DUAL QUADRANGLES IN THE PLANE

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Abstract. We consider quadrangles of perimeter 2 in the plane with marked directed edge. To such quadrangle Q a two-dimensional plane $\Pi \in \mathbb{R}^4$ with orthonormal base is corresponded. Orthogonal plane $\Pi^\perp$ defines a plane quadrangle Q° of perimeter 2 and with marked directed edge. This quadrangle is defined uniquely (up to rotation and symmetry). Quadrangles Q and Q° will be called dual to each other. The following properties of duality are proved: a) duality preserves convexity, non convexity and self-intersection; b) duality preserves the length of diagonals; c) the sum of lengths of corresponding edges in Q and Q° is 1.

1. Introduction

We follow the work [1] (see also the bibliography there). Let Q be a quadrangle with perimeter 2 and with marked directed edge in plane $\mathbb{R}^2$. It means that we indicate the first vertex and the direction of going around of Q.

Remark 1.1. If perimeter of a quadrangle is not 2, then we made a dilation with some positive α.

Let $Q = ABCD$ and A be the first vertex. Vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ we will consider as complex numbers z_1, z_2, z_3 and z_4, respectively. Then

$$z_1 + z_2 + z_3 + z_4 = 0, \quad \text{and} \quad |z_1| + |z_2| + |z_3| + |z_4| = 2.$$

Remark 1.2. The above complex description of Q is invariant with respect to a translation.

In what follows we will consider only *non degenerate* quadrangles (with one exception in Section 4), i.e. quadrangles with non-collinear successive edges.

Let's define complex numbers u_1, u_2, u_3, u_4 in the following way: a) $u_k^2 = z_i$, k=1,2,3,4; b) u_1 we choose arbitrarily; c) the rotation from u_k to u_{k+1}, $k = 1, 2, 3$ is in the same direction as the rotation from z_k to z_{k+1}. Let $u_k = a_k + i\,b_k$, $k = 1, 2, 3, 4$, then

$$\sum_k (a_k^2 + b_k^2) = 2 \quad \text{and} \quad \sum_k [a_k^2 - b_k^2) + 2i\, a_k b_k] = 0,$$

i.e. $\bar{a} = (a_1, a_2, a_3, a_4)$ and $\bar{b} = (b_1, b_2, b_3, b_4)$ are a pair of orthonormal vectors in $\mathbb{R}^4$. Let $\Pi = \langle \bar{a}, \bar{b} \rangle$ be the linear hull. The two-dimensional plane Π uniquely defines its orthogonal complement — the two-dimensional plane $\Pi^\perp$. An orthonormal base $(\bar{c}, \bar{d})$ of $\Pi^\perp$ in its turn defines a quadrangle Q° of perimeter 2, which will be called the quadrangle *dual* to the quadrangle Q.

We will prove the following properties of the quadrangle duality.

- The dual quadrangle Q° is defined uniquely up to rotation and reflection (Theorem 2.1).
- The change of the first vertex and the direction of the going around of Q does not change the dual quadrangle Q° (Theorem 2.2).
- The duality preserves: a) convexity (Corollary 5.1); b) non-convexity (Theorem 5.1); c) self-intersection (Theorem 4.1).
- The sum of lengths of corresponding edges (in the sense of Section 3) of Q and Q° is 1 (Theorem 6.1).
- The lengths of the corresponding diagonals of Q and Q° are equal (Theorem 7.1).
- Parallelograms are self-dual (Theorem 8.1).

2. General remarks

Our definition of the dual quadrangle Q° is not strictly correct, because the base $(\bar{c},\bar{d})$ of $\Pi^\perp$ is not unique: it is defined up to a rotation and up to the order of base vectors.

Theorem 2.1. *The plane $\Pi^\perp$ uniquely defines the dual quadrangle up to a rotation and up to a reflection.*

Proof. Let a base $(\bar{e},\bar{f})$ of $\Pi^\perp$ be obtained by the rotation of $(\bar{c},\bar{d})$ on an angle α. Thus,

$$e_k = c_k\cos(\alpha) - d_k\sin(\alpha), \quad f_k = c_k\sin(\alpha) + d_k\cos(\alpha),\ k=1,2,3,4,$$

i.e.

$$e_k + i\,f_k = (c_k + i\,d_k)e^{i\alpha} \Rightarrow (e_k + i\,f_k)^2 = (c_k + i\,d_k)^2 e^{2i\alpha},\ k=1,2,3,4.$$

Hence, the rotation of base of $\Pi^\perp$ on angle α implies the rotation of Q° on angle 2α.

