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Criteria on existence of horseshoes near homoclinic tangencies of arbitrary orders

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ABSTRACT
Consider $(m + 1)$-dimensional, $m \geq 1$, diffeomorphisms that have a saddle fixed point O with m-dimensional stable manifold $W^s(O)$, one-dimensional unstable manifold $W^u(O)$, and with the saddle value σ different from 1. We assume that $W^s(O)$ and $W^u(O)$ are tangent at the points of some homoclinic orbit and we let the order of tangency be arbitrary. In the case when $\sigma < 1$, we prove necessary and sufficient conditions of existence of topological horseshoes near homoclinic tangencies. In the case when $\sigma > 1$, we also obtain the criterion of existence of horseshoes under the additional assumption that the homoclinic tangency is simple.

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1. Introduction

Homoclinic orbit, or *Poincaré homoclinic orbit*, is an orbit that is bi-asymptotic to a saddle periodic trajectory. By its definition, any homoclinic orbit belongs to the intersection $W^s \cap W^u$ of the stable and unstable manifolds of the corresponding periodic orbit. Depending on whether this intersection is transverse or non-transverse, the homoclinic orbit is called *transverse* or *non-transverse*, respectively. The latter case is called also the *homoclinic tangency*.

In fact, the possibility of existence of homoclinic orbits in multidimensional dynamical systems was discovered by H.Poincaré as early as in the end of the nineteenth century. Now, more than a century after Poincaré, the presence of transverse Poincaré homoclinic orbits is regarded as the universal criterion for chaos, i.e. for presence of complicated dynamics. Actually, the set Λ of nonwandering orbits from a small neighbourhood U of a transverse homoclinic orbit has nontrivial structure: in particular, it contains countably many periodic and homoclinic orbits, continuum Poisson stable orbits, etc. The collection of such results can be referred to as the Poincaré–Birkhoff–Smale–Shilnikov theory. One of the main results of this theory states that the set Λ is *uniformly hyperbolic* and admits a *complete description* in terms of symbolic dynamics [30,32], and moreover, it is locally maximal [30], i.e. the largest closed invariant set in U.

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The problem of studying behaviour of trajectories near a nontransversal homoclinic orbit becomes much more complicated and, moreover, it cannot be solved as complete description, especially when nearby systems are considered. The point here is that arbitrarily small smooth perturbations of any system with homoclinic tangency can lead to the appearance of homoclinic and periodic orbits of any order of degeneracy, see e.g. [15,16]. Therefore, a special attention should be paid to those problems which are related to the study of characteristic features of homoclinic structures and their principal bifurcations.

A problem of such type was studied first in the famous paper by N.K. Gavrilov and L.P. Shilnikov [1]. In that paper, three-dimensional (3D) flows (and thus, two-dimensional (2D) diffeomorphisms) with quadratic homoclinic tangencies were considered and many important homoclinic phenomena were discovered. In particular, all quadratic tangencies with $\sigma < 1$ were partitioned into two classes with respect to the type of the structure of the set Λ of trajectories from a small neighbourhood $U = U(O \cup \Gamma_0)$ of the orbit Γ_0 homoclinic to the saddle O: namely, the first class corresponds to the trivial structure of Λ (in which case $\Lambda = O \cup \Gamma_0$), and for the second class, there are nontrivial hyperbolic subsets in Λ. In [1], it has been developed an effective technique based on geometric and analytical conditions for the existence of such subsets. This technique also allows one to describe these subsets in terms of symbolic dynamics. In [4,13], these methods were generalized for multidimensional systems with homoclinic tangencies of arbitrary finite orders in the sectionally dissipative case (i.e. with $\sigma < 1$). In particular, in [13], the classification results for homoclinic tangencies (with respect to the structure of the set Λ) were presented, and in [4], nontrivial non-uniformly hyperbolic subsets were proved to exist in Λ and their symbolic dynamics was described.[1] As for topological horseshoes, certain conditions on their existence near arbitrary homoclinic tangencies were established in [21,22] for the case of 2D dissipative diffeomorphisms.

In this paper, we examine multidimensional diffeomorphisms with homoclinic tangencies to a saddle fixed point whose unstable invariant manifold is *one-dimensional* (1D). We consider the problem on the structure of the set Λ of orbits lying entirely in a sufficiently small neighbourhood of a *non-transverse* homoclinic orbit.

It is worth saying that this problem, unlike the one in the case of transverse homoclinic orbit, has several fundamental peculiarities as follows. *First:* the problem does not allow a simple univalent answer (in contrast to the case of transverse homoclinic orbit); actually, we can distinguish and classify two totally different situations: either the set Λ has *trivial structure*, or Λ contains *infinitely many horseshoes*. *Second:* if one proposes additional conditions in topological terms (e.g. assuming the homoclinic tangency to be isolated or one-sided, see Definition 2.2 from Section 2), then one may obtain classification results only in the cases when $\sigma < 1$ (see Theorem 2.3 and Corollary 2.4), or when $\sigma > 1$ for 2D diffeomorphisms (see Corollary 2.5). As the example from Section 2.5 shows, in multidimensional cases with $\sigma > 1$, some additional assumptions are required. Therefore, we suppose that the homoclinic tangencies under consideration are *simple* by terminology of [18,19], see Section 2.2. This additional assumption allows us to obtain some important classification results for the case $\sigma > 1$ (see Theorems 2.12, 2.13).

Throughout this paper, we assume the following. Let f be an $(m+1)$-dimensional C^r-diffeomorphism, $r \geq 2$, with a hyperbolic fixed point O which has multipliers $\lambda_1, \dots, \lambda_m, \gamma$ such that

$$0 < |\lambda_m| \leq \cdots \leq |\lambda_2| \leq |\lambda_1| < 1 < |\gamma| \tag{1}$$

and

$$\sigma \equiv |\lambda_1||\gamma| \neq 1,$$

σ here is called the *saddle value*. We assume also that f has a homoclinic orbit Γ_0 to O at which the m-dimensional stable manifold, W^s, and 1D unstable manifold, W^u, of O are tangent, and this tangency can be arbitrary (in particular, we let the tangency be of any finite or even infinite order, see statements of Theorems 2.3, 2.12 and 2.13).

Let U be a small neighbourhood of the contour $O \cup \Gamma_0$ and Λ be the set of f-orbits lying entirely in U. The main problem is to study the structure of the set Λ.

We consider the following two different general cases:[2]

(1) the *sectionally dissipative case*, where $\sigma < 1$, and
(2) the *sectionally saddle case*, where $\sigma > 1$.

The sectionally dissipative case is a very popular topic in chaotic dynamics, and many relevant results have been obtained for this case. Among numerous papers concerning it, we would like to mention especially the pioneering papers by Gavrilov and Shilnikov [1] and Newhouse [25] which have caused actually a big interest in the study of the corresponding problems. One of the most important results here is that bifurcations of such systems can lead to appearance of periodic attractors, see [1,3,19,25,28]. Moreover, the famous *Newhouse phenomenon* [25] takes place here: namely, there exist open regions (called Newhouse regions) in which systems with infinitely many periodic attractors are dense.

Note that the Newhouse phenomenon can be observed not only in the sectionally dissipative case. In general, the Newhouse regions (i.e. open regions in the space of C^r-dynamical systems, $r \geq 2$, in which systems with homoclinic tangencies are dense) exist in any neighbourhood of any system with homoclinic tangency, see [18,26,28,29]. In particular, in the sectionally saddle case, the Newhouse phenomenon is sometimes related to existence of infinitely many stable invariant curves [17] and even of infinitely many strange attractors, e.g. discrete Lorenz-like attractors [6,10–12]. Actually, the condition $\sigma < 1$ means that the so called effective dimension d_e of the system[3] equals 1 [34] (in this case, all k-dimensional volumes with $k \geq 2$ are contracted exponentially under iterations in a neighbourhood of the homoclinic trajectory), and therefore, only 1D dynamical behaviour is expected here. In contrast, when $\sigma > 1$, the effective dimension may be arbitrary [34].

The content of the paper is as follows. In Section 2, we define and discuss principal notions and state main results: Theorems 2.3–2.13. In Section 2.1, we discuss the problem on existence of topological horseshoes in the sectionally dissipative case and state the main results with respect to this topic (Theorem 2.3 and Corollaries 2.4 and 2.5). In Section 2.2, we introduce the notion of simple homoclinic tangency and of simple homoclinic tangency of order n. These definitions can be regarded as generalizations of the notion of simple quadratic (with $n = 1$) homoclinic tangency introduced in [19]. We also give necessary technical details as preliminaries for proofs of main results: namely, properties of local and global maps are considered in Section 2.3. In Section 2.4, we discuss the problem on existence of topological horseshoes near simple homoclinic tangencies by paying a special attention to the sectionally saddle case (see Theorems 2.12 and 2.13). In Section 3, we prove Theorems 2.3–2.13.

2. Definitions and statements of main results

Let us recall first some facts and results concerning homoclinic tangencies.

Definition 2.1: We say that

(i) f possesses trivial dynamics near the homoclinic orbit $\Gamma_0 \subset \{\Gamma\}$ if $\Lambda = O \cup \{\Gamma\}$, where $\{\Gamma\}$ is the set of homoclinic to O orbits which are close to Γ_0;

(ii) f has a topological horseshoe if Λ contains an f-invariant subset $\widetilde{\Lambda}$ such that $f|_{\widetilde{\Lambda}}$ is topologically semi-conjugate to a subshift of finite type with positive topological entropy;

(iii) f has a hyperbolic horseshoe if $\widetilde{\Lambda}$ from (ii) contains a uniformly hyperbolic nontrivial closed invariant set (here 'nontrivial' means that the restriction of f on this set has positive topological entropy).

Let M^+ and M^- be a pair of points of Γ_0 such that $M^+ \in W^s_{loc} \cap U_0$, $M^- \in W^u_{loc} \cap U_0$. Let Π^+ and Π^- be sufficiently small neighbourhoods of the points M^+ and M^-, respectively, and let $M^+ = f^q(M^-)$ for some positive integer q. Denote the map $f\big|_{U_0}$ by T_0 and the map $f^q\big|_{\Pi^-}$ by T_1 (thus, $T_1(M^-) = M^+$). The map T_0 is called the *local map* because it is defined in a small neighbourhood of O; while the map T_1 is called the *global map* because it acts along the global piece of the orbit Γ_0.

