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Bethe Subalgebras in Braided Yangians and Gaudin-Type Models

Dimitri Gurevich[1,2], **Pavel Saponov**[3,4], **Alexey Slinkin**[3,5]

[1] Université Polytechnique Hauts-de-France, LAMAV, 59313 Valenciennes, France.
E-mail: gurevich@ihes.fr
[2] Interdisciplinary Scientific Center Poncelet (ISCP, UMI 2615 du CNRS), Moscow, Russian Federation 119002
[3] National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow, Russian Federation 101000. E-mail: Pavel.Saponov@ihep.ru
[4] Institute for High Energy Physics, NRC "Kurchatov Institute", Protvino, Russian Federation 142281
[5] University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA. E-mail: slinalex@live.unc.edu

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Abstract: In Gurevich and Saponov (J Geom Phys 138:124–143, 2019) the notion of braided Yangians of Reflection Equation type was introduced. Each of these algebras is associated with an involutive or Hecke symmetry R. In these algebras quantum analogs of certain symmetric polynomials (elementary symmetric ones, power sums) were defined. In the present paper we show that these quantum symmetric polynomials commute with each other and consequently generate a commutative Bethe subalgebra. As an application, we get some Gaudin-type models.

1. Introduction

Let V be a finite dimensional vector space over the ground field $\mathbb{C}$, $\dim_{\mathbb{C}} V = N$. Consider a linear operator $R \in \mathrm{End}(V^{\otimes 2})$ which is a solution of the so-called braid relation

$$R_1 R_2 R_1 = R_2 R_1 R_2. \tag{1.1}$$

Both sides of this relation belong to $\mathrm{End}(V^{\otimes 3})$. Also, we use the shorthand notation $R_k = R_{k\,k+1}$ for the operator R acting in the positions k and $k+1$ of the tensor product $V^{\otimes p}$, $p \geq k+1$. Thus, the factors in (1.1) are $R_1 = R \otimes I$ and $R_2 = I \otimes R$. Hereafter, I is the identity operator or matrix.

An operator subject to (1.1) will be called a *braiding*.

In addition, we assume all braidings R, we are dealing with, to obey either the Hecke condition

$$(R - q\, I)(R + q^{-1}\, I) = 0, \quad q \in \mathbb{C}\backslash\{0, \pm 1\}$$

or the condition

$$R^2 = I.$$

Respectively, we will call such an operator R a *Hecke symmetry* or an *involutive symmetry*.

Given a Hecke or involutive symmetry R, we can construct some current (i.e. depending on parameters) operators

$$R(u,v) = R - \frac{(q-q^{-1})\,u\,I}{u-v}, \tag{1.2}$$

provided R is a Hecke symmetry, and

$$R(u,v) = R - \frac{I}{u-v} \tag{1.3}$$

provided R is involutive. These current operators meet the *braid relation with parameters*

$$R_1(u,v)R_2(u,w)R_1(v,w) = R_2(v,w)R_1(u,w)R_2(u,v). \tag{1.4}$$

The correspondence $R \mapsto R(u,v)$ is called the *Baxterization procedure*[1] and the resulting operator $R(u,v)$ is called a *current braiding*. More precisely, the operator (1.2) will be referred to as the *trigonometric* braiding and (1.3) as the *rational* one.

Since the current operator $R(u,v)$ depends on the ratio of parameters $x = u/v$ in the trigonometric case and on their difference $x = u - v$ in the rational case, we also use the notation $R(x)$ for both cases. So, we respectively have

$$R(x) = R - \frac{(q-q^{-1})x}{x-1}\,I, \qquad R(x) = R - \frac{1}{x}\,I. \tag{1.5}$$

We hope this does not lead to a misunderstanding.

Note, that in the literature one often uses the operator $\mathcal{R}(u,v) = PR(u,v)$, where P stands for the usual flip or its matrix. This operator is called the *quantum R-matrix* and meets the Quantum Yang–Baxter Equation:

$$\mathcal{R}_{12}(u,v)\mathcal{R}_{13}(u,w)\mathcal{R}_{23}(v,w) = \mathcal{R}_{23}(v,w)\mathcal{R}_{13}(u,w)\mathcal{R}_{12}(u,v).$$

Since the Baxterization procedure can be applied to *any* involutive or Hecke symmetry, it enables one to contract a large set of current braidings not coming from the affine Quantum Groups $U_q(\widehat{sl(m)})$. A big family of the corresponding symmetries was constructed in [G]. These symmetries lead to the R-symmetric and R-skew-symmetric algebras with unusual Poincaré–Hilbert series and, therefore, they cannot be obtained by a deformation (quantization) of a classical (or super-)algebras. For these symmetries no object of the Quantum Group type is known.

Nevertheless, any of the current braiding (1.2) or (1.3) enables us to introduce a Yangian-like algebra via the following system of quadratic relations

$$R_1(u,v)L_1(u)L_2(v) - L_1(v)L_2(u)R_1(u,v) = 0. \tag{1.6}$$

The *generating matrix* $L(u)$ is assumed to be a formal series in the parameter u^{-1}:

$$L(u) = \sum_{k=0}^{\infty} L[k]\,u^{-k}, \tag{1.7}$$

[1] Note that the Baxterization of the Hecke algebras was considered in [Jo], where this term was introduced.

where $N\times N$ matrices $L[k]$ are called the Laurent coefficients. Consequently, the equality (1.6) leads to an infinite system of relations on the matrix elements of the Laurent coefficients $L[k]$.

In case $R = P$ the system (1.6) gives rise to the Drinfeld's Yangian $\mathbf{Y}(gl(N))$ provided $L[0] = I$. By the initial definition, $\mathbf{Y}(gl(N))$ is a unital associative algebra, generated by entries of the Laurent coefficients $L[k]$, $k \geq 1$, whereas the entries of the matrix $L(u)$ are treated as elements of $\mathbf{Y}(gl(N))[[u^{-1}]]$.

If R is the Hecke symmetry coming from the Quantum Group $U_q(sl(N))$, the corresponding Yangian-like algebra is sometimes called a *q-Yangian*, provided $L[0]$ is subject to some additional conditions (see [M]).

Introduced in [GS1,GS2] were some other current algebras associated with current braidings (1.2) and (1.3). These algebras are called *braided Yangians*.

