. The proof is the same as the one of Lemma 3.13. □

Corollary 4.10. *Under the identifications of Lemma 4.9 we have*

$$(\widetilde{\mathcal{P}}^2_\nu)_{(1,1)} = \mathcal{P}_\nu,\ (\widetilde{\mathcal{P}}^2_\nu)_{(1,0)}|_{(\overline{T}_\mu \cap \overline{S}_\nu) \times \overline{S}_\mu} = \underline{\mathbb{C}} \boxtimes \mathcal{P}_\mu,\ (\widetilde{\mathcal{P}}^2_\nu)_{(0,1)}|_{(\overline{T}_\mu \cap \overline{S}_\nu) \times \overline{S}_\mu} = \underline{\mathbb{C}} \boxtimes \mathcal{P}_\mu,$$

$$(\widetilde{\mathcal{P}}^2_\nu)_{(0,0)}|_{(\overline{S}_\nu \cap \overline{T}_{\mu_1}) \times (\overline{S}_{\mu_1} \cap \overline{T}_{\mu_2}) \times \overline{S}_{\mu_2}} = \underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} \boxtimes \mathcal{P}_{\mu_2}.$$

Let us fix a cocharacter $\mu \in \Lambda, \mu \leqslant \nu$. Recall the family $\Upsilon^{2,\mathrm{princ}}_\nu \colon \mathrm{Vin}\, \overline{S}^{\mathrm{princ}}_\nu \to \mathbb{A}^1$ in § 4.2.2 and the families $\overline{\mathfrak{Y}}^{\nu-\mu,\mathrm{princ}}, \overline{\mathfrak{Y}}^{\nu,\mu,\mathrm{princ}}$ in § 2.5.6 and § 2.5.8.

Lemma 4.11. (a) *The closure of the family* $(\overline{T_\mu} \cap \overline{S}_\nu) \times \overline{S}_\mu \times \mathbb{G}_m \to \mathbb{G}_m \times \{0\}$ *in the family* $(\Upsilon^{2,\mathrm{princ}}_\nu)^{-1}(\mathbb{A}^1 \times \{0\}) \to \mathbb{A}^1$ *is isomorphic to the family* $\overline{\mathfrak{Y}}^{\nu,\mu,\mathrm{princ}} \times \overline{S}_\mu \to \mathbb{A}^1$ *on the level of reduced schemes.*

(b) *The closure of the family* $(\overline{T_\mu} \cap \overline{S}_\nu) \times \overline{S}_\mu \times \mathbb{G}_m \to \{0\} \times \mathbb{G}_m$ *in the family* $(\Upsilon_\nu^{2,\mathrm{princ}})^{-1}(\{0\} \times \mathbb{A}^1) \to \mathbb{A}^1$ *is isomorphic to the family* $(\overline{T_\mu} \cap \overline{S}_\nu) \times \mathrm{Vin}\, \overline{S}_\mu^{\mathrm{princ}} \to \mathbb{A}^1$ *on the level of reduced schemes.*

Proof. To prove (a) let us construct a closed embedding $\varkappa\colon \overline{\mathfrak{Y}}^{\nu,\mu,\mathrm{princ}} \times \overline{S}_\mu \hookrightarrow \mathrm{Vin}^2\, \overline{S}_\nu^{\mathrm{princ}}|_{\mathbb{A}^1 \times \{0\}}$ of families over $\mathbb{A}^1$. It sends an S-point

$$((\mathcal{F}_1, \mathcal{F}_2, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}), (\mathcal{F}_3, \eta'_{\lambda^\vee}, \zeta'_{\lambda^\vee})) \in (\overline{\mathfrak{Y}}^{\nu,\mu,\mathrm{princ}} \times \overline{S}_\mu)(S):$$

$$(\mathcal{O}_{S\times X}(-\langle \lambda^\vee, \nu\rangle \cdot (S \times x)) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_1}^{\lambda^\vee} \xrightarrow{\varphi_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_2}^{\lambda^\vee} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}(-\langle \lambda^\vee, \mu\rangle \cdot (S \times x)),$$

$$\mathcal{O}_{S\times X}(-\langle \lambda^\vee, \mu\rangle \cdot (S \times x)) \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_3}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{S\times X})$$

to the point $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}, \eta'_{\lambda^\vee} \circ \zeta_{\lambda^\vee}, \eta_{\lambda^\vee}, \zeta'_{\lambda^\vee}) \in \mathrm{Vin}^2\, \overline{S}_\nu^{\mathrm{princ}}|_{\mathbb{A}^1 \times \{0\}}(S)$:

$$\mathcal{O}_{S\times X}(-\langle \lambda^\vee, \nu\rangle \cdot (S \times x)) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_1}^{\lambda^\vee} \xrightarrow{\varphi_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_2}^{\lambda^\vee} \xrightarrow{\eta'_{\lambda^\vee} \circ \zeta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_3}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{S\times X}.$$

It follows that the morphism $\varkappa$ induces the isomorphism from $\overline{\mathfrak{Y}}^{\nu,\mu,\mathrm{princ}} \times \overline{S}_\mu$ to the closure of $(\overline{T_\mu} \cap \overline{S}_\nu) \times \overline{S}_\mu \times \mathbb{G}_m$ in the family $(\Upsilon_\nu^{2,\mathrm{princ}})^{-1}(\mathbb{A}^1 \times \{0\})$.

To prove (b), we construct a closed embedding $\varpi\colon (\overline{T}_\mu \cap \overline{S}_\nu) \times \mathrm{Vin}\, \overline{S}_\mu^{\mathrm{princ}} \hookrightarrow \mathrm{Vin}^2\, \overline{S}_\nu^{\mathrm{princ}}|_{\{0\}\times\mathbb{A}^1}$ of families over $\mathbb{A}^1$. It sends an S-point

$$((\mathcal{F}_1, \eta_{\lambda^\vee}, \zeta_{\lambda^\vee}), (\mathcal{F}_2, \mathcal{F}_3, \varphi_{\lambda^\vee}, \tau_{\mu^\vee}, \eta'_{\lambda^\vee}, \zeta'_{\lambda^\vee})) \in ((\overline{T}_\mu \cap \overline{S}_\nu) \times \mathrm{Vin}\, \overline{S}_\mu^{\mathrm{princ}})(S):$$

$$(\mathcal{O}_{S\times X}(-\langle \lambda^\vee, \nu\rangle \cdot (S \times x)) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_1}^{\lambda^\vee} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}(-\langle \lambda^\vee, \mu\rangle \cdot (S \times x)),$$

$$\mathcal{O}_{S\times X}(-\langle \lambda^\vee, \mu\rangle \cdot (S \times x)) \xrightarrow{\eta'_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_2}^{\lambda^\vee} \xrightarrow{\varphi_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_3}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{S\times X})$$

to the point $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \eta'_{\lambda^\vee} \circ \zeta_{\lambda^\vee}, \varphi_{\lambda^\vee}, \eta_{\lambda^\vee}, \zeta'_{\lambda^\vee})$:

$$\mathcal{O}_{S\times X}(-\langle \lambda^\vee, \nu\rangle \cdot (S \times x)) \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_1}^{\lambda^\vee} \xrightarrow{\eta'_{\lambda^\vee} \circ \zeta_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_2}^{\lambda^\vee} \xrightarrow{\varphi_{\lambda^\vee}} \mathcal{V}_{\mathcal{F}_3}^{\lambda^\vee} \xrightarrow{\zeta'_{\lambda^\vee}} \mathcal{O}_{S\times X}.$$

Here we use the identification of the scheme $\overline{T}_\mu \cap \overline{S}_\nu$ with $\overline{\mathfrak{Z}}^{\nu,\mu}$. We see that the morphism ϖ induces an isomorphism from $(\overline{T}_\mu \cap \overline{S}_\nu) \times \mathrm{Vin}\, \overline{S}_\mu^{\mathrm{princ}}$ to the closure of the family $(\overline{T_\mu} \cap \overline{S}_\nu) \times \overline{S}_\mu \times \mathbb{C}^\times \to \mathbb{C}^\times$ in the family $(\Upsilon_\nu^{2,\mathrm{princ}})^{-1}(\{0\} \times \mathbb{A}^1) \to \mathbb{A}^1$. □

Let us consider the cospecialization morphism

$$H_c^{\langle 2\rho^\vee, \nu\rangle}((\mathrm{Vin}^2\, \overline{S}_\nu)_{(0,0)}, (\widetilde{\mathcal{P}}_\nu^2)_{(0,0)}) \to H_c^{\langle 2\rho^\vee, \nu\rangle}((\mathrm{Vin}^2\, \overline{S}_\nu)_{(1,1)}, (\widetilde{\mathcal{P}}_\nu^2)_{(1,1)}).$$