The following definition is related to topological classification of (homoclinic) tangencies.

Definition 2.2: The homoclinic tangency is isolated if, for some neighbourhoods Π^+ and Π^-, the point M^+ is a unique intersection point of the curve $l_u = T_1(W^u_{loc} \cap \Pi^-) \cap \Pi^+$ with W^s_{loc}. We say that the **isolated** homoclinic tangency is one-sided if W^s_{loc} divides Π^+ into two half-parts and the curve l_u belongs as a whole to the closure of exactly one half of Π^+; otherwise, we will call the isolated tangency topologically crossing. We say that a one-sided tangency is from below if the point M^- is not an accumulation point of the curves $T_0^i(l_u)$, $i = 0, 1, \ldots$, i.e.,

$$M^- \notin \overline{\bigcup_{i\geq 0} \{T_0^i(l_u)\}}; \tag{2}$$

and is from above otherwise, i.e. if

$$M^- \in \overline{\bigcup_{i\geq 0} \{T_0^i(l_u)\}}; \tag{3}$$

(see Figure 1 for illustration).

2.1. Topological horseshoes in the sectionally dissipative case

In this section, we observe topological properties of orbits lying entirely in a small neighbourhood of the homoclinic orbit Γ_0 in the case when $\sigma < 1$.

Note first that in the sectionally dissipative case ($\sigma < 1$), condition (2) can be regarded as a certain criterion of trivial dynamics, and the following result confirms this fact.

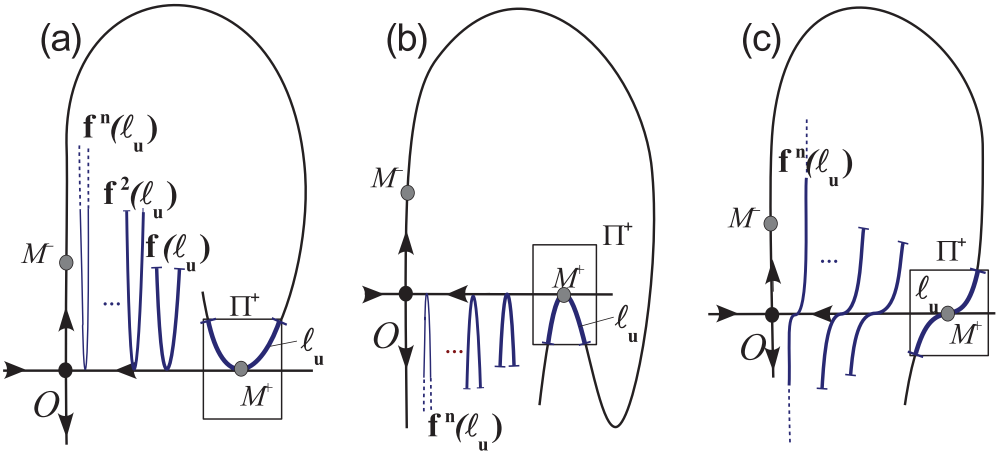

Figure 1. Examples of one-sided homoclinic tangencies: (a) from above, (b) from below, and (c) topologically crossing.

Theorem 2.3 ([On topological horseshoes in the sectionally dissipative case])**:** *Let a diffeomorphism f have a homoclinic tangency and* $\sigma < 1$. *Then the following holds*

(1) *if the tangency satisfies condition* (2) (*in particular, when the tangency is from below*), *then f possesses the trivial dynamics near* Γ_0;
(2) *otherwise, if condition* (3) *holds* (*in particular, when the tangency is from above*), *then f has infinitely many topological horseshoes near* Γ_0.

Note that Theorem 2.3 can be regarded as a criterion of existence of topological horseshoes in a small neighbourhood of a homoclinic tangency for the case $\sigma < 1$. In particular, it implies that if $\gamma < 0$, then horseshoes may not exist only when a part of the manifold $W^u(O)$ lies in $W^s_{\text{loc}}(O)$.

Also note that sometimes without additional assumptions for f in the case $\sigma < 1$, we can say even more than Theorem 2.3 states. For example, the geometry of the problem is such that infinitely many first return maps $T_i = T_1 T_0^i : \Pi^+ \to \Pi^- \to \Pi^+, i = \bar{k}, \bar{k}+1$, are defined here. Indeed, the domain of the map $T_0^i : \Pi^+ \to \Pi^-$ is some strip $\sigma_i^0 \subset \Pi^+$ and $T_0^i(\sigma_i^0) = \sigma_i^1 \subset \Pi^-$ (see Figure 5 and Section 2.3). In turn, the strip σ_i^1 is transformed under T_1 into horseshoe-shaped figure $T_1\sigma_i^1$, see Figure 2.

If the tangency is one-sided from above, see Figure 2(a), each of these first return maps is, in fact, a (topological) horseshoe map; if the tangency is a topologically crossing, then every pair of the first return maps may be associated with an appropriate horseshoe map, see Figure 2(b). Though Theorem 2.3 says nothing about hyperbolicity, nevertheless, one can apply some indirect facts, namely Katok Theorem, in order to deduce the following.

Corollary 2.4: *In the case* 2 *of Theorem* 2.3 *the set* Λ *contains infinitely many hyperbolic horseshoes.*

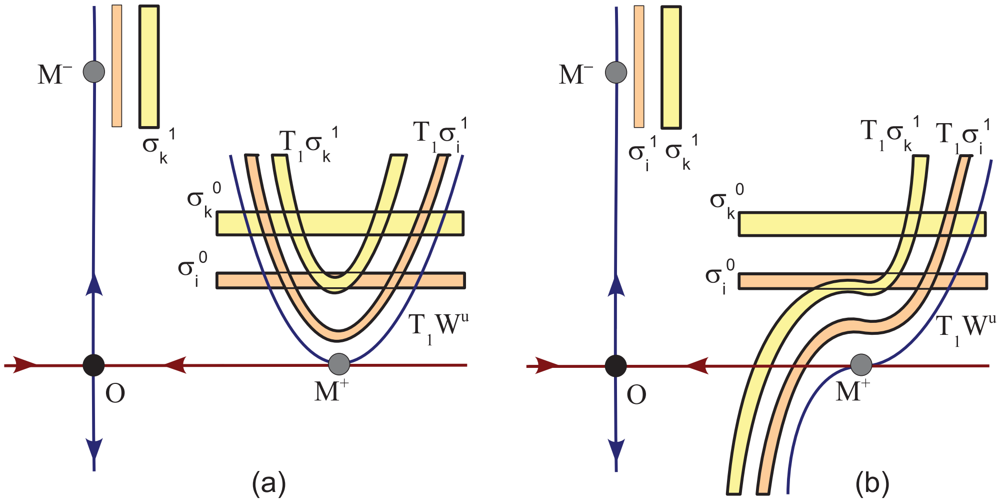

Figure 2. Geometrical horseshoes in the cases of (a) one-sided tangency; (b) topologically crossing tangency.

Proof: In the paper [23] by Katok it was proved (see Corollary 4.3 from [24]) that any $C^{1+\alpha}$ diffeomorphism on 2D manifold with positive topological entropy has a uniformly hyperbolic nontrivial invariant set such that the restriction of the diffeomorphism to this set is topologically conjugate to a topological Markov chain (a horseshoe). We now may apply this result to the sectionally dissipative case, i.e. when $\sigma < 1$. Indeed, by Theorem 2.3, we get that the restriction of f to each topological horseshoe has positive topological entropy. The latter means that there are f-orbits having the first Lyapunov exponent $\mathcal{L}_1$ positive. Now the sectional dissipativity also implies that the first return maps near Γ_0 contract exponentionally all 2D volumes. Therefore, all remaining Lyapunov exponents must be negative. Remark that in the paper [23] by Katok, the proof of Corollary 4.3 for 2D diffeomorphisms is based just on multidimensional result (Corollary 4.1) on existence of topological horseshoes provided that the positive and negative Lyapunov exponents exist and there are no zero Lyapunov exponents. As we have shown, in the case under consideration, zero Lyapunov exponents are absent due to the sectional dissipativity.[4] □

Thus, in the sectionally dissipative case, relation (2) provides, in fact, the criterion for trivial dynamics near homoclinic tangency. In contrast, in the case $\sigma > 1$ one has totally different horseshoe geometry, even in dimension two, see Figure 3 for illustration. However, Theorem 2.3 can be evidently applied to the case $\sigma > 1$ for 2D diffeomorphisms, since we can always consider f^{-1} instead of f. Then, the following condition

$$M^+ \notin \overline{\bigcup_{i\geq 0} \{T_0^{-i}(l_s)\}}, \tag{4}$$

where l_s is the curve on Π^- defined as $l_s = T_1^{-1}(W^s_{loc} \cap \Pi^+) \cap \Pi^-$, must be considered instead (2). Thus, we have the following

Corollary 2.5 ([On topological horseshoes in dimension 2 and $\sigma > 1$])**:** *Let f be t2D and $\sigma > 1$. Then the following holds*