Definition 1. A braided Yangian $\mathbf{Y}(R)$ is a unital associative algebra generated by a countable set of generators $l_i^j[k]$, $1 \leq i, j \leq N$, $k \geq 1$ which are subject to the system of quadratic relations

$$R_1(u,v)L_{\overline{1}}(u)L_{\overline{2}}(v) - L_{\overline{1}}(v)L_{\overline{2}}(u)R_1(u,v) = 0, \tag{1.8}$$

where the generating $N\times N$ matrix $L(u)$ reads

$$L(u) = I + \sum_{k=1}^{\infty} L[k]u^{-k}, \qquad L[k] = \|l_i^j[k]\|_1^N.$$

In the formula (1.8) the following notation is used

$$L_{\overline{1}}(u) = L_1(u), \quad L_{\overline{k+1}}(u) = R_k L_{\overline{k}}(u) R_k^{-1}, \quad k \geq 1. \tag{1.9}$$

Note, that in a braided Yangian $\mathbf{Y}(R)$ we impose condition $L[0] = I$.

Let us emphasize an important property (1.9) of the above definition: pushing forward the matrix $L_1(u)$ to higher positions $L_{\overline{k}}(u)$ is performed with the use of the initial symmetry R instead of the usual flip P as is done in the Yangian (1.6). Using (1.9) one can rewrite the defining system (1.8) in the form:

$$R_1(u,v)L_1(u)R_1L_1(v) - L_1(v)R_1L_1(u)R_1(u,v) = 0.$$

Remark 2. The braided Yangians can be defined in a more general way which is based on the notion of compatible braidings, introduced in [IOP]. Consider an ordered pair of braidings (R, F). The braidings R and F are called *compatible* if they are subject to the relations

$$R_1F_2F_1 = F_2F_1R_2, \qquad R_2F_1F_2 = F_1F_2R_1. \tag{1.10}$$

Given such a pair with an involutive or Hecke symmetry R, we define the corresponding braided Yangian by the same system (1.8) but the matrix copies $L_{\overline{k}}$ should be constructed with the use of the braiding F:

$$L_{\overline{1}} = L_1, \qquad L_{\overline{k+1}}(u) = F_k L_{\overline{k}}(u) F_k^{-1}, \quad k \geq 1.$$

In fact, we extend the construction of the Quantum Matrix Algebras corresponding to parameterless braidings (see [IOP]) to the Yangian-like algebras.

Note that for any compatible braidings R and F the relations (1.8) can be pushed forward to higher positions:

$$R_k(u,v)L_{\overline{k}}(u)L_{\overline{k+1}}(v) = L_{\overline{k}}(v)L_{\overline{k+1}}(u)R_k(u,v), \quad \forall k \geq 1. \tag{1.11}$$

Namely this property is the main "raison d'être" for the notion of compatible braidings. The braided Yangians in this more general setting are considered in [GST].

In the present paper we mainly consider the braided Yangian $\mathbf{Y}(R)$ defined by (1.8). It corresponds to the compatible pair (R, R) and will be called the braided Yangian of *Reflection Equation (RE) type*.[2] The Yangian-like algebras defined by (1.6) are called the *Yangians of RTT type*.

Certain properties of the braided Yangians of RE type were studied in [GS1]. The present paper is a sequel of the cited one. In particular, in [GS1] the quantum counterparts of elementary symmetric polynomials and power sums were introduced. The main objective of the current paper is to prove that they commute with each other and thus generate a commutative Bethe subalgebra. Note that our method of proving this commutativity differs from that used for the Drinfeld Yangians $\mathbf{Y}(gl(N))$ and for other Yangians of RTT type.[3] One of the main dissimilarities consists in using the braided trace Tr_R associated with the initial symmetry R instead of the usual trace.

As an application of the mentioned results we define some Gaudin-type models associated with involutive symmetries (we call these models *braided Gaudin ones*). By applying the Talalaev's method from [T] we construct Bethe subalgebras for these models.

The paper is organized as follows. The Bethe subalgebras in the braided Yangians of RE type are constructed in the next section. Their commutativity is established in Sect. 3. In Sect. 4 we introduce braided counterparts of the rational Gaudin model and construct the corresponding Bethe subalgebras.

2. Bethe Subalgebras Via Quantum Symmetric Polynomials

We begin this section with the following observation. The current braidings $R(x)$, we are dealing with, are non-singular except for some isolated values of parameters. Namely, in the rational and trigonometric cases we respectively have

$$R^{-1}(x) = \frac{x^2}{x^2-1}\, R(-x), \quad x \neq \pm 1,$$
$$R^{-1}(x) = \frac{(x-1)^2}{(x-1)^2 - \lambda^2 x}\, R(x^{-1}), \quad x \neq q^{\pm 2}, \quad \lambda \equiv q - q^{-1}.$$

[2] This terminology is motivated by a similarity of the system (1.8) and that defining an RE algebra

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R,$$

where the overlined indices have the same meaning (1.9) as above. Note that this system can be equivalently written as $R_1L_1R_1L_1 = L_1R_1L_1R_1$. Stress that here we use a constant (i.e. parameterless) braiding R and a constant generating matrix. As for current RE algebras, they have many different forms and we do not consider them here.

[3] The problem of constructing the Bethe subalgebra in the q-Yangians or affine Quantum Group was considered in numerous publications going back to L. Faddeev school and seminal works by Drinfeld and Jimbo (see [KS,D,J]). An extensive list of references to other pioneering works can be found in the book [CP].

Let us introduce a family of skew-symmetrizers $A^{(k)}_{1\dots k}(R) \in \mathrm{End}(V^{\otimes n})$, $n \ge k$, associated with a Hecke symmetry R (see [Gy,J,G]) by the following recurrent relations:

$$A^{(1)} = I, \qquad A^{(k+1)}_{1\dots k+1} = \frac{k_q}{(k+1)_q}\, A^{(k)}_{1\dots k}\Big(\frac{q^k}{k_q}\, I - R_k\Big) A^{(k)}_{1\dots k}, \quad k \ge 1, \tag{2.1}$$

where $k_q = (q^k - q^{-k})/(q - q^{-1})$ is the standard notation for q-numbers. The parameter q is assumed to be generic: $q^k \ne 1$, $\forall k \in \mathbb{Z}_+$, so all q-integers are nonzero. The low indices indicate the positions where the operators $A^{(k)}$ are located.

