4 Measurement of solid angles. About regular polygons and bodies

The surface of a sphere, like a circle, is divided into 400 equal parts by planes, going through a radius. These parts are called *degrees* and are divided in tenths, hundredths and so on.

Part of the surface of a sphere, formed by planes, passing through the center, and whose magnitude is measured in degrees, is called a *solid angle*. Planes, forming a solid angle, will be called *sides* of the angle and depending on the number of these planes it is called *three-sided, four-sided* and so on.

Like polygons could be formed by adding and subtracting triangles, multi-sided solid angles can be formed from three-sided angles. Because measuring solid angles in general boils down to measuring only three-sided angles.

The intersection of a plane and the surface of a sphere results in a circle.

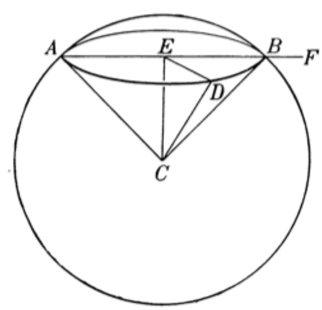

Fig. 11

The perpendicular CE (Fig. 11) from the center of the sphere C to the plane ADB, which intersect the surface of the sphere in the lines ADB, could not fall out of the sphere, for example at point F, because otherwise the line AF drawn on the plane inside the sphere, would have two points of intersection A and B on the surface of the sphere, which together with the center C would form an isosceles triangle, and in it, one could draw a perpendicular to the line AF in addition to CF. Two points A and D, taken arbitrarily on the line ADB, form together with the center two triangles ACE and DCE, which will be placed next to each other on one plane, keeping the line CE common, will form an isosceles triangle, where the line CE from the apex will be perpendicular to the the base, and therefore $AE = ED$. As all similar lines AE and ED are equal, then the line ADB is a circle, and the point E is its center.

All solid angles bounded on the surface of the sphere by circles whose radii are equal with the radius of sphere and therefore those circles have the greatest radii. For this reason the circles whose centers coincides with the center of the sphere are called the greatest circles.

The *apex* of a solid angle is the common point of intersection of the sides. The extension of the sides beyond the point of intersection form another solid angle, which, together with the first angle, are called *apex solid angles. Isosceles* and *equilateral* [three-sided] solid angles are those which have two or all three sides equal.

The apex solid angles are equal. It is necessary to show this only for three-sides angles, because the planes, dividing such an angle into three-sided angles, divide the extention beyond the apex – another angle – into the same number of three-sided angles.

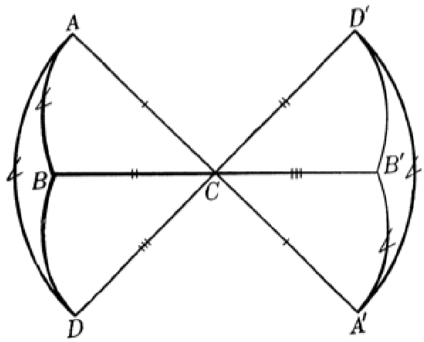

Fig. 12

If in the three-sided solid angle $CABD$ (Fig. 12) the sides ACB and ACD are equal, then their apex angle $CA'B'D'$ is equal; because putting $\angle ACB$ over $\angle A'CD'$, the equality of the plane angles between CAB and CAD will make the plane ACD to cover the plane $A'CB'$ and also $\angle ACD = \angle A'CB'$, $\angle ACB = \angle A'CD'$, then the line CB will coincide with CD', CD with CB', so one angle will completely fill the other.