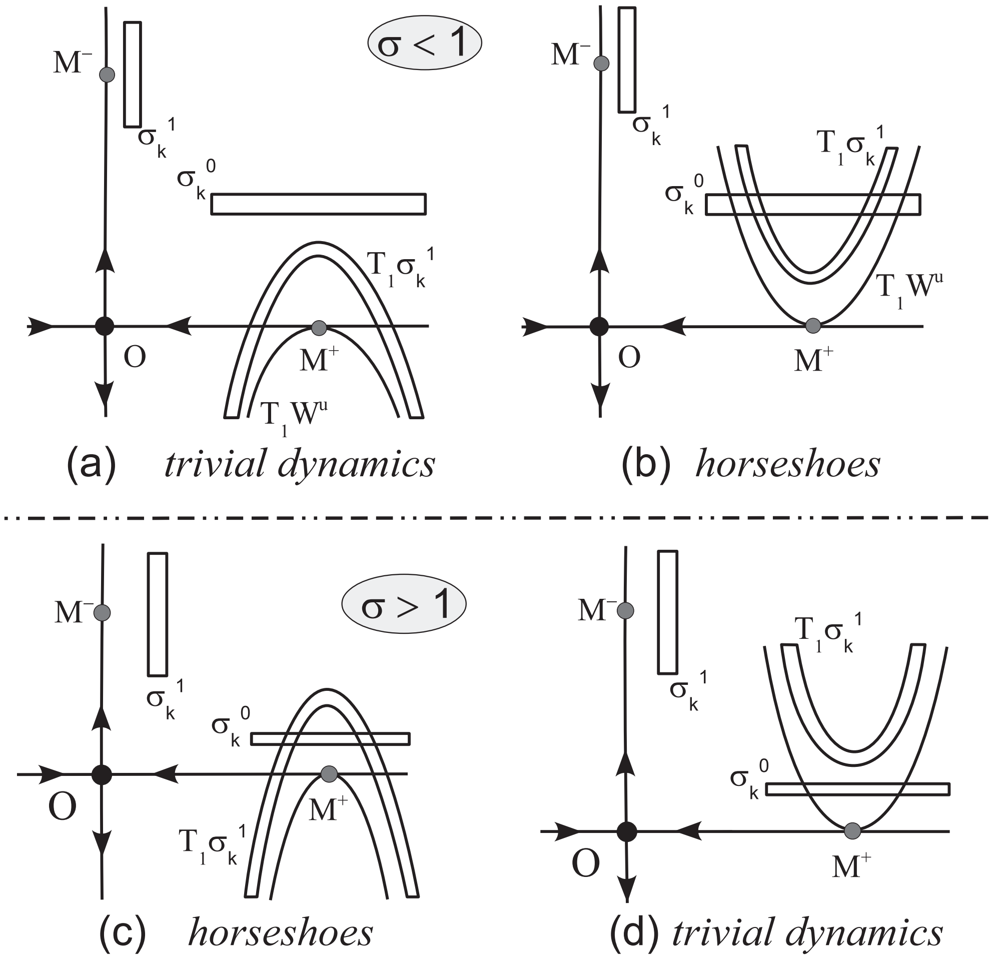

Figure 3. Geometry of horseshoes in the case $\sigma < 1$ (a),(b) and in the case $\sigma > 1$ (c),(d).

(1) *if the tangency satisfies condition* (4)*, then f possesses the trivial dynamics near* Γ_0*;*
(2) *otherwise, if condition* (4) *is not satisfied, f has infinitely many topological horseshoes near* Γ_0*; moreover, the set* Λ *contains infinitely many hyperbolic horseshoes.*

Unfortunately, this approach with changing f to f^{-1} is not suitable in higher dimensions since the unstable manifold of the point O for f^{-1} will have dimension greater than 1, i.e. such diffeomorphisms do not belong to the class of systems under consideration. Therefore, multidimensional case with $\sigma > 1$ requires another technique and certain specification of the problem, see next subsection.

2.2. Definition of a simple homoclinic tangency

In this section (as in [17,19]), we assume that f satisfies some additional general conditions.

First, we make some assumptions on the multipliers of O. Let these multipliers $\lambda_1, \dots,$ λ_m, γ be ordered by the rule (1). We call *leading* (or weak) those multipliers of O that are equal to $|\lambda_1|$ in absolute value. Accordingly, other stable multipliers (less than $|\lambda_1|$ in modulus) are called *nonleading* (or strong stable). We consider the following generic assumption:

(A) the leading stable multipliers of O are simple.

Then two different types of saddle fixed (periodic) points are defined. Namely,

(A1) the point O is a *saddle*, i.e. the multiplier λ_1 is real and $|\lambda_1| > |\lambda_j|$ for $j = 2, \dots, m$;

(A2) the point O is a *saddle-focus*, i.e. the multipliers λ_1 and λ_2 are complex conjugate, $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = \lambda e^{-i\varphi}$, $0 < \lambda < 1$, $0 < \varphi < \pi$ and $\lambda > |\lambda_j|$ for $j = 3, \dots, m$.

When the point O has non-leading stable multipliers, we need more assumptions. Recall the following facts.

First, W^s_{loc} contains the invariant C^r-smooth *strong stable manifold* $W^{ss}_{\text{loc}} \subset W^s_{\text{loc}}$ which touches at O the eigenspace of Df corresponding to the non-leading multipliers λ_i. Hence, $\dim\ W^{ss}_{\text{loc}} = m - 1$ if A1 holds, and $\dim\ W^{ss}_{\text{loc}} = m - 2$ if A2 holds. Also, it is well known (see e.g. [20,32]) that W^s_{loc} admits invariant C^r-smooth *strong stable foliation* F^{ss} containing W^{ss}_{loc} as a leaf.

Another fact we need (see, for example, [20,27,32]) is that the manifold $W^u(O)$ is a part of the so-called *extended unstable manifold* $W^{ue}(O)$. It is a smooth (at least $C^{1+\epsilon}$) invariant manifold which is tangent at O to the eigenspace of Df corresponding to the leading stable multipliers, thus, W^{ue} is two- or 3D depending on whether O is a saddle or saddle-focus, respectively. Though manifold W^{ue} is not uniquely defined, any such a manifold contains W^u_{loc}, and moreover, any two such manifolds are tangent to each other at points of W^u_{loc}. Thus, at the homoclinic point $M^- \in W^u_{\text{loc}}$, the tangent space to W^{ue}, denoted by $\mathcal{T}_{M^-}W^{ue}$, is uniquely defined. Since $M^+ = T_1(M^-)$, we can extend W^{ue} up to the homoclinic point M^+. Denote the tangent space to W^{ue} at M^+ by $\mathcal{T}_{M^+}W^{ue}$. Evidently, $\mathcal{T}_{M^+}W^{ue} = DT_1\,(\mathcal{T}_{M^-}W^{ue}))$, where DT_1 denotes the linear part of the global map $T_1 \equiv f^q\colon \Pi^- \to \Pi^+$ at the point M^-.

We also consider the following general conditions (the same as in [18,19]):

(B) $M^+ \notin W^{ss}_{\text{loc}}$ and

(C) $\mathcal{T}_{M^-}W^{ue}$ is transverse to $F^{ss}(M^+)$ at M^+, where $F^{ss}(M^+)$ is the leaf of the foliation F^{ss} containing the point M^+.

Definition 2.6: A homoclinic tangency satisfying conditions A–C is called simple.

The notion of simple quadratic homoclinic tangency was introduced in [18,19] as a 'homoclinic version' of quasi-transversal intersection [27]. Thus, Definition 2.6 is an extension of this notion to tangencies of any orders when $\dim W^u(O) = 1$.

See Figure 4(a) for illustration of conditions A(1), B, and C in the case when the point O is a saddle with 2D stable manifold. In Figure 4(b) and (c), two main cases of non-simple homoclinic tangencies are shown when condition C is violated. The condition of simple homoclinic tangency in the coordinate form are shown in Section 2.3, see formulas (11) and (12).

Thus, we generalize the notion of simple quadratic homoclinic tangency introduced in [19]. We can adapt also this definition to arbitrary homoclinic tangencies and, in particular, to homoclinic tangencies of finite order.

Definition 2.7: Let f be a C^r-diffeomorphism under consideration and n be an integer with $1 \le n < r$. We say that the homoclinic tangency at M^+ is of order n if there exist local (near M^+) C^r-coordinates $(x_1, \dots, x_m, y)$ such that the corresponding local piece of W^s is written as $y = 0$ and the equation for the piece of W^u near M^+ can be written (in the parameter

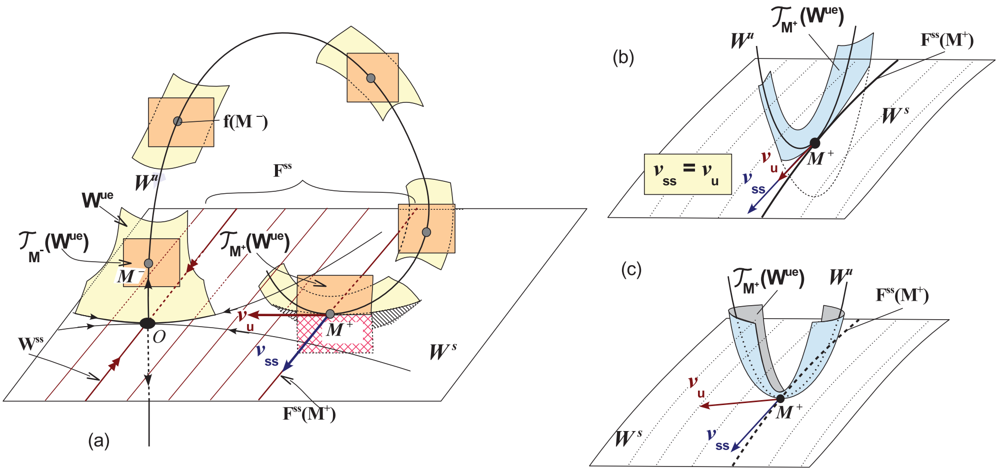

Figure 4. (a) An illustration to definition of simple homoclinic tangency for a 3D diffeomorphism (satisfying conditions A–C). (b),(c) Two main cases of non-simple homoclinic tangencies when condition C is violated: (b) $\mathcal{T}_{M^-}W^{ue}$ is transverse to $W^s_{loc}(O)$ and touches $F^{ss}(M^+)$ and (b) $\mathcal{T}_{M^-}W^{ue}$ touches $W^s_{loc}(O)$. Here v_u is a tangent vector to $W^u(O)$ at the point M^+ and v_{ss} is a tangent vector to $F^{ss}(M^+)$ at M^+.

form) as follows

$$x_i = b_i t + O(t^2),\ \ (i = 1, \dots, m),\ \ y = g(t),$$

where $g(t)$ is C^r and

$$g(0) = \frac{dg(0)}{dt} = \dots = \frac{d^n g(0)}{dt^n} = 0,\ \frac{d^{n+1}g(0)}{dt^{n+1}} = (n+1)!\, d \neq 0,$$

where t is a parameter varying near zero, b_i and d are some constants and $\Sigma|b_i| \neq 0$.

If all derivatives $d^i g(0)/dt^i$ vanish for $i = 0, \dots, r$, we say that the tangency is of indefinite order, and for the case $r = \infty$ this tangency is called flat.

By definition, tangencies of odd orders are one-sided, while, tangencies of even orders correspond to topologically crossing tangencies. Tangencies of small orders have special names: *quadratic* for $n = 1$, *cubic* for $n = 2$, and *quadric* for $n = 3$. Note that the order of tangency may depend on the change of coordinates: for instance, quadratic tangency can be transformed into a tangency of indefinite order by C^1-change. Nevertheless, some properties are invariant under coordinate change: the tangency remains isolated, one-sided, topologically crossing, etc.