From Corollary 4.10, it follows that $H_c^{\langle 2\rho^\vee, \nu\rangle}((\mathrm{Vin}^2\, \overline{S}_\nu)_{(0,0)}, (\widetilde{\mathcal{P}}_\nu^2)_{(0,0)}) = \bigoplus_{\nu \geqslant \mu_1 \geqslant \mu_2} H_c^{\langle 2\rho^\vee, \nu-\mu_1\rangle}(\overline{S}_\nu \cap \overline{T}_{\mu_1}, \underline{\mathbb{C}}) \otimes H_c^{\langle 2\rho^\vee, \mu_1-\mu_2\rangle}(\overline{S}_{\mu_1} \cap \overline{T}_{\mu_2}, \underline{\mathbb{C}}) \otimes H_c^{\langle 2\rho^\vee, \mu_2\rangle}(\overline{S}_{\mu_2}, \mathcal{P})$, and

$$H_c^{\langle 2\rho^\vee, \nu\rangle}((\mathrm{Vin}^2\, \overline{S}_\nu)_{(1,1)}, (\widetilde{\mathcal{P}}_\nu^2)_{(1,1)}) = H_c^{\langle 2\rho^\vee, \nu\rangle}(\overline{S}_\nu, \mathcal{P}).$$

Thus we get the morphisms

$$H_c^{\langle 2\rho^\vee, \nu-\mu_1\rangle}(\overline{S}_\nu \cap \overline{T}_{\mu_1}, \underline{\mathbb{C}}) \otimes H_c^{\langle 2\rho^\vee, \mu_1-\mu_2\rangle}(\overline{S}_{\mu_1} \cap \overline{T}_{\mu_2}, \underline{\mathbb{C}}) \otimes H_c^{\langle 2\rho^\vee, \mu_2\rangle}(\overline{S}_{\mu_2}, \mathcal{P}) \to H_c^{\langle 2\rho^\vee, \nu\rangle}(\overline{S}_\nu, \mathcal{P}).$$

Note that the following diagram is commutative:

$$\begin{array}{ccc} H_c^{\langle 2\rho^\vee, \nu\rangle}((\overline{S}_\nu \cap \overline{T}_{\mu_1}) \times (\overline{S}_{\mu_1} \cap \overline{T}_{\mu_2}) \times \overline{S}_{\mu_2}, \underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} \boxtimes \mathcal{P}) & \longrightarrow & H_c^{\langle 2\rho^\vee, \nu\rangle}((\overline{S}_\nu \cap \overline{T}_{\mu_2}) \times \overline{S}_{\mu_2}, \underline{\mathbb{C}} \boxtimes \mathcal{P}) \\ \downarrow & & \downarrow \\ H_c^{\langle 2\rho^\vee, \nu\rangle}((\overline{S}_\nu \cap \overline{T}_{\mu_1}) \times \overline{S}_{\mu_1}, \underline{\mathbb{C}} \boxtimes \mathcal{P}) & \longrightarrow & H_c^{\langle 2\rho^\vee, \nu\rangle}(\overline{S}_\nu, \mathcal{P}), \end{array} \tag{4.12}$$

where the morphisms in the diagram are the cospecialization morphisms. Indeed, both compositions are equal to the cospecialization morphism from the fiber over (0,0) to the fiber over (1,1).

From Lemma 4.11 and Lemma 3.16 it follows that the diagram (4.12) is equal to the diagram (4.8). Hence (4.8) is commutative and Proposition 4.7 is proved. □

4.3. *Compatibility of the coproduct with the tensor structure*

Let $\mathcal{P}, \mathcal{Q} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$. We set $V := H^\bullet(\mathrm{Gr}_G, \mathcal{P}),\ W := H^\bullet(\mathrm{Gr}_G, \mathcal{Q}) \in \mathrm{Rep}(G^\vee)$. We need to check that the diagram

$$\begin{array}{ccc} \mathcal{A} \otimes V \otimes W & \xrightarrow{\mathrm{act}_{V\otimes W}} & V \otimes W \\ \downarrow{\scriptstyle \Delta\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathcal{A}\otimes\mathcal{A}\otimes V\otimes W & \xrightarrow{(\mathrm{act}_V\otimes \mathrm{act}_W)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})} & V\otimes W \end{array} \tag{4.13}$$

commutes, where the morphism $\tau\colon \mathcal{A}\otimes V \to V\otimes\mathcal{A}$ sends $a\otimes b$ to $b\otimes a$.

Fix $\theta\in\Lambda$. Let $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta,X}$ be the following moduli space: it associates to a scheme S

(1) an S-point $x\in X(S)$ of the curve X;
(2) an S-point $(\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee})$ of $\mathrm{VinBun}^{\mathrm{princ}}_G$;
(3) for every $\lambda^\vee\in\Lambda^{\vee+}$, morphisms of sheaves $\eta_{\lambda^\vee}$,

$$\eta_{\lambda^\vee}: \mathcal{O}_{S\times X}(-\langle\lambda^\vee,\theta\rangle\cdot\Gamma_x)\to\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}$$

and rational morphisms $\zeta_{\lambda^\vee}: \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}\to\mathcal{O}_{S\times X}$ regular on $(S\times X)\setminus\{\Gamma_x\}$, satisfying the same conditions as in § 3.1.2.

Fix $\theta_1,\theta_2\in\Lambda$. Let $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}$ be the following moduli space: it associates to a scheme S

(1) a pair of S-points $(x_1,x_2)\in X^2(S)$ of the curve X;
(2) an S-point $(\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee})\in\mathrm{VinBun}^{\mathrm{princ}}_G(S)$ of $\mathrm{VinBun}^{\mathrm{princ}}_G$;
(3) for every $\lambda^\vee\in\Lambda^{\vee+}$, morphisms of sheaves $\eta_{\lambda^\vee}$,

$$\eta_{\lambda^\vee}: \mathcal{O}_{S\times X}(-\langle\lambda^\vee,\theta_1\rangle\cdot\Gamma_{x_1}-\langle\lambda^\vee,\theta_2\rangle\cdot\Gamma_{x_2})\to\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_1}$$

and rational morphisms $\zeta_{\lambda^\vee}: \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_2}\to\mathcal{O}_{S\times X}$ regular on $(S\times X)\setminus\{\Gamma_{x_1}\cup\Gamma_{x_2}\}$, satisfying the same conditions as in § 3.1.2.

We have a projection $\pi^{\mathrm{Vin}}_{\theta_1,\theta_2}\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\to X^2$ that forgets the data of $\mathcal{F}_1,\mathcal{F}_2,\varphi_{\lambda^\vee},\tau_{\mu^\vee},\eta_{\lambda^\vee},\zeta_{\lambda^\vee}$. Let

$$\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\hookrightarrow\mathrm{Gr}_{G,X^2}\underset{X^2}{\times}\mathrm{Gr}_{G,X^2}\times\mathbb{A}^1$$

be the restriction of the closed embedding ϑ in § 3.1.7 to $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\subset\mathrm{VinGr}_{G,X^2}$. The morphism $\Upsilon^{\mathrm{princ}}_{\theta_1,\theta_2}\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\to\mathbb{A}^1$ is defined as the composition of morphisms $\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\hookrightarrow\mathrm{VinGr}^{\mathrm{princ}}_{G,X^2}\to\mathbb{A}^1$.

4.3.1. *The three-parameter deformation.* Let us consider the morphism

$$\Upsilon^{\mathrm{princ}}_{\theta_1,\theta_2}\times\pi_{\theta_1,\theta_2}\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\to X^2\times\mathbb{A}^1.$$

Recall the closed embedding

$$\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}\colon\ \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}\hookrightarrow\mathrm{Gr}_{G,X^2}\underset{X^2}{\times}\mathrm{Gr}_{G,X^2}\times\mathbb{A}^1.$$

Recall the sheaf $\mathcal{P} \underset{X}{\star} \mathcal{Q}$ on Gr_{G,X^2} in § 2.1.4. Set

$$(\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2} := (\underline{\mathbb{C}} \boxtimes (\mathcal{P} \underset{X}{\star} \mathcal{Q}) \boxtimes \underline{\mathbb{C}})|_{\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2}}.$$

It follows from Corollary 4.15 that the complex $(\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2}$ has finite-dimensional support. Recall the embeddings $\Delta_X \hookrightarrow X^2 \hookleftarrow U$ in § 2.1.4.

