J. Math. Fluid Mech. (2019) 21:25

https://doi.org/10.1007/s00021-019-0430-9

Journal of Mathematical Fluid Mechanics

Large Oceanic Gyres: Lagrangian Description

Anatoly Abrashkin

Communicated by A. Constantin

Abstract. A hydrodynamic model of an oceanic gyre is proposed. The fluid motion is considered in the leading-order shallow-water approximation in the spherical Lagrangian coordinates. Motion of liquid particles at the spherical surfaces is studied versus latitude and longitude as unknown variables. The boundary condition at the edge of the gyre is not formulated. An approximation of the "averaged latitude" is introduced when the coefficients of the momentum equation are replaced by constant values corresponding to the latitude of the gyre's center. It is shown that the resulting set of equations is similar to the equations of plane hydrodynamics. Its analytical solutions containing two arbitrary functions and two arbitrary constants (time frequencies) are obtained. The trajectories of liquid particles represent a superposition of two rotational motions, and their general properties are discussed. A family of the gyres with invariable shape in time is selected. Their outer boundaries either remain motionless or rotate uniformly. An example of the unsteady gyre both rotating and deforming in its shape is studied numerically.

Mathematics Subject Classification. 76U05.

Keywords. Oceanic gyres, Lagrangian spherical coordinates, Trajectories of liquid particles.

1. Introduction

A circulation of the World Ocean waters is a system of planetary gyres arranged versus similar wind cycles in the atmosphere. Except for the Antarctic Circumpolar Current all of ocean waters are involved into the macro-scale anticyclonic circulations corresponding to the subtropical maxima of the atmospheric pressure. There are five such subtropical whirlpools in the World Ocean: the North Atlantic, the South Atlantic, the North Pacific, the South Pacific and the Indian Ocean. All of these large gyres move circularly in the clockwise direction in the Northern hemisphere and in the opposite direction in the Southern hemisphere. Beside there are two types of gyres of a smaller scale—tropical near the Equator and subpolar in the polar region.

In this paper the flow inside the gyres is studied in the framework of hydrodynamics equations of an ideal homogeneous fluid. Effects of flow stratification, viscosity and turbulence were not taken into account. Our main goal is to show how the Euler equation may result in a solution that recovers the essential structure of a gyre on the surface of the planet directly. For a description of large-scale gyres, a β-plane is not a good approximation to the spherical geometry of the Earth [1–4]. The gyres are generated by the interaction between the wind stress and the effect of the Earth's rotation, and the reliance on the β-plane approximation leads to a neglect of dependence on longitude although significant longitudinal variations of the pressure are known to occur in mid-latitudes (see the discussions in [5,6]). Thus fluid flows have to be considered in the spherical coordinates in the shallow-water approximation. We neglect the vertical motion of the fluid. Liquid particles move at the spherical surfaces and their location is determined by latitude and longitude only. This approach corresponds to a zero approximation with respect to the small parameter of the ratio of the ocean's depth to the radius of the Earth.

As the gyres are limited by continental coasts (land areas) they can be qualified as stable monopole vortex structures with the center of rotation. Fluid motion inside the gyres is formed by a global wind

Published online: 03 April 2019

patterns. We study a family of eddies induced by self-consistent pressure distributions at the free surface. The boundary condition at the edge of the vortex is not set which seems to be natural and valid as the gyres are not free eddies. They exist within certain continental boundaries that maintain their stable form. Indeed, floating debris often remains for decades in a gyre, and for that which reaches the centre, there is no escape. This occurs because a typical gyre circles around large areas—of the order of 10^6 km^2 in the case of the North Pacific Gyre—of essentially stationary water. In [4] the model of stationary gyres in the Eulerian variables was studied for a number of particular representations of vorticity. Explicit expressions for the stream function were obtained and flow properties were studied.

Each of the gyres is surrounded by meridional (coastal) and zonal currents. Obviously they interact with the gyres and affect them. This impact occurs in two ways. The first one is the external path, when the edge of the gyre is deformed due to the boundary conditions for pressure and velocity of external flows. The second way is internal, connected with instability of zonal and meridional currents and energy transfer inside the gyre. This process is supported both by the wave propagation [7–13] and by vortex penetration, which are often generated by unstable meandering of an intense current like the Gulf Stream [14,15]. However, mesoscale eddies are generated inside the gyre as a result of the baroclinic instability, thus maintaining a high level of the mesoscale variability [16–21]. All of these feathures allow us to consider gyres as large-scale non-stationary inhomogeneously vortical long-lived formations in the ocean.

A two-dimensional model of the gyre aimed at the study of the trajectories of liquid particles is suggested below. In Sect. 2 the Lagrangian formulation of govering equations in the rotating reference frame is introduced. The momentum equations are written in a form different from the monography [22]. After the pressure elimination the expressions for three Lagrangian invariants are obtained. In the absence of rotation they are currently referred to the Cauchy invariants [22–27]. The consistency of the Cauchy invariants is less known than the famous Kelvin's theorem of conservation of velocity circulation. However, both conservation laws have the same meaning [28]. The difference between them is that Kelvin's theorem concerns conservation of the integral quantity, i.e. velocity circulation, where the Cauchy invariant is local but expressing the same constancy. The invariants we found are a generalization of the Cauchy invariants to the case of rotating fluid.

In Sect. 3 derivation of the hydrodynamic equations of ideal fluid in the spherical Lagrangian coordinates for uniformly rotating fluid is described in the form not cited in the literature yet. In Sect. 4 approximations used in the analysis of the gyre's flows are presented. The approximation of the "average latitude" is proposed here along with the traditional approach of the shallow water. It means that the latitude-dependent coefficients in the nonlinear momentum equation are replaced by the constant values of the coefficients in the center of the gyre. It is shown that the approximate system of equations of spherical hydrodynamics coincides with the system of equations of plane hydrodynamics with a specific replacement of unknown functions. In Sect. 5 a class of their exact solutions containing two arbitrary functions and two time frequencies is found. Motions of liquid particles represent a superposition of two rotations with uniform angular velocity that is why may be named as Ptolemaic. A family of the gyres with the stable form was selected. Such gyres can be either stationary or could rotate uniformly over their center. In a general case the Ptolemaic gyres are deformed in a rather complex way in addition to its rotation.