Note that the condition on finite order of homoclinic tangency is needed in order to establish more detailed properties of the system than those which give us Theorems 2.3 and 2.13.[5]

Note that conditions B and C are independent of the choice of homoclinic points M^+ and M^- because of invariance of the involved manifolds and foliations. Moreover, conditions A–C make very important dynamical sense. Namely, if these conditions are satisfied then (see e.g. [17])

- *f has a global smooth invariant center manifold W^c which contains the orbits O and Γ_0 as well as all orbits lying entirely in U.*

According to [21], this manifold is normally hyperbolic, since f is exponentially contracting along directions transverse to W^c, which correspond, at O, to the u-directions. Therefore, either $\dim W^c = 2$ (if O is a saddle and so, A1 holds) or $\dim W^c = 3$ (if O is a saddle-focus and so, A2 holds). Thus, the problem under consideration allows *dimension reduction* to dimension two or three depending on whether O is a saddle or a saddle-focus, respectively.

It is worth noting that, in general, one can guarantee only $C^{1+\varepsilon}$-smoothness for W^c (more precisely, its smoothness does not exceed the integer part of $\ln|\lambda_i|(\ln|\lambda_1|)^{-1}$, where $i = 2$ in case A1, and $i = 3$ in case A2). Thus, condition D on finite order of homoclinic tangency makes no sense when we consider the restricted system on W^c (formally speaking, the initial tangency becomes here indefinite). Besides, it is hard to check condition D unlike conditions (A)–(C), which involve only first derivatives. Instead, in our approach in the present paper,[6] we will assume that the homoclinic tangency is either *simple one-sided* or *simple topologically crossing*. In these cases, we will obtain certain meaningful results on existence of horseshoes (topological or even hyperbolic) when $\sigma > 1$, see Section 2.4.

2.3. On coordinate forms of T_0 and T_1

For next explanations and calculations, we need 'good' analytical expressions for both the local and global maps $T_0 \equiv f\big|_U$ and $T_1 \equiv f^q\big|_{\Pi^-}$. Especially, this is important for the local map T_0, since we consider orbits which are iterated under T_0 arbitrarily long and, thus, we need expressions for T_0^i with arbitrary large i. Of course, best local coordinates would be those in which the map T_0 has linear form, however, sufficiently smooth linearizing coordinates not always exist. Therefore, we will use other coordinates (possessing main properties of linear coordinates) as stated in the following lemma.

Lemma 2.8 **([17]):** *Let f be C^r ($r \geq 2$) and O have multipliers $\lambda_1, \dots, \lambda_m, \gamma$ satisfying (1). Then the local map T_0 can be written, in some C^r-coordinates (x, u, y) on U, as follows:*

$$(\bar{x}, \bar{u}, \bar{y}) = \Big(\hat{A}x + h_1(x, u, y), \hat{B}u + h_2(x, u, y), \gamma y + h_3(x, u, y)\Big), \tag{5}$$

where eigenvalues of the matrix $\hat{A}$ are equal to $|\lambda_1|$ in absolute value, whereas eigenvalues of $\hat{B}$ are smaller. Moreover, the functions h_1, h_2, h_3 satisfy the identities

$$\begin{aligned} &h_1(0,0,y) \equiv 0,\ \ h_2(0,0,y) \equiv 0,\ \ h_3(x,y,0) \equiv 0,\\ &h_1(x,u,0) \equiv 0,\ \ h_3(0,0,y) \equiv 0,\\ &\frac{\partial h_1}{\partial x}\Big|_{x=0,u=0} \equiv 0,\ \ \frac{\partial h_2}{\partial x}\Big|_{x=0,u=0} \equiv 0\ \ \frac{\partial h_3}{\partial y}\Big|_{y=0} \equiv 0. \end{aligned} \tag{6}$$

The form (5) (with identities (6) satisfied) is called the *main normal form of saddle map*.

Remark 2.9: If condition A holds, we have in Lemma 2.8 that either $x \in \mathbb{R}^1$ and $\hat{A} = \lambda_1$ in case A1; or $x \in \mathbb{R}^2$, $\hat{A} = \lambda R_\varphi$, where $\lambda = |\lambda_1|$ and $R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, in case A2.

Remark 2.10: If f is 2D, then form (5) of T_0 is written as

$$\bar{x} = \lambda_1 x + h_1(x, y)xy, \quad \bar{y} = \gamma y + h_3(x, y)xy,$$

where $h_1(0, y) \equiv 0$, $h_3(x, 0) \equiv 0$.

However, we need more information about iterations T_0^k, especially, for k large. If T_0 is linear, i.e. $\bar{x} = \hat{A}x, \ \bar{u} = \hat{B}u, \ \bar{y} = \gamma y$, then the situation is trivial. We can write the map $T_0^k : \sigma_k^0 \mapsto \sigma_k^1$ either in the direct form $x_1 = \hat{A}^k x_0, \ u_1 = \hat{B}^k u_0, \ y_1 = \gamma^k y_0$ or in the so-called *cross-form* $x_1 = \hat{A}^k x_0, \ u_1 = \hat{B}^k u_0, \ y_0 = \gamma^{-k} y_1$. Evidently, these two forms are equivalent. Similar cross-form for the map T_0^k exists also in the nonlinear case [30], and if the map T_0 is written in the main normal form (given by Lemma 2.8), then the corresponding cross-form of T_0^k is close to the linear cross-form, as the following lemma shows.

Lemma 2.11 **([17]):** *Let $(x_k, u_k, y_k) = T_0^k(x_0, u_0, y_0)$ and the local map T_0 be written in the form (5) with identities (6). Then the following relations take place for large k:*

$$\begin{array}{rcl} x_k - \hat{A}^k x_0 &=& \hat{\lambda}^k \xi_k(x_0, u_0, y_k), \\ u_k &=& \hat{\lambda}^k \hat{\xi}_k(x_0, u_0, y_k), \\ y_0 - \gamma^{-k} y_k &=& \hat{\gamma}^{-k} \eta_k(x_0, u_0, y_k), \end{array} \tag{7}$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are some constants such that $0 < \hat{\lambda} < |\lambda_1|$, $\hat{\gamma} > |\gamma|$ and the functions $\xi_k, \eta_k, \hat{\xi}_k, \hat{\eta}_k$ are uniformly bounded for all k, as well as derivatives up to order $(r-2)$.[7]

In the coordinates from Lemma 2.8, the manifolds $W^s_{\text{loc}}(O)$, $W^u_{\text{loc}}(O)$ as well as W^{ss}_{loc} are straightened, i.e. they have the following equations:

$$W^s_{\text{loc}}(O) \ : \ \{y = 0\}, \quad W^u_{\text{loc}}(O) \ : \ \{(x, u) = 0\}, \quad W^{ss}_{\text{loc}} \ : \ \{x = 0, y = 0\}.$$

Therefore, we assume that the homoclinic points M^+ and M^- have the following coordinates: $M^+ = (x^+, u^+, 0)$ and $M^-(0, 0, y^-)$, where $y^- > 0$. Condition B means that $\|x^+\| \neq 0$. Thus, in case A1, we have that $x^+ \neq 0$, since $x \in \mathbb{R}^1$ (and $u \in \mathbb{R}^{m-1}$), and without loss of generality, we assume that $x^+ > 0$ here; in case A2 we have that $x^+ = (x_1^+, x_2^+)$ and $|x_1^+| + |x_2^+| \neq 0$, since $x \in \mathbb{R}^2$ (and $u \in \mathbb{R}^{m-2}$).

In what follows, we will use in U local coordinates given by Lemma 2.8. Consider in U small rectangle neighbourhoods $\Pi^+ = \{\|(x - x^+, u - u^+)\| \leq \varepsilon_0, |y| \leq \varepsilon_0\}$ and $\Pi^- = \{\|(x, u)\| \leq \varepsilon_1, |y - y^-| \leq \varepsilon_1\}$ of the homoclinic points $M^+(x^+, u^+, 0)$ and $M^-(0, 0, y^-)$, respectively. We assume that the constants $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ are sufficiently small, so, e.g. $T_0\Pi^+ \cap \Pi^+ = \varnothing$ and $T_0^{-1}\Pi^- \cap \Pi^- = \emptyset$.

Evidently, we can propose that any orbit of Λ, except for O, has intersection points with Π^+ and Π^- (as Γ_0 intersects Π^+ and Π^-, any orbit close to Γ_0 must also intersect Π^+ and Π^-). Note that points from Π^+ can reach Π^- under iterations of the local map T_0. The set Σ_0 of such initial points on Π^+ consists of infinitely many disjoint "strips" $\sigma_k^0 \subset \Pi^+$, $k = k_0, k_0 + 1, \ldots$, that are defined as $\sigma_k^0 = T_0^{-k}(\Pi^-) \cap \Pi^+$. Then, we can say that the so-called successor map $T_0' : \Pi^+ \mapsto \Pi^-$ by orbits of the local map T_0 is defined on the set Σ_0, and the range of T_0' is the set Σ_1 consisting of infinitely many disjoint "strips" $\sigma_k^1 \subset \Pi^-$, $k = k_0, k_0 + 1, \ldots$, such that $\sigma_k^1 = T_0^k \sigma_k^0$ (one can also write that $\sigma_k^1 = T_0^k(\Pi^+) \cap \Pi^-$), see

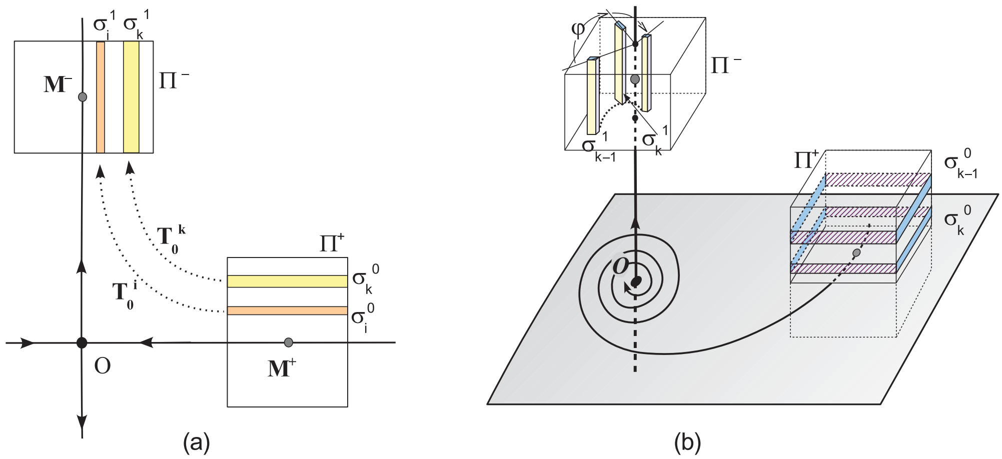

Figure 5. Towards construction of the strips σ_k^0 and $\sigma_k^1 = T_0^k(\sigma_k^0)$: (a) for 2D case and (b) for 3D case, where the point O is a saddle-focus.