Lemma 4.14. (a) *The restriction of the morphism $\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}$ of the families over $X^2 \times \mathbb{A}^1$*

$$\begin{array}{ccc} \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2} & \xrightarrow{\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}} & \mathrm{Gr}_{G,X^2} \underset{X^2}{\times} \mathrm{Gr}_{G,X^2} \times \mathbb{A}^1 \\ & \searrow \quad \swarrow & \\ & X^2 \times \mathbb{A}^1 & \end{array}$$

to the open subvariety $U \times \mathbb{A}^1 \subset X^2 \times \mathbb{A}^1$ is isomorphic to

$$\begin{array}{ccc} (\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,X} \underset{\mathbb{A}^1}{\times} \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_2,X})|_{(U\times\mathbb{A}^1)} & \xrightarrow{\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}|_{(U\times\mathbb{A}^1)}} & ((\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U \underset{U}{\times} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U) \times \mathbb{A}^1 \\ & \searrow \quad \swarrow & \\ & U \times \mathbb{A}^1 & \end{array}$$

where the morphism $\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}|_{(U\times\mathbb{A}^1)}$ is given by

$$\left(\vartheta^{(1)}_{\theta_1}, \vartheta^{(1)}_{\theta_2}, \vartheta^{(2)}_{\theta_1}, \vartheta^{(2)}_{\theta_2}, \vartheta^{(3)}_{\theta_1} = \vartheta^{(3)}_{\theta_2}\right),$$

and $\vartheta^{(i)}_{\theta_1}, \vartheta^{(i)}_{\theta_2}$ are the corresponding components of the morphisms

$$\vartheta^{\mathrm{princ}}_{\theta_1}\colon \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,X} \hookrightarrow \mathrm{Gr}_{G,X} \underset{X}{\times} \mathrm{Gr}_{G,X} \times \mathbb{A}^1, \quad \vartheta^{\mathrm{princ}}_{\theta_2}\colon \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_2,X} \hookrightarrow \mathrm{Gr}_{G,X} \underset{X}{\times} \mathrm{Gr}_{G,X} \times \mathbb{A}^1.$$

(b) *The restriction of the morphism $\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}$ of the families over $X^2 \times \mathbb{A}^1$ to the closed subvariety $\Delta_X \times \mathbb{A}^1 \subset X^2 \times \mathbb{A}^1$ is isomorphic to*

$$\begin{array}{ccc} \mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1+\theta_2,X} & \xrightarrow{\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2}|_{(\Delta_X\times\mathbb{A}^1)}} & \mathrm{Gr}_{G,X} \underset{X}{\times} \mathrm{Gr}_{G,X} \times \mathbb{A}^1 \\ & \searrow \quad \swarrow & \\ & \Delta_X \times \mathbb{A}^1 & \end{array}$$

where the morphism $(\vartheta^{\mathrm{princ}}_{\theta_1,\theta_2})|_{\Delta_X\times\mathbb{A}^1}$ coincides with the morphism $\vartheta^{\mathrm{princ}}_{\theta_1+\theta_2}$.

Proof. It follows from Proposition 3.6. □

Corollary 4.15. *Under the identifications of Lemma 4.14 and Proposition 4.1 we have:*

(a) $((\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2})_{((x,x),0)}|_{(\overline{T}_\mu \cap \overline{S}_{\theta_1+\theta_2})\times \overline{S}_\mu} = \underline{\mathbb{C}} \boxtimes (\mathcal{P} \star \mathcal{Q})_\mu$ *for $x \in X$;*

(b) $((\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2})_{((x,y),0)}|_{((\overline{T}_{\mu_1} \cap \overline{S}_{\theta_1})\times(\overline{T}_{\mu_2} \cap \overline{S}_{\theta_2}))\times \overline{S}_{\mu_1}\times\overline{S}_{\mu_2}} = \underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} \boxtimes \mathcal{P}_{\mu_1} \boxtimes \mathcal{Q}_{\mu_2}$ *for $(x,y) \in U$;*

(c) $\big((\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2}\big)_{((x,x),1)} = (\mathcal{P} \star \mathcal{Q})_{\theta_1+\theta_2}$ *for* $x \in X$;

(d) $\big((\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2}\big)_{((x,y),1)} = \mathcal{P}_{\theta_1} \boxtimes \mathcal{Q}_{\theta_2}$ *for* $(x,y) \in U$.

4.3.2. *Proof of compatibility with the tensor structure.* Let us fix two distinct points $x, y \in X$. Let us consider the cospecialization morphism

$$\begin{array}{c} H_c^{\langle 2\rho^\vee, \theta_1+\theta_2\rangle}\big((\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2})_{((x,x),0)}, ((\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2})_{((x,x),0)}\big) \\ \downarrow \\ H_c^{\langle 2\rho^\vee, \theta_1+\theta_2\rangle}\big((\mathrm{Vin}\,\overline{S}^{\mathrm{princ}}_{\theta_1,\theta_2})_{((x,y),1)}, ((\widetilde{\mathcal{P}} \underset{X}{\star} \widetilde{\mathcal{Q}})_{\theta_1,\theta_2})_{((x,y),1)}\big) \end{array}$$

from the point $((x,x),0)$ to the point $((x,y),1)$. It may be obtained in two ways: by first cospecializing from $((x,x),0)$ to $((x,x),1)$ and then to $((x,y),1)$ or first cospecializing from $((x,x),0)$ to $((x,y),0)$ and then to $((x,y),1)$. For $\nu_1,\nu_2 \in \Lambda$, let $\mathrm{p}_{\nu_1,\nu_2}\colon (V\otimes W)_{\nu_1+\nu_2} \twoheadrightarrow V_{\nu_1} \otimes W_{\nu_2}$ be the natural projection. Using Corollary 4.15 and Lemma 4.14 we see that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_{\theta_1+\theta_2-\mu} \otimes (V\otimes W)_\mu & \xrightarrow{\mathrm{act}_{V\otimes W}} & (V\otimes W)_{\theta_1+\theta_2} \\ \Big\downarrow{\scriptstyle \oplus\Delta_{\mu_1,\mu_2}\otimes \mathrm{p}_{\theta_1-\mu_1,\theta_2-\mu_2}} & & \Big\downarrow{\scriptstyle \mathrm{p}_{\theta_1,\theta_2}} \\ \bigoplus_{\mu_1+\mu_2=\theta_1+\theta_2-\mu} \mathcal{A}_{\mu_1}\otimes \mathcal{A}_{\mu_2} \otimes V_{\theta_1-\mu_1}\otimes W_{\theta_2-\mu_2} & \xrightarrow{(\mathrm{act}_V\otimes \mathrm{act}_W)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})} & V_{\theta_1}\otimes W_{\theta_2} \end{array}.$$

Summing over all $\theta_1, \theta_2, \mu \in \Lambda$ such that $\mu \leqslant \theta_1+\theta_2$ we get the diagram (4.13).

LEMMA 4.16. *Let us fix* $\theta \in \Lambda^{\mathrm{pos}}$, $a \in \mathcal{A}_\theta \setminus \{0\}$. *There exists* $\nu \in \Lambda^+$ *such that the operator* $\mathrm{act}_{V^\nu}(a)$ *is nonzero.*

Proof. It follows from [6, Proposition 6.4] that there exists $\lambda \in \Lambda^+$ such that $\overline{S}_{\theta-\lambda} \cap \overline{T}_{-\lambda}$ is contained inside $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap \overline{S}_{\theta-\lambda}$. It follows that $V^{-w_0(\lambda)}_{-\lambda+\theta} = H_c^{\langle 2\rho^\vee,\theta\rangle}(\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap \overline{S}_{\theta-\lambda}, \underline{\mathbb{C}})$ embeds into $\mathcal{A}_\theta = H_c^{\langle 2\rho^\vee,\theta\rangle}(\overline{S}_{\theta-\lambda}\cap\overline{T}_{-\lambda}, \underline{\mathbb{C}})$. Indeed, the latter space has a basis formed by the fundamental classes of irreducible components of $\overline{S}_{\theta-\lambda}\cap\overline{T}_{-\lambda}$, while the former space has a basis formed by the fundamental classes of irreducible components of $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap \overline{S}_{\theta-\lambda}$, and each of those irreducible components is an irreducible component of $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap \overline{S}_{\theta-\lambda}$. Thus the vertical arrows of the following commutative diagram are embeddings:

$$\begin{array}{ccc} \mathcal{A}_\theta \otimes V^{-w_0(\lambda)}_{-\lambda} & \xrightarrow{\mathrm{act}_{V^{-w_0(\lambda)}}} & V^{-w_0(\lambda)}_{-\lambda+\theta} \\ \downarrow & & \downarrow \\ \mathcal{A}_\theta \otimes \mathcal{A}_0 & \xrightarrow{\mathbf{m}} & \mathcal{A}_\theta. \end{array} \tag{4.17}$$

Commutativity of the diagram (4.17) follows from the definitions of $\mathrm{act}, \mathbf{m}$ using the identifications $\overline{S}_{\theta-\lambda}\cap\overline{T}_{-\lambda} \simeq \overline{S}_\theta \cap \overline{T}_0$, $\overline{\mathfrak{Y}}^\theta \simeq \overline{\mathfrak{Y}}^{\theta-\lambda,-\lambda}$ (see Lemma 3.16) and the natural closed embedding $\overline{\mathfrak{Y}}^{\theta-\lambda,-\lambda} \hookrightarrow \mathrm{Vin}\,\overline{S}_{\theta-\lambda}$. Note that $\mathbf{m}(a\otimes 1) = a$ is nonzero. It follows that the operator $\mathrm{act}_{V^{-w_0(\lambda)}}(a)$ is nonzero. $\square$

4.4. *Schieder conjecture*

According to [**22**, Question 6.6.1], the bialgebra $\mathcal{A}$ is expected to be isomorphic to the universal enveloping algebra $U(\mathfrak{n}^\vee)$. We will construct such an isomorphism.

