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# LEARNING INTRANSITIVITY: FROM INTRANSITIVE GEOMETRICAL OBJECTS TO “RHIZOMATIC” INTRANSITIVITY

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*A new class of intransitive objects — geometrical and mathematical constructions forming intransitive cycles A > B > C > A — are presented. In contrast to the famous intransitive dice, lotteries, etc., they show deterministic (not probabilistic) intransitive relations. The simplest ones visualize intransitivity that can be understood at a qualitative level and does not require quantitative reasoning. They can be used as manipulatives for learning intransitivity. Classification of the types of situations in which the transitivity axiom does and does not work is presented. Four levels of complexity of intransitivity are introduced, from simple combinatorial intransitivity to a “rhizomatic” one. A possible version of the main educational message for students in teaching and learning transitivity-intransitivity is presented.*

## INTRODUCTION

In decision making, many researchers consider the transitivity axiom (if *A* > *B* and *B* > *C* then *A* > *C*, where “>” means “is preferable to”) as a key component of rational thinking. The authors of the Comprehensive Assessment of Rational Thinking (CART) declare that if “you have violated the transitivity axiom, ... you are not instrumentally rational. The content of A, B, and C do not matter to the axiom” (Five Minutes with Keith E. Stanovich, Richard F. West, and Maggie E. Toplak, 2016). “Any claim of empirical violations of transitivity by individual decision makers requires evidence beyond a reasonable doubt”, according to Regenwetter et al. [2011]. These statements are contrary to numerous studies in an adjacent area — math research of various intransitive objects and the intransitive cycles between them. The intransitive cycle of superiority is characterized by such binary relations between *A*, *B*, and *C* that *A* is superior to *B*, *B* is superior to *C*, and *C* is superior to *A* (i.e., *A* > *B* > *C* > *A*, in contrast to transitive relations *A* > *B* > *C*). Various sets of intransitive objects (intransitive dice, lotteries, playing cards, etc.) have been invented and many studies of intransitive cycles emerging between such objects have been conducted (see e.g., [Conrey et al., 2016; Gardner, 1970; 1974; Grime, 2017; Pegg, 2005; Trybuła, 1961]). They show that, contrary to the CART authors’ opinion, the content of *A*, *B*, and *C* does matter (some *A*, *B*, and *C* are in transitive relations of superiority, some others are in intransitive ones, and it depends on their content). While choosing between intransitive dice, one should prefer dice *A* to dice *B* in the pair *A*-*B*, *B* to *C* in pair *B*-*C*, and *C* to *A* in pair *A*-*C*. Currently, numerous educational videos can be found via Internet searches for the terms intransitive dice and non-transitive dice (e.g., [Lawler, 2017]). Various problems ranging in com-


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plexity are designed to promote intransitivity understanding in students of various ages and educational levels (from secondary to higher school settings) in different areas including not only math but also biology, sociology, etc. [Beardon, 1999/2011; Scheinerman, 2012; Stewart, 2010; Strogatz, 2015]. One should agree with T. Roberts, who writes:

> Transitivity and intransitivity are fascinating concepts that relate both to mathematics and to the real world we live in. A couple of lessons devoted to this topic are almost certain to interest and engage students of almost any age, as they seek to discover which relationships are transitive, and which are not, and further to try to discover any general rules that might distinguish between the two [Roberts, 2004].

The only clarification that can be made is that a couple of lessons may be enough for students' primary engagement in the topic, but hardly enough for its detailed analysis — see, for example, the analysis of intransitive dice by Fields medalist T. Gower [2017] in the pages of his Polymath project. If P. C. Fishburn's [1991] analogy between an advanced understanding of intransitivity and non-Euclidean geometry is right (we agree with it), the levels of complexity of the issue can very high. Yet the initial levels, even related to exact reasoning, can be (unexpectedly) simple. Let us consider this in more detail.

All of the intransitive math objects presented in math studies and in problems for students deal with numbers, mostly with probabilities which are not evident and must be counted. In this article we present geometrical and mechanical constructions in intransitive relations of superiority. From a mathematical view, it is a new class of intransitive objects. From an educational view, they can be considered in the framework of the Vygotskian theory of cultural tools (e.g., [Erickson, 1999]) including manipulatives. "Manipulatives are tools students use to support meaningful learning" and to "construct new insights" [Cramer, Wyberg, 2009]. Our manipulatives, intransitive geometrical and mechanical constructions, show deterministic (not probabilistic) intransitive relations in an evident way. The objects vary in complexity from very simple to advanced. The simplest ones demonstrate such intransitivity that can be understood at a qualitative level and does not require quantitative reasoning.