Figure 5(a) which illustrates the 'strips' in 2D case. In Figure 5(b), such a picture is shown for 3D case, where the point O is a saddle-focus.

In the coordinates of Lemma 2.8, the global map $T_1 \equiv f^q\colon \Pi^- \to \Pi^+$ can be written as follows

$$(\bar{x} - x^+, \bar{u} - u^+) \;=\; F(x_1, u_1, y_1 - y^-), \quad \bar{y} \;=\; G(x_1, u_1, y_1 - y^-), \tag{8}$$

where C^r-functions F and G are defined on Π^- and $F(0) = 0$, $G(0) = 0$, $G_y(0) = 0$. Then we can write the map T_1 in the following form

$$\begin{aligned} &(\bar{x} - x^+, \bar{u} - u^+) = ax + \hat{a}u + b(y - y^-) + O\left(\|(x, u)\|^2 + (y - y^-)^2\right), \\ &\bar{y} = cx + \hat{c}u + \Psi(y - y^-) + O\left(\|(x, u)\|^2 + \|(x, u)\||y - y^-|\right), \end{aligned} \tag{9}$$

where $\Psi(0) = 0$ and $\Psi'(0) = 0$, since the curve $T_1(W^u_{\text{loc}})$ touches W^s_{loc} at the point M^+, and

$$\det \begin{pmatrix} a, & \hat{a}, & b \\ c, & \hat{c}, & 0 \end{pmatrix} \neq 0, \tag{10}$$

Note that the matrix (10) is the Jacobian matrix for map (9) at the point $M^-(x = 0, u = 0, y = y^-)$. It is an $(m + 1 \times (m + 1)$-matrix, so its determinant is well defined.

Note also that, in the coordinates of Lemma 2.8, the foliation F^{ss} has the form $\{x = \text{const}, y = 0\}$ and the tangent space $\mathcal{T}_M W^{ue}$ to W^{ue} at any point $M \in W^u_{\text{loc}}$ is the plane $u = 0$. Then condition C means, by (9), that the planes $\mathcal{T}_{M^+} W^{ue}: \{(\bar{x} - x^+, \bar{u} - u^+) = ax + b(y - y^-), \ \ \bar{y} = cx\}$ and $F^{ss}(M^+): \left\{\bar{x} = x^+, \bar{y} = 0\right\}$ are transverse (here, the plane $\mathcal{T}_{M^+} W^{ue}$ is given in a parameter form where x and $(y - y^-)$ are parameters). It means that the system $(0, u - u^+) = ax + b(y - y^-)$, $0 = cx$ has the unique solution. Thus, condition C reads as

$$b_1 \neq 0, \quad c \neq 0 \ \text{ in the case A1} \tag{11}$$

and as

$$b_1^2 + b_2^2 \neq 0, \quad c_1^2 + c_2^2 \neq 0 \;\text{ in the case A2.} \tag{12}$$

2.4. On simple homoclinic tangencies in the sectionally saddle case

In this section, we consider, essentially, the multidimensional sectionally saddle case $\sigma > 1$. Concerning the homoclinic tangencies, we assume that they are simple, i.e. conditions A–C are fulfilled, and, besides, they are *isolated* and, accordingly, either *one-sided* or *topologically crossing* tangencies. Note that we do not assume here condition D to be satisfied.

In the case of a topologically crossing tangency, the diffeomorphism f has, of course, infinitely many topological horseshoes. However,

- if the topologically crossing tangency is simple and the point O is a saddle, i.e. if conditions A1, B, and C hold, then in U there exist infinitely many *hyperbolic horseshoes*.

This fact follows from the Katok theorem, since the problem allows the reduction to dim $= 2$ in this situation. Indeed, the 2D global invariant center manifold W^c exists here (see Section 2.2) and it is $C^{1+\epsilon}$-smooth and, hence, the restricted system $f\big|_{W^c}$ is smooth and 2D.[8]

The case of *one-sided homoclinic tangency* (with $\sigma > 1$) is more interesting here, because the systems under conditions A–C can possess both trivial and nontrivial dynamics near Γ_0 depending on the type of tangency. In order to prove the corresponding results, we introduce the so-called 'index of one-sided tangency' that can take value +1 or -1 and is defined as follows. For given diffeomorphism f, we take a pair of the homoclinic points $M^+(x^+, 0) \in W^s_{\text{loc}}$ and $M^-(0, y^-) \in W^u_{\text{loc}}$ and their sufficiently small neighbourhoods Π^+ and Π^-. Next, we consider the piece $T_1(W^u_{\text{loc}}) \cap \Pi^+$ of $W^u(O)$ which, by (9), has the equation

$$(\bar{x} - x^+, \bar{u} - u^+) = b(y - y^-) + O\left((y - y^-)^2\right), \quad \bar{y} = \Psi(y - y^-), \tag{13}$$

written in the parametric form, where $t = y - y^-$ is a parameter. Since $\Psi(0) = 0$, $\Psi'(0) = 0$, the curve (13) touches the plane $y = 0$ at $t = 0$. Let this tangency be one-sided, then we introduce the index ν_0 as follows:

$$\nu_0 \;=\; \text{sign}\ \Psi(\xi) \;\text{ at } \xi \neq 0. \tag{14}$$

Thus, for the case when $\gamma > 0$, the homoclinic tangency (one-sided and isolated) is 'from below' if $\nu_0 = -1$, and it is 'from above' if $\nu_0 = +1$. For the case when $\gamma < 0$, the value of index ν_0 depends on the choice of homoclinic points: for example, $\nu_0(M^+) = -\nu_0(f(M^+))$; it means that we can always take pairs of the homoclinic points so that $\nu_0 = +1$. Note that if the tangency is of even order, then $\nu_0 = \text{sign}\, d$. This fact follows directly from Definition 2.2.

Theorem 2.12 ([Simple one-sided homoclinic tangencies with trivial dynamics])**:** *Let f have a one-sided homoclinic tangency satisfying conditions A1, B, and C and such that*

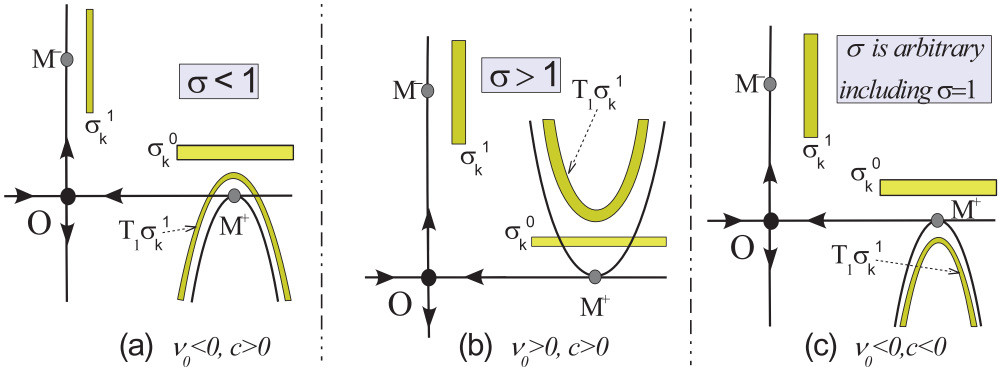

Figure 6. Examples of simple homoclinic tangencies with trivial dynamics for $\lambda_1 > 0, \gamma > 0$.

(1) *either* $\gamma > 0$, $\nu_0 < 0$ *in the case* $\sigma < 1$;
(2) *or* $\lambda_1 > 0$, $c\nu_0 > 0$ *in the case* $\sigma > 1$;
(3) *or* $\lambda_1 > 0$, $\gamma > 0$, $c < 0$, $\nu_0 < 0$ *independently on* σ.
Then f possesses trivial dynamics near Γ_0, *i.e.* $\Lambda = O \cup \Gamma_0$.

In Figure 6, we show some examples of 2D diffeomorphisms with trivial dynamics near a simple one-sided homoclinic tangency. We can see that, in all cases, the image $T_1 T_0^k(\sigma_k^0)$ of any strip σ_k^0 (with sufficiently large k) under the corresponding first return map $T_1 T_0^k$ does not intersect with σ_k^0. It means that any first return maps near homoclinic orbits have no nonwandering points and, moreover, we can easily deduce from this, that only the orbits O and Γ_0 are nonwandering.

Concerning the existence of nontrivial dynamics near one-sided homoclinic tangencies with $\sigma > 1$, one can deduce some sufficient conditions like those that are presented in the following theorem.

Theorem 2.13 ([Simple one-sided homoclinic tangencies with nontrivial dynamics at $\sigma > 1$]): *Let f have a one-sided homoclinic tangency satisfying conditions A–C and let* $\sigma > 1$.

(1) *If the point O is a saddle-focus, i.e. conditions A2 holds, then the set* Λ *contains infinitely many topological horseshoes.*
(2) *If the point O is a saddle, i.e. A1 holds, then in all such cases, except for the case* $\lambda_1 > 0$, $c\nu_0 > 0$, *the set* Λ *contains infinitely many hyperbolic horseshoes.*

We prove Theorem 2.12 in Section 3.2 and Theorem 2.13 in Section 3.3.

2.5. An example of diffeomorphism with a non-simple homoclinic tangency in the case $\sigma > 1$

As one can see from the proof Theorem 2.3, in the sectionally dissipative case $\sigma < 1$, conditions A, B, C are not used. In contrast, in the sectionally saddle case $\sigma > 1$, these conditions seem to be necessary, even for the existence of topological horseshoes. We will illustrate this fact by considering the following model example of a 3D diffeomorphism g_0.