For an involutive symmetry R the skew-symmetrizers are defined by the same formula (2.1) where we should set $q = 1$ and use k instead of k_q.

Hereafter, we confine ourselves to even skew-invertible symmetries. The bi-rank of these symmetries (Hecke or involutive) is assumed to be $(m|0)$.[4] Precise definitions of even symmetries, bi-rank, and a review of properties of the corresponding symmetries can be found, for example, in [O,GPS,GS1]. Here we give a short summary of necessary facts to be used below:

- For any braiding under consideration there exists an $N \times N$ matrix C with the following properties:

$$\mathrm{Tr}_{(2)} R_{12} C_2 = I_1, \qquad R_k C_k C_{k+1} = C_k C_{k+1} R_k \quad \forall k \ge 1, \qquad \mathrm{Tr}\, C = \frac{m_q}{q^m}. \tag{2.2}$$

- With the matrix C we define the R-*trace* of a matrix X (of an appropriate size) with entries from an algebra $\mathcal{A}$:

$$\mathrm{Tr}_{R(12\dots k)} X = \mathrm{Tr}_{(12\dots k)}(C_1 C_2 \dots C_k\, X), \quad X \in \mathrm{Mat}_{N^k}(\mathcal{A}). \tag{2.3}$$

The second relation in (2.2) allows us to prove an important *cyclic property* of the R-trace:

$$\mathrm{Tr}_{R(12\dots k)}(f(R_1, R_2 \dots R_{k-1})X) = \mathrm{Tr}_{R(12\dots k)}(X f(R_1, R_2 \dots R_{k-1})) \tag{2.4}$$

for any polynomial f in braidings and for any matrix X as above.

- The following property of the R-trace is of the great importance in what follows:

$$\mathrm{Tr}_{R(k+1)}(R_k^{\pm 1} X_{1\dots k} R_k^{\mp 1}) = I_k \mathrm{Tr}_{R(k)}(X_{1\dots k}), \tag{2.5}$$

where $X_{1\dots k}$ is an operator (a matrix) in the space $V^{\otimes k}$.

- The skew-symmetrizers (2.1) are idempotents

$$A^{(k)} A^{(k)} = A^{(k)} \quad \forall k \ge 1, \tag{2.6}$$

and they vanish for $k \ge m+1$:

$$A^{(m+1)} \equiv 0, \qquad \mathrm{rank}\, A^{(m)} = 1. \tag{2.7}$$

Now, we introduce quantum analogs of the elementary symmetric polynomials defined via generating matrix $L(u)$ of a braided Yangian of RE type $\mathbf{Y}(R)$ in the trigonometrical case.

[4] In general $m \ne N$. However, if R equals P or is its deformation, then $m = N$.

Definition 3. The elements

$$e_k(u) = \mathrm{Tr}_{R(12\ldots k)}\, A^{(k)}_{12\ldots k} L_{\overline{1}}(u) L_{\overline{2}}(q^{-2}u)\ldots L_{\overline{k}}(q^{-2(k-1)}u), \quad k \geq 1, \tag{2.8}$$

are called (quantum) elementary symmetric polynomials in the Yangian of RE type $\mathbf{Y}(R)$.

The subalgebra generated in $\mathbf{Y}(R)$ by the elementary symmetric polynomials is called the *Bethe subalgebra*.

It should be emphasized that this definition (as well as the definition (2.10) of the power sums) does not assume R to be even. However, if R is even of bi-rank $(m|0)$, then in virtue of (2.7) there exists the highest non-zero elementary symmetric polynomial $e_m(u)$ which is called the *quantum determinant*. As was shown in [GS1] it is always central element in the corresponding braided Yangian $\mathbf{Y}(R)$ (contrary to the case of RTT type Yangians where the centrality of the quantum determinant depends on R).

In the next section we show that the elementary symmetric polynomials commute with each other and consequently, the Bethe subalgebra is commutative. To this end we need the following lemma.

Lemma 4. *The following matrix identity holds:*

$$\mathrm{Tr}_{R(k+1\ldots k+p)} A^{(p)}_{k+1\ldots k+p} L_{\overline{k+1}}(u) L_{\overline{k+2}}(q^{-2}u)\ldots L_{\overline{k+p}}(q^{-2(p-1)}u) = I_{12\ldots k}\, e_p(u), \tag{2.9}$$

where $e_p(u)$ is an elementary symmetric polynomial defined in (2.8).

Proof. First of all, note that due to the definition (1.9) of $L_{\overline{k}}(u)$ and the braid relation for R we have

$$R_i^{\pm 1} L_{\overline{k}}(u) = L_{\overline{k}}(u) R_i^{\pm 1} \qquad \forall\, i \notin \{k-1, k\}.$$

Taking into account this property we can write (we omit the arguments of the matrices L)

$$L_{\overline{k+1}} L_{\overline{k+2}} = R_k L_{\overline{k}} R_k^{-1} L_{\overline{k+2}} = R_k L_{\overline{k}} L_{\overline{k+2}} R_k^{-1} = R_k R_{k+1} L_{\overline{k}} L_{\overline{k+1}} R_{k+1}^{-1} R_k^{-1}.$$

Then we continue as follows

$$L_{\overline{k+1}} L_{\overline{k+2}} L_{\overline{k+3}} = R_k R_{k+1} L_{\overline{k}} L_{\overline{k+1}} L_{\overline{k+3}} R_{k+1}^{-1} R_k^{-1} = R_k R_{k+1} R_{k+2} (L_{\overline{k}} L_{\overline{k+1}} L_{\overline{k+2}}) R_{k+2}^{-1} R_{k+1}^{-1} R_k^{-1}$$

and so on till the following intermediate result:

$$L_{\overline{k+1}} \ldots L_{\overline{k+p}} = R_k \ldots R_{k+p-1} (L_{\overline{k}} \ldots L_{\overline{k+p-1}}) R_{k+p-1}^{-1} \ldots R_k^{-1}.$$

Now, with the use of the relation (2.1) and the braid relation for R we can show that

$$A^{(p)}_{k+1\ldots k+p} R_k R_{k+1} \ldots R_{k+p-1} = R_k R_{k+1} \ldots R_{k+p-1} A^{(p)}_{k\ldots k+p-1}.$$