2. Lagrangian Formulation of Governing Equations in Rotating Reference Frame

Let us consider the motion of a homogeneous incompressible fluid in the reference frame moving with constant angular velocity $\vec{\Omega}$. The equations of hydrodynamics in the Eulerian variables are written in the following form [9]:

$$div\,\overrightarrow{R_t} = 0, \tag{1}$$

$$\overrightarrow{R_{tt}} + 2\vec{\Omega} \times \overrightarrow{R_t} = -\frac{1}{\rho}\nabla p - \nabla \Phi_* - \overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \overrightarrow{R}). \tag{2}$$

Here, $\overrightarrow{R}(X,Y,Z)$ is the radius vector of elementary liquid volume (X,Y,Z are the Cartesian coordinates), $\overrightarrow{R_t}$ and $\overrightarrow{R_{tt}}$ are its velocity and acceleration respectively, t is time, ρ is density, p is pressure, and Φ_* is the potential of external forces. Expression (1) is the continuity equation. Vector momentum equation (2) is the expression of the Newton's second law taking into account an action of the Coriolis and centrifugal forces on a liquid particle. The latter has a gradient character, so Eq. (2) can be written as

$$\overrightarrow{R_{tt}} + 2\overrightarrow{\Omega} \times \overrightarrow{R_t} = -\frac{1}{\rho}\nabla H;$$
$$H = \frac{p}{\rho} + \Phi_* + \Phi_c, \nabla\Phi_c = -\frac{1}{2}\nabla\left(\overrightarrow{\Omega} \times \overrightarrow{R}\right)^2, \tag{3}$$

where Φ_c is the potential of the centrifugal forces.

Let us find representation of Eqs. (1) and (3) in terms of the Lagrangian variables $\{a_i\} = \{a,b,c\}$. In this assumption the vector $\overrightarrow{R}$ is considered as a function of these variables. The continuity equation in these variables has a form of the time-independent Jacobian transition from the Eulerian to the Lagrangian variables and could be written in the form [22]:

$$\frac{D(X,Y,Z)}{D(a,b,c)} = S_0(a,b,c). \tag{4}$$

If the initial positions of the fluid particles are expressed in the Lagrangian variables, i.e.

$$X_0 = a,\ Y_0 = b,\ Z_0 = c, \tag{5}$$

then $S_0 = 1$. In a general case, it is a function of the Lagrangian variables. For one-to-one correspondence between the coordinates of fluid particles X,Y,Z and their labels a,b,c this function should not turn to zero within the flow region.

The scalar multiplication of Eq. (3) by $\overrightarrow{R_{a_i}}$ yields the momentum equations in the Lagrangian coordinates

$$\overrightarrow{R_{tt}}\overrightarrow{R_{a_i}} + 2\left(\overrightarrow{\Omega}\overrightarrow{R_t}\overrightarrow{R_{a_i}}\right) = -\nabla H_{ai},\ i = 1,2,3. \tag{6}$$

Equations (4) and (6) form the set of hydrodynamic equations for an ideal incompressible fluid in the Lagrangian variables in rotating reference frame.

We will omit the gradient term from Eq. (6) by taking its cross derivatives and calculating their difference:

$$\overrightarrow{R_{tta_j}}\overrightarrow{R_{a_i}} - \overrightarrow{R_{tta_i}}\overrightarrow{R_{a_j}} + 2\left(\overrightarrow{\Omega}\overrightarrow{R_{ta_j}}\overrightarrow{R_{a_i}}\right) - 2\left(\overrightarrow{\Omega}\overrightarrow{R_{ta_i}}\overrightarrow{R_{a_j}}\right) = 0. \tag{7}$$

With allowance for

$$\overrightarrow{R_{tta_j}}\overrightarrow{R_{a_i}} - \overrightarrow{R_{tta_i}}\overrightarrow{R_{a_j}} = \left(\overrightarrow{R_{ta_j}}\overrightarrow{R_{a_i}} - \overrightarrow{R_{ta_i}}\overrightarrow{R_{a_j}}\right)_t,$$
$$\left(\overrightarrow{\Omega}\overrightarrow{R_{ta_j}}\overrightarrow{R_{a_i}}\right) - \left(\overrightarrow{\Omega}\overrightarrow{R_{ta_j}}\overrightarrow{R_{a_i}}\right) = \left(\Omega\overrightarrow{R_{a_i}}\overrightarrow{R_{ta_j}}\right)_t,$$

Equation (7) is now rewritten in the form

$$\frac{\partial}{\partial t}\left[\overrightarrow{R_{ta_j}}\overrightarrow{R_{a_i}} - \overrightarrow{R_{ta_i}}\overrightarrow{R_{a_j}} + 2\left(\overrightarrow{\Omega}\overrightarrow{R_{a_j}}\overrightarrow{R_{a_i}}\right)\right] = 0$$

which is equivalent to the conditions of conservation of three invariants S_1, S_2, S_3:

$$\overrightarrow{R_{tb}}\overrightarrow{R_c} - \overrightarrow{R_{tc}}\overrightarrow{R_b} + 2\left(\overrightarrow{\Omega}\overrightarrow{R_b}\overrightarrow{R_c}\right) = S_1(a,b,c), \tag{8}$$
$$\overrightarrow{R_{tc}}\overrightarrow{R_a} - \overrightarrow{R_{ta}}\overrightarrow{R_c} + 2\left(\overrightarrow{\Omega}\overrightarrow{R_c}\overrightarrow{R_a}\right) = S_2(a,b,c), \tag{9}$$
$$\overrightarrow{R_{ta}}\overrightarrow{R_b} - \overrightarrow{R_{tb}}\overrightarrow{R_a} + 2\left(\overrightarrow{\Omega}\overrightarrow{R_a}\overrightarrow{R_b}\right) = S_3(a,b,c), \tag{10}$$

which are the functions of the Lagrangian coordinates only. Equations (8)–(10) follow from the momentum equations.

If there is no rotation ($\mathit{\Omega} = 0$), then the system (8)–(10) reduces to the following set of equations

$$\overrightarrow{R_{tb}}\overrightarrow{R_c} - \overrightarrow{R_{tc}}\overrightarrow{R_b} = S_{10}\left(a, b, c\right),$$
$$\overrightarrow{R_{tc}}\overrightarrow{R_a} - \overrightarrow{R_{ta}}\overrightarrow{R_c} = S_{20}\left(a, b, c\right),$$
$$\overrightarrow{R_{ta}}\overrightarrow{R_b} - \overrightarrow{R_{tb}}\overrightarrow{R_a} = S_{30}\left(a, b, c\right).$$

The subscript "0" indicates that the invariants are calculated in a non-rotating fluid. Those expressions were discovered by Cauchy, and the functions S_{10}, S_{20}, S_{30} are referred to as the Cauchy invariants [22–27]. Expressions (8)–(10) are a generalization of the Cauchy invariants for flows formed within uniformly rotating fluid.

3. Lagrangian Equations in Spherical Coordinates

We introduce a set of (right-handed) spherical coordinates $(R, \mathit{\Theta}, \mathit{\Phi})$, where R is the distance (radius) from the centre of a sphere, $\mathit{\Theta}$ (with $0 \leq \mathit{\Theta} \leq \pi$) is the polar angle; $\mathit{\Phi}$ (with $0 \leq \mathit{\Phi} < 2\pi$) is the azimuthal angle.