Corollary 4.18. *Schieder bialgebra $\mathcal{A}$ is isomorphic to $U(\mathfrak{n}^\vee)$.*

Proof. Let $\mathcal{A}^{\mathrm{prim}} \subset \mathcal{A}$ stand for the subspace of primitive elements equipped with the natural structure of Lie algebra. We will denote this Lie algebra $\mathfrak{a}$. We have a canonical morphism $\epsilon\colon U(\mathfrak{a}) \to \mathcal{A}$. We have to check that ϵ is an isomorphism and to construct an isomorphism $\varepsilon\colon \mathfrak{a} \xrightarrow{\sim} \mathfrak{n}^\vee$. According to § 4.3, $\mathfrak{a}$ acts on the fiber functor $H^\bullet(\mathrm{Gr}_G, \bullet)\colon \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G) \to \mathrm{Vect}$, and hence we obtain a homomorphism of Lie algebras $\mathfrak{a} \to \mathfrak{g}^\vee$, see [**17**, Lemma 23.69]. This homomorphism clearly lands into $\mathfrak{n}^\vee \subset \mathfrak{g}^\vee$ (see 4.1.1). This is the desired homomorphism ε.

The vector space $\mathcal{A}^{\mathrm{prim}}$ was computed in [**12**, Proposition 4.2]: it is a direct sum of lines $\bigoplus_{\alpha\in R^+} \mathcal{A}^{\mathrm{prim}}_\alpha$. Note that while the definition of *algebra* structure on $\mathcal{A}$ in [**12**] is *different* from Schieder's one in § 2.6.3, the definition of *coalgebra* structure on $\mathcal{A}$ in [**12**, 2.11] is manifestly the same as Schieder's one in § 2.6.2. Thus the character of the Λ^{pos}-graded vector space $\mathfrak{a} = \bigoplus_{\alpha\in R^+} \mathfrak{a}_\alpha$ coincides with the character of $\mathfrak{n}^\vee$ as elements in $\mathbb{Z}[\Lambda^{\mathrm{pos}}]$.

It follows from Lemma 4.16 that for $i \in I$, the action of $\mathfrak{a}_{\alpha_i}$ on the fiber functor is not trivial, and hence $\varepsilon(\mathfrak{a}_{\alpha_i}) \neq 0$, that is $\varepsilon(\mathfrak{a}_{\alpha_i}) = \mathfrak{n}^\vee_{\alpha_i}$. Since $\bigoplus_{i\in I} \mathfrak{n}^\vee_{\alpha_i}$ generates $\mathfrak{n}^\vee$, we see that ε is surjective. Since the characters of $\mathfrak{a}$ and $\mathfrak{n}^\vee$ coincide, ε is an isomorphism. Since the action of $U(\mathfrak{a}) \simeq U(\mathfrak{n}^\vee)$ on the fiber functor is effective and factors through the action of $\mathcal{A}$, we conclude that $\epsilon\colon U(\mathfrak{a}) \to \mathcal{A}$ is injective. One last comparison of characters shows that ϵ is an isomorphism, and hence $U(\mathfrak{n}^\vee) \xrightarrow{\sim} U(\mathfrak{a}) \xrightarrow{\sim} \mathcal{A}$ (the first arrow is ε^{-1}, the second one is ϵ). ☐

4.5. *Comparison with the Ext-algebra of* [**12**]

There is another construction of the algebra structure in $\mathcal{A}$ going back to [**12**]. Let us denote it by $\mathcal{A} \ni a, b \mapsto a \circ b$. In this Section we identify this product with Schieder's product.

Let us denote the shriek extension of the constant sheaf on $T_\nu \cap \overline{\mathrm{Gr}}^\lambda_G$ by $\mathcal{R}^\lambda_\nu$. As λ grows, these sheaves form an inverse system, and we denote its inverse limit by $\mathcal{R}_\nu$. By Braden's Theorem, $\mathrm{Ext}^{\langle 2\rho^\vee, \nu-\mu\rangle}(\mathcal{R}^\lambda_\nu, \mathcal{R}^\lambda_\mu) = H^{\langle 2\rho^\vee, \nu-\mu\rangle}(\iota^*_{\nu,-} r^!_{\nu,-} r_{\mu,-,!}\underline{\mathbb{C}}) = H^{\langle 2\rho^\vee, \nu-\mu\rangle}(\iota^!_{\nu,+} r^*_{\nu,+} r_{\mu,-,!}\underline{\mathbb{C}}) = H_c^{\langle 2\rho^\vee, \nu-\mu\rangle}(T_\mu \cap S_\nu \cap \overline{\mathrm{Gr}}^\lambda_G)$. As λ grows, this compactly supported cohomology forms an inverse system that eventually stabilizes; the stable value will be denoted by $\mathrm{Ext}^{\langle 2\rho^\vee, \nu-\mu\rangle}(\mathcal{R}_\nu, \mathcal{R}_\mu) = H_c^{\langle 2\rho^\vee, \nu-\mu\rangle}(T_\mu \cap S_\nu) = \mathcal{A}_{\nu-\mu}$.

The composition of the above Ext groups defines the desired product $(a \in \mathcal{A}_{\theta_1}, b \in \mathcal{A}_{\theta_2}) \mapsto a \circ b \in \mathcal{A}_{\theta_1+\theta_2}$. Furthermore, for $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ we have $\Phi_\nu(\mathcal{P}) = \mathrm{Ext}^{\langle 2\rho^\vee, \nu\rangle}(\mathcal{R}_\nu, \mathcal{P})$ (the right-hand side is again defined as the limit of a direct system that eventually stabilizes to $\iota^*_{\nu,-} r^!_{\nu,-}\mathcal{P}$). Hence the composition of Ext groups also defines an action $(a \in \mathcal{A}_{\nu-\mu}, \phi \in \Phi_\mu(\mathcal{P})) \to a \circ \phi \in \Phi_\nu(\mathcal{P})$.

Lemma 4.19. (a) *The $\circ$-product on $\mathcal{A}$ coincides with Schieder's product* $\mathbf{m}$.
(b) *The $\circ$-action of $\mathcal{A}$ on Φ coincides with the action of* 4.1.1.

Proof. (a) follows from (b). To prove (b), due to the fact that Φ_μ is represented by $\mathcal{R}_\mu$, it suffices to compare the two actions of $\mathcal{A}_{\nu-\mu}$ on $\mathrm{Id} \in \Phi_\mu(\mathcal{R}_\mu)$. That is we have to check that the cospecialization morphism (§ 4.1.1) $\mathcal{A}_{\nu-\mu} = \mathcal{A}_{\nu-\mu} \otimes \mathbb{C} = \mathcal{A}_{\nu-\mu} \otimes H^0_c(S_\mu, \mathcal{R}_\mu) \to H_c^{\langle 2\rho^\vee, \nu-\mu\rangle}(S_\nu, \mathcal{R}_\mu) = H_c^{\langle 2\rho^\vee, \nu-\mu\rangle}(S_\nu \cap T_\mu) = H_c^{\langle 2\rho^\vee, \nu-\mu\rangle}(\overline{S}_\nu \cap T_\mu) = \mathcal{A}_{\nu-\mu}$ is the identity morphism. This follows from the fact that the Drinfeld–Gaitsgory interpolation of $\overline{S}_\nu \cap T_\mu$ is trivial. ☐

4.6. *Integral form*

Note that the bialgebra $\mathcal{A}$ comes equipped with a natural basis of fundamental classes of irreducible components of $\overset{\circ}{\mathfrak{Z}}{}^\theta$, and the structure constants of multiplication in this basis belong to $\mathbb{Z}$. Hence $\mathcal{A}$ acquires an integral form $\mathcal{A}_\mathbb{Z}$. The algebra $U(\mathfrak{n}^\vee)$ also has the integral Chevalley–Kostant form $U(\mathfrak{n}^\vee)_\mathbb{Z}$, see, for example, [**23**, Chapter 2].

PROPOSITION 4.20. *The isomorphism* $U(\mathfrak{n}^\vee) \xrightarrow{\sim} \mathcal{A}$ *of Corollary* 4.18 *induces an isomorphism* $U(\mathfrak{n}^\vee) \supset U(\mathfrak{n}^\vee)_\mathbb{Z} \xrightarrow{\sim} \mathcal{A}_\mathbb{Z} \subset \mathcal{A}$.