So, we have

$$\begin{aligned}\mathrm{Tr}_{R(k+1\ldots k+p)} \quad & A^{(p)}_{k+1\ldots k+p} L_{\overline{k+1}} L_{\overline{k+2}} \ldots L_{\overline{k+p}} \\ & = \mathrm{Tr}_{R(k+1\ldots k+p)} R_k \ldots R_{k+p-1} (A^{(p)}_{k\ldots k+p-1} L_{\overline{k}} \ldots L_{\overline{k+p-1}}) R_{k+p-1}^{-1} \ldots R_k^{-1}.\end{aligned}$$

At last, applying the relation (2.5) p times we get

$$\mathrm{Tr}_{R(k+1\ldots k+p)} A^{(p)}_{k+1\ldots k+p} L_{\overline{k+1}} L_{\overline{k+2}} \ldots L_{\overline{k+p}} = I_k \mathrm{Tr}_{R(k\ldots k+p-1)} A^{(p)}_{k\ldots k+p-1} L_{\overline{k}} L_{\overline{k+1}} \ldots L_{\overline{k+p-1}}.$$

Thus, we have shifted down by one the numbers of spaces where the R-trace is applied. The rest of the proof is evident: applying the same steps as above, we finally come to (2.9). $\square$

Remark 5. Note that the relation (2.9) is still valid for a braided Yangian associated with a compatible pair (R, F), where R is a Hecke symmetry.

It is natural to ask whether there exist analogs of other symmetric polynomials in the algebra $\mathbf{Y}(R)$? We are able to give the positive answer for the so-called power sums and complete symmetric polynomials. As for the quantum analogs of the Schur polynomials, we do not know their consistent definition.

Definition 6. The elements

$$p_k(u) = \mathrm{Tr}_{R(1\ldots k)} R_{k-1} \ldots R_2 R_1 L_{\overline{1}}(q^{-2(k-1)}u) L_{\overline{2}}(q^{-2(k-2)}u) \ldots L_{\overline{k}}(u), \quad k \geq 1, \tag{2.10}$$

are called the *quantum power sums*.

The sets of elementary symmetric polynomials and power sums are related by a series of the following *quantum Newton identities* [GS1]:

$$\begin{aligned} k_q e_k(u) - q^{k-1} p_1(q^{-2(k-1)}u) e_{k-1}(u) + q^{k-2} p_2(q^{-2(k-2)}u) \\ e_{k-2}(u) + \ldots + (-1)^k p_k(u) = 0. \end{aligned} \tag{2.11}$$

Similarly to the classical case, these formulae enable us to express power sums via the elementary symmetric polynomials and visa versa. Thus, we can conclude that the power sums also generate the Bethe subalgebra of $\mathbf{Y}(R)$.

In the rational case we respectively define quantum elementary symmetric polynomials and quantum power sums as follows

$$\begin{aligned} e_k(u) &= \mathrm{Tr}_{R(1\ldots k)} A^{(k)}_{12\ldots k} L_{\overline{1}}(u) L_{\overline{2}}(u-1) \ldots L_{\overline{k}}(u-k+1), \\ p_k(u) &= \mathrm{Tr}_{R(1\ldots k)} R_{k-1} \ldots R_2 R_1 L_{\overline{1}}(u-k+1) L_{\overline{2}}(u-k+2) \ldots L_{\overline{k}}(u). \end{aligned} \tag{2.12}$$

They are related by the quantum Newton identities of the form

$$k e_k(u) - p_1(u-k+1) e_{k-1}(u) + p_2(u-k+2) e_{k-2}(u) + \cdots + (-1)^k p_k(u) = 0.$$

Also, in the algebras $\mathbf{Y}(R)$ some analogs of the Cayley-Hamilton identity are valid (see [GS1]) but we do not need them here.

3. Commutativity of Bethe Subalgebra

In this section we prove that the family of elements $e_k(u)$, $1 \leq k \leq m$, generate a commutative subalgebra in $\mathbf{Y}(R)$. If we expand each $e_k(u)$ in a series in the inverse powers of u, we get countable set of polynomials in entries of the matrices $L[k]$ commuting with each other.

We begin with some technical results. Let us introduce a convenient notation for specific chains of current braidings:

$$\left[R_{i\to j}(u)\right]^{(\pm)} = \begin{cases} R_i(u) R_{i+1}(q^{\pm 2}u) \ldots R_j(q^{\pm 2(j-i)}u), & \text{if} \quad j \geq i \\ R_i(u) R_{i-1}(q^{\pm 2}u) \ldots R_j(q^{\pm 2(i-j)}u), & \text{if} \quad i \geq j. \end{cases} \tag{3.1}$$

Analogous notation will be applied to the similar products of $R^{-1}(u)$. Note, that the above chains of current braidings are non-singular except for a finite number of isolated values of the parameter u and the inverse chain is given by the formula:

$$\left\{\Big[R_{i\to j}(u)\Big]^{(\pm)}\right\}^{-1}=\Big[R^{-1}_{j\to i}(q^{\pm 2|i-j|}u)\Big]^{(\mp)}. \tag{3.2}$$

Also, note that the middle term in formula (2.1) can be written as follows

$$\frac{q^k}{k_q}\,I-R_k=-R_k(q^{2k}).$$

This allows us to get an explicit presentation for the skew-symmetrizer $A^{(k)}$ as a product of chains of braidings. Though this result is well-known, we remind it in the Lemma 7 below for the reader convenience.

Lemma 7. *The skew-symmetrizers $A^{(k)}$ defined by* (2.1) *can be written in the following equivalent forms:*

$$A^{(k)}_{1\dots k}=\frac{(-1)^{\frac{k(k-1)}{2}}}{k_q!}\Big[R_{1\to k-1}(q^2)\Big]^{(+)}\Big[R_{1\to k-2}(q^2)\Big]^{(+)}\cdots R_1(q^2), \tag{3.3}$$

$$A^{(k)}_{1\dots k}=\frac{(-1)^{\frac{k(k-1)}{2}}}{k_q!}\Big[R_{k-1\to 1}(q^2)\Big]^{(+)}\Big[R_{k-1\to 2}(q^2)\Big]^{(+)}\cdots R_{k-1}(q^2), \tag{3.4}$$

$$A^{(k)}_{1\dots k}=\frac{(-1)^{\frac{k(k-1)}{2}}}{k_q!}\,R_1(q^2)\Big[R_{2\to 1}(q^4)\Big]^{(-)}\dots\Big[R_{k-1\to 1}(q^{2(k-1)})\Big]^{(-)}, \tag{3.5}$$

$$A^{(k)}_{1\dots k}=\frac{(-1)^{\frac{k(k-1)}{2}}}{k_q!}\,R_{k-1}(q^2)\Big[R_{k-2\to k-1}(q^4)\Big]^{(-)}\dots\Big[R_{k-1\to 1}(q^{2(k-1)})\Big]^{(-)}. \tag{3.6}$$

Here, we use the standard notation $k_q!=1_q2_q\dots k_q$.