Let g_0 have a saddle fixed point O with multipliers $\lambda_1, \lambda_2, \gamma$ such that $0 < \lambda_2 < \lambda_1 < 1 < \gamma$ and $\lambda_1\gamma > 1$. Let g_0 have a quadratic homoclinic tangency at the points of some homoclinic orbit Γ_0. We assume that the local map T_0 is linear and the global map T_1 is model, i.e. T_0 has the form $(\bar{x}, \bar{u}, \bar{y}) = (\lambda_1 x, \lambda_2 u, \gamma y)$ and T_1 is as follows:

$$(\bar{x} - x^+, \bar{u} - u^+, \bar{y}) = \left(b_1(y - y^-),\ a_{21}x + b_2(y - y^-),\ \hat{c}u + d(y - y^-)^2\right). \tag{15}$$

If $b_1 a_{21}\hat{c} \neq 0$, the map T_1 is diffeomorphism. However, we see that condition C is violated in this case: the model T_1 corresponds to the global map (9) with $c = 0$, i.e. (11) is not satisfied. We assume that $d < 0$, i.e. the quadratic homoclinic tangency is 'from below'. Then it is not very hard to establish the following dynamical properties of g_0.

Proposition 2.14:

(*i*) *If* $\hat{c} < 0$, *then* $\Lambda(g_0)$ *is trivial, i.e.* $\Lambda(g_0) = O \cup \Gamma_0$.
(*ii*) *If* $\hat{c} > 0$ *and* $\lambda_2\gamma < 1$, *then* $\Lambda(g_0)$ *is trivial, i.e.* $\Lambda(g_0) = O \cup \Gamma_0$
(*iii*) *If* $\hat{c} > 0$ *and* $\lambda_2\gamma > 1$, *then* $\Lambda(g_0)$ *contains infinitely many topological horseshoes* Ω_k.

Proof: As T_0 is linear, the equations of $W^s_{\text{loc}}(O)$ and $W^u_{\text{loc}}(O)$ are $y = 0$ and $(x = 0, u = 0)$, respectively. We choose a pair of homoclinic points: $M^+(x^+, u^+, 0) \in W^s_{\text{loc}}$ and $M^-(0, 0, y^-) \in W^u_{\text{loc}}$ assuming that $u^+ > 0, y^- > 0$. We consider sufficiently small rectangle neighbourhoods $\Pi^+\{|x - x^+| \leq \varepsilon_0, |u - u^+| \leq \varepsilon_0, |y| \leq \varepsilon_0)\}$ and $\Pi^-\{|x| \leq \varepsilon_1, |u| \leq \varepsilon_1, |y - y^-| \leq \varepsilon_1)\}$ of the points M^+ and M^-, respectively. In the case under consideration, the map T_0^k can be written in the form $x_k = \lambda_1^k x_0,\ u_k = \lambda_2^k u_0,\ y_0 = \gamma^{-k} y_k$. Then, we can write the exact formulas for the strips $\sigma_k^0 = T_0^{-k}(\Pi^-) \cap \Pi^+$ and $\sigma_k^1 = T_0^k(\Pi^+) \cap \Pi^-$, where

$$\begin{aligned} \sigma_k^0 &= \{(x, u, y) \big| |x - x^+| \leq \varepsilon_0, |u - u^+| \leq \varepsilon_0, |y - \gamma^{-k}y^-| \leq \gamma^{-k}\varepsilon_1\}, \\ \sigma_k^1 &= \{(x, u, y) \big| |x - \lambda_1^k x^+| \leq \lambda_1^k \varepsilon_0, |u - \lambda_2^k u^+| \leq \lambda_2^k \varepsilon_0, |y - y^-| \leq \varepsilon_1\}. \end{aligned} \tag{16}$$

Consider the horseshoe $T_1(\sigma_k^1)$. By (15), it has the coordinate $\bar{y}$ in Π^+ as follows:

$$\bar{y} = \hat{c}\lambda_2^k u^+(1 + \ldots) + d(y - y^-)^2. \tag{17}$$

Note also that the coordinate y on the strip σ_j^0, by (16), can be written as

$$y = \gamma^{-j} y^-(1 + \ldots) > 0 \tag{18}$$

In the case (i) $\hat{c} < 0$, we obtain from (17), since $d < 0$, that $\bar{y} < 0$ for all horseshoes $T_1(\sigma_k^1)$. Thus, by (18), we have that $T_1(\sigma_k^1) \cap \sigma_j^0 = \emptyset$ for all sufficiently large i and j, see Figure 7(a).

In the case (ii) $\hat{c} > 0$ and $\lambda_2\gamma < 1$, we obtain from (17), since $d < 0$, that $\bar{y} \geq \hat{c}\lambda_2^k u^+(1 + \varepsilon_0)$ for the coordinate $\bar{y}$ on $T_1(\sigma_k^1)$. Thus, if $T_1(\sigma_k^1) \cap \sigma_j^0 \neq \emptyset$, then

$$\hat{c}\lambda_2^k u^+(1 + \varepsilon_0) \geq \gamma^{-j} y^-(1 - \varepsilon_1).$$

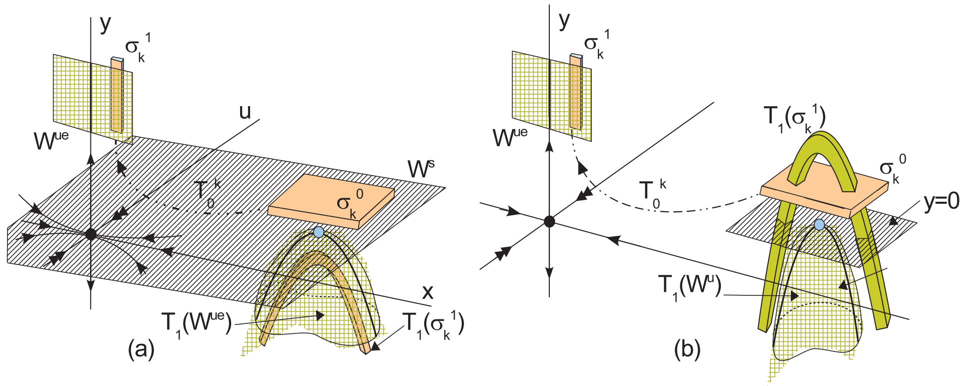

Figure 7. Examples of 3D diffeomorphisms with a non-simple quadratic homoclinic tangency (the case where $T_1(W^{ue})$ touches W^s_{loc}) when (a) Λ is trivial and (b) Λ contains horseshoes.

As $\lambda_2\gamma < 1$, this inequality means that $j \gg k$ for large j and k. Thus, $T_1(\sigma_k^1)$ can intersect the strip σ_j^0 with only number j greater than k, in turn, $T_1(\sigma_j^1)$ can intersect the strip σ_i^0 only if $i > j$, etc. Evidently, this implies that $\Lambda(g_0) = O \cup \Gamma_0$.

In the case (iii) $\hat{c} > 0$ and $\lambda_2\gamma > 1$, we obtain from (17), since $d < 0$, that the top of the horseshoe $T_1(\sigma_k^1)$ (it corresponds to $y = y^-$) has coordinate $\bar{y}$ satisfying $\bar{y} \geq \hat{c}\lambda_2^k u^+(1-\varepsilon_0)$. Thus, the top of $T_1(\sigma_k^1)$ is posed from above the strip σ_k^0, since $\gamma^{-k} \ll \lambda_2^k$ for large k and $\lambda_2\gamma > 1$. On the other hand, the bottom of the horseshoe $T_1(\sigma_k^1)$ lies below W^s_{loc} because its y-coordinate equals $\bar{y} = \hat{c}\lambda_2^k u^+ + d\varepsilon_1^2$, which is negative for large k. Thus, we have here topological horseshoes Ω_k for all sufficiently large k, see Figure 7(b). □

3. Proofs of main results

3.1. Proof of Theorem 2.3

Now we assume only that a diffeomorphism f has a homoclinic tangency of the invariant manifolds of a saddle fixed point O with multipliers $\lambda_1, \dots, \lambda_m, \gamma$ ordered by the rule (1) and such that $\sigma \equiv |\lambda_1||\gamma| < 1$.

In the case under consideration, by Lemma 2.8, we can write the local map T_0 in the following form

$$\bar{x} = \mathcal{A}x + \hat{h}(x,y), \quad \bar{y} = \gamma y + h_3(x,y), \tag{19}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^1$, $\mathcal{A}$ is an $(m \times m)$-matrix with eigenvalues $\lambda_1, \dots, \lambda_m$. Let us clarify that comparing (19) with (5) the coordinate x here is (x, u) in (5), the matrix $\mathcal{A} = \text{diag}\,(\hat{A}, \hat{B})$ and $\hat{h} = (h_1, h_2)$. Then, by (8), the global map T_1 takes the form

$$\begin{aligned} \bar{x} - x^+ &= F(x, y - y^-), \\ \bar{y} &= G(x, y - y^-) \equiv cx + \Psi(y - y^-) + O(\|x\|^2 + \|x\||y - y^-|) \end{aligned} \tag{20}$$

where $(x, y) \in \Pi^-$, $(\bar{x}, \bar{y}) \in \Pi^+$, $F(0,0) = 0$, $G(0,0) = 0$ and $\Psi(0) = \Psi'(0) = 0$.

If the homoclinic tangency is isolated then the function $\Psi(y - y^-)$ vanishes only at $y = y^-$. Besides, in the case of one-sided tangency, we have that either $\Psi \geq 0$ (the tangency from above') or $\Psi \leq 0$ (the tangency "from below" with $\gamma > 0$).

Again, by Lemma 2.11, the map $T_0^k : \sigma_k^0 \subset \Pi^+ \to \sigma_k^1 \subset \Pi^-$ can be written in the following cross-form:

$$x_k - \mathcal{A}^k x_0 = \hat{\lambda}^k \tilde{\xi}_k(x_0, y_k), \quad y_0 - \gamma^{-k} y_k = \hat{\gamma}^{-k} \tilde{\eta}_k(x_0, y_k), \tag{21}$$

where $(x_0, y_0) \in \sigma_k^0$, $(x_1, y_1) \in \sigma_k^1$ and $0 < \hat{\lambda} < |\lambda_1|$, $\hat{\gamma} > |\gamma|$.