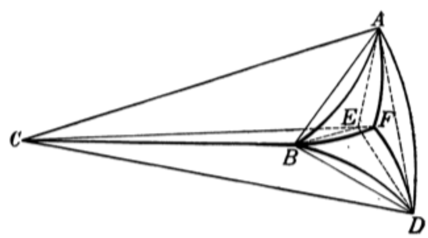

Fig. 13

If the solid angle $CABD$ (Fig.13) is not isosceles, then we draw a plane through the points A, B, D, and also draw to it a perpendicular CE, then the triangles AEC, BEC, DEC will be formed with equal angles ACE, BCE, DCE; because they, having a common line CE, and being placed one next to another in one plane, form an isosceles triangle, in which the line through the apex is perpendicular to the base. It follows from here that CE, when extended to F on the surface of the sphere, gives equal arcs AF, BF and DF, i.e., the solid angle ABD is divided into three isosceles solid angles AFB, BFD, DFA, which should be added or they should be subtracted from others, depending on whether F is inside or outside the solid angle. Now, when the solid angle is divided into three isosceles triangles with the help of three planes, the corresponding to it apex angle will be divided by the same planes into equal angles, and therefore the apex solid angles are equal.

Imagine a part of the surface of a sphere, cut off by the planes of the three-sided solid angle, whose apex is at the center of the sphere. The lines a, b, c (Fig. 14) are lines of intersection of the planes with the surface of the sphere, points A, B, C will be the points of intersection of the lines between themselves, taken in such a way that A will be against a, B against b, C against c. We extend a in both direction on the sphere's surface, until we complete a whole circle. We extend the arcs b and c beyond the point of their intersections A, until they reach the circle of a. We obtain four solid angles X, Y, Z, and $\curlywedge ABC$. Denote by $\measuredangle A$ the plane angle in $\curlywedge ABC$ which is opposite the arc a, and denote by $\measuredangle B$, $\measuredangle C$ two other angles opposite the arcs b and c. It is easy to see, that

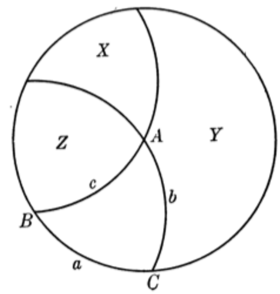

Fig. 14

$$\curlywedge ABC + Y = \measuredangle B;$$

$$\curlywedge ABC + Z = \measuredangle C.$$

The same applies to the apex solid angle with the angle X, and as a result instead of it the angle X itself alone gives

$$\curlywedge ABC + X = \measuredangle A.$$

At the end the sum of four solid angles X, Y, Z, and $\curlywedge ABC$ is 200°, which added to the previous three equations gives:

$$\curlywedge ABC = \frac{\measuredangle A + \measuredangle B + \measuredangle C - 200^\circ}{2}.$$

In this way, we define the solid three-sided angle with the help of plane angles.

From here it is not difficult to conclude how to define any solid angle from the plane angles which are forming it. If we divide the solid angle into three-sided, following the above method, we see, that for a number of sides n the magnitude of the solid angle will be:

$$\frac{1}{2} \times \{\text{sum of plane angles} - (n-2)\,200^\circ\}.$$

From here we draw important conclusions about regular bodies. But in order to fully cover the subject first we will consider *regular polygons*. We call regular polygons those, whose sides and angles are equal. We can form a regular polygon with n sides, when we divide a circle into n equal parts and connect the points of intersection by lines. Then the center of the circle is called also *center* of the regular polygon. On the contrary, in any regular polygon

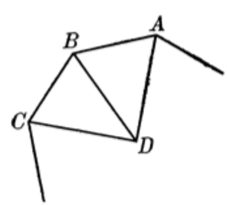

Fig. 15

there is a center. Divide one of the angles by a line [into two equal parts]. Such a line will pass through the whole polygon and will divide it into two equal parts. From here it follows, that another similar line will intersect the first one inside the polygon. Let AD divide the angle A into two equal parts (Fig.15), BD divides the angle B, which is equal to A, into two equal part, and D is their point of intersection. In $\triangle ABD$ the two sides AD and BD will be equal, the third line CD forms a $\triangle CBD$, equal to $\triangle ABD$, as long as the angle $C = B = A$ and the line $BC = BA$. Because, putting $\triangle ABD$ on $\triangle DBC$, leaving BD a common side, the line BA must fail on BC with point A on C, therefore $AD = DC$ and $\angle DCB = \angle DAB = \frac{1}{2}\angle A$. So all distances from D to the apexes of the angles in a regular polygon will be equal to AD; then D is the center of the polygon.