The lemma can be easily proved by induction in k with the help of (2.1).

Then, with the use of the braid relation (1.4) we can find the following rule for the permutation of R-chains:

$$\Big[R_{1\to k}(u)\Big]^{(+)}\Big[R_{1\to k-1}(q^2)\Big]^{(+)}=\Big[R_{2\to k}(q^2)\Big]^{(+)}\Big[R_{1\to k-1}(q^2u)\Big]^{(+)}R_k(u).$$

Note, that in the right hand side we have a cyclic permutation of the set of u-depending parameters of the corresponding R-chain:

$$\{u,q^2u,\dots,q^{2(k-1)}u\}\to\{q^2u,\dots,q^{2(k-1)}u,u\}.$$

This observation with formulae of Lemma 7 allows us to prove the following result.

Lemma 8. *The R-chains commute with the skew-symmetrizers in accordance with the rule:*

$$\begin{aligned}
\Big[R_{1\to k}(q^{-2(k-1)}u)\Big]^{(+)} A^{(k)}_{1\dots k} &= A^{(k)}_{2\dots k+1}\Big[R_{1\to k}(u)\Big]^{(-)},\\
\Big[R^{-1}_{1\to k}(u)\Big]^{(-)} A^{(k)}_{1\dots k} &= A^{(k)}_{2\dots k+1}\Big[R^{-1}_{1\to k}(q^{-2(k-1)}u)\Big]^{(+)},\\
A^{(k)}_{1\dots k}\Big[R_{k\to 1}(u)\Big]^{(-)} &= \Big[R_{k\to 1}(q^{-2(k-1)}u)\Big]^{(+)} A^{(k)}_{2\dots k+1},\\
A^{(k)}_{1\dots k}\Big[R^{-1}_{k\to 1}(q^{-2(k-1)}u)\Big]^{(+)} &= \Big[R^{-1}_{k\to 1}(u)\Big]^{(-)} A^{(k)}_{2\dots k+1}.
\end{aligned}$$

Let us point out the inverse order of the parameters of the R-chains in the left and right hand sides of the above relations, as well as the shift of the spaces in which the skew-symmetrizer acts.

The last auxiliary result is connected with the chains of generating matrices $L(u)$. First of all, applying (1.11) we get

$$\begin{aligned}
&\Big[R_{k-1\to 1}(q^{2(k-1)})\Big]^{(-)} L_{\overline{1}}(u)L_{\overline{2}}(q^{-2}u)\,\cdots\, L_{\overline{k-1}}(q^{-2(k-2)}u)L_{\overline{k}}(q^{-2(k-1)}u)\\
&\qquad = L_{\overline{1}}(q^{-2}u)L_{\overline{2}}(q^{-4}u)\,\cdots\, L_{\overline{k-1}}(q^{-2(k-1)}u)L_{\overline{k}}(u)\Big[R_{k-1\to 1}(q^{2(k-1)})\Big]^{(-)}.
\end{aligned}$$

Then, using Lemma 7 we come to the following result:

$$\begin{aligned}
&A^{(k)}_{1\dots k} L_{\overline{1}}(u)L_{\overline{2}}(q^{-2}u)\dots L_{\overline{k}}(q^{-2(k-1)}u)\\
&\quad = L_{\overline{1}}(q^{-2(k-1)}u)L_{\overline{2}}(q^{-2(k-2)}u)\dots L_{\overline{k}}(u)A^{(k)}_{1\dots k}.
\end{aligned} \tag{3.7}$$

Now, we are ready to prove the main theorem of this paper.

Proposition 9. *The elementary symmetric functions* (2.8) *commute with each other:*

$$e_k(u)e_p(v) = e_p(v)e_k(u), \qquad \forall\, k, p \ge 1, \quad \forall\, u, v. \tag{3.8}$$

Proof. In order to simplify formulae we introduce a shorthand notation for a chain of the matrices $L(u)$ with shifted arguments:

$$\big[L_{\overline{i}\to\overline{j}}(u)\big]^{(\pm)} = L_{\overline{i}}(u)L_{\overline{i+1}}(q^{\pm 2}u)\dots L_{\overline{j}}(q^{\pm 2(j-i)}u).$$

Using this notation and the identity (2.9) we can write the left hand side of (3.8) in the form:

$$e_k(u)e_p(v) = \mathrm{Tr}_{R(1\dots k+p)} A^{(k)}_{1\dots k} A^{(p)}_{k+1\dots k+p}\big[L_{\overline{1}\to\overline{k}}(u)\big]^{(-)}\big[L_{\overline{k+1}\to\overline{k+p}}(v)\big]^{(-)}. \tag{3.9}$$

Then, we permute the u-depending chain of L matrices with v-depending chain with the help of the braided Yangian relations (1.11), where $F = R$. At the first step of this process we move the utmost right element $L_{\overline{k}}(q^{-2(k-1)}u)$ through the v-chain of the matrices L. Relations (1.11) give:

$$L_{\overline{k}}(q^{-2(k-1)}u)L_{\overline{k+1}}(v) = R^{-1}_k(q^{-2(k-1)}u/v)L_{\overline{k}}(v)L_{\overline{k+1}}(q^{-2(k-1)}u)R_k(q^{-2(k-1)}u/v).$$

Repeating this procedure we find:

$$L_{\overline{k}}\ (q^{-2(k-1)}u)\big[L_{\overline{k+1}\to\overline{k+p}}(v)\big]^{(-)} = \Big[R^{-1}_{k\to k+p-1}(q^{-2(k-1)}x)\Big]^{(+)}$$
$$\big[L_{\overline{k}\to\overline{k+p-1}}(v)\big]^{(-)}L_{\overline{k+p}}(q^{-2(k-1)}u)\Big[R_{k+p-1\to k}(q^{-2(k-p)}x)\Big]^{(-)},$$

where $x = u/v$.