Let us now consider the base $(\bar{d},\bar{c})$, instead of the base $(\bar{c},\bar{d})$, then

$$\begin{array}{l} \mathrm{Re}\left((d_k + i\,c_k)^2\right) = -\mathrm{Re}\left((c_k + i\,d_k)^2\right) \\ \mathrm{Im}\left((d_k + i\,c_k)^2\right) = \mathrm{Im}\left((c_k + i\,d_k)^2\right) \end{array} \qquad k=1,2,3,4,$$

i.e. this change of base implies the reflection of Q° with respect to the axis OY. □

Let $ABCD$ be the quadrangle Q, where A is the first vertex and the order $ABCD$ defines the direction of going around. Let $(\bar{c},\bar{d})$ be the base of $\Pi^\perp$ and $KLMN$ be vertices of Q° (K is the first vertex and the order $KLMN$ defines the direction of going around).

Theorem 2.2. *The dual of quadrangle Q does not depend on the choice of the first vertex and on the direction of the going around.*

Proof. Let us consider the going around of $Q = ABCD$ in the same direction, but the first vertex be B, i.e. $Q = BCDA$. Complex numbers z_1, z_2, z_3, z_4 are the same, but in order z_2, z_3, z_4, z_1. Complex numbers u_2, u_3, u_4 are the same, but the last one may be u_1 or $-u_1$. If the last number is u_1, then $\Pi = \langle (a_2,a_3,a_4,a_1),(b_2,b_3,b_4,b_1)\rangle$ and $\Pi^\perp = \langle (c_2,c_3,c_4,c_1),(d_2,d_3,d_4,d_1)\rangle$. Thus, if $KLMN$ is the original dual quadrangle, then $LMNK$ is the new one, but the same. If the last number is $-u_1$, then $\Pi = \langle (a_2,a_3,a_4,-a_1),(b_2,b_3,b_4,-b_1)\rangle$ and $\Pi^\perp = \langle (c_2,c_3,c_3,-c_1),(d_2,d_3,d_4,-d_1)\rangle$, i.e. the result is the same because $(-c_1 - i\,d_1)^2 = (c_1 + i\,d_1)^2$.

Now let us consider the going around in the opposite direction, hence, $Q = ADCB$. In this case we must consider complex numbers $-z_4, -z_3, -z_2, -z_1$ and their square roots $\pm i\, u_4$, $\pm i\, u_3$, $\pm i\, u_2$, $\pm i\, u_1$. Thus

$$\Pi = \langle \mp(a_4, a_3, a_2, a_1), \pm(b_4, b_3, b_2, b_1)\rangle$$

and

$$\Pi^{\perp} = \langle (c_4, c_3, c_2, c_1), (d_4, d_3, d_2, d_1)\rangle,$$

i.e. $Q^{\circ} = KNML$. □

Remark 2.1. The rotation of Q does not change the plane Π.

Corollary 2.1. *Let Q° is dual to Q and $(Q^{\circ})^{\circ}$ is dual to Q°, then $Q = (Q^{\circ})^{\circ}$ up to a rotation and up to a reflection.*

3. The main construction

In this section, using the knowledge of lengths of edges and angles of the quadrangle Q we will construct the base of the plane $\Pi^{\perp}$.

Let $Q = ABCD$ be positioned in the following way: A is at the origin, B is in the positive real axis, C and D are in the upper half-plane. Let $|AB| = s_1$, $|BC| = s_2$, $|CD| = s_3$ and $|DA| = s_4$. 4-dimensional vectors $\bar{a} = (a_1, a_2, a_3, a_4)$ and $\bar{b} = (0, b_2, b_3, b_4)$ will be considered as quaternions a and b. Let $v = (0, a_2, a_3, a_4)$, $|v|^2 = a_2^2 + a_3^2 + a_4^2 = 1 - a_1^2 = 1 - s_1$. We consider quaternion products $g = a \cdot v = (-a_2^2 - a_3^2 - a_4^2, a_1a_2, a_1a_3a_1a_4) = (s_1 - 1, a_1a_2, a_1a_3, a_1a_4)$ and $h = b \cdot v = (0, b_3a_4 - b_4a_3, b_4a_2 - b_2a_4, b_2a_3 - b_3a_4)$. The corresponding vectors $\bar{g}$ and $\bar{h}$ constitute an orthogonal base of $\Pi^{\perp}$ (not orthonormal, because $|g| = |h| = \sqrt{1 - s_1}$).