The Cartesian coordinates X, Y, Z have the following form

$$X = R\ sin\ \mathit{\Theta}\ cos\ \mathit{\Phi}, \quad Y = R\ sin\ \mathit{\Theta}\ sin\ \mathit{\Phi}, \quad Z = R\ cos\ \mathit{\Phi}.$$

The Cartesian (a, b, c) and the spherical (r, θ, φ) Lagrangian variables are related by analogous formulas:

$$a = r\,sin\,\theta\,cos\,\varphi, b = r\,sin\,\theta\,sin\,\varphi, c = r\,cos\,\theta,$$

the latter have the same meaning with $(R, \mathit{\Theta}, \mathit{\Phi})$.

The equation of continuity is obtained via the Jacobian multiplication rule

$$\frac{D\left(X, Y, Z\right)}{D\left(a, b, c\right)} = \frac{D\left(X, Y, Z\right)}{D\left(R, \mathit{\Theta}, \mathit{\Phi}\right)}\frac{D\left(R, \mathit{\Theta}, \mathit{\Phi}\right)}{D\left(r, \theta, \varphi\right)}\frac{D\left(r, \theta, \varphi\right)}{D\left(a, b, c\right)} = S_0\left(r, \theta, \varphi\right).$$

When taking into account the following equalities

$$\frac{D\left(X, Y, Z\right)}{D\left(R, \mathit{\Theta}, \mathit{\Phi}\right)} = R^2\,sin\,\mathit{\Theta}; \frac{D\left(r, \theta, \varphi\right)}{D\left(a, b, c\right)} = \left(r^2\,sin\,\theta\right)^{-1},$$

we obtain

$$R^2\ sin\,\Theta\ \frac{D\left(R, \Theta, \mathit{\Phi}\right)}{D\left(r, \theta, \varphi\right)} = r^2\,S_0\,sin\,\theta = S_0^*\left(r, \theta, \varphi\right) \tag{11}$$

This expression represents the continuity equation in spherical Lagrangian coordinates.

Let us return now to the derivation of the momentum equations of motion. Let write the Eq. (6) in an expanded form

$$\overrightarrow{R}_{tt}\overrightarrow{R}_a + 2(\overrightarrow{\mathit{\Omega}}\,\overrightarrow{R}_t\,\overrightarrow{R}_a) = -H_a, \tag{12}$$
$$\overrightarrow{R}_{tt}\overrightarrow{R}_b + 2(\overrightarrow{\mathit{\Omega}}\,\overrightarrow{R}_t\,\overrightarrow{R}_b) = -H_b, \tag{13}$$
$$\overrightarrow{R}_{tt}\overrightarrow{R}_c + 2(\overrightarrow{\mathit{\Omega}}\,\overrightarrow{R}_t\,\overrightarrow{R}_c) = -H_c. \tag{14}$$

Using the transformation rules of the Jacobians, we obtain the partial derivatives of the arbitrary function Q with respect to the spherical Lagrangian variable r :

$$\frac{\partial Q}{\partial r} = \frac{D\left(Q, \theta, \varphi\right)}{D\left(r, \theta, \varphi\right)} = \frac{D\left(Q, \theta, \varphi\right)}{D\left(a, b, c\right)}\frac{D\left(a, b, c\right)}{D\left(r, \theta, \varphi\right)}$$
$$= \left(r^2 sin\theta\right)^{-1}\left[\frac{\partial Q}{\partial a}\frac{D\left(\theta, \varphi\right)}{D\left(b, c\right)} + \frac{\partial Q}{\partial b}\frac{D\left(\theta, \varphi\right)}{D\left(c, a\right)} + \frac{\partial Q}{\partial c}\frac{D\left(\theta, \varphi\right)}{D\left(a, b\right)}\right].$$

The derivative with respect to the corresponding Lagrangian coordinate is linear in each of the terms of Eqs. (10)–(12). Multiplying the equations of this system by $\frac{D(\theta,\varphi)}{D(b,c)}$, $\frac{D(\theta,\varphi)}{D(c,a)}$, $\frac{D(\theta,\varphi)}{D(a,b)}$ respectively and adding them we get

$$\overrightarrow{R}_{tt}\overrightarrow{R}_r + 2(\overrightarrow{\Omega}\overrightarrow{R}_t\overrightarrow{R}_r) = -H_r. \tag{15}$$

Two other equations could be obtained by analogy

$$\overrightarrow{R}_{tt}\overrightarrow{R}_\theta + 2(\overrightarrow{\Omega}\overrightarrow{R}_t\overrightarrow{R}_\theta) = -H_\theta, \tag{16}$$

$$\overrightarrow{R}_t t\overrightarrow{R}_\varphi + 2(\overrightarrow{\Omega}\overrightarrow{R}_t\overrightarrow{R}_\varphi) = -H_\varphi. \tag{17}$$

The momentum equations have a similar form in the Cartesian and the spherical coordinates. This is due to the Lagrangian coordinates act as labels for liquid particles, and they may be chosen arbitrarily. The limitation is the condition of one-to-one correspondence between the labels and the coordinates of the particles (under the condition of non-vanishing of the Jacobian of the transformation). Equations (15)–(17) were not cited yet. In the monography [22] the left-side of the momentum equation is remained invariable but a gradient of the function H is expressed through the derivatives with respect to the Lagrangian variables.

Equations (15)–(17) together with Eq. (11) form the complete simultaneous hydrodynamic equations of an ideal incompressible fluid in the spherical Lagrangian coordinates. In a vector form the momentum equations will be written rather compactly. But their representation via unknown functions R, Θ, Φ looks extremely cumbersome. So we will not give their full formulation here. But Eqs. (15)–(17) will be used to determine the pressure of specific vortex flows further.

It is more convenient to use Eqs. (8)–(10) instead of momentum equations for investigations of gyre's features as these equations are of the same type. In each of them the differentiation with respect to the certain pair of the Lagrangian variables is fulfilled. In terms of unknown functions R, Θ, Φ Eqs. (8)–(10) are written as follows

$$\begin{aligned}&\frac{D\left(R_t, R\right)}{D\left(b,c\right)} + R^2\frac{D\left(\Theta_t, \Theta\right)}{D\left(b,c\right)} + R^2 sin^2\Theta\frac{D\left(\Phi_t, \Phi\right)}{D\left(b,c\right)} + 2R\Theta_t\frac{D\left(R, \Theta\right)}{D\left(b,c\right)}\\ &+2\Omega\left(R^2 sin^2\Theta\frac{D\left(R, \Phi\right)}{D\left(b,c\right)} + R^2 sin\Theta cos\Theta\frac{D\left(\Theta, \Phi\right)}{D\left(b,c\right)}\right) = S_1,\end{aligned} \tag{18}$$

$$\begin{aligned}&\frac{D\left(R_t, R\right)}{D\left(c,a\right)} + R^2\frac{D\left(\Theta_t, \Theta\right)}{D\left(c,a\right)} + R^2 sin^2\Theta\frac{D\left(\Phi_t, \Phi\right)}{D\left(c,a\right)} + 2R\Theta_t\frac{D\left(R, \Theta\right)}{D\left(c,a\right)}\\ &+2\Omega\left(R^2 sin^2\Theta\frac{D\left(R, \Phi\right)}{D\left(c,a\right)} + R^2 sin\Theta cos\Theta\frac{D\left(\Theta, \Phi\right)}{D\left(c,a\right)}\right) = S_2,\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{D\left(R_t, R\right)}{D\left(a,b\right)} + R^2\frac{D\left(\Theta_t, \Theta\right)}{D\left(a,b\right)} + R^2 sin^2\Theta\frac{D\left(\Phi_t, \Phi\right)}{D\left(a,b\right)} + 2R\Theta_t\frac{D\left(R, \Theta\right)}{D\left(a,b\right)}\\ &+2\Omega\left(R^2 sin^2\Theta\frac{D\left(R, \Phi\right)}{D\left(a,b\right)} + R^2 sin\Theta cos\Theta\frac{D\left(\Theta, \Phi\right)}{D\left(a,b\right)}\right) = S_3.\end{aligned} \tag{20}$$

The vector $\overrightarrow{\Omega}$ is considered to be oriented along Z-axis here.