A note on terminology: in the math literature, the terms "intransitive" and "non-transitive" (e.g., "intransitive dice" and "non-transitive dice") are used as synonyms in spite of some difference between the logical terms "intransitive relation" and "non-transitive relation". In this article we will use the term "intransitive" as explicitly related to the concept of intransitive cycles.

## DESCRIPTION OF INTRANSITIVE GEOMETRICAL AND MECHANICAL CONSTRUCTIONS

All of the objects are designed as Condorcet-like compositions, in correspondence with the structure of the Condorcet paradox (or the voting paradox; [Beardon, 1999/2011]).

Our geometrical interpretation of the paradox is that we use chains of geometrical elements ordered like elements in the Condorcet paradox (originally, voters' preferences, but it does not matter here): *ABC*, *BCA*, *CAB*. One can see that the first element of any set moves to the last position in the next set and moves all the other elements one position to the left without changing the sequence.

As an example, let us consider such counter-intuitive objects as intransitive double gears (or friction wheels). The notation of elements will be the following: *X* is a larger gear (a larger wheel), *Y* is a smaller gear (a smaller wheel), and *Z* is an empty part of a shaft (without any gear or wheel on it).

Then, in correspondence with the Condorcet paradox:

- the first double-gear (*A*) will have the element sequence *X*, *Y*, *Z*;
- the second double-gear (*B*) will have the element sequence *Z*, *X*, *Y*; and
- the third double-gear (*C*) will have the element sequence *Y*, *Z*, *X*.

Figure 1(*c*) shows that, if joined in pairs, *A*'s rotational speed is higher than *B*'s in the pair *A*-*B*; the rotational speed of *B* is higher than that of *C* in the pair *B*-*C*; but the rotational speed of *C* is higher than the rotational speed of *A* in pair *A*-*C* [Poddiakov, 2010; Poddiakov, Valsiner, 2013].

The same principle of design is applied to other objects. Let us consider three geometrical blocks modeling tractors with different shapes of towing couplers (see Fig. 1*d*). Tractor *A* has a triangle lug at the front to be coupled as a trailer by another tractor, and a square hole from behind to couple another tractor as a trailer. Tractor *B* has a square lug at the front to be coupled as a trailer by another tractor, and a circle hole from behind to couple another tractor as a trailer. Tractor *C* has a circle lug at the front to be coupled as a trailer by another tractor, and a triangle hole from behind to couple another tractor as a trailer. A driver stands near the tractors. Which tractor should the driver choose as a leading one to sit in it if s/he has an aim to bring:

- — couple *A*-*B*;
- — couple *B*-*C*;
- — couple *A*-*C*

to a destination point?

One can see that the driver should choose *A* in couple *A*-*B*, *B* in couple *B*-*C*, and *C* in couple *A*-*C*. This model of intransitive relations does not require quantitative comparisons, counting, an understanding of probability, or other operations required to understand more complex intransitive objects like intransitive dice or playing cards. Distinction and comparison of geometrical shapes is all that is necessary here (besides an understanding of the task statement).