Let Q be a convex quadrangle

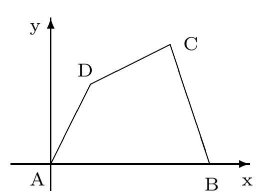

Figure 1

Then

$$z_1 = s_1,\ z_2 = s_2 \exp(\pi - \beta_2),\ z_3 = s_3 \exp(2\pi - \beta_2 - \beta_3),\ z_4 = s_4 \exp(\pi + \beta_1),$$

where $\angle DAB = \beta_1$, $\angle ABC = \beta_2$, $\angle BCD = \beta_3$ and $\angle CDA = \beta_4$. Thus,

$$\begin{aligned} \bar{a} &= [\sqrt{s_1}, \sqrt{s_2}\sin(\gamma_2), -\sqrt{s_3}\cos(\gamma_2 + \gamma_3), -\sqrt{s_4}\sin(\gamma_1)], \\ \bar{b} &= [0, \sqrt{s_2}\cos(\gamma_2), \sqrt{s_3}\sin(\gamma_2 + \gamma_3), \sqrt{s_4}\cos(\gamma_1)], \end{aligned}$$

where $\gamma_k = \beta_k/2$, $k = 1, 2, 3, 4$, and

$$\begin{aligned} \bar{g} &= [s_1 - 1, \sqrt{s_1 s_2}\sin(\gamma_2), -\sqrt{s_1 s_3}\cos(\gamma_2 + \gamma_3), -\sqrt{s_1 s_4}\sin(\gamma_1)], \\ \bar{h} &= [0, -\sqrt{s_3 s_4}\cos(\gamma_4), \sqrt{s_2 s_4}\sin(\gamma_1 + \gamma_2), -\sqrt{s_2 s_3}\cos(\gamma_3)] \end{aligned} \tag{1}$$

If our quadrangle Q is non convex

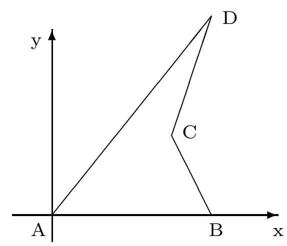

Figure 2

then

$$z_1 = s_1,\ z_2 = s_2 \exp(\pi - \beta_2),\ z_3 = s_3 \exp(\beta_3 - \beta_2),\ z_4 = s_4 \exp(180 + \beta_1),$$

where $\angle DAB = \beta_1$, $\angle ABC = \beta_2$, $\angle BCD = \beta_3$ and $\angle CDA = \beta_4$. And

$$\begin{array}{l} \bar{a} = [\sqrt{s_1}, \sqrt{s_2}\sin(\gamma_2), \sqrt{s_3}\cos(\gamma_3 - \gamma_2), -\sqrt{s_4}\sin(\gamma_1)], \\ \bar{b} = [0, \sqrt{s_2}\cos(\gamma_2), \sqrt{s_3}\sin(\gamma_3 - \gamma_2), \sqrt{s_4}\cos(\gamma_1)]. \end{array}$$

Thus

$$\begin{array}{l} \bar{g} = [s_1 - 1, \sqrt{s_1 s_2}\sin(\gamma_2), \sqrt{s_1 s_3}\cos(\gamma_3 - \gamma_2), -\sqrt{s_1 s_4}\sin(\gamma_1)], \\ \bar{h} = [0, -\sqrt{s_3 s_4}\cos(\gamma_4), \sqrt{s_2 s_4}\sin(\gamma_1 + \gamma_2), \sqrt{s_2 s_3}\cos(\gamma_3)]. \end{array} \tag{2}$$