Finally,we have to turn from a pair of the Cartesian Lagrangian coordinates to a pair of the spherical Lagrangian variables. Let's discuss the transition process by means of a simple example and suppose simultaneous equations to be given

$$\frac{D\left(A,B\right)}{D\left(b,c\right)} = C_1, \frac{D\left(A,B\right)}{D\left(c,a\right)} = C_2, \frac{D\left(A,B\right)}{D\left(a,b\right)} = C_3, \tag{21}$$

where the right-hand sides are the Lagrangian invariants. The form of Eq. (21) in the spherical coordinates r, θ, φ is required to be found here.

So we write the first equation of this system as follows

$$\frac{D\left(A,B\right)}{D\left(b,c\right)}=\frac{D\left(A,B,a\right)}{D\left(b,c,a\right)}=\frac{D\left(A,B,a\right)}{D\left(a,b,c\right)}=\frac{D\left(A,B,a\right)}{D\left(r,\theta,\varphi\right)}\frac{D\left(r,\theta,\varphi\right)}{D\left(a,b,c\right)}=C_1,$$

or in an expanded form

$$\frac{D\left(A,B\right)}{D\left(\theta,\varphi\right)}a_r+\frac{D\left(A,B\right)}{D\left(\varphi,r\right)}a_\theta+\frac{D\left(A,B\right)}{D\left(r,\theta\right)}a_\varphi=C_1r^2\,sin\,\theta. \tag{22}$$

By analogy one can obtain

$$\frac{D\left(A,B\right)}{D\left(\theta,\varphi\right)}b_r+\frac{D\left(A,B\right)}{D\left(\varphi,r\right)}b_\theta+\frac{D\left(A,B\right)}{D\left(r,\theta\right)}b_\varphi=C_2r^2\,sin\,\theta, \tag{23}$$

$$\frac{D\left(A,B\right)}{D\left(\theta,\varphi\right)}c_r+\frac{D\left(A,B\right)}{D\left(\varphi,r\right)}c_\theta+\frac{D\left(A,B\right)}{D\left(r,\theta\right)}c_\varphi=C_3r^2\,sin\,\theta. \tag{24}$$

Expressions (22)–(24) are the expansions of the right-hand sides via elements of one of the rows of the determinant

$$\frac{D\left(a,b,c\right)}{D\left(r,\theta,\varphi\right)}=\begin{vmatrix} a_r & a_\theta & a_\varphi \\ b_r & b_\theta & b_\varphi \\ c_r & c_\theta & c_\varphi \end{vmatrix}. \tag{25}$$

Multiplying the relations (22)–(24) by the corresponding minors of the determinant (25), we obtain

$$\frac{D\left(A.B\right)}{D\left(\theta,\varphi\right)}=r^2sin^2\theta\,\left(C_1cos\varphi+C_2sin\varphi+C_3\right)=C_1^*\left(r,\theta,\varphi\right);$$

$$\frac{D\left(A,B\right)}{D\left(\varphi,r\right)}=rsin\,\theta\,\left[\left(C_1cos\varphi+C_2sin\varphi\right)-C_3sin\theta\right]=C_2^*\left(r,\theta,\varphi\right);$$

$$\frac{D\left(A,B\right)}{D\left(r,\theta\right)}=\left(C_2-C_1\right)rsin\varphi=C_3^*\left(r,\theta,\varphi\right).$$

The Eqs. (18)–(20) represent a linear combination of the Jacobians of the same type. Therefore we can apply the proven rule to each of them when replacing pairs of the Cartesian coordinates by pairs of the spherical coordinates. The Lagrangian equations in the spherical coordinates could be written as follows:

$$\begin{aligned}&\frac{D\left(R_t,R\right)}{D\left(\theta,\varphi\right)}+R^2\frac{D\left(\Theta_t,\Theta\right)}{D\left(\theta,\varphi\right)}+R^2sin^2\Theta\frac{D\left(\Phi_t,\Phi\right)}{D\left(\theta,\varphi\right)}+2R\Theta_t\frac{D\left(R,\Theta\right)}{D\left(\theta,\varphi\right)}\\&+2\Omega\left(R^2sin^2\Theta\frac{D\left(R,\Phi\right)}{D\left(\theta,\varphi\right)}+R^2sin\Theta cos\Theta\frac{D\left(\Theta,\Phi\right)}{D\left(\theta,\varphi\right)}\right)=S_1^*\left(r,\theta,\varphi\right);\end{aligned} \tag{26}$$

$$\begin{aligned}&\frac{D\left(R_t,R\right)}{D\left(\varphi,r\right)}+R^2\frac{D\left(\Theta_t,\Theta\right)}{D\left(\varphi,r\right)}+R^2sin^2\Theta\frac{D\left(\Phi_t,\Phi\right)}{D\left(\varphi,r\right)}+2R\Theta_t\frac{D\left(R,\Theta\right)}{D\left(\varphi,r\right)}\\&+2\Omega\left(R^2sin^2\Theta\frac{D\left(R,\Phi\right)}{D\left(\varphi,r\right)}+R^2sin\Theta cos\Theta\frac{D\left(\Theta,\Phi\right)}{D\left(\varphi,r\right)}\right)=S_2^*\left(r,\theta,\varphi\right);\end{aligned} \tag{27}$$

$$\begin{aligned}&\frac{D\left(R_t,R\right)}{D\left(r,\theta\right)}+R^2\frac{D\left(\Theta_t,\Theta\right)}{D\left(r,\theta\right)}+R^2sin^2\Theta\frac{D\left(\Phi_t,\Phi\right)}{D\left(r,\theta\right)}+2R\Theta_t\frac{D\left(R,\Theta\right)}{D\left(r,\theta\right)}\\&+2\Omega\left(R^2sin^2\Theta\frac{D\left(R,\Phi\right)}{D\left(r,\theta\right)}+R^2sin\Theta cos\Theta\frac{D\left(\Theta,\Phi\right)}{D\left(r,\theta\right)}\right)=S_3^*\left(r,\theta,\varphi\right),\end{aligned} \tag{28}$$

here S_1^*,S_2^*,S_3^*—invariants dependent on the Lagrangian spherical coordinates.

