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MATHEMATICS

On the Existence and Uniqueness of a Solution of a Nonlinear Integral Equation

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Abstract—A nonlinear integral equation arising in the spatial model of biological communities developed by Austrian scientists Ulf Dieckmann and Richard Law is studied. Sufficient conditions for the existence of a solution of this equation (a fixed point of the integral operator) are found. The uniqueness of the solution is also analyzed.

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1. INTRODUCTION

In this paper, we study a nonlinear integral equation arising in the spatial biological model of adaptive dynamics developed by Dieckmann and Law [1, 2]. This model deals with a self-structured community of biological species. A short description of the model is given in Section 2. A more detailed exposition can be found in [1–4]. Then we describe the mathematical formulation of a problem related to the above-mentioned integral equation and analyze its well-posedness. More specifically, sufficient conditions for the existence of a solution of this nonlinear integral equation are found and its uniqueness is proved.

2. DESCRIPTION OF THE BIOLOGICAL MODEL

Consider a single-species community inhabiting an area $A \subset \mathbb{R}^n$. The biological environment is characterized by several homogeneous parameters, namely, the natural death rate d, the competition death rate d', and the birth rate b, and by two radial symmetric functions m and ω known as the dispersal (at birth) and competition kernels, respectively, which satisfy the conditions

$$\forall x \in \mathbb{R}^n \quad m(x) \ge 0, \quad \omega(x) \ge 0.$$

$$\lim_{\|x\| \to +\infty} m(x) = \lim_{\|x\| \to +\infty} \omega(x) = 0.$$

$$\int_{\mathbb{R}^n} m(x)dx = \int_{\mathbb{R}^n} \omega(x)dx = 1.$$

On every time interval, the state of the community is characterized by three spatial moments (unknown functions): $N(t)$ is the mean density of individuals; $C(x,t)$ is the mean density of pairs of individuals, where x is the shift of the second individual with respect to the first one; and $T(x,y,t)$ is the mean density of triplets of individuals, where x and y are the respective shifts of the second and third individuals with respect to the first one.

In this paper, we study an equilibrium state of the community described by a stationary point of the system of spatial dynamics equations [2]

$$\begin{aligned} \frac{dN}{dt}(t) &= (b-d)N(t) - d'\int_{\mathbb{R}^n} C(\xi,t)w(\xi)d\xi, \\ \frac{\partial C}{\partial t}(\xi,t) &= bm(\xi)N(t) + \int_{\mathbb{R}^n} bm(\xi')C(\xi+\xi',t)d\xi' \\ &- (d + d'\omega(\xi))C(\xi,t) - \int_{\mathbb{R}^n} d'\omega(\xi')T(\xi,\xi',t)d\xi'. \end{aligned} \quad (1)$$

3. EQUILIBRIUM OPERATOR

Following [4], we consider a parametric power-2 closure of the third spatial moment:

$$T_\alpha(\xi,\xi') = \frac{\alpha}{2}\left(\frac{C(\xi)C(\xi')}{N} + \frac{C(\xi)C(\xi'-\xi)}{N} + \frac{C(\xi')C(\xi'-\xi)}{N} - N^3\right) + (1-\alpha)\frac{C(\xi)C(\xi')}{N}.$$

Substituting this expression into system (1) and setting the derivatives to zero, after some algebra (see, e.g., [4]), we obtain the nonlinear integral equation

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$$\left(\overline{\omega} + b - \frac{\alpha}{2}\left(b - d - \frac{d'(b-d)}{Y}\right)\right)Q = \frac{Y\overline{m}}{b-d} - \overline{\omega} + [\overline{m} * Q] - \alpha\frac{b-d}{2Y}((Q+2)[\overline{\omega} * Q] + [\overline{\omega}Q * Q]), \quad (2)$$

where $Y = Y(Q) = \langle\overline{\omega}, Q + 1\rangle$, $\overline{\omega} = d'\omega$, and $\overline{m} = bm$. In what follows, this equation is studied in operator form.