Now, we should move the next factor $L_{\overline{k-1}}(q^{-2(k-2)}u)$ to the right position (it is possible since this factor commutes with the chain of R^{-1} braidings appeared in the above formula) and so on. Thus, we arrive at the expression:

$$\begin{aligned}&\big[L_{\overline{1}\to\overline{k}}(u)\big]^{(-)}\big[L_{\overline{k+1}\to\overline{k+p}}(v)\big]^{(-)}\\ &= \overleftarrow{\prod_{1\le s\le k}}\Big[R^{-1}_{s\to s+p-1}(q^{-2(s-1)}x)\Big]^{(+)}\big[L_{\overline{1}\to\overline{p}}(v)\big]^{(-)}\big[L_{\overline{p+1}\to\overline{p+k}}(u)\big]^{(-)}\\ &\quad\overrightarrow{\prod_{1\le r\le k}}\Big[R_{r+p-1\to r}(q^{-2(r-p)}x)\Big]^{(-)},\end{aligned}\tag{3.10}$$

where we use the following notation for an ordered product of noncommutative factors:

$$\overrightarrow{\prod_{1\le i\le k}} Q_i \equiv Q_1Q_2\dots Q_k, \qquad \overleftarrow{\prod_{1\le i\le k}} Q_i \equiv Q_kQ_{k-1}\dots Q_1.$$

So, with the use of (3.10) we rewrite (3.9) in the following form

$$\begin{aligned}e_k(u)e_p(v) = \mathrm{Tr}_{R(1\dots k+p)}\Big\{&A^{(k)}_{1\dots k}A^{(p)}_{k+1\dots k+p}\overleftarrow{\prod_{1\le s\le k}}\Big[R^{-1}_{s\to s+p-1}(q^{-2(s-1)}x)\Big]^{(+)}\big[L_{\overline{1}\to\overline{p}}(v)\big]^{(-)}\\ &\times\big[L_{\overline{p+1}\to\overline{p+k}}(u)\big]^{(-)}\overrightarrow{\prod_{1\le r\le k}}\Big[R_{r+p-1\to r}(q^{-2(r-p)}x)\Big]^{(-)}\Big\}.\end{aligned}\tag{3.11}$$

Now, we are going to prove that the ordered products of chains of R and R^{-1} braidings in the above formula cancel each other under the trace operation. Consequently, the right hand side turns into the product $e_p(v)e_k(u)$.

Note that except for the simplest case $k = p = 1$ this cancellation is not straightforward since in (3.11) there is a pair of skew-symmetrizers between two chains of braidings. First, we permute the skew-symmetrizers with the braidings which leads to the change of their ordering. Then, we have to make some identical transformations in order to guarantee that the parameters of the current braidings and inverse current braidings fit each other in a proper way (see (3.2)).

So, we begin with moving the skew-symmetrizers through the chains of R^{-1} braidings with the use of the relations of Lemma 8. For the skew-symmetrizer $A^{(p)}_{k+1\dots k+p}$ this procedure is straightforward (it suffices to use the second line of Lemma 8 k times):

$$A^{(p)}_{k+1\dots k+p}\overleftarrow{\prod_{1\le s\le k}}\Big[R^{-1}_{s\to s+p-1}(q^{-2(s-1)}x)\Big]^{(+)} = \overleftarrow{\prod_{1\le s\le k}}\Big[R^{-1}_{s\to s+p-1}(q^{-2(s-p)}x)\Big]^{(-)}A^{(p)}_{1\dots p}.$$

In order to move the second skew-symmetrizer $A^{(k)}_{1\dots k}$ we have to reorder the chains of braidings using the following evident identity:

$$\prod^{\leftarrow}_{1\le s\le k}\Big[R^{-1}_{s\to s+p-1}(q^{-2(s-p)}x)\Big]^{(-)} = \prod^{\rightarrow}_{1\le s\le p}\Big[R^{-1}_{s+k-1\to s}(q^{-2(s+k-p-1)}x)\Big]^{(+)}.$$

Now, we can move $A^{(k)}_{1\dots k}$ with the help of the last line formula of Lemma 8:

$$A^{(k)}_{1\dots k}\prod^{\rightarrow}_{1\le s\le p}\Big[R^{-1}_{s+k-1\to s}(q^{-2(s+k-p-1)}x)\Big]^{(+)} = \prod^{\rightarrow}_{1\le s\le p}\Big[R^{-1}_{s+k-1\to s}(q^{-2(s-p)}x)\Big]^{(-)}A^{(k)}_{p+1\,\dots\,p+k}.$$

So, we come to the relation:

$$e_k(u)e_p(v) = \mathrm{Tr}_{R(1\dots k+p)}\Big\{\prod^{\rightarrow}_{1\le s\le k}\Big[R^{-1}_{k+1-s\to k+p-s}(q^{-2(s-p)}x)\Big]^{(-)}A^{(p)}_{1\dots p}A^{(k)}_{p+1\dots k+p}\big[L_{\overline{1}\to\overline{p}}(v)\big]^{(-)}$$
$$\times\big[L_{\overline{p+1}\to\overline{p+k}}(u)\big]^{(-)}\prod^{\rightarrow}_{1\le r\le k}\Big[R_{r+p-1\to r}(q^{-2(r-p)}x)\Big]^{(-)}\Big\},$$

where we also reordered the chains of inverse braidings with the use of identity:

$$\prod^{\rightarrow}_{1\le s\le p}\Big[R^{-1}_{s+k-1\to s}(q^{-2(s-p)}x)\Big]^{(-)} = \prod^{\rightarrow}_{1\le s\le k}\Big[R^{-1}_{k+1-s\to k+p-s}(q^{-2(s-p)}x)\Big]^{(-)}.$$

Next, using the projector property (2.6) we can write

$$A^{(p)}_{1\dots p}A^{(k)}_{p+1\dots p+k} = A^{(p)}_{1\dots p}A^{(k)}_{p+1\dots p+k}A^{(p)}_{1\dots p}A^{(k)}_{p+1\dots p+k}$$

and then move the second pair of skew-symmetrizers to the rightmost position. In so doing, we apply (3.7) to pass through the chains of L matrices and Lemma 8 to permute $A^{(k)}A^{(p)}$ with the chains of current braidings.