Proof. We consider the perverse sheaves with integral coefficients $\mathcal{I}_!(\lambda, \mathbb{Z}) \xrightarrow{\sim} \mathcal{I}_{!*}(\lambda, \mathbb{Z})$ of [**2**, Lemma 11.5]. The fiber functor applied to them gives the integral Weyl modules $\Phi(\mathcal{I}_!(\lambda,\mathbb{Z})) \simeq V^\lambda_\mathbb{Z} \in \mathrm{Rep}(G^\vee_\mathbb{Z})$. These are free $\mathbb{Z}$-modules with the bases given by the fundamental classes of the irreducible components of $\mathrm{Gr}^\lambda_G \cap S_\nu$, see [**2**, Proposition 11.1]. For a fixed $\theta \in \Lambda^{\mathrm{pos}}$, and a dominant coweight $\lambda \gg 0$, the action of $\mathcal{A}$ on the fundamental class of $\mathrm{Gr}^\lambda \cap S_{w_0(\lambda)}$ induces an isomorphism of the weight spaces $\mathcal{A}_\theta \xrightarrow{\sim} \Phi_{\theta+w_0\lambda}(\mathrm{IC}(\overline{\mathrm{Gr}}{}^\lambda_G)) \simeq V^\lambda_{\theta+w_0\lambda}$. Moreover, it follows from the proof of Lemma 4.16 that this isomorphism respects the integral forms: $\mathcal{A}_{\mathbb{Z},\theta} \xrightarrow{\sim} V^\lambda_{\mathbb{Z},\theta+w_0\lambda}$. However, according to [**23**, Corollary 1 of Theorem 2], the action of $U(\mathfrak{n}^\vee)$ on the lowest weight vector of $V^\lambda_\mathbb{Z}$ also induces an isomorphism of the weight spaces respecting the integral forms $U(\mathfrak{n}^\vee)_{\mathbb{Z},\theta} \xrightarrow{\sim} V^\lambda_{\mathbb{Z},\theta+w_0\lambda}$.

The proposition follows. □

5. *Arbitrary reductive groups and symmetric Kac–Moody Lie algebras*

5.1. *Configurations*

In general, we may have $\Lambda^{\mathrm{pos}} := \bigoplus_{i\in I} \mathbb{N}\alpha_i \subsetneq \Lambda_{\geqslant 0} := \{\alpha \in \Lambda : \langle\lambda^\vee, \alpha\rangle \geqslant 0 \ \ \forall \lambda^\vee \in \Lambda^{\vee+}\}$. It happens, for example, for $G = \mathrm{PGL}_2$. Recall that for $\theta \in \Lambda_{\geqslant 0}$, a point $D \in X^\theta$ is a collection of effective divisors $D_{\lambda^\vee} \in X^{(\langle\lambda^\vee,\theta\rangle)}$ for $\lambda \in \Lambda^{\vee+}$ such that for every $\lambda_1^\vee, \lambda_2^\vee \in \Lambda^{\vee+}$ we have $D_{\lambda_1^\vee} + D_{\lambda_2^\vee} = D_{\lambda_1^\vee+\lambda_2^\vee}$. For $\theta = \sum_{i\in I} n_i\alpha_i \in \Lambda^{\mathrm{pos}}$ we define $X^{(\theta)} = \prod_{i\in I} X^{(n_i)}$. We have a closed ('diagonal') embedding $X^{(\theta)} \hookrightarrow X^\theta$ (it fails to be an isomorphism, for example, for $G = \mathrm{PGL}_2$).

The complement $X^\theta \setminus X^{(\theta)}$ accounts for some undesirable components of VinBun_G if we use the naive definition for arbitrary G. The reason lies in the 'bad' components of $\overline{\mathrm{Bun}}_B$. Schieder explains in [**19**, § 7] how to get rid of the 'bad' components using a central extension

$$1 \to \mathcal{Z} \to \widehat{G} \to G \to 1 \tag{5.1}$$

such that $\mathcal{Z}$ is a (connected) central torus in $\widehat{G}$, and the derived subgroup $[\widehat{G}, \widehat{G}] \subset \widehat{G}$ is simply connected. The same remedy works in our setup.

5.2. *Example for* $G = \mathrm{PGL}_2$

We identify Λ with $\mathbb{Z}$, so that the simple root α corresponds to 2. We have $\Lambda^{\mathrm{pos}} = 2\mathbb{N}$, $\Lambda_{\geqslant 0} = \mathbb{N}$. Let $X = \mathbb{P}^1$ with homogeneous coordinates x, y. We identify $G = \mathrm{PGL}_2 = \mathrm{SO}_3$ and consider the fiber $\mathfrak{B}$ of $\overline{\mathrm{Bun}}{}^1_B$ over the G-bundle $\mathcal{V} = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1)$ (with the evident symmetric self-pairing $\mathcal{V} \otimes \mathcal{V} \to \mathcal{O}$). In other words, $\mathfrak{B}$ is the projectivization of $\mathrm{Hom}^0(\mathcal{O}(-1), \mathcal{V})$, where Hom^0 is formed by the isotropic homomorphisms. Thus $\mathrm{Hom}^0(\mathcal{O}(-1), \mathcal{V}) = \{(P_0, P_1, P_2) : P_0P_2 - P_1^2 = 0\}$, where P_i is a homogeneous polynomial in x, y of degree i. Clearly, $\mathrm{Hom}^0(\mathcal{O}(-1), \mathcal{V})$ consists of two irreducible three-dimensional components: the first one given by $P_0 = P_1 = 0$, and the second one given by (away from the first one) $P_0 \neq 0$, $P_2 = P_1^2 P_0^{-1}$. We denote the projectivization of the first (respectively, second) component by $\mathfrak{B}_{\text{bad}}$ (resp. $\mathfrak{B}_{\text{good}}$).

We also consider the 'shifted zastava' space $Z^{1,-1}$: the open part of $\mathfrak{B}$ formed by those $\mathcal{O}(-1) \hookrightarrow \mathcal{V}$ whose composition with the projection $\mathcal{V} \twoheadrightarrow \mathcal{O}(1)$ is nonzero. In other words, it is given by the condition $P_2 \neq 0$. The factorization projection $Z^{1,-1} \to X^\alpha$ is given by $(P_0, P_1, P_2) \mapsto P_2$. We see that $Z^{1,-1}_{\text{bad}}$ projects isomorphically onto X^α, while $Z^{1,-1}_{\text{good}}$ projects onto $X^{(\alpha)} \hookrightarrow X^\alpha$.

The existence of 'bad' components in the zastava spaces implies the existence of 'bad' components in the local models Y^θ in § 2.5.3, and hence the existence of 'bad' components in VinBun_G.

5.3. *Modified constructions*

Let us redenote $\overline{\mathrm{Bun}}_B$ by $\overline{\mathrm{Bun}}_B^{\text{naive}}$. In order to get rid of the 'bad' components of $\overline{\mathrm{Bun}}_B^{\text{naive}}$, Schieder modified the definition as

$$\overline{\mathrm{Bun}}_B := \mathrm{Maps}_{\text{gen}}(X, G\backslash\overline{\widehat{G}/N}/\widehat{T} \supset G\backslash(\widehat{G}/N)/\widehat{T} = \mathrm{pt}/B),$$

where $\widehat{G}$ is chosen as in (5.1), and $1 \to \mathcal{Z} \to \widetilde{T} \to T \to 1$ is the corresponding extension of Cartan tori. Schieder proved that $\overline{\mathrm{Bun}}_B$ is canonically independent of the choice of central extension (5.1), that is, $\overline{\mathrm{Bun}}_B$ is well defined. Note also that $G\backslash\overline{\widehat{G}/N}/\widehat{T} = G\backslash(\overline{\widehat{G}/N}/\mathcal{Z})/T$.

Following Schieder, we modify Definition 2.7 of $\mathrm{Vin}_G^{\text{naive}}$ as $\mathrm{Vin}_G := \mathrm{Vin}_{\widetilde{G}}/\mathcal{Z}$. Accordingly, we modify Definition 2.13 of $\mathrm{VinBun}_G^{\text{naive}}$ as

$$\mathrm{VinBun}_G := \mathrm{Maps}_{\text{gen}}(X, \mathrm{Vin}_G/(G \times G) \supset {}_0\mathrm{Vin}_G/(G \times G))$$

(instead of $\mathrm{Maps}_{\text{gen}}(X, \mathrm{Vin}_G^{\text{naive}}/(G \times G) \supset {}_0\mathrm{Vin}_G^{\text{naive}}/(G \times G))$).

With this understanding, the definition of VinGr_{G,X^n} in § 3.1.6 stays intact, as well as the construction of the action of Schieder bialgebra $\mathcal{A}$ on the geometric Satake fiber functor.