To introduce an "equilibrium operator," we rewrite (2) as $\mathcal{A}Q = Q$, where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}f = \frac{\frac{Y\overline{m}}{b-d} - \overline{\omega} + [\overline{m} * f] - \alpha\frac{b-d}{2Y}((f+2)[\overline{\omega} * f] + [\overline{\omega}f * f])}{\overline{\omega} + b - \frac{\alpha}{2}\left(b - d - \frac{d'(b-d)}{Y}\right)}, \quad (3)$$

and consider the problem of finding a fixed point of this operator. The difficulties in studying operator (3) are associated with the fact that it is neither contractive nor compact. We represent it as the sum of a compact and a noncompact part: $\mathcal{A} = \mathcal{K} + \mathcal{S}$. Here,

$$\mathcal{K}f = \frac{\frac{Y\overline{m}}{b-d} - \overline{\omega} + [\overline{m} * f] - \alpha\frac{b-d}{Y}[\overline{\omega} * f]}{\overline{\omega} + b - \frac{\alpha}{2}\left(b - d - \frac{d'(b-d)}{Y}\right)},$$

$$\mathcal{S}f = -\alpha\frac{b-d}{2Y} \cdot \frac{f[\overline{\omega} * f] + [\overline{\omega}f * f]}{\overline{\omega} + b - \frac{\alpha}{2}\left(b - d - \frac{d'(b-d)}{Y}\right)}.$$

In what follows, we additionally assume that the functions m and ω are everywhere continuous.

The definitions of the above-introduced operators involve fractions with denominators depending on f (via Y). This fact may cause difficulties in studying the compactness of the operators. Nevertheless, the following lemmas hold.

Lemma 1. *Suppose that* $R < \frac{1}{\|\omega\|_C}$ *and* $d' > 0$. *Then the fraction* $\frac{1}{Y}$ *is bounded away from zero and infinity uniformly in f for all* $f \in B(R)$.

Lemma 2. *Let* $b > d \geq 0, d' > 0, \alpha \in [0;1]$. *Assume that* $R < \frac{1}{\|\omega\|_C}$. *Then the function*

$$g_Y(x) = \frac{1}{\overline{\omega}(x) + b - \frac{\alpha}{2}\left(b - d - \frac{d'(b-d)}{Y}\right)}$$

is continuous and bounded away from zero and infinity uniformly in f for all $f \in B(R)$.

Next, using Fubini's classical theorems and Riesz's criterion, we can prove the compactness of the "blocks" involved in the operator $\mathcal{K}$, namely, the following assertions hold true.

Lemma 3. *The operators* $\mathcal{B}_\omega f = [\omega * f]$ *and* $\mathcal{B}_m f = [m * f]$ *are compact as operators from* $L_1(\mathbb{R})$ *to* $L_1(\mathbb{R})$.

Lemma 4. *The operator* $\mathcal{C}f = \varphi(x)\int_{\mathbb{R}}\omega(y)f(y)dx + \psi(x)$, *where* φ, ψ *are continuous summable functions, is compact as an operator from* $L_1(\mathbb{R})$ *to* $L_1(\mathbb{R})$.

By using these lemmas, we can find conditions guaranteeing the compactness of the entire operator $\mathcal{K}$. They are stated in the form of the following theorem.

Theorem 1. *Let* $b > d \geq 0, d' > 0, \alpha \in [0;1]$. *Assume that* $R < \frac{1}{\|\omega\|_C}$. *Then* $\mathcal{K}$ *is defined as an operator from* $B(R)$ *to* $L_1(\mathbb{R})$ *and is compact.*

The following result is proved using the Leray–Schauder principle for the existence of a fixed point of a compact operator [5].

Theorem 2. *Under the conditions of Theorem* 1, *if* $\rho = 1 - R\|\omega\|_C > 0$ *and* $\alpha > 0$, *then there exists* $d' \in \left(0; \frac{3}{4}\rho\right)$ *such that the operator* $\mathcal{K}$ *has a fixed point in* $B(R)$.

Now, we use the fact that, for $\alpha > 0$, the image of $B(R)$ under the operator $\mathcal{K}$ is a closed subball $B' \subset B(R)$ such that $d(\partial B', \partial B(R)) > 0$. Here, by $d(A, B)$, we mean the distance between the sets A and B in the metric generated by the norm of $L_1(\mathbb{R})$.

The second part of the equilibrium operator (i.e., the operator $\mathcal{S}$) is also defined as an operator from $B(R)$ to $L_1(\mathbb{R})$ (with the same condition imposed on R as in Theorem 1), but this operator is not compact. This can be proved, for example, by constructing a sequence of functions $f_n \in B(R)$ whose image (under the operator $\mathcal{S}$) does not contain a fundamental subsequence.

To complete the proof of the existence of a fixed point for the equilibrium operator, we need the following result from [5].