So, it is possible to have regular polygons with any number of sides and there is a center in every regular polygon.

A *regular body* is called a body which is bounded by regular polygons and whose solid and plane angles are equal. It is needed also that all polygons of the body are equal, and that equal number of polygons fit in all solid angles of the body.

Solid angles whose plane angles are equal and have equal sides can be called *regular solid angles.* In them the planes, dividing the plane angles ino two equal parts, intersect in common lines – the *axes* of the angle, and each of them divides the solid angle into two equal parts, and therefore, coming out of the solid angle, it either coincides with the edge or divides the side into two equal parts. The axes are inclined to the edges under equal angles.

In the regular body the plane, dividing one of the plane angles into two equal parts, gives when intersects the surface of the body either a regular polygon, or such a polygon where the sides going through one side of the body and the angles through one angle of the body are equal. In the first case it is not difficult to see that the center of the polygon will be a *center* of a regular body, that is, such a point, which is situated at equal distance from the apexes of the solid angles. In the second case, connecting the ends of every two adjacent sides by a line we again get a regular polygon, whose center will be the center of a regular body.

Let the number of the sides of a regular body is n, every side is a regular polygon with m sides, in every solid angle fit t polygons.

Drawing a sphere from the center of a regular body, then each side will correspond at the center a regular solid angle with m sides, in which every plane angle = $\frac{400°}{t}$, and so the whole solid angle = $\frac{1}{2}\{m\times\frac{400°}{t}-(m-2)\times 200°\}$. As such an angle should be together with nth part of 400°, then

$$n = \frac{4t}{2m-(m-2)\times t}.$$

Numbers n, t, m should be positive integers and since $m \geq 3$ than also $t \geq 3$. Let $m = 3 + p$, where therefore $p = 0$, or $p > 0$, then

$$n = \frac{4t}{6-t-(t-2)p}.$$

From here it is obvious, that t should be less than 6. If we take $m = 6 + q$, then n is

$$\frac{4t}{12 - 4t - q(t-2)},$$

and since $12 - 4t$ could be only 0 or a negative number, and $t - 2$ is always a positive number, then q must be negative, that is, m is always smaller than 6.

And so all suggested numbers m and t are described in the table:

m=3	t=3	n=4	body	is called	*tetrahedron* (four-sided)
m=3	t=4	n=8	"	"	*octahedron* (eight-sided)
m=3	t=5	n=20	"	"	*icosahedron* (twenty-sided)
m=4	t=3	n=6	"	"	*cube* (six-sided)
m=4	t=4	denominator in n becomes negative, so such a body is impossible			
m=5	t=3	n=12	body is called *dodecahedron* (twelfe-sided)		
m=5	t=4	denominator in n is negative and therefore such a body is impossible			
m=5	t=5	n is also negative and such a body is impossible			

Regular polygons, as we saw, could be with any number of sides. On the contrary, regular bodies can be only five.

What is also different in regular polygons, compared to regular bodies, is that the number of their sides is equal to the number of their angles, whereas in regular bodies this is not so. Let r denote the number of angles in a regular body, keeping the previous notations, $n \times m \frac{400°}{t}$ will represent the sum of all plane angles, which will be formed, when from the center of the regular body we draw planes through all sides of the polygon. On the other hand, this sum should give $r \times 400°$, therefore $r = \frac{nm}{t}$. So

angles in tetrahedron........................	4
angles in cube	8
angles in octahedron	6
angles in dodecahedron	20
angles in icosahedron	12