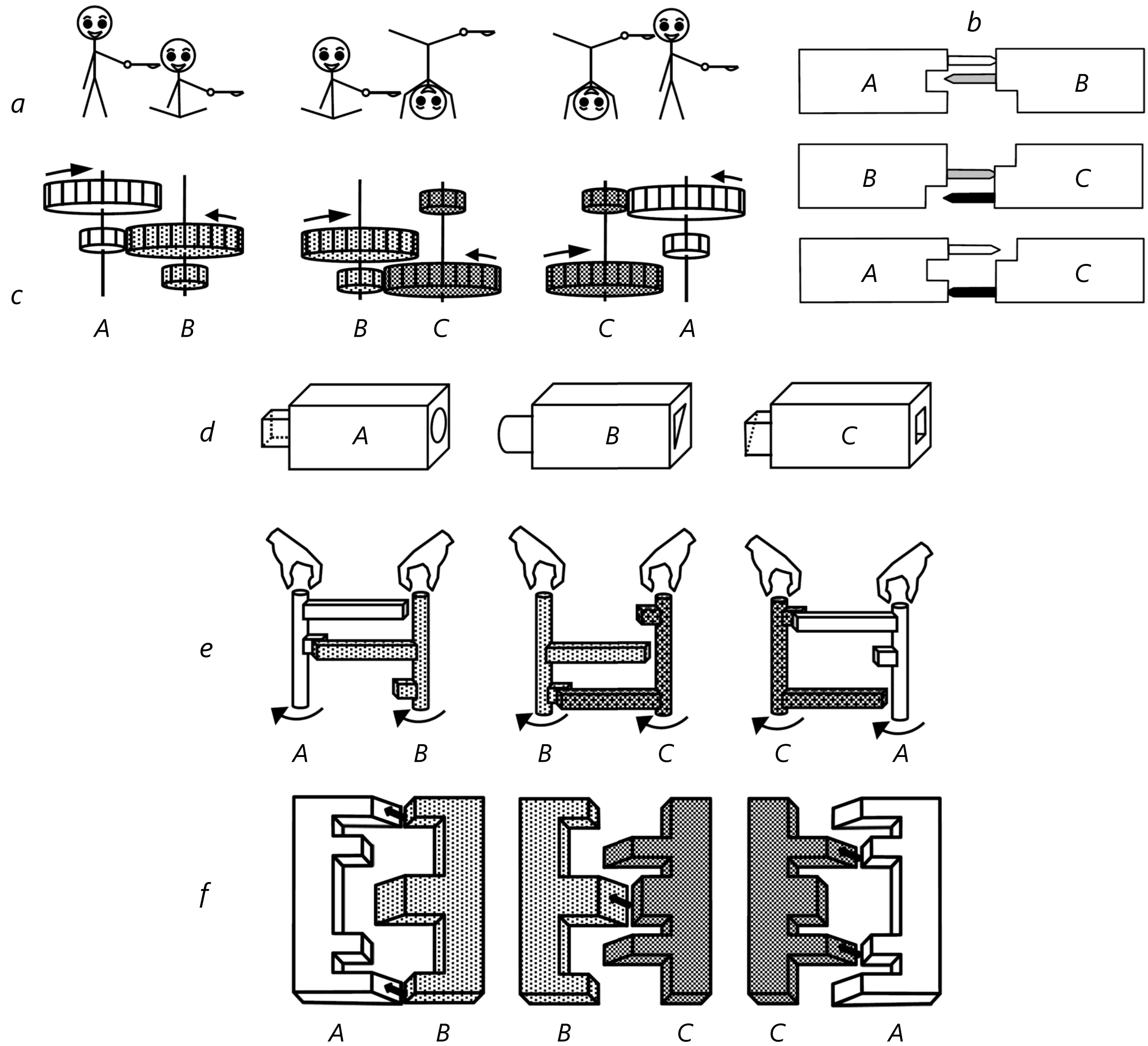


***Fig. 1.*** Examples of intransitive geometrical Condorcet-like compositions:
(*a*) toy Monkeys feeding one another;
(*b*) stylized plastic Mobile Assault Towers marking one another with inserted felt-tip pens;
(*c*) Intransitive Double Gears with intransitive speeds of rotation;
(*d*) stylized Tractors with intransitive towing couplers;
(*e*) Intransitive Double Levers (with the same rotation force applied to the shaft, Lever *A* will overpower Lever *B*, Lever *B* will overpower Lever *C* and Lever *C* will overpower Lever *A*);
(*f*) stylized Combs with Intransitive Ramps (Comb *A* can serve as a ramp for Comb *B* and lift it but not vice versa, Comb *B* can lift Comb *C* but not vice versa, and Comb *C* can lift Comb *A* but not vice versa).