5.4. *A Coulomb branch construction*

In case G is almost simple simply laced, the relative compactification $\overline{Z}^\theta \supset Z^\theta$ [**14**, § 7.2] of zastava space for $X = \mathbb{A}^1$ was constructed in [**7**, Remark 3.7] in the course of study of Coulomb branches of $3d$ $\mathcal{N} = 4$ supersymmetric quiver gauge theories. The diagonal fibers of the factorization morphism $\pi\colon \overline{Z}^\theta \to \mathbb{A}^{(\theta)}$ are identified with $\overline{\mathfrak{Z}}^{\theta,0} = \overline{S}_\theta \cap \overline{T}_0$, see Remark 2.21. The construction of [**7**] proceeds in terms of θ-dimensional representations of the Dynkin graph Q of G equipped with an orientation. Given an arbitrary quiver Q with a set Q_0 of vertices and without loop edges, and a dimension vector $\theta \in \mathbb{N}^{Q_0}$, this construction produces an affine scheme $\pi\colon Z^\theta_{\mathfrak{g}_Q} \to \mathbb{A}^{(\theta)}$ and its relative projectivization $\overline{\pi}\colon \overline{Z}^\theta_{\mathfrak{g}_Q} \to \mathbb{A}^{(\theta)}$. Here we denote by $\mathfrak{g}_Q$ the corresponding symmetric Kac–Moody algebra, and we expect $Z^\theta_{\mathfrak{g}_Q}$ (respectively, $\overline{Z}^\theta_{\mathfrak{g}_Q}$) to play the role of (compactified) zastava space for $\mathfrak{g}_Q$ (cf. [**7**, Remark 3.26.(1)]). In particular, we hope that $\pi, \overline{\pi}$ enjoy the factorization property, and $\dim \overline{\mathfrak{Z}}^\theta_{\mathfrak{g}_Q} = \dim \mathfrak{Z}^\theta_{\mathfrak{g}_Q} = |\theta|$, where $\overline{\mathfrak{Z}}^\theta_{\mathfrak{g}_Q}$ (respectively, $\mathfrak{Z}^\theta_{\mathfrak{g}_Q}$) stands for the diagonal fiber $\overline{\pi}^{-1}(\theta \cdot 0)$ (respectively, $\pi^{-1}(\theta \cdot 0)$).

Furthermore, the construction of [**7**, Remark 3.2] equips $Z^\theta_{\mathfrak{g}_Q}$ and $\overline{Z}^\theta_{\mathfrak{g}_Q}$ with an action of the Cartan torus $T_Q = \mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^{Q_0}]$. We expect the set of T_Q-fixed points in $\overline{\mathfrak{Z}}^\theta_{\mathfrak{g}_Q}$ to be discrete and parametrized by $\{\gamma \in \mathbb{N}^{Q_0} : \gamma \leqslant \theta\}$. Instead of T_Q-action we will consider a $\mathbb{C}^\times$-action arising from a regular cocharacter $\mathbb{C}^\times \to T_Q$. We expect the attractor (respectively, repellent) to a point γ to be isomorphic to $\mathfrak{Z}^{\theta-\gamma}_{\mathfrak{g}_Q}$ (respectively, $\mathfrak{Z}^\gamma_{\mathfrak{g}_Q}$).

Finally, we define $\mathcal{A}^Q_\theta := H_c^{2|\theta|}(\mathfrak{Z}^\theta_{\mathfrak{g}_Q})$, and $\mathcal{A}^Q := \bigoplus_{\theta \in \mathbb{N}^{Q_0}} \mathcal{A}^Q_\theta$. It is equipped with a bialgebra structure: a comultiplication arising from factorization as in § 2.6.2, and a multiplication arising from the Drinfeld–Gaitsgory interpolation associated to the above $\mathbb{C}^\times$-action as in § 2.6.3. We conjecture that $\mathcal{A}^Q$ is isomorphic to $U(\mathfrak{g}_Q^+)$.

Appendix. *Proofs of Propositions* 2.16 *and* 3.17

By Dennis Gaitsgory

A.1. *Proof of Proposition* 2.16

We will need two lemmas.

Lemma A.1. *Let $Z \to Z'$ be a morphism of affine schemes, equipped with actions of algebraic groups H and H', compatible under a surjective homomorphism $H \to H'$ with finite kernel that lies in the center of H. Let ${}_0Z' \subset Z'$ be an open H'-invariant subscheme, and let ${}_0Z \subset Z$ be its preimage. Assume that the morphism ${}_0Z \to {}_0Z'$ is finite. Then the resulting morphism*

$$b\colon \operatorname{Maps}_{\mathrm{gen}}(X, Z/H \supset {}_0Z/H) \to \operatorname{Maps}_{\mathrm{gen}}(X, Z'/H' \supset {}_0Z'/H')$$

is finite.

Warning: Note the morphism b is not necessarily schematic; rather its base change by a scheme yields a Deligne–Mumford stack. However, the notion of finiteness makes sense in this context as well: finite means proper + finite fibers over geometric points.

Proof. It is easy to see that b is quasi-finite (that is, every geometric point has a finite preimage). Hence, it suffices to show that it is proper.

The morphism b is the composition of the morphisms

$$\operatorname{Maps}_{\mathrm{gen}}(X, Z/H \supset {}_0Z/H) \to \operatorname{Maps}_{\mathrm{gen}}(X, Z'/H \supset {}_0Z'/H) \to \operatorname{Maps}_{\mathrm{gen}}(X, Z'/H' \supset {}_0Z'/H').$$

The first of these morphisms is schematic, and the second is not. We will show that both these morphisms are proper.

First we check that $\operatorname{Maps}_{\mathrm{gen}}(X, Z/H \supset {}_0Z/H) \to \operatorname{Maps}_{\mathrm{gen}}(X, Z'/H \supset {}_0Z'/H)$ is proper. We will do so by checking the valuative criterion.

Denote by ξ the generic point of X. Let $\mathcal{D}$ be the spectrum of a discrete valuation ring, and let $\overset{\circ}{\mathcal{D}} \subset \mathcal{D}$ be the spectrum of its fraction field. Given a square

$$\begin{array}{ccc} \overset{\circ}{\mathcal{D}} \times X & \longrightarrow & Z/H \\ \downarrow & & \downarrow \\ \mathcal{D} \times X & \longrightarrow & Z'/H \end{array}$$

such that the composition $\mathcal{D} \times \xi \to \mathcal{D} \times X \to Z'/H$ factors through the open embedding ${}_0Z'/H \hookrightarrow Z'/H$, we have to lift $\mathcal{D} \times X \to Z'/H$ to a morphism $\mathcal{D} \times X \to Z/H$.

A morphism $\overset{\circ}{\mathcal{D}} \times X \to Z/H$ (respectively, $\mathcal{D} \times X \to Z'/H$) is the same as an H-bundle $\overset{\circ}{\mathcal{F}}$ on $\overset{\circ}{\mathcal{D}} \times X$ (respectively, $\mathcal{F}$ on $\mathcal{D} \times X$) and an H-equivariant morphism $\overset{\circ}{\mathcal{F}} \to Z$ (respectively, $\mathcal{F} \to Z'$). Thus we have a square

$$\begin{array}{ccc} \overset{\circ}{\mathcal{F}} & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow \\ \mathcal{F} & \longrightarrow & Z' \end{array}$$

such that the composition

$$\mathcal{F}|_{\mathcal{D}\times\xi} \to \mathcal{F} \to Z' \tag{A.2}$$

factors through the open embedding ${}_0Z' \hookrightarrow Z'$ and we have to lift $\mathcal{F} \to Z'$ to an H-equivariant morphism $\mathcal{F} \to Z$.

We now use the assumption that ${}_0Z \to {}_0Z'$ is proper. This implies that the morphism ${}_0Z/H \to {}_0Z'/H$ is proper. Since $\mathcal{D}\times\xi$ is the spectrum of a DVR with generic point $\overset{\circ}{\mathcal{D}}\times\xi$, in the diagram

$$\begin{array}{ccc} \overset{\circ}{\mathcal{D}}\times\xi & \longrightarrow & Z/H \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}\times\xi & \longrightarrow & Z'/H, \end{array}$$

the map $\mathcal{D}\times\xi \to Z'/H$ lifts to a map $\mathcal{D}\times\xi \to Z/H$. Hence, the morphism (A.2) lifts to an H-equivariant morphism $\mathcal{F}|_{\mathcal{D}\times\xi} \to {}_0Z$.

Thus we obtain an H-equivariant morphism from an open subset $\mathcal{F}|_U \subset \mathcal{F}$ to Z, such that $\operatorname{codim}_{\mathcal{F}}(\mathcal{F}|_{(\mathcal{D}\times X)\setminus U}) = 2$. Since Z is affine and $\mathcal{F}$ is normal, this morphism extends to the whole of $\mathcal{F}$.