4. Statement of the Problem and Approximations Used

We are interested in the fluid dynamics details inside the oceanic gyres. For this purpose we will use the spherical coordinates, where R are the distances from the Earth centre, Θ (and θ) are the polar angles and Φ (and φ) are the azimuthal angles, i.e. the angles of longitude. The values $\pi/2 - \Theta$ and $\pi/2 - \theta$ are the conventional angles of latitude, so that the North Pole corresponds to $\Theta, \theta = 0$, the South Pole to $\Theta, \theta = \pi$, and the Equator to $\Theta, \theta = \pi/2$. The vector $\overrightarrow{\Omega}$ is directed from South to North; $\Omega \approx 7.29 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ is the constant rate of rotation of the Earth. The gravity potential is $\Phi_* = g\,(R - R_0)$, where g is the acceleration due to gravity.

(a) *Thin-layer approximation* Let us consider the Earth to be a solid sphere of the radius R_0 covered with a water layer of the thickness h. The variable R is equal to $R = R_0 + r$, where $0 \le r \le h$. In the zero order with respect to the small parameter h/R_0 in Eqs. (26)–(28) R may be replaced by R_0 (formally $h = 0$).

For the thin-layer approximations it is typical that the velocity component across the layer is small compared with the speeds along the layer. For motions observed in ocean gyres typical horizontal velocities are of the order of $0.01\,\mathrm{m\,s^{-1}}$ and the ratio of the vertical speed to the horizontal speeds is of about 10^{-4} to 10^{-5}; see [29,30]. We will neglect the vertical motion of the fluid. Fluid particles should move along the spherical surface $r = const$. The flows on the spherical surfaces will be similar, and the non-leakage condition is satisfied automatically at the solid surface $r = 0$.

Let us consider the coordinates of the trajectory Θ, Φ to be independent of the variable r. So Eqs. (26)–(28) could be rewritten as

$$sin\Theta \frac{D\,(\Theta, \Phi)}{D\,(\theta, \varphi)} = R_0^{-2} S_0^*\,(\theta, \varphi)\,, \tag{29}$$

$$\frac{D\,(\Theta_t, \Theta)}{D\,(\theta, \varphi)} + sin^2\Theta \frac{D\,(\Phi_t, \Phi)}{D\,(\theta, \varphi)} + 2\Omega\, sin\,\Theta\, cos\Theta \frac{D\,(\Theta, \Phi)}{D\,(\theta, \varphi)} = R_0^{-2} S_1^*\,(\theta, \varphi)\,. \tag{30}$$

The left part of Eq. (29) is entirely included in one of the terms of Eq. (30). This fact will be used in our calculations further.

Equations (29) and (30) do not include pressure. From Eq. (15) it follows that

$$H = \frac{p}{\rho} + gr - \frac{1}{2}\Omega^2\,(R_0 + r)^2\, sin^2\Theta = H\,(\theta, \varphi)\,, \tag{31}$$

where $H\,(\theta, \varphi)$—the function to be defined. The following condition

$$p\,|_{r=h} = P_s\,(\theta, \varphi)\;,$$

should be satisfied at the free surface, where $P_s\,(\theta, \varphi)$ is the pressure inducing the studied vortical motion. In the present problem this is a non-uniform air pressure (wind) above the water free surface. The function $P_s\,(\theta, \varphi)$ is determined by Eqs. (16) and (17) via the known solution of the set of Eqs. (29) and (30). The function $H\,(\theta, \varphi)$ is connected with it by means of the next relation

$$H\,(\theta, \varphi) = \frac{P_s\,(\theta, \varphi)}{\rho} + gh - \frac{1}{2}\Omega^2\left(R_0^2 + 2R_0 h\right) sin^2\Theta. \tag{32}$$

The flow velocity in this approximation $\overrightarrow{V}(0, R_0\Theta_t, R_0\Phi_t sin\Theta)$ has only two components—a meridional and an azimuthal. Since both of them are independent of the radius, only the radial component of the vorticity will be non-zero, and thus could be written in the form:

$$\begin{aligned}\omega_R &= \frac{1}{R_0 sin\Theta}\left[\frac{\partial\,(V_\Phi sin\Theta)}{\partial\Theta} - \frac{\partial V_\Theta}{\partial\Phi}\right] = \frac{1}{sin\Theta}\left[\frac{D\,(\Phi_t sin^2\Theta, \Phi)}{D\,(\Theta, \Phi)} + \frac{D\,(\Theta_t, \Theta)}{D\,(\Theta, \Phi)}\right] \\ &= \frac{R_0^2}{S_0^*}\left[\frac{D\,(\Theta_t, \Theta)}{D\,(\theta, \varphi)} + sin^2\Theta \frac{D\,(\Phi_t, \Phi)}{D\,(\theta, \varphi)} + 2\Phi_t sin\Theta cos\Theta \frac{D\,(\Theta, \Phi)}{D\,(\theta, \varphi)}\right].\end{aligned} \tag{33}$$

The formulation of the final expression was based on Eq. (29). The vorticity formula (33) is similar to the left-hand side of Eq. (30), which represents the Cauchy integral in the thin-layer approximation. The value Ω was replaced by the function Φ_t in the third term in square brackets of Eq. (33). Comparing Eqs. (29), (30) and (33) it is possible to obtain the relation for the vorticity via this function:

$$\omega_R + 2\left(\Omega - \Phi_t\right) cos\Theta = \frac{S_1^*}{S_0^*}. \tag{34}$$

The vorticity of liquid particles is not preserved in a general case. It forms an integral of motion when summing the "spin vorticity" [4] with the term that takes into account an azimuthal motion of the fluid.

(b) *Average latitude approximation* Let us introduce some new unknown functions

$$X = \lambda\Phi, \Theta^* = \frac{\pi}{2} - \Theta, Y = cos\Theta = sin\Theta^*, \tag{35}$$

and the variable

$$\theta^* = \frac{\pi}{2} - \theta,$$

here λ is a constant, and Θ^*, θ^* are latitudes in the Eulerian and Lagrangian description (both are positive in the Northern hemisphere and negative in the Southern one). Simultaneous Eqs. (29) and (30) could be reformulated as follows

$$\frac{D\left(X, Y\right)}{D\left(\varphi, \theta^*\right)} = \lambda R_0^{-2} S_0^*, \tag{36}$$

$$\lambda^{-2} cos^2\theta^* \frac{D\left(X_t, X\right)}{D\left(\varphi, \theta^*\right)} + cos^{-2}\theta^* \frac{D\left(Y_t, Y\right)}{D\left(\varphi, \theta^*\right)} + 2\Omega R_0^{-2} S_0^*\, sin\theta^* = R_0^{-2} S_1^*. \tag{37}$$

The suggested substitution of variables was fulfilled within Jacobians only (the reason will be clarified below). The set of Eqs. (36) and (37) is extremely difficult for its solution. For simplicity we propose to replace it with another nonlinear system with possible exact solutions.