If at last our quadrangle Q is self-intersecting

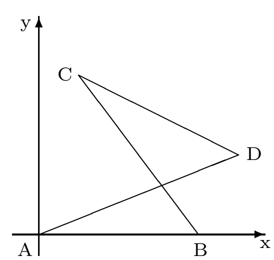

Figure 3

then

$$z_1 = s_1,\ z_2 = s_2 \exp(\pi - \beta_2),\ z_3 = s_3 \exp(\beta_3 - \beta_2), z_4 = s_4 \exp(\beta_1 - \pi),$$

where $\angle ABC = \beta_2$, $\angle BCD = \beta_3$, $\angle CDA = \beta_4$, $\angle DAB = \beta_1$. And

$$\begin{array}{l} \bar{a} = [\sqrt{s_1}, \sqrt{s_2}\sin(\gamma_2), \sqrt{s_3}\cos(\gamma_3 - \gamma_2), \sqrt{s_4}\sin(\gamma_1)], \\ \bar{b} = [0, \sqrt{s_2}\cos(\gamma_2), \sqrt{s_3}\sin(\gamma_3 - \gamma_2), -\sqrt{s_4}\cos(\gamma_1)]. \end{array}$$

Thus,

$$\begin{array}{l} \bar{g} = [s_1 - 1, \sqrt{s_1 s_2}\sin(\gamma_2), \sqrt{s_1 s_3}\cos(\gamma_3 - \gamma_2), \sqrt{s_1 s_4}\sin(\gamma_1)], \\ \bar{h} = [0, \sqrt{s_3 s_4}\cos(\gamma_4), -\sqrt{s_2 s_4}\sin(\gamma_1 + \gamma_2), \sqrt{s_2 s_3}\cos(\gamma_3)]. \end{array} \tag{3}$$

4. Self-intersecting quadrangles

Theorem 4.1. *The quadrangle, dual to a self-intersecting quadrangle, is also self-intersecting.*

Proof. Let Q be a self-intersecting quadrangle $ABCD$ (see Figure 3) and $Q^\circ = KLMN$ — its dual. As z_2 belongs to the upper half-plane, then $a_2 > 0$ and $g_2 > 0$. As $a_3b_4 - a_4b_3 < 0$, because of the clockwise turn from z_3 to z_4, then $h_2 > 0$ and $(g_2 + i\,h_2)^2$ belongs to the upper half-plane. Thus, M also belongs to the upper half-plane (K is at origin and $L = (1 - s_1, 0)$).

As the turn from z_2 to z_3 is clockwise, then $a_2b_3 - a_3b_2 < 0$ and $h_4 > 0$. As $g_4 > 0$, then $(g_4 + i\,h_4)^2$ belongs to the upper half-plane. Thus, N belongs to the lower half-plane (the direction of the vector $\overline{NK}$ is up), i.e. Q° cannot be convex — M and N belong to different half-planes with respect to KL.

Now we will demonstrate that Q° is self-intersecting. Let us consider the following highly symmetric quadrangle $Q_0 = ABCD$

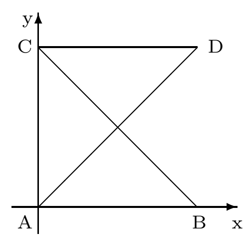

Figure 4

where all angles β_k, $k = 1, 2, 3, 4$, are $\pi/3$ (see Figure 3). Its dual $Q_0^\circ = KLMN$

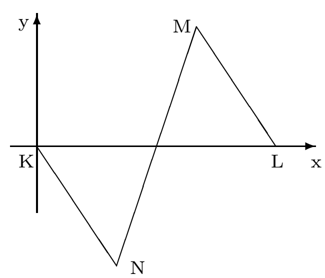

Figure 5

is the same quadrangle, rotated clockwise on $\pi/3$. Let us assume that there exists a self-intersecting quadrangle Q_1 with non-convex dual (see below)

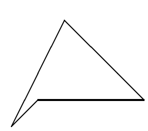

Figure 6

Let Q_t, $0 \leqslant t \leqslant 1$, be a continuous family of non-degenerate self-intersecting quadrangles, that connects Q_0 with Q_1. We construct this family by moving vertices B, C and D. Then the continuous family Q_t° connects Q_0° with Q_1°. Hence, for some α, $0 < \alpha < 1$, the dual quadrangle Q_α° must be degenerate:

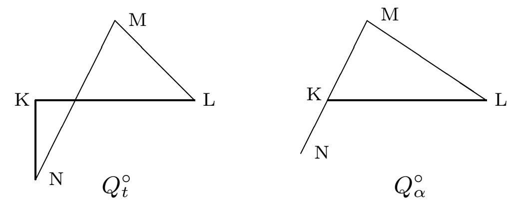

Figure 7

Now we will consider quadrangles $(Q_t^\circ)^\circ$ and take the limit for $t \to \alpha$. Let us consider the quadrangle Q_t° in the left part of Figure 7 and let (with some abusing of the notation) $|KL| = s_1$, $|LM| = s_2$, $|MN| = s_3$, $|NK| = s_4$, $\angle NKL = \beta_1$, $\angle KLM = \beta_2$, $\angle LMN = \beta_3$, $\angle MNK = \beta_4$. Then

$$\begin{aligned}\bar{a}_t &= [\sqrt{s_1}, \sqrt{s_2}\sin(\gamma_2), -\sqrt{s_3}\cos(\gamma_2+\gamma_3), \sqrt{s_4}\sin(\gamma_2+\gamma_3-\gamma_4)],\\ \bar{b}_t &= [0, \sqrt{s_2}\cos(\gamma_2), \sqrt{s_3}\sin(\gamma_2+\gamma_3), \sqrt{s_4}\cos(\gamma_2+\gamma_3-\gamma_4)].\end{aligned}$$

When $t \to \alpha$, then Q_t° on the left (Figure 7) is transformed into Q_α° on the right, with angles $\angle MKL = \tilde\beta_1$, $\angle KLM = \tilde\beta_2$ and $\angle LMK = \tilde\beta_3$. When $t \to \alpha$, then $\beta_4 \to 0$, $\beta_2 \to \tilde\beta_2$, $\beta_3 \to \tilde\beta_3$ and $\beta_1 \to \pi - \tilde\beta_1$. Hence,

$$\begin{aligned}\bar{a}_\alpha &= [\sqrt{s_1}, \sqrt{s_2}\sin(\tilde\gamma_2), -\sqrt{s_3}\sin(\tilde\gamma_1), \sqrt{s_4}\cos(\tilde\gamma_1)],\\ \bar{b}_\alpha &= [0, \sqrt{s_2}\cos(\tilde\gamma_2), \sqrt{s_3}\cos(\tilde\gamma_1), \sqrt{s_4}\sin(\tilde\gamma_1)].\end{aligned}$$

Thus,

$$\begin{aligned}\bar{g}_\alpha &= [\sqrt{1-s_1}, \sqrt{s_1s_2}\sin(\tilde\gamma_2), -\sqrt{s_1s_3}\sin(\tilde\gamma_1), \sqrt{s_1s_4}\cos(\tilde\gamma_1)],\\ \bar{h}_\alpha &= [0, \sqrt{s_3s_4}, -\sqrt{s_2s_4}\cos(\tilde\gamma_1+\tilde\gamma_2), -\sqrt{s_2s_3}\sin(\tilde\gamma_1+\tilde\gamma_2)].\end{aligned}$$

The quadrangle, constructed with the use of vectors $\bar{g}_\alpha$ and $\bar{h}_\alpha$, belongs to our family Q_t (Corollary 2.1). Now let us consider complex numbers $(g_\alpha)_3 + i(h_\alpha)_3$, $(g_\alpha)_4 + i(h_\alpha)_4$ and compute the product

$$\frac{(h_\alpha)_3}{(g_\alpha)_3} \cdot \frac{(h_\alpha)_4}{(g_\alpha)_4} =$$
$$= \frac{\sqrt{s_2s_4}\cos(\tilde\gamma_1+\tilde\gamma_2)}{\sqrt{s_1s_3}\sin(\tilde\gamma_1)} \cdot -\frac{\sqrt{s_2s_3}\sin(\tilde\gamma_1+\tilde\gamma_2)}{\sqrt{s_1s_4}\cos(\tilde\gamma_1)} = -\frac{s_2\sin(\tilde\beta_3)}{s_1\sin(\tilde\beta_1)} = -1$$

(because in triangle the ratio of an edge to the sine of the opposite angle is constant and equal to the diameter of the circumscribed circle). As this product is -1 then the corresponding vectors are orthogonal. Thus the squaring of this complex numbers produces collinear vectors. Hence, the quadrangle $(Q_\alpha^\circ)^\circ$ is degenerate. But it cannot be so, because it belongs to our non-degenerate family. $\square$

5. Non-convex quadrangles

Theorem 5.1. *If Q is non-convex quadrangle, then its dual is also non convex.*

Proof. Let Q be a quadrangle in Figure 2. As z_2 belongs to the upper half-plane, then $a_2 > 0$, thus $g_2 > 0$. As $a_3b_4 - a_4b_3 > 0$, because of the counter clockwise turn from z_3 to z_4, then $h_2 < 0$, i.e. $(g_2 + i\,h_2)^2$ belongs to the lower half-plane. Thus, the vertex M lies in the lower half-plane.