It remains to show that the morphism

$$\mathrm{Maps}_{\mathrm{gen}}(X, Z'/H \supset {}_0Z'/H) \to \mathrm{Maps}_{\mathrm{gen}}(X, Z'/H' \supset {}_0Z'/H')$$

is proper. It suffices to check that the morphism $\mathrm{Maps}(X, Z'/H) \to \mathrm{Maps}(X, Z'/H')$ is proper. Indeed, the following diagram is Cartesian:

$$\begin{array}{ccc} \mathrm{Maps}(X, Z'/H) & \longrightarrow & \mathrm{Maps}(X, Z'/H') \\ \big\downarrow & & \big\downarrow \\ \mathrm{Bun}_H & \longrightarrow & \mathrm{Bun}_{H'} \end{array}$$

so the desired properness of $\mathrm{Maps}(X, Z'/H) \to \mathrm{Maps}(X, Z'/H')$ follows from the properness of the morphism $\mathrm{Bun}_H \to \mathrm{Bun}_{H'}$. (Note, however, that the latter morphism is not schematic; its fibers are isomorphic to $\mathrm{Bun}_{\ker(H\to H')}$.) □

For $\Lambda^{\vee+} \ni \lambda^\vee \leqslant \mu^\vee \in \Lambda^\vee$, we denote by $V^{\lambda^\vee,\mu^\vee}$ the irreducible representation of Vin_G such that $V^{\lambda^\vee,\mu^\vee}|_G$ coincides with $V^{\lambda^\vee}$, and the center $Z_{G_{\mathrm{enh}}} = T$ acts on $V^{\lambda^\vee,\mu^\vee}$ via the character $\mu^\vee$. Let $\lambda_j^\vee \in \Lambda^{\vee+}, j \in J$ be a collection of dominant weights that generate $\Lambda^\vee \otimes \mathbb{Q}$. We have a natural morphism $\omega\colon \mathrm{Vin}_G \to \prod_{j\in J} \mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})$. Note that the preimage $\omega^{-1}(\prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee}) \setminus \{0\}))$ is exactly ${}_0\mathrm{Vin}_G$. We denote by ${}_0\omega$ the restriction of ω to ${}_0\mathrm{Vin}_G$.

Lemma A.3. *The morphism ${}_0\omega\colon {}_0\mathrm{Vin}_G \to \prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee}) \setminus \{0\})$ is finite.*

Proof. It follows from the Tannakian description of Vin_G in § 2.2.8 that the morphism ${}_0\omega$ is quasi-finite. Hence, it is enough to show that it is proper. We have the action $T \curvearrowright \mathrm{Vin}_G$ as in § 2.2.2. We also have the action of a torus $T' := \prod_{j\in J}\mathbb{C}^\times$ on $\prod_{j\in J}\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})$ via dilations. The morphism $T \twoheadrightarrow T'$ given by $t \mapsto \prod_{j\in J}\lambda_j^\vee(t)$ defines an action of T on $\prod_{j\in J}\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})$. The morphism ω is T-equivariant. The following diagram is cartesian:

$$\begin{array}{ccc} {}_0\mathrm{Vin}_G & \xrightarrow{{}_0\omega} & \prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\}) \\ \downarrow & & \downarrow \\ {}_0\mathrm{Vin}_G/T & \xrightarrow{{}_0\omega/T} & (\prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\}))/T. \end{array}$$

Note that ${}_0\mathrm{Vin}_G/T$ is proper, while $(\prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\}))/T$ is separated (as the stack quotient of a smooth scheme $\prod_{j\in J}\mathbb{P}\,\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})$ by a finite group $\mathrm{Ker}(T\to T')$). Hence the morphism ${}_0\omega/T$ is proper, and thus the morphism ${}_0\omega$ is proper. □

Now we are in a position to finish the proof of Proposition 2.16. We apply Lemma A.1 to $Z := \mathrm{Vin}_G$, $Z' := \prod_{j\in J}\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})$, $H := G\times G\times T$, $H' := G\times G\times T'$, ${}_0Z' := \prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\})$. Note that the preimage $\omega^{-1}({}_0Z') = {}_0Z$ coincides with ${}_0\mathrm{Vin}_G$. It follows from Lemma A.3 that the morphism

$${}_0\omega\colon {}_0\mathrm{Vin}_G = {}_0Z \to {}_0Z' = \prod_{1\leqslant i\leqslant r}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\})$$

is finite. So to show that the stack $\overline{\mathrm{Bun}}_G$ is proper over $\mathrm{Bun}_G\times\mathrm{Bun}_G$ it is enough to prove the properness of the stack

$$\mathrm{Maps}_{\mathrm{gen}}\left(X, \prod_{j\in J}\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})/(G\times G\times T') \supset \prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\})/(G\times G\times T')\right)$$

over $\mathrm{Bun}_G\times\mathrm{Bun}_G$. Since $\prod_{j\in J}(\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})\setminus\{0\})/T' \simeq \prod_{j\in J}\mathbb{P}\,\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})$, it follows that the stack in question is isomorphic to the stack of quasi-maps from X to $\prod_{j\in J}\mathbb{P}\,\mathrm{End}(V^{\lambda_j^\vee,\lambda_j^\vee})/(G\times G)$ which is known to be proper over $\mathrm{Bun}_G\times\mathrm{Bun}_G$. Proposition 2.16 is proved.

A.2. *Proof of Proposition* 3.17

The family ${}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x}$ can be identified with the following fiber product:

$$\mathrm{Maps}_{\mathbb{A}^1}(X\times\mathbb{A}^1,\ {}_0\mathrm{Vin}^{\mathrm{princ}}_G/(G\times G)) \underset{\mathrm{Maps}_{\mathbb{A}^1}((X\setminus\{x\})\times\mathbb{A}^1,\ {}_0\mathrm{Vin}^{\mathrm{princ}}_G/(G\times G))}{\times} \mathbb{A}^1.$$

Let us consider the following $\mathbb{C}^\times$-action on the group G:

$$c\mapsto (g\mapsto 2\rho(c)\cdot g\cdot 2\rho(c^{-1})),\ c\in\mathbb{C}^\times,\ g\in G.$$

Let us denote by $\widetilde{G}\to\mathbb{A}^1$ the corresponding Drinfeld–Gaitsgory interpolation. Note that $\widetilde{G}$ is a group scheme over $\mathbb{A}^1$. It follows from [**11**, § 2.4, D.6] that the stack ${}_0\mathrm{Vin}^{\mathrm{princ}}_G/(G\times G)$ over $\mathbb{A}^1$ is isomorphic to the stack $\mathbb{A}^1/\widetilde{G}$ over $\mathbb{A}^1$. Thus the family ${}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x}$ is identified with

$$\mathrm{Maps}_{\mathbb{A}^1}(X\times\mathbb{A}^1,\mathbb{A}^1/\widetilde{G}) \underset{\mathrm{Maps}_{\mathbb{A}^1}((X\setminus\{x\})\times\mathbb{A}^1,\mathbb{A}^1/\widetilde{G})}{\times} \mathbb{A}^1.$$

Let us denote this family by $\mathrm{Gr}_{\widetilde{G}}$. Note that $\mathrm{Gr}_{\widetilde{G}}$ is the affine Grassmannian for the group-scheme $\widetilde{G}$ over $\mathbb{A}^1$.

Let us construct the sought-for morphism of ind-schemes over $\mathbb{A}^1$:

$$\eta : \mathrm{Gr}_{\widetilde{G}} \to \widetilde{\mathrm{Gr}}_G. \tag{A.4}$$

We first construct a morphism of (pre)stacks over $\mathbb{A}^1$:

$$\alpha : \mathbb{A}^1/\widetilde{G} \to \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{X}, \mathbb{A}^1/G)^{\mathbb{C}^\times}. \tag{A.5}$$

We start with the evaluation morphism of group-schemes over $\mathbb{A}^1$

$$\mathbb{X} \underset{\mathbb{A}^1}{\times} \widetilde{G} \to \mathbb{A}^1 \times G,$$

which gives rise to a map of stacks over $\mathbb{A}^1$:

$$\eth \colon \mathbb{X} \underset{\mathbb{A}^1}{\times} (\mathbb{A}^1/\widetilde{G}) \to \mathbb{A}^1 \times (\mathrm{pt}/G).$$

Note that the above morphism $\eth$ is $\mathbb{C}^\times$-equivariant with respect to the $\mathbb{C}^\times$-action on $\mathbb{X} \underset{\mathbb{A}^1}{\times} (\mathbb{A}^1/\widetilde{G})$ via the action of $\mathbb{C}^\times$ on $\mathbb{X}$ and trivial action on $\mathbb{A}^1 \times (\mathrm{pt}/G)$. Hence, it defines a point of

$$\mathrm{Maps}_{\mathbb{A}^1}\left(\mathbb{X} \underset{\mathbb{A}^1}{\times} (\mathbb{A}^1/\widetilde{G}), \mathbb{A}^1/G\right)^{\mathbb{C}^\times} \simeq \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{A}^1/\widetilde{G}, \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{X}, \mathbb{A}^1/G)^{\mathbb{C}^\times}),$$

that is, we obtain a map of (pre)stacks over $\mathbb{A}^1$:

$$\mathbb{A}^1/\widetilde{G} \to \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{X}, \mathbb{A}^1/G)^{\mathbb{C}^\times},$$

which is the desired α.