## DISCUSSION

In spite of a rich tradition of math studies of various intransitive objects, there is no appropriate tradition of studies of understanding (misunderstanding) intransitive objects in cognitive and educational psychology. Owing to the brilliant Piagetian works in the area of cognitive and developmental psychology, the main trend is related to studies of abilities to make transitive inferences (if $A > B$ and $B > C$ then $A > C$) about transitive options (e.g., lengths of sticks) [Andrews, Halford 1998; Andrews, Hewitt-Stubbs, 2015; Camarena et al., 2018; Mou, Province, Luo, 2014; Shultz, Vogel, 2004]. Naturally, for such options violations of transitivity are a fallacy. Before math studies of intransitivity, this approach could have seemed universal. Even opponents of Piaget like Trabasso [Bryant, Trabasso, 1971] questioned not the status of transitivity as a normative rule and its violations as fallacies, but only age and conditions in which, for example, children already demonstrate that they can master transitivity and not violate it. Yet how can people understand mathematical intransitive objects? How are they solving intransitivity problems designed by math educators? More generally: how are abilities to reveal non-evident intransitive relations and to make inferences about objective intransitivity (e.g., about intransitivity of intransitive dice, athletes' teams, game strategies, etc.) developing in different domains (or as a general complex)? How are these abilities related to abilities to make "classical" transitive inferences? These questions have not yet been answered.

A possible theoretical framework which can include both 1) beliefs about the transitivity of superiority as an axiom with a ban on its violations and 2) beliefs about intransitivity as an objective property of complex (systems) interactions between multi-variable objects involves the distinction of four types of situations [Poddiakov, 2010].

(1) Relations of superiority between objects are objectively transitive (e.g., in case of three sticks), and a problem solver makes correct conclusions about their transitivity.

(2) Relations are objectively transitive, but a problem solver wrongly considers them as intransitive. Most studies are conducted in this paradigm.

(3) Relations of superiority between objects are objectively intransitive (e.g., relations between three or more sets, each of which contains three or more sticks having lengths equal to numbers on sides of intransitive dice, are intransitive — in contrast to the situation of a comparison of just three sticks), and a problem solver makes correct conclusions about their intransitivity.

(4) Relations of superiority between objects are objectively intransitive, but a problem solver wrongly considers them as transitive (e.g., because of taking the transitivity axiom for granted).

Here one can roughly distinguish between four levels of complexity of intransitive relations of superiority. This classification is not exhaustive and serves to mark some reference points.

(a) Simple combinatorial intransitivity between non-interacting objects (e.g., in intransitive dice sets, intransitive sets of sticks, etc.). Each object can be exactly described by a few parameters (like numbers on the sides of dice). The parameters are additive, without interactions: the sticks' intransitive sets do not interact with one another, only the sticks' lengths are compared, and comparisons are possible even without immediate touching. Information about the objects is complete.

(b) Interactive intransitivity without qualitative transformations of the objects participating in the intransitive relations. Information about the objects and their interactions is complete. An example is the intransitivity between interacting geometrical and mechanical objects described above. The intransitive gears are rotating at different speeds as a result of the intransitive interactions, but there are no qualitative transformations of the gears.

(c) Interactive intransitivity with qualitative transformations of the objects participating in the intransitive relations. Information about the objects and their interactions is complete. This intransitivity can be observed between pieces' positions in strategy games like chess. Position *A* for White is preferable to Position *B* for Black (i.e., when offered a choice, one should choose *A*), Position *B* for Black is preferable to Position *C* for White, which is preferable to Position *D* (Black) — but the latter is preferable to Position *A* (White) [Poddiakov, 2017]. The positions qualitatively transform after each move.

(d) Interactive "rhizomatic" (multiple, intertwining) intransitivity of superiority in real complex systems. A body of biological studies is devoted to the complex intransitive competitions of various species and individuals in ecological niches; for a review see Permogorskiy [2015]. Such competition transforms participants. Information about the participants, their features and interactions is incomplete for the participants and for observers (researchers) because of complexity and the multiplicity of interactions and permanent changes of the participants themselves and their strategies.

## CONCLUSION

Let us get back to the statement that "if you have violated the transitivity axiom, you are not instrumentally rational, and the content of *A*, *B*, and *C* do not matter" in an educational context. The main message for students in teaching and learning transitivity-intransitivity can be more multi-dimensional and not so straightforward. In complex and multi-variable situations, intransitive choices are perfectly rational because the choice options are in intransitive relations of superiority (like intransitive dice). That is, transitive choices of intransitive options are a fallacy. Here any attempts of linear, transitive ordering of options lead to a loss. By contrast, in situations of objective transitivity, any intransitive cycle of choices of options ends in a loss, and one must solve problems related to the building of linear hierarchies. Various educational tools can be used to support students' understanding of these different types of situations. The aim of our future research will be testing opportunities that use of some of the objects described above to support intransitivity understanding.

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