The fluid particles in gyres moved around a fixed center. Let's denote its position by coordinates $\Phi_0 = \varphi_0, \Theta_0^* = \theta_0^*$. All gyres are located within a limited latitude band excluding areas near the North and South poles ($\Theta^* = \pm\frac{\pi}{2}$), where the coefficients of the Jacobians turn to zero and to an infinity respectively. In addition, all of the gyres lie on one side of the equator so that $\sin\Theta^*$ (and the third term in the left side of Eq. (37)) does not change the sign within the fluid flow. Taking this into account let us make the following replacement in the trigonometric multipliers of Eq. (37)

$$\Theta^* = \Theta_0^*, \tag{38}$$

and assume $\lambda = cos^2\Theta_0^*$ as well. Then Eq. (37) has the following form

$$\frac{D\left(X_t, X\right)}{D\left(\varphi, \theta^*\right)} + \frac{D\left(Y_t, Y\right)}{D\left(\varphi, \theta^*\right)} = \left(S_1^* - 2\Omega S_0^*\, sin\Theta_0^*\right) R_0^{-2} cos^2\Theta_0^* = S_1^{**}. \tag{39}$$

Condition (38) can be called "the averaged latitude approximation" and Eq. (39) is the equation for the radial Cauchy invariant in the average latitude approximation.

There are two types of terms in Eq. (37). The first two terms in the square brackets (including Jacobians) describe the vortex dynamics related indirectly to the influence of the Earth's rotation. The third term already contains Ω and therefore takes into account the effect of planet rotation. The averaging procedure in this term seems to be the most natural. As spin vorticity changes with latitude monotonically, then replacing it with the average value we assume that the effect of rotation acts similarly throughout the latitude band. This property of our approximation is similar of the f-plane approach. But we will stay within the framework of the spherical coordinate system where functions of latitude and longitude are unknown. Here the continuity equation (36) is used in the exact formulation. The approximation is applied to the equation for the Cauchy integral only. Averaging of the Jacobian's coefficients means that the weights of the meridional and azimuthal components in the Cauchy integral will change. But since the Jacobians are included in Eqs. (36) and (39) in the form of the same linear combinations (with different weight factors) it is worth to be assumed that the solutions of the set of approximate equations will not be much different from the solutions of Eqs. (36) and (37). In mathematical sense we replace coefficients of the nonlinear equation, which are monotonic and sign-constant functions, with their averaged values. Obviously the approximate solution will be more accurate the narrower the band of latitude is within the gyre and the closer a part of the flow lays to the gyre's center.

Equations (36) and (39) coincide with the equations of two-dimensional dynamics of an ideal incompressible fluid in the Cartensian Lagrangian coordinates [22,31,32]. The functions X, Y are similar to the horizontal and vertical coordinates of the trajectory of a liquid particle respectively, and φ, θ^* relates to its Lagrangian variables.

By introducing complex coordinates of the particle trajectory $W = X + iY$, $\bar{W} = X - iY$ and complex Lagrangian coordinates $\chi = \varphi + i\theta^*, \bar{\chi} = \varphi - i\theta^*$ (the overline in $\bar{\chi}$ denotes the complex conjugate of χ) we can write Eqs. (36) and (39) in the form [33]:

$$\frac{D\left(W, \bar{W}\right)}{D\left(\chi, \bar{\chi}\right)} = R_0^{-2} S_0^* cos^2 \Theta_0^* = S_0^{**}\left(\chi, \bar{\chi}\right), \tag{40}$$

$$\frac{D\left(W_t, \bar{W}\right)}{D\left(\chi, \bar{\chi}\right)} = \frac{i}{2} S_1^{**}\left(\chi, \bar{\chi}\right). \tag{41}$$

The functions S_0^{**}, S_1^{**} are independent of time. The first one determines the dependence of the initial position of liquid particles in the Lagrangian coordinates.

The sign of the function does not change in the flow region due to one-to-one mapping between particle coordinates and their Lagrangian labels. We assume for definiteness $S_0^{**} \geq 0$. When writing Eq. (41) we took into account that time derivative of Eq. (40) equals to zero. Note also that the unknown function W will always represent the solution of Eq. (40) when satisfying Eq. (41) [34].

5. Ptolemaic Eddies on the Sphere

Equations (40) and (41) have an exact solution [33]:

$$W = W_0\left(t\right) + G\left(\chi - \chi_0\right) e^{i\delta t} + F\left(\bar{\chi} - \overline{\chi_0}\right) e^{i\mu t}, \tag{42}$$

here $W_0\left(t\right)$ is a complex fuction of time, G, F are analytical functions of the corresponding arguments, χ_0 is complex constant, and δ, μ are real ones. It is convenient to assume that the function W_0 defines the position of the center of the gyre, which remains fixed in our consideration, and therefore W_0 is constant and equals to

$$W_0^* = \Phi_0 cos^2 \Theta_0^* + isin\Theta_0^*. \tag{43}$$

Let us the complex constant in the form $\chi_0 = \varphi_0 + i\theta_0^*$, where φ_0, θ_0^* are the Lagrangian coordinates of the center of the gyre ($\varphi_0 = \Phi_0, \theta_0^* = \Theta_0^*$) and require $G\left(0\right) = F\left(0\right) = 0$ so that $W\left(\chi_0, \overline{\chi_0}, t\right) = W_0^*$.

In the plane X, Y the motion of the liquid particle will represent a superposition of two circular rotations. The circular motion relatively to the point W_0^* with the radius $|G|$ and the frequency δ is superimposed by a circle rotation with the radius $|F|$ and the frequency μ. The particle trajectories for this solution are either epicycloids ($\delta\mu > 0$) or hypocycloids ($\delta\mu < 0$). In the Ptolemaic picture of the world the planets move along such trajectories; that is why these flows were named Ptolemaic [33].

Let us introduce the notation $\chi' = \chi - \chi_0$ and write the expression (42) as follows

$$W - W_0^* = G(\chi')e^{i\delta t} + F\left(\bar{\chi}'\right) e^i \mu t \tag{44}$$

For fixed χ' Eq. (44) gives a parametric representation of the trajectory of a liquid particle. The first term on the right side of Eq. (44) describes rotation over a circle of the radius $|G|$. We will consider that $|G| > |F|$. The value of this radius is set by selecting the range of variation of χ'. Let it be a circle $|\chi'| \leq \alpha$. In the case $|G| > |\mu F/\delta|$ the trajectory of the particle will represent a shorten epi- or hypocycloid. In the case of equality $|G| = |\mu F/\delta|$ it will be an ordinary epi- or hypocycloids with points of sharpness. Finally, in the case of the opposite inequality the trajectory of the fluid particle will be a prolonged epi- or hypocycloid having self-intersection points. If the ratio of frequincies μ and δ is integer, then the trajectory will be a closed curve with $|(\mu/\delta)-1|$ cusps. For a rational ratio of frequencies the trajectory will be closed through a finite number of revolutions. In the case of μ to δ ratio is irrational the trajectory will remain open.