As z_4 belongs to the lower half-plane, then $a_4 < 0$, $b_4 > 0$, thus $g_4 < 0$. As $a_2b_3 - a_3b_2 < 0$, because of the clockwise turn from z_2 to z_3, then $h_4 > 0$, i.e. $(g_4 + i\,h_4)^2$ belongs to the lower half-plane. Thus, the vertex N lies in the upper half-plane, i.e. vertices M and N lie in different half-planes with respect to edge KL. Hence, the quadrangle $KLMN$ cannot be convex. But by Theorem 4.1. it cannot be self-intersecting, so it is non-convex. □

Corollary 5.1. *The dual to a convex quadrangle is also convex.*

6. Edges

Theorem 6.1. *Let $Q = ABCD$ be a convex quadrangle and $Q^\circ = KLMN$ be its dual. Then $|AB| + |KL| = 1$, $|BC| + |LM| = 1$, $|CD| + |MN| = 1$ and $|DA| + |NK| = 1$.*

Proof. As $|g| = |h| = \sqrt{1 - s_1}$, then have to prove that $|(g_2 + i\,h_2)^2| = g_2^2 + h_2^2 = (1 - s_1)(1 - s_2)$. Let $|AC| = l$, then

$$
\begin{aligned}
g_2^2 + h_2^2 &= s_1s_2\sin^2(\gamma_2) + s_3s_4\cos^2(\gamma_4) = \\
&[s_1s_2 - s_1s_2\cos(\beta_2) + s_3s_4 + s_3s_4\cos(\beta_4)]/2 = \\
&\left[s_1s_2 + (l^2 - s_1^2 - s_2^2)/2) + s_3s_4 - (l^2 - s_3^2 - s_4^2)/2\right]/2 = \\
&[(s_3 + s_4)^2 - (s_1 - s_2)^2]/4 = [(s_3 + s_4 + s_1 - s_2)(s_3 + s_4 - s_1 + s_2)] = \\
&= (1 - s_2)(1 - s_1).
\end{aligned}
$$

The same reasoning proves that $|NK| = 1 - s_4$. As perimeters of Q and Q° are 2, then $|MN| = 1 - s_3$. □

Remark 6.1. The same reasoning proves the theorem for non-convex and self-intersecting quadrangles.

7. Diagonals

Theorem 7.1. *Let $Q = ABCD$ be a convex quadrangle and $Q^\circ = KLMN$ be its dual, then $|AC| = |KM|$ and $|BD| = |LN|$, i.e. the duality preserves lengths of diagonals.*

Proof. Let $l = |AC| = |z_1 + z_2| = |z_3 + z_4|$. We will prove, that $|g_1^2 + (g_2 + i\,h_2)^2| = (1 - s_1)l$. At first we will find the real part of the complex number $(g_2 + i\,h_2)^2$:

$$
\begin{aligned}
\mathrm{Re}(g_2 + i\,h_2)^2 = g_2^2 - h_2^2 &= s_1s_2\sin^2(\gamma_2) - s_3s_4\cos^2(\gamma_4) = \\
&[s_1s_2(1 - \cos(\beta_2) - s_3s_4(1 + \cos(\beta_4))]/2 = \\
&= [2s_1s_2 + l^2 - s_1^2 - s_2^2 - 2s_3s_4 + l^2 - s_3^2 - s_4^2]/4 = \\
&= [2l^2 - (s_1 - s_2)^2 - (s_3 + s_4)^2]/4.
\end{aligned}
$$