Proof of item 1 of Theorem 2.3: Take some strip $\sigma_k^1 \subset \Pi^-$. Then coordinates (x, y) on σ_k^1 satisfy, by (21), the following inequalities:

$$\|x - \mathcal{A}^k x^+\| \leq \|\mathcal{A}^k\| \varepsilon_0 + O(\hat{\lambda}^k), \quad |y - y^-| \leq \varepsilon_1$$

Let the corresponding horseshoe $T_1(\sigma_k^1)$ intersect a strip σ_i^0 for some i. By (21), the coordinates (x, y) on σ_i^o satisfy the following inequalities

$$\|x - x^+\| \leq \varepsilon_0, \quad \gamma^{-i}(y^- - \varepsilon_1) + O(\hat{\gamma}^{-i}) \leq y \leq \gamma^{-i}(y^- + \varepsilon_1) + O(\hat{\gamma}^{-i})$$

Then system (20) with $(x, y) \in \sigma_k^1$ must have a solution $(\bar{x}, \bar{y}) \in \sigma_i^0$. Evidently, there is a constant $\tilde{\lambda}$ bigger than $|\lambda_1|$, such that $|\tilde{\lambda}\gamma| < 1$ and $\|\mathcal{A}^k\| < \tilde{\lambda}^k$. Then since $T_1(\sigma_k^1) \cap \sigma_i^0 \neq \emptyset$, it follows from (20) that the equation

$$\gamma^{-i}\bar{y} + O(\hat{\gamma}^{-i}) = \alpha_k(x, y) + \Psi(y - y^-), \tag{22}$$

where $\|\alpha_k\| < \tilde{\lambda}^k$, $y \in [y^- - \varepsilon_1, y^- + \varepsilon_1]$, $\|x - x^+\| \leq \varepsilon_0$, has a solution $\bar{y} \in [y^- - \varepsilon_1, y^- + \varepsilon_1]$.

Let the homoclinic tangency satisfy condition (2). If $\gamma > 0$ (the main case), it follows that $\Psi(y_1 - y^-) \leq 0$. Then the Equation (22) can have a solution only if

$$\gamma^{-i}\bar{y} + O(\hat{\gamma}^{-i}) - \alpha_k(x, y) \leq 0$$

Since $\gamma > 0$, $\hat{\gamma} > \gamma$, $\|\alpha_k\| < \tilde{\lambda}^k$ and $|\tilde{\lambda}\gamma| < 1$, the inequality above can hold only in the case where $i \gg k$. Thus, any horseshoe $T_1(\sigma_k^1)$ can intersect only those strips σ_i^0 whose numbers i are much larger than k, see Figure 8(a). In its turn, the horseshoe $T_1(\sigma_i^1)$, again, can intersect only some strips σ_j^0 with $j > i$, etc. It implies that some backward iterations of any point from Π^+ must leave U, except, of course, for homoclinic points $\{\Gamma\} \cap \Pi^+$. Thus, this implies that $\Lambda = O \cup \{\Gamma\}$ (if Γ_0 is isolated, then $\{\Gamma\} = \Gamma_0$).

If $\gamma < 0$, condition (2) implies the identity $\Psi(y_1 - y^-) \equiv 0$. Evidently, the Equation (22) can have a solution in this case again only for $i \gg k$. □

Proof of item 2 of Theorem 2.3: Now let condition (3) hold. By (20), the equation of the curve $T_1(W^u_{\text{loc}}) \cap \Pi^+$ has the following parametric form:

$$\bar{x} - x^+ = F(0, y - y^-), \quad \bar{y} = \Psi(y - y^-),$$

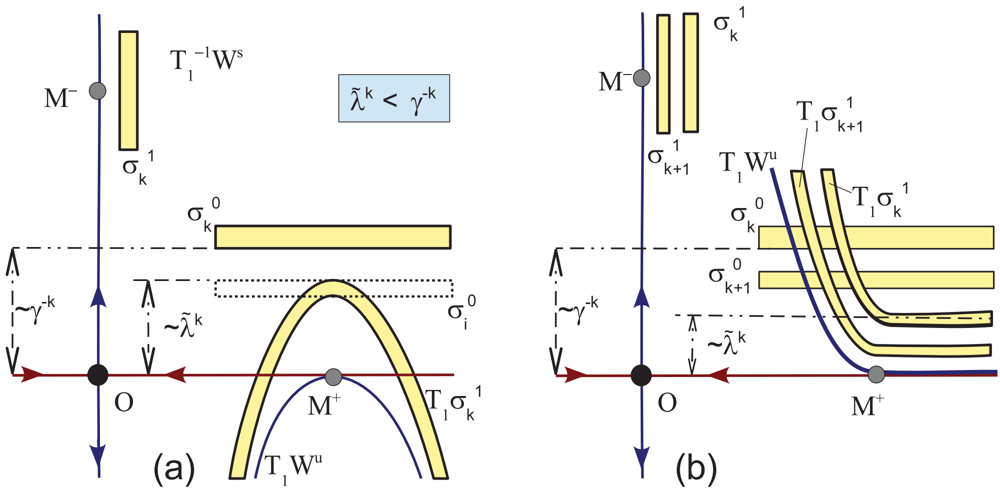

Figure 8. Towards the proof of Theorem 2.3: (a) to item 1 of Theorem 2.3; and (b) to item 2 of Theorem 2.3 for some degenerate case (for general case see Figure 3(b)).

where $(\bar{x},\bar{y}) \in \Pi^+$, $y^- > 0$ and $y \in [y^- - \varepsilon_1, y^- + \varepsilon_1]$ is the parameter. Consider first the case $\gamma > 0$. Since (3) holds, it follows that there exist $\hat{y} \in [y^- - \varepsilon_1, y^- + \varepsilon_1]$ and $\delta_0 > 0$ such that $\Psi(\hat{y} - y^-) > \delta_0$. Consider a strip $\sigma_k^0 \subset \Pi^+$. It is posed in Π^+ at distance ρ_k from the plane $W^s_{\text{loc}} : \{y = 0\}$, where $\gamma^{-k}(y^- - \varepsilon_1) \le \rho_k \le \gamma^{-k}(y^- + \varepsilon_1)$. Let k be large enough, so that $\delta_0 > \gamma^{-k}$. Then the horseshoe $T_1(\sigma_k^1)$ has a non-empty intersection with the strip σ_k^0. Indeed, the segment $y = \hat{y}$ of the strip σ_k^1 is mapped under T_1 into the segment $\bar{y}(\hat{y}) = \Psi(\hat{y} - y^-) + \alpha_k(x, \hat{y})$ of the horseshoe $T_1(\sigma_k^1)$ and thus, $\bar{y}(\hat{y}) > \delta_0 - \tilde{\lambda}^k$ for large k. On the other hand, the top of the horseshoe $T_1(\sigma_k^1)$, i.e. the T_1-image of the segment $y = y^-$ of σ_k^1, has the coordinate $\bar{y} = \bar{y}(y^-) = \alpha_k(x, y^-) < \tilde{\lambda}^k$ for large k. Thus, we have that $\bar{y}(\hat{y}) > \rho_k \sim \gamma^{-k} > \bar{y}(y^-) \sim \tilde{\lambda}^k$ as $k \to \infty$, since $\tilde{\lambda}\gamma < 1$.

This implies that in the case when the tangency is one-sided from above, infinitely many of the first return maps $T_k = T_1 T_0^k : \sigma_k^0 \to \sigma_k^0$ will have topological horseshoes, see Figure 3(b). If the tangency is more degenerate, e.g. as in Figure 8(b) (here $\Psi(y - y^-) \equiv 0$ at $y > y^-$), one can construct topological 'bi-horseshoe', using, for example, the strips $\sigma_k^0, \sigma_{k+1}^0$ and their horseshoes $T_1(\sigma_k^1)$, $T_1(\sigma_{k+1}^1)$.

Evidently, the above arguments cover also the case $\gamma < 0$, since we may consider here only even numbers k.

Concerning the case of topologically crossing tangency, we note that the curve $T_1 W^u_{loc}$ intersects infinitely many strips σ_k^0. Moreover, the strips σ_j^1 accumulate to W^u_{loc}. It implies that (infinitely many) strips σ_k^0 intersect horseshoes $T_1(\sigma_j^1)$ for all sufficiently large k and j. □

3.2. Proof of Theorem 2.12

As we said before, item (1) of Theorem 2.12 (related to the sectionally dissipative case $\sigma < 1$) follows directly from Theorem 2.3. Thus, we need to consider the items (2) and (3) of the theorem.

Proof of item 2: Let now f be a diffeomorphism having a one-sided homoclinic tangency satisfying conditions A1, B, and C with $\lambda_1 > 0$, $c\nu_0 > 0$ in the case $\sigma > 1$.