From α we obtain the morphism

$$\begin{aligned}
\mathrm{Gr}_{\widetilde{G}} &= \mathrm{Maps}_{\mathbb{A}^1}(X \times \mathbb{A}^1, \mathbb{A}^1/\widetilde{G}) \underset{\mathrm{Maps}_{\mathbb{A}^1}\left((X\setminus\{x\})\times\mathbb{A}^1, \mathbb{A}^1/\widetilde{G}\right)}{\times} \mathbb{A}^1 \\
&\xrightarrow{\alpha} \mathrm{Maps}_{\mathbb{A}^1}(X \times \mathbb{A}^1, \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{X}, \mathbb{A}^1/G)^{\mathbb{C}^\times}) \underset{\mathrm{Maps}_{\mathbb{A}^1}\left((X\setminus\{x\})\times\mathbb{A}^1, \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{X},\mathbb{A}^1/G)^{\mathbb{C}^\times}\right)}{\times} \mathbb{A}^1 \\
&\simeq \mathrm{Maps}_{\mathbb{A}^1}(\mathbb{X}, \mathrm{Maps}_{\mathbb{A}^1}(X \times \mathbb{A}^1, \mathbb{A}^1/G))^{\mathbb{C}^\times} \underset{\mathrm{Maps}_{\mathbb{A}^1}\left(\mathbb{X}, \mathrm{Maps}_{\mathbb{A}^1}((X\setminus\{x\})\times\mathbb{A}^1, \mathbb{A}^1/G)\right)^{\mathbb{C}^\times}}{\times} \mathbb{A}^1 = \widetilde{\mathrm{Gr}}_G.
\end{aligned}$$

This is the desired morphism η in (A.4). The diagram (3.18) is commutative by construction.

Note that the composition

$$\mathrm{Gr}_{\widetilde{G}} \xrightarrow{\eta} \widetilde{\mathrm{Gr}}_G \xrightarrow{\gamma} \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$$

coincides with the restriction of the morphism $\vartheta\colon \mathrm{VinGr}^{\mathrm{princ}}_{G,x} \to \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$ to the open ind-subscheme ${}_0\mathrm{VinGr}^{\mathrm{princ}}_{G,x} \simeq \mathrm{Gr}_{\widetilde{G}}$. Note that the morphism $\gamma \circ \eta\colon \mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 = \mathrm{Gr}_{G\times G\times\mathbb{A}^1}$ also coincides with the morphism induced by the closed embedding $\widetilde{G} \hookrightarrow G \times G \times \mathbb{A}^1$ of group schemes over $\mathbb{A}^1$.

It follows from Lemma 3.7 that the morphism $\gamma \circ \eta$ is a locally closed embedding. The morphism γ is a locally closed embedding by [**10**, § 2.5.11]. So the morphism η is a locally closed embedding. It follows from Remark 3.14, § 3.1.2, § 3.2.5 and § 3.2.4 that the morphism η is bijective on the level of $\mathbb{C}$-points. Thus η is an isomorphism between the corresponding reduced ind-schemes.

To show that η is an isomorphism of ind-schemes let us note that the ind-scheme $\mathrm{Gr}_{\widetilde{G}}$ is formally smooth over $\mathbb{A}^1$ and the morphism η induces the isomorphism between scheme fibers of the families $\mathrm{Gr}_{\widetilde{G}}$ and $\widetilde{\mathrm{Gr}}_G$ over $\mathbb{A}^1$.

Proposition 3.17 is proved.

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References

1. P. Baumann, J. Kamnitzer and A. Knutson, 'The Mirković-Vilonen basis and Duistermaat-Heckman measures', Preprint, 2019, arXiv:1905.08460. (With appendix by A. Dranowski, J. Kamnitzer, and C. Morton-Ferguson.)

2. P. Baumann and S. Riche, 'Notes on the geometric Satake equivalence', *Relative aspects in representation theory, Langlands functoriality and automorphic forms*, Lecture Notes in Mathematics 2221 (eds V. Heiermann and D. Prasad; Springer, Cham, 2018) 1–134.

3. A. Beilinson and V. Drinfeld, *Quantization of Hitchin's Hamiltonians and Hecke eigensheaves*, Preprint, 2000, http://math.uchicago.edu/∼mitya/langlands/hitchin/BD-hitchin.

4. T. Braden, 'Hyperbolic localization of intersection cohomology', *Transform. Groups* 8 (2003) 209–216.

5. A. Braverman, M. Finkelberg and D. Gaitsgory, 'Uhlenbeck spaces via affine Lie algebras', *Progr. Math.* 244 (2006) 17–135. (See arXiv:0301176 for erratum.)

6. A. Braverman, M. Finkelberg, D. Gaitsgory and I. Mirković, 'Intersection cohomology of Drinfeld's compactifications', *Selecta Math.* 8 (2002) 381–418; erratum, *Selecta Math.* 10 (2004) 429–430.

7. A. Braverman, M. Finkelberg and H. Nakajima, 'Coulomb branches of $3d\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian', *Adv. Theor. Math. Phys.* 23 (2019) 75–166. (With appendices by Alexander Braverman, Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes.)

8. A. Braverman and D. Gaitsgory, 'Geometric Eisenstein series', *Invent. Math.* 150 (2002) 287–384.

9. M. Brion, 'Group completions via Hilbert schemes', *J. Algebraic Geom.* 12 (2003) 605–626.

10. V. Drinfeld and D. Gaitsgory, 'On a theorem of Braden', *Transform. Groups* 19 (2014) 313–358.

11. V. Drinfeld and D. Gaitsgory, 'Geometric constant term functor(s)', *Selecta Math.* (*N.S.*) 22 (2016) 1881–1951.

12. B. Feigin, M. Finkelberg, A. Kuznetsov and I. Mirković, 'Semi-infinite flags. *II*. Local and global intersection cohomology of quasimaps' spaces', *Amer. Math. Soc. Transl. Ser.* (2) 194 (1999) 113–148.

13. M. Finkelberg and V. Ginzburg, 'Calogero-Moser space and Kostka polynomials', *Adv. Math.* 172 (2002) 137–150.

14. D. Gaitsgory, 'Twisted Whittaker model and factorizable sheaves', *Selecta Math.* (*N.S.*) 13 (2008) 617–659.

15. D. Gaitsgory and D. Nadler, 'Spherical varieties and Langlands duality', *Mosc. Math. J.* 10 (2010) 65–137.

16. G. Lusztig, 'Semicanonical bases arising from enveloping algebras', *Adv. Math.* 151 (2000) 129–139.

17. J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics 170 (Cambridge University Press, Cambridge, 2017) xvi+644pp.

18. I. Mirković and K. Vilonen, 'Geometric Langlands duality and representations of algebraic groups over commutative rings', *Ann. of Math.* (2) 166 (2007) 95–143; erratum, *Ann. of Math.* (2) 188 (2018) 1017–1018.

19. S. Schieder, 'The Harder-Narasimhan stratification of the moduli stack of G-bundles via Drinfeld's compactifications', *Selecta Math.* (*N.S.*) 21 (2015) 763–831.

20. S. Schieder, 'Picard-Lefschetz oscillators for the Drinfeld-Lafforgue-Vinberg degeneration for SL_2', *Duke Math. J.* 167 (2018) 835–921.

21. S. Schieder, 'Geometric Bernstein asymptotics and the Drinfeld-Lafforgue-Vinberg degeneration for arbitrary reductive groups', Preprint, 2016, arXiv:1607.00586.

22. S. Schieder, 'Monodromy and Vinberg fusion for the principal degeneration of the space of G-bundles', Preprint, 2017, arXiv:1701.01898.

23. R. Steinberg, *Lectures on Chevalley groups*, University Lecture Series 66 (American Mathematical Society, Providence, RI, 2016) xi+160pp.

24. E. B. Vinberg, 'On reductive algebraic semigroups', *Amer. Math. Soc. Transl. Ser.* (2) 169 (1995) 145–182.

25. E. B. Vinberg, The asymptotic semigroup of a semisimple Lie group, *Semigroups in algebra, geometry and analysis*, De Gruyter Expositions in Mathematics 20 (eds K. H. Hofmann, J. D. Lawson and Ernest B. Vinberg; de Gruyter, Berlin, 1995) 293–310.

26. G. Wilson, 'Collisions of Calogero-Moser particles and an adelic Grassmannian', *Invent. Math.* 133 (1998) 1–41.

27. X. Zhu, 'An introduction to affine Grassmannians and the geometric Satake equivalence', *Geometry of moduli spaces and representation theory*, IAS/Park City Mathematics Series 24 (American Mathematical Society, Providence, RI, 2017) 59–154.

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