All of these statements will qualitatively remain valid at the Earth's surface according to the formulas:

$$\Phi = \Phi_0 + cos^{-2}\Theta_0^* Re\left[G(\chi^{'})e^{i\delta t} + F\left(\bar{\chi}'\right)e^{i\mu t}\right]$$
$$\Theta^* = \arcsin\left\{sin\Theta_0^* + Im\left[G(\chi^{'})e^{i\delta t} + F\left(\bar{\chi}'\right)e^{i\mu t}\right]\right\}. \tag{45}$$

In the plane Φ, Θ^* the trajectories of fluid particles are nonuniformly stretched and represent the deformed epi- or hypocycloids. In this sense one can speak about the qualitative analogy of the fluid motion in the auxiliary plane of variables X, Y and in the physical plane of meridians and parallels Φ, Θ^*. Formulas (44) and (45) describe the superposition of two rotational motions. The first one is performed with the period $2\pi/\delta$ corresponding to the rotation of the liquid particle in a large circle. The second one with the period $2\pi/\mu$ corresponds to the perturbation of the finite amplitude superimposed with the background flow. It is naturally to associate this perturbation with mesoscale eddies formed at the gyre's boundaries. The size of mesoscale eddies is significantly smaller than the characteristic scale of the gyres, and their velocities are much higher (of the order of 1 m/s). That is why it is natural to consider $\delta \ll \mu$, so the trajectory of the liquid particle has a large number of cusps.

Equations (44) and (45) determine the dynamics of the vortex flow corresponding to the circle of the radius α within the plane of the Lagrangian variable χ'. Its value is determined depending on the position of the gyre at the Earth's surface. Obviously α should not exceed the minimum distance from the center of the gyre Φ_0, Θ_0^* to the shoreline. The shape of the gyre depends on two functions G and F. This means that Eqs. (44) and (45) give a solution for a fairly wide class of initial conditions: the shape of the vortical region and its inner velocity distribution at the initial time. The only restriction for the choice of functions G and F is the constant sign of the right side of the continuity equation (40) within the flow [see explanation to Eq. (4)]. Let us assume for certainty that $S_0^{**} \geq 0$. In this case the following condition must be met

$$|G'| - |F'| \geq 0, |\chi'| \leq \alpha. \tag{46}$$

The equality $|\chi'| = \alpha$ is allowed at the gyre's edge.

Turning to the specific examples let's start with the analysis of stationary gyres. The fluid's motion is known to be stationary if the Lagrangian values are invariant under time translation [22]. For a two-dimensional flow this means that coordinates of a fluid particle depend on two variables $q, s+ut$ only, where q, s are the Lagrangian variables, and u is a constant velocity. The coordinate s changes along the streamline, which coincides with the trajectory of the fluid particle in a stationary flow, and the coordinate q changes from one streamline to another. In our case these variables are convenient to be chosen in the following form:

$$q = |\chi'|, s = \arg\chi',$$

the function G is linear, and the function F is a power one:

$$G\left(\chi'\right) = \chi' = q^{is}, \left(F\chi'\right) = \beta\overline{\chi}'^{n} = \beta q^n e^{-ins}, \tag{47}$$

here n is an integer positive number, and β is a constant with the dimension α^{1-n}. From the condition (46) it follows that $\beta \leq \left(n\alpha^{n-1}\right)^{-1}$.

In the new variables the expression (44) will be written as follows:

$$W - W_0^* = qexp\left[i\left(s+\delta t\right)\right] + \beta q^n exp\left[-in\left(s - \frac{\mu}{n}t\right)\right]. \tag{48}$$

When assuming $\mu = -n\delta$ the flow (48) is stationary, since it depends on two variables $q, s+\delta t$ only. Trajectories of liquid particles in the plane W and hence the stream functions represent hypocycloids with the number of the cusps $n+1$. The fluid streamlines represent deformed hypocycloids in the meridian-to-latitude Cartesian grid. It is simple to represent the qualitative form of streamlines for such eddies. They form a system of closed wavy lines around the center of the gyre with the coordinates Φ_0, Θ_0^*. The number of waves stowed along each of them equals to $n+1$.

In the case of an arbitrary ratio of frequencies δ and μ the fluid motion is unsteady according to Eq. (48). The latter expression can be written as follows:

$$W - W_0^* = \{qexp\,(i\Delta) + \beta q^n exp\,(-in\Delta)\}\, exp\left[i\left(\frac{\mu + n\delta}{n+1}\right)t\right],$$
$$\Delta = s + \frac{\delta - \mu}{n+1}t. \tag{49}$$

When $\mu = -n\delta$ the expression in the braces describes the steady flow. The exponential multiplier outside the braces relates to the uniform rotation of the fluid. Then solutions (48) and (49) describe a hypocycloidal gyre rotating at an angular frequency $(\mu + n\delta)\,/\,(n+1)$ without any variations in its shape.

For the considered flows the stream function is introduced by the following formulas

$$V_\Theta = R_0\Theta_t = \frac{1}{sin\Theta}\psi_\Phi, V_\Phi = R_0\Phi_t sin\Theta = -\psi_\Theta.$$

A total differential for this function has the form

$$d\psi = \psi_\Phi d\Phi + \psi_\Theta d\Theta = \frac{R_0}{2i\lambda}\frac{D\left(W, \bar{W}\right)}{D\left(\chi', t\right)}d_\chi + c.c.$$

From another side, $d\psi = \psi_{\chi'}d\chi' + \psi_{\overline{\chi}'}d\chi' = \psi_{\chi'}d\chi' + c.c.$, therefore

$$\psi_{\chi'} = \frac{R_0}{2i\lambda}\frac{D\left(W, \bar{W}\right)}{D\left(\chi', t\right)},$$

so the stream function for the flows (44) is written as:

$$\psi = -\frac{R_0}{\lambda}\left\{\delta\left|G\right|^2 + \mu\left|F\right|^2 + (\delta + \mu)\, Re\left[G\bar{F}\exp i\,(\delta - \mu)\,t\right]\right\}. \tag{50}$$

For hypocycloidal gyres (47) in the case $\mu = -n\delta$ the stream function depends on q and $s + \delta t$ only as it should be for steady flows. It is easy to show that the lines $\psi = const$ correspond to this curve by means of the parametric representation of the hypocycloid. In any other cases expression (50) corresponds to unsteady flows.