Since A1 holds, the local map T_0, by virtue of Lemma 2.8, takes the form

$$(\bar{x}, \bar{u}, \bar{y}) = \big(\lambda_1 x + h_1(x, u, y), \hat{B}u + h_2(x, u, y), \gamma y + h_3(x, u, y)\big),$$

where $x, y \in \mathbb{R}^1$, $u \in \mathbb{R}^{m-1}$ and the matrix $\hat{B}$ has eigenvalues $\lambda_2, \dots, \lambda_m$. Then, by Lemma 2.11, the map $T_0^k : \sigma_k^0 \to \sigma_k^1$ can be written in the following cross-form (compare with (7))

$$x_k = \lambda_1^k x_0 + O(\hat{\lambda}^k), \quad u_k = O(\hat{\lambda}^k), \quad y_0 = \gamma^{-k} y_k + O(\hat{\gamma}^{-k}), \tag{23}$$

where $(x_0, u_0, y_0) \in \sigma_k^0 \subset \Pi^+$, $(x_1, u_1, y_1) \in \sigma_k^1 \subset \Pi^-$, $0 < \hat{\lambda} < |\lambda_1|$ and $\hat{\gamma} > |\gamma|$. Then, using (9), we can write the first return map $T_k = T_1 T_0^k : \sigma_k^0 \mapsto \Pi^+$ as follows:

$$\begin{aligned} &(\bar{x} - x^+, \bar{u} - u^+) = b(y - y^-) + O\left(|\lambda_1|^k \|(x, u)\| + (y - y^-)^2\right), \\ &\bar{y} = c_1 \lambda_1^k x + \Psi(y - y^-) + O\left(\hat{\lambda}^k \|(x, u)\| + |\lambda_1^k| \|(x, u)\| |y - y^-|\right), \end{aligned} \tag{24}$$

Let $\bar{y}$ be the y-coordinate of some strip $\sigma_j^0 \subset \Pi^+$. Then we can write, by (23), that $\bar{y} = \gamma^{-j} y_1 + O(\hat{\gamma}^{-j})$, where $y_1 \in [y^- - \varepsilon_1, y^- + \varepsilon_1]$. Introduce the new coordinates

$$\xi = x - x^+, \quad \zeta = u - u^+, \quad \eta = y - y^-.$$

Then we can write the second equation from (24) in the form

$$\gamma^{-j}(y^- + \bar{\eta}) + O(\hat{\gamma}^{-j}) = c_1 \lambda_1^k (x^+ \xi) + \Psi(\eta) + |\lambda_1|^k O(\eta) + O(\hat{\lambda}^k). \tag{25}$$

Since ξ, η are small, $x^+ > 0$, $y^- > 0$ as well as $\lambda_1 > 0$ and $c_1 \nu_0 > 0$ (recall that ν_0 is the index defined by (14)), we obtain that Equation (25) can have solutions only in the case when $|\gamma|^{-j} \geq \lambda_1^k$. Since $|\lambda_1 \gamma| > 1$, this inequality can be fulfilled only if $k \gg j$. Thus, any horseshoe $T_1(\sigma_k^1)$ can intersect only those strips σ_i^0 whose numbers are strictly less than k, see Figure 6(b). It implies that some forward iteration of any point from Π^+ must leave U. Thus, only two orbits, O and Γ_0, will stay always in U. □

Proof of item 3: In the case $\lambda_1 > 0, \gamma > 0, c_1 < 0, \nu_0 < 0$, Equation (25) has no solutions at all, independently of σ, since $\Psi(\eta) \leq 0$ in this case, and, thus, the left-hand and right-hand sides of (25) have opposite signs. It means that the horseshoes $T_1(\sigma_k^1)$ do not intersect the strips σ_i^0, see Figure 6(c), i.e. the dynamics is trivial. □

3.3. Proof of Theorem 2.13

Here f has a one-sided simple homoclinic tangency with $\sigma > 1$. The simplicity of tangency means that conditions A–C hold.

Proof of item (1): We consider the case where A2 holds, i.e. the point O is a saddle-focus. Then, by Lemma 2.8, the local map T_0 has the form

$$(\bar{x}, \bar{u}, \bar{y}) = \big(\lambda R_\varphi x + h_1(x, u, y), \hat{B}u + h_2(x, u, y), \gamma y + h_3(x, u, y)\big),$$

where $x = (x_1, x_2)$ and $R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is the rotation matrix on the angle $\varphi \in (0, \pi)$. The global map $T_1 \equiv f^q \colon \Pi^- \to \Pi^+$ has now the form (9), where $c = (c_1, c_2)$, $b = (b_1, b_2)^\top$ and inequalities (12) hold, as well as the function $\Psi(y - y^-)$ is a sign-determined function for $y \neq y^-$.

Consider the first return map $T_k = T_1 T_0^k : \sigma_k^0 \mapsto \Pi^+$ which can be written now as

$$\begin{aligned}(\bar{x} - x^+, \bar{u} - u^+) &= b(y - y^-) + O\left(\lambda^k \|(x, u)\| + (y - y^-)^2\right), \\ \bar{y} &= \lambda^k \left((c_1 \cos k\varphi + c_2 \sin k\varphi)x_1 + (c_2 \cos k\varphi - c_1 \sin k\varphi)x_2\right) \\ &\quad + \Psi(y - y^-) + O\left(\hat{\lambda}^k \|(x, u)\| + \lambda^k \|(x, u)\| |y - y^-|\right). \end{aligned} \qquad (26)$$

Let us show that, for infinitely many values of k, these maps T_k are geometrically horseshoe maps. Introduce new x-coordinates as $\xi_1 = x_1 - x_1^+$, $\xi_2 = x_2 - x_2^+$. Then, the second equation from (26) can be written as

$$\begin{aligned}\bar{y} &= \lambda^k \left(\hat{C}\cos(k\varphi + \theta) + O(\|\xi\|)\right) + \Psi(y - y^-) \\ &\quad + O\left(\hat{\lambda}^k \|(\xi, u)\| + \lambda^k \|(\xi, u)\| |y - y^-|\right), \end{aligned} \qquad (27)$$

where $\hat{C} = \sqrt{(c_1^2 + c_2^2)((x_1^+)^2 + (x_2^+)^2)}$ and $\theta \in [0, 2\pi)$ is an angle such that $\cos\theta = (c_1 x_1^+ + c_2 x_2^+)/\hat{C}$, $\sin\theta = (c_2 x_1^+ - c_1 x_2^+)/\hat{C}$. Note that $\hat{C} > 0$, since conditions B and C imply, respectively, that $(x_1^+)^2 + (x_2^+)^2 \neq 0$ and $c_1^2 + c_2^2 \neq 0$.

Consider first the case when $\Psi(s) \geq 0$ at $|s| \leq \varepsilon_1$ and $\gamma > 0$. Then $\bar{y}$ from (27) runs from $\bar{y}_{\min} = \lambda^k(\hat{C}\cos(k\varphi + \theta) + O(\|\xi\|)) + O(\hat{\lambda}^k)$ to $\bar{y}_{\max} = \max_{|s| \leq \varepsilon_1} \Psi(s) + O(\lambda^k)$. However, values of the coordinate y on the strip σ_k^0 satisfy the inequality

$$\gamma^{-k}(y^- - \varepsilon_1) < y < \gamma^{-k}(y^- + \varepsilon_1).$$

Evidently, there are such $\delta_0 > 0$ and $\delta_1 > 0$ that (i) $\bar{y}_{\max} > \delta_0 > 0$ for all sufficiently large k and (ii) for any $0 < \varphi < \pi$, there are infinitely many such k that $\bar{y}_{\min} < -\lambda^k \delta_1$ (since $\hat{C} > 0$ and $\|\xi\|$ is small). Thus, the first return map $T_k = T_1 T_0^k$ for such k transforms the strip σ_k^0 into the horseshoe $T_k(\sigma_k^0)$ such that its top is posed from below σ_k^0 and the images of upper and below sides of the strip σ_k^0 are posed from above σ_k^0. Thus, we have that f possesses infinitely many topological horseshoes. □

Other three cases ($\Psi(s) \leq 0$ and $\gamma > 0$, $\Psi(s) \geq 0$ and $\gamma < 0$, $\Psi(s) \leq 0$ and $\gamma < 0$) are considered similarly.

Proof of item (2) of Theorem 2.13: Since λ_1 is real here, the first return map $T_k = T_1 T_0^k$: $\sigma_k^0 \mapsto \Pi^+$ can be written now as

$$\begin{aligned}(\bar{x} - x^+, \bar{u} - u^+) &= (b_1, b_2)^\intercal (y - y^-) + O\left(|\lambda_1|^k \|(x, u)\| + (y - y^-)^2\right), \\ \bar{y} &= c\lambda_1^k x + \Psi(y - y^-) + \\ &\quad + O\left(\hat{\lambda}^k \|(x, u)\| + |\lambda_1|^k \|(x, u)\| |y - y^-|\right). \end{aligned} \tag{28}$$

Introduce new x-coordinates as $\xi = x - x^+$. Then the second equation from (28) can be written as

$$\bar{y} = c\lambda_1^k \left(x^+ + O(|\xi|) + O([\hat{\lambda}/\lambda_1]^k)\right) + \Psi(y - y^-).$$

Consider the model equation $\gamma^{-k} y^- = c\lambda_1^k x^+ + \Psi(s)$ where $s \in [-\varepsilon_1, \varepsilon_1]$ and for some $\hat{\delta} > 0$ $\Psi(s) \in [0, \hat{\delta}]$ or $\Psi(s) \in [-\hat{\delta}, 0]$ and $\nu_0 = \text{sign}\Psi(s)_{s \neq 0}$. Since $|\lambda_1 \gamma| > 1$, this model equation has no solution only in the case where $\lambda_1 > 0$ and $c\nu_0 > 0$. In other cases, at least two solutions exist. Evidently, it gives us the sought result. □

Notes

1. Note that methods of [1,4] were applied for detection and description of hyperbolic invariant sets like half-orienable and classical Smale horseshoes in two- and three-dimensional generalized Hénon maps [7,9,24].
2. We do not consider here the case $\sigma = 1$ because it is very specific and has been considered in details, see e.g. [5,14].
3. The effective dimension of a system generically coincides with the maximal codimension of admissible bifurcations (for example, it is possible that there appear periodic trajectories with d_e multipliers each equal to +1, while more than d_e such multiplies cannot appear).
4. We thank Dmitry Turaev for letting us know the interesting fact that Katok Theorem can be directly applied to the sectionally dissipative case. In fact, the corresponding multidimensional version of Katok Theorem for diffeomorphisms contracting two-dimensional areas was proposed in [31]. In fact, we repeat here the arguments from [31].
5. For example (see [4,8]), these theorems can be extended to provide infinitely many Smale horseshoes, and moreover, to provide existence of non-uniformly hyperbolic sets which almost coincide with the whole Λ (and even coincide with Λ for a dense set of systems under consideration).
6. Condition D was essentially used in our papers [4,8] to obtain hyperbolic properties of systems with homoclinic tangencies.
7. Concerning derivatives of order $(r - 1)$, the following estimates hold as $k \to +\infty$:

$$\|x_k, u_k\|_{C^{r-1}} = o(|\lambda_1|^k), \quad \|y_0\|_{C^{r-1}} = o(|\gamma|^{-k}),$$

while the derivatives of order r are estimated as follows:

$$\|x_k, u_k, y_0\|_{C^r} = o(1)_{k \to \infty} .$$

8. Note, however, if the point O is a saddle-focus, i.e. the conditions A2, B and C hold, we cannot apply the Katok theorem directly, since dim $W^c = 3$ and, formally, we have on W^c the

sectionally saddle case again (moreover, the map $f|_{W^c}$ has the saddle-focus fixed point with $\dim W^s = 2$).

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