At last, with the help of (2.4) we cyclically move the chains of braidings and the second pair of skew-symmetrizers. Finally, we get the following result:

$$e_k(u)e_p(v) = \mathrm{Tr}_{R(1\dots k+p)}\Big\{\prod^{\rightarrow}_{1\le r\le k}\Big[R_{r+p-1\to r}(q^{-2(k-r)}x)\Big]^{(+)}A^{(k)}_{1\dots k}A^{(p)}_{k+1\dots k+p}$$
$$\times\prod^{\leftarrow}_{1\le s\le k}\Big[R^{-1}_{s\to s+p-1}(q^{-2(k-p+1-s)}x)\Big]^{(-)}A^{(k)}_{p+1\dots p+k}A^{(p)}_{1\dots p}$$
$$\times\big[L_{\overline{1}\to\overline{p}}(q^{-2(p-1)}v)\big]^{(+)}\big[L_{\overline{p+1}\to\overline{p+k}}(q^{-2(k-1)}u)\big]^{(+)}\Big\}.$$

Note, that in the above expression the chains of current braidings are exactly inverse to each other (see (3.2)). The only obstacle for their cancellation is the pair of skew-symmetrizers between them. To remove these skew-symmetrizers, we permute the *outer* pair $A^{(k)}_{p+1\dots p+k}A^{(p)}_{1\dots p}$ with the chain of R^{-1} braidings, use the above projector property and then move the pair of skew-symmetrizers back through the chain of inverse braidings.

Due to such a double permutation the chains of inverse current braidings are not changed and all R chains cancel each other. Finally, we get

$$e_k(u)e_p(v) = \mathrm{Tr}_{R(1\ldots k+p)}\Big\{A^{(p)}_{1\ldots p}A^{(k)}_{p+1\ldots p+k}\big[L_{\overline{1}\to\overline{p}}(q^{-2(p-1)}v)\big]^{(+)}\big[L_{\overline{p+1}\to\overline{p+k}}(q^{-2(k-1)}u)\big]^{(+)}\Big\}$$
$$= \mathrm{Tr}_{R(1\ldots k+p)}\Big\{\big[L_{\overline{1}\to\overline{p}}(v)\big]^{(-)}\big[L_{\overline{p+1}\to\overline{p+k}}(u)\big]^{(-)}A^{(p)}_{1\ldots p}A^{(k)}_{p+1\ldots p+k}\Big\} = e_p(v)e_k(u).$$

The proof is completed. $\square$

As was noticed above, on expanding $e_k(u)$ and $e_p(v)$ in a series in u^{-1} and v^{-1} we can reformulate this commutativity in terms of polynomials in entries of the matrices $L[k]$.

4. Braided Gaudin Models

In the present section we deal with a Gaudin-like model, arising from the braided Yangians considered above. First, we recall the classical version of the rational Gaudin model.

Let $M(k)$, $k = 1, 2, \ldots, K$, be $m \times m$ matrices subject to the following relations

$$M_1(k)M_2(l) = M_2(l)M_1(k), \quad k \neq l, \tag{4.1}$$

and

$$M_1(k)M_2(k) - M_2(k)M_1(k) = M_1(k)P - M_2(k)P. \tag{4.2}$$

Then the elements

$$H_k = \sum_{l\neq k} \mathrm{Tr}\,\frac{M(k)M(l)}{u_k - u_l}, \tag{4.3}$$

where u_k, $k = 1, 2, \ldots, K$, are pairwise distinct complex numbers, commute with each other.

This claim results from the fact that the elements H_k are quadratic Hamiltonians of the rational Gaudin model. The higher Hamiltonians of this model have been constructed by Talalaev [T]. His result can be presented as follows.

Let $L(u) = \sum_{k\geq 1} L[k]u^{-k}$ be an $m \times m$ matrix subject to the relation

$$[L_1(u), L_2(v)] = \Big[\frac{P}{u-v}, L_1(u) + L_2(v)\Big]. \tag{4.4}$$

Then the elements

$$QH_k(u) = \mathrm{Tr}_{(1\ldots m)}A^{(m)}_{1\ldots m}(L_1(u) - I\,\frac{d}{du})(L_2(u) - I\,\frac{d}{du})\ldots(L_k(u) - I\,\frac{d}{du}) \triangleright 1 \tag{4.5}$$

commute with each other:

$$QH_k(u)QH_l(v) = QH_l(v)QH_k(u) \quad \forall\, u, v \quad \text{and} \quad \forall\, k, l.$$

Here the notation $\triangleright$ means that the above differential operator is applied to the unit. Thus, all terms of the resulting differential operator $\sum_p^k Q_p(u)\partial^p$ disappear except for that which does not contain the derivative.

Now, by using the fact that the matrix $\sum_{i=1}^{K}\frac{M_i}{u-u_i}$ satisfies the relation (4.4) (a proof of this fact is straightforward) we get a family of Hamiltonians for the rational Gaudin model.

Observe that the system (4.2) means that entries of each matrix $M(k)$ generate a Lie algebra $gl(m)$, while the entries of matrices $M(k)$ and $M(l)$ with $k\neq l$ commute with each other due to (4.1). Thus, the vector space spanned by the entries of all matrices $M(k)$ is endowed with the structure of the Lie algebra $gl(m)^{\oplus K}$.

Now, in the relations (4.1) and (4.2) we replace all indices by their overlined counterparts according to (1.9) provided that R is an involutive symmetry. Namely, we consider the following relations

$$M_{\overline{1}}(k)M_{\overline{2}}(l)=M_{\overline{2}}(l)M_{\overline{1}}(k),\quad k\neq l, \tag{4.6}$$

and

$$M_{\overline{1}}(k)M_{\overline{2}}(k)-M_{\overline{2}}(k)M_{\overline{1}}(k)=M_{\overline{1}}(k)R-M_{\overline{2}}(k)R. \tag{4.7}$$

Note, that the relations (4.7) define the enveloping algebra of a braided Lie algebra (denoted $gl(V_R)$) generated by entries of the matrix $M(k)$ as was defined in [GPS]. The relation (4.6) means that the copies of such braided Lie algebras commute with each other in a braided way.

Our present purpose is to construct a braided counterpart of the family (4.5) and thus to get Hamiltonians for the corresponding *braided Gaudin model*.

In order to do so we follow the main lines of the Talalaev's approach but instead of the Drinfeld's Yangian we consider the braided Yangian arising from an involutive symmetry R of bi-rank $(m|0)$.