Now the real part of $g_1^2+(g_2+i\,h_2)^2$ is

$$\begin{gathered}(1-s_1)^2+[2l^2-(s_1-s_2)^2-(s_3+s_4)^2]/4=\\ =[4(1-s_1)^2+2l^2-(s_1-s_2)^2-(s_3+s_4)^2]/4=\\ =[2(1-s_1)^2+2l^2+(1-2s_1+s_2)(1-s_2)+\\ +(1-s_1+s_3+s_4)(1-s_1-s_3-s_4)]/4=\\ =[2(1-s_1)^2+2l^2+(1-2s_1+s_2)(1-s_2)+(s_2-1)(3-2s_1-s_2)]/4=\\ =[2(1-s_1)^2+2l^2-2(1-s_2)^2]/4=[l^2-(s_1-s_2)(s_3+s_4)]/2.\end{gathered}$$

Now we will find the square of the imaginary part of $g_1^2+(g_2+i\,h_2)^2$:

$$\begin{aligned}&4s_1s_2s_3s_4\sin^2(\gamma_2)\cos^2(\gamma_4)=\\ &\qquad=s_1s_2(1-\cos(\beta_2)s_3s_4(1+\cos(\beta_4))=\\ &\qquad=(2s_1s_2+l^2-s_1^2-s_2^2)(2s_3s_4+s_3^2+s_4^2-l^2)/4=\\ &\qquad=(l^2-(s_1-s_2)^2)((s_3+s_4)^2-l^2)/4.\end{aligned}$$

At last we can find $|g_1^2+(g_2+i\,h_2)^2|^2$:

$$\begin{aligned}&[(l^2-(s_1-s_2)(s_3+s_4))^2+(l^2-(s_1-s_2)^2)((s_3+s_4)^2-l^2)]/4=\\ &\qquad=[l^2(-2(s_1-s_2)(s_3+s_4)+(s_3+s_4)^2+(s_1-s_2)^2)]/4=\\ &\qquad=l^2[(s_3+s_4-s_1+s_2)^2]/4=l^2(1-s_1)^2.\end{aligned}$$

Analogously, we can prove that $|g_1^2+(g_4+i\,h_4)^2|=(1-s_1)\cdot|BD|$. ☐

Remark 7.1. The statement of this theorem is also valid for non-convex and self-intersecting quadrangles. The reasoning is the same.

8. Special cases

Theorem 8.1. *The dual to a trapezoid is a trapezoid.*

Proof. Let $Q=ABCD$ be a trapezoid, where $AB\parallel CD$:

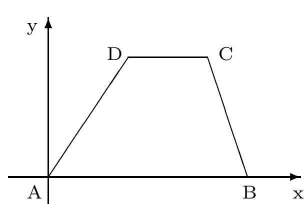

Figure 8

Here z_3 is a negative real number, hence, $u_3=\alpha\,i$, $\alpha>0$, hence, $g_3=0$, hence $(g_3+i\,h_3)^2$ is a negative real number. ☐

Theorem 8.2. *The dual to a parallelogram is the same parallelogram.*

Proof. Let $Q=ABCD$ be a parallelogram. As $|AB|+|BC|=1$, then $|KL|=|BC|$ and $|LM|=|AB|$. It remains to note that $|AC|=|KM|$. ☐

9. The geometric construction

Given a convex quadrangle Q it is easy to construct the dual Q°, using ruler and compass.

Let $Q = ABCD$ be a convex quadrangle

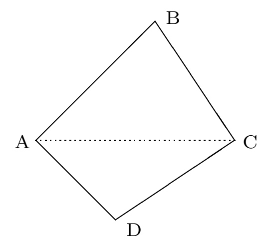

Figure 9

with diagonal AC. Let $|AB| = s_1$, $|BC| = s_2$, $|CD| = s_3$ and $|DA| = s_4$. Using compass we construct the point B_1: a) it is in the same half-plane (with respect to AC) as point B; b) $|B_1A| = (s_2+s_3+s_4-s_1)/2$; c) $|B_1C| = (s_1+s_3+s_4-s_2)/2$. In the same way we construct the point D_1: a) it is in the same half-plane (with respect to AC) as point D; b) $|D_1A| = (s_1+s_2+s_3-s_4)/2$; c) $|D_1C| = (s_1+s_2+s_4-s_3)/2$. Then AB_1CD_1 will be the required dual.

References

[1] J.Cantarella, T.Needham, C.Shonkwiler & G.Stewart, *Random triangles and polygons in the plane*, The American Mathematical Monthly, 126(2), 2019, 113-134.

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