Theorem (on fixed points of a perturbed compact operator). *Let A be a compact operator defined on a* *domain G of a Banach space. Assume that A has a nonzero rotation on the boundary of G and maps G to a subdomain $H \subset G$ such that $d(\partial G, \partial H) = \delta > 0$. If A is perturbed by a Lipschitz operator whose norm does not exceed δ, then the perturbed operator has fixed points in G.*

Under the conditions of Lemmas 1 and 2, the operator $\mathcal{S}$ satisfies all the assumptions of this theorem, i.e., $\mathcal{S}$ is a Lipschitz operator with a constant $L = L(d')$ and its norm vanishes as $d' \to 0 + 0$.

Relying on the above argument, we can prove the following important result.

Theorem 3. *Suppose that the conditions of Theorems* 1 *and* 2 *are satisfied. If* $\alpha > 0$, *then, for sufficiently small* d', *the operator* $\mathcal{A}$ *has a fixed point in* $B(R)$.

An immediate consequence of this theorem is the existence of a solution of Eq. (2) and a biologically interesting fact that, if $\frac{d'\bar{m}}{b-d} - \bar{\omega} \not\equiv 0$, then the fixed point of the operator $\mathcal{A}$ is nonzero.

4. UNIQUENESS OF THE FIXED POINT

Now, we find sufficient conditions under which the fixed point of the operator $\mathcal{A}$ is unique. For this purpose, we need to prove the following assertion.

Lemma 5. *Under the conditions of Theorem* 1, *there is* $b_0 > 0$ *such that, for all* $b \in (0; b_0)$ *and all* $d \in [0; b)$, *there exists* $d'_0 = d'_0(b, d) > 0$ *such that, for all* $d' \in (0; d'_0)$, $\mathcal{H}$ *is a Lipschitz operator with a Lipschitz constant* $L < 1$.

By using the fact that a Lipschitz operator with a Lipschitz constant $L < 1$ can have at most one fixed point, we prove the following result.

Theorem 4. *Under the conditions of Theorem* 3 *and Lemma* 5, *there exist constants* $b > d \geq 0$ *and a small number* $d' > 0$ *such that the operator* $\mathcal{A}$ *has a unique fixed point in the ball* $B(R)$.

5. CONCLUSIONS

In this paper, we have studied the well-posedness of a problem related to a nonlinear integral equation derived by applying a parametric power-2 closure of the third spatial moment. Sufficient conditions for the existence and uniqueness of a solution of this equation were found in the case when the competition and dispersal kernels are continuous. Note that the linear integral equation derived by using the closure with $\alpha = 0$ (so-called asymmetric power-2 closure) was intensively studied in [6–8]. Specifically, it was shown that, for $d \neq 0$, this equation can have only the trivial solution, while, for $d = 0$, it additionally has nontrivial solutions, which can be found, for example, by applying iterative Neumann series. The stability of the considered problem remains an open question.

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REFERENCES

1. U. Dieckmann and R. Law, "Moment approximations of individual-based models," *The Geometry of Ecological Interactions: Simplifying Spatial Complexity*, Ed. by U. Dieckmann, R. Law, and J. Metz (Cambridge Univ. Press, Cambridge, 2000), pp. 252–270.
2. U. Dieckmann and R. Law, "Relaxation Projections and the Method of Moments," *The Geometry of Ecological Interactions: Simplifying Spatial Complexity*, Ed. by U. Dieckmann, R. Law, and J. Metz (Cambridge Univ. Press, Cambridge, 2000), pp. 412–455.
3. A. G. Bodrov and A. A. Nikitin, Dokl. Math. **89** (2), 210–213 (2014).
4. A. A. Nikitin and M. V. Nikolaev, Moscow Univ. Comput. Math. Cybern. **42** (3), 105–1139 (2018).
5. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations* (Gostekhteorizdat, Moscow, 1956; Pergamon, New York, 1964).
6. A. A. Davydov, V. I. Danchenko, and M. Yu. Zvyagin, Proc. Steklov Inst. Math. **267**, 40–49 (2009).
7. A. A. Davydov, V. I. Danchenko, and A. A. Nikitin, "Integral equation for stationary distributions of biological communities," *Problems of Dynamic Control* (Fak. Vychisl. Mat. Mat. Fiz. Mosk. Gos. Univ., Moscow, 2009), pp. 15–29 [in Russian].
8. A. G. Bodrov and A. A. Nikitin, Moscow Univ. Comput. Math. Cybern. **39** (4), 157–162 (2015).

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