More precisely, we work with the braided Yangian defined by (1.8) with the following current braiding:

$$R(u,v)=R-\frac{h\,I}{u-v}, \tag{4.8}$$

which differs from the initial rational current braiding by a numerical parameter h in the numerator. We denote the corresponding braided Yangian $\mathbf{Y}(R)_h$.

Also, we pass to a new generating matrix $\tilde{L}(u)=\|\tilde{l}_i^{\,j}(u)\|_{1\le i,j\le m}$ connected with the initial one by a linear shift:

$$L(u)=I+h\tilde{L}(u)\quad\Leftrightarrow\quad l_i^{\,j}(u)=\delta_i^{\,j}+h\,\tilde{l}_i^{\,j}(u). \tag{4.9}$$

Consequently, we get the following system for the generating matrix $\tilde{L}(u)$:

$$\Big(R-\frac{h\,I}{u-v}\Big)\tilde{L}_1(u)R\tilde{L}_1(v)-\tilde{L}_1(v)R\tilde{L}_1(u)\Big(R-\frac{h\,I}{u-v}\Big)=-\Big[R,\frac{\tilde{L}_1(u)-\tilde{L}_1(v)}{u-v}\Big].$$

In the limit $h\to 0$ we get an algebra defined by

$$[\tilde{L}_{\overline{1}}(u),\tilde{L}_{\overline{2}}(v)]=\Big[\frac{R}{u-v},\tilde{L}_{\overline{1}}(u)+\tilde{L}_{\overline{2}}(v)\Big]. \tag{4.10}$$

This relation defines an analog of the first Sklyanin structure and can be treated in terms of the braided Lie algebras (see [GST]).

In the algebra $\mathbf{Y}(R)_h$ we consider the following elements

$$\hat{e}_k(u) = \mathrm{Tr}_{R(12\dots m)}\, A^{(m)}_{12\dots m} \tilde{L}_{\overline{1}}(u)\tilde{L}_{\overline{2}}(u-h)\dots \tilde{L}_{\overline{k}}(u-h(k-1)), \quad k \geq 1. \quad (4.11)$$

We state that these elements commute with each other in the algebra $\mathbf{Y}(R)_h$. Indeed, their structure is the same as that of the elements $e_k(u)$ defined in (2.12), the additional factors h in arguments of generating matrices appear due to our modification of the current braiding (4.8). Since the generating matrices $L(u)$ and $\tilde{L}(u)$ are connected by the shift (4.9), then the elements (4.11) are linear combinations of similar commutative elements constructed via the matrix $L(u)$. A detailed proof of this fact in the case $R = P$ is given in [GS2]. The general case is similar.

Next, we introduce the following linear combinations

$$\tau_k(u) = \sum_{p=0}^{k} (-1)^{k-p} \frac{k!}{p!(k-p)!}\, \hat{e}_p(u).$$

If we expand the elements $\hat{e}_k(u)$ in a series in h then the corresponding h-expansion of $\tau_k(u)$ turns out to begin with a term proportional to h^k (see [T, GS2]):

$$\tau_k(u) = h^k(QH_k(u) + o(h)).$$

Since the set of elements $\tau_k(u)$ is obviously commutative, the above expansion ensures the commutativity of $QH_k(u)$. Now, applying the method from [T], we can find an explicit form (4.12) of the elements $QH_k(u)$ in the algebra (4.10). Therefore, the following proposition holds.

Proposition 10. *The elements*

$$QH_k(u) = \mathrm{Tr}_{R(1\dots m)} A^{(m)}_{1\dots m} (\tilde{L}_{\overline{1}}(u) - I\,\frac{d}{du})(\tilde{L}_{\overline{2}}(u) - I\,\frac{d}{du})\dots(\tilde{L}_{\overline{k}}(u) - I\,\frac{d}{du}) \triangleright 1 \quad (4.12)$$

commute with each other in the algebra defined by (4.10).

Also, observe that the matrix

$$\sum_{k=1}^{K} \frac{M(k)}{u-u_k},$$

meets the system (4.10), provided that the matrices $M(k)$ are subject to (4.6) and (4.7). A verification of this property is straightforward.

Consequently, the mutual commutativity of the elements $QH_k(u)$ remains valid if we replace the matrix $\tilde{L}_1(u)$ by $\sum_k \frac{M(k)}{u-u_k}$ in (4.12). The corresponding set of commutative Hamiltonians can be constructed in the usual way via computing the residues at the points u_k. For example, the quadratic Hamiltonians are

$$H_k = \sum_{l \neq k} \mathrm{Tr}_R \frac{M(k)\,M(l)}{u_k - u_l}. \quad (4.13)$$

Emphasize, that by contrast with (4.3) the trace in this formula is associated with the symmetry R.

Unfortunately, this method does not lead to new interesting systems when it is applied to the braided Yangians in the trigonometric case. Indeed, in [GS2] there was applied a similar change of the generating matrix $L(u)$ of the braided Yangian $\mathbf{Y}(R)$ arising from a Hecke symmetry R_q. However, the role of the parameter h in a subsequent expansion was attributed to $\log q$. Let us assume that the Hecke symmetry R_q tends to an involutive symmetry R while $q \to 1$. Then we get the limit algebra defined by the following system

$$[\tilde{L}_{\overline{1}}(u),\ \tilde{L}_{\overline{2}}(v)] = \Big[\frac{R}{u-v}, u\tilde{L}_{\overline{1}}(u) + v\tilde{L}_{\overline{2}}(v)\Big], \tag{4.14}$$

which does not depend on q. In [GS2] the case $R = P$ was considered.

By using the Talalaev's method [T], it is not difficult to construct a Bethe subalgebra in the algebra defined by (4.14). Then, by applying the map

$$\tilde{L}(u) \mapsto \sum_{k=1}^{K} \frac{M(k)\, u_k}{u - u_k},$$

it is possible to conclude that the elements

$$H_k = \sum_{l \neq k} \mathrm{Tr}_R \frac{M(k)\, M(l)\, u_l}{u_k - u_l} \tag{4.15}$$

commute with each other, provided the matrices $M(k)$ are subject to (4.6) and (4.7).

We refer the reader for details to the paper [GS2], where a particular case is considered. However, the same reasoning is valid in general. Unfortunately, this model is not novel, since the change of parameters $u_k \mapsto u_k^{-1}$ reduces the family (4.15) to (4.13).

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