Figure 1 shows the dynamics of the boundary of the gyre under the following conditions

$$G = \chi', F = -0.2\left(\frac{\bar{\chi}'}{\alpha}\right)^3 + 0.1\left(\frac{\bar{\chi}'}{\alpha}\right)^4, |\chi'| \le \frac{\pi}{15'},$$
$$\Phi_0 = \frac{\pi}{4}, \Theta_0^* = \frac{\pi}{6}, \mu = 6\delta.$$

The gyre corresponds to a circle with a radius 12° on the Lagrangian plane χ'. The center of the gyre is selected at the point corresponding to 45° of the West longitude and 30° of the North latitude. It is located inside the North-Atlantic gyre just to the East of the Sargasso Sea. The sign of the angular frequency of rotation of liquid particles over the large circle δ is positive, which corresponds to the clockwise periodic rotation of the gyre with the period . The trajectories of the liquid particles are the closed curves with five cusps. Figure 1 shows the dynamics of the gyre boundary at different moments of time corresponding to eight values of the parameter δt. In addition to rotational motion the gyre is deformed in a rather complex way as well. There are four cusps at the boundary of the gyre. They change their shape over time, so we can say that an unsteady azimuthal wave propagates along the gyre's boundary.

It is interesting to compare the solution (44), (45) with the solution for the Ptolemaic vortices in the traditional fluid dynamics. The velocity field of a two-dimensional plane vortex should decrease inversely with respect to the distance from its center at infinity. As a result, when the vortex motion is matched

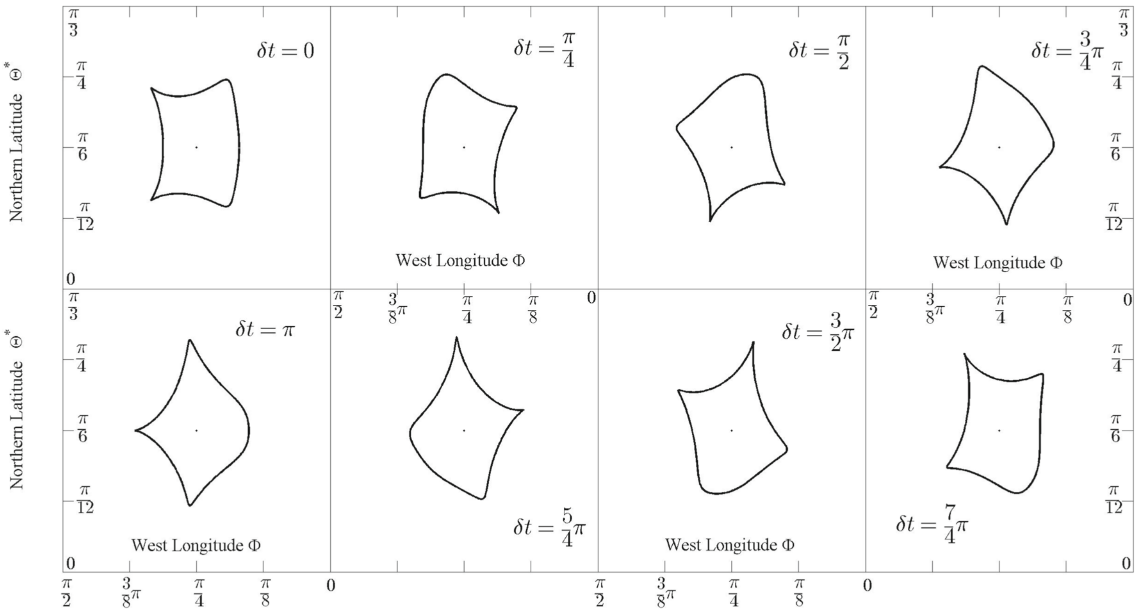

Fig. 1. Dynamics of the boundary of the gyre

with an external potential flow the frequency μ is assumed to be zero and trajectories of liquid particles inside the vortex are circular [33]. In our case $\mu \neq 0$, so liquid particles have more complex trajectories.

In [35] the stability of the Ptolemaic vortices with respect to short-wave perturbations was investigated. The problem was analyzed for solutions (44) in case of the linear function G and arbitrary F, δ, μ. It is not quite correct to transfer the results of [35] to our problem. But bearing in mind the possibility of their qualitative comparison we indicate that the plane Ptolemaic vortices are stable in a very narrow band of μ/δ and $|F'|$ only. Outside this interval they are unstable for the three-dimensional short-wave perturbations. This result being applied to the studied gyres fits well into the general representations of the fluid motion inside ones. Our solutions correspond to averaged long-period two-dimensional motions of liquid particles around their generic center. A structure of an averaged drift of the fluid is marked by arrows on the map of ocean currents. This large-scale circulation generates three-dimensional mesoscale eddies as a result of its instability, thus representing a source of the mesoscale variability [16–21].

The Lagrangian invariants of the Ptolemaic flows on the sphere's surface (44), (45) are written as follows:

$$S_0^* = \frac{R_0^2}{cos^2\Theta_0^*}\left(|G'|^2 - |F'|^2\right), S_1^{**} = 2\left(\delta\,|G'|^2 - \mu\,|F'|^2\right),$$

$$S_1^* = \frac{R_0^2}{cos^2\Theta_0^*}\left[(\delta + \Omega)\,|G'|^2 - (\mu + \Omega)\,|F'|^2\right],$$

and the expression for the vorticity (33) turns to the form

$$\omega_R = \frac{S_1^*}{S_0^*} - 2\left(\Omega - cos^2\Theta_0^* ReW_t\right) ImW.$$

In addition to the component depending on the Lagrangian variables only it contains the non-stationary terms with the frequencies $\delta, \mu, 2\delta, 2\mu, \mu + \delta, \mu - \delta$ as well.

The vortex motions in study are induced by the pressure $P_s\,(\theta, \varphi)$ at the free surface. The pressure distribution being found from the set of Eqs. (14) and (15) and taking into account Eq. (32) could be written as follows

$$\Theta_{tt}\Theta_\theta + sin^2\Theta\Phi_{tt}\Phi_\theta + cos\Theta sin\Theta\left[2\Theta_t\ \Phi_t\Phi_\theta - \Theta_\theta\Phi_t^2 \right.$$
$$\left. + 2\Omega\left(\Theta_t\Phi_\theta - \Theta_\theta\Phi_t\right) - \Omega^2\left(1 + \frac{2h}{R_0}\right)\right] = -\frac{1}{\rho R_0^2}\left(P_s\right)_\theta, \tag{51}$$
$$\Theta_{tt}\Theta_\varphi + sin^2\Theta\Phi_{tt}\Phi_\varphi + cos\Theta sin\Theta\left[2\Theta_t\ \Phi_t\Phi_\varphi - \Theta_\varphi\Phi_t^2\right.$$
$$\left. + 2\Omega\left(\Theta_t\Phi_\varphi - \Theta_\varphi\Phi_t\right)\right] = -\frac{1}{\rho R_0^2}\left(P_s\right)_\varphi. \tag{52}$$

For calculation of the function P_s it is necessary to find the expressions for variables Θ, Φ from Eq. (44), then to substitute them to the set of Eqs. (51) and (52) with its further integration. The left parts of Eqs. (51) and (52) represent extremely bulky expressions. It is impossible to calculate the integral in explicit form even in the approximation of the averaged latitude. But in each case (for the certain functions G and F) the pressure distribution at the free surface can be obtained by means of numerical integration. The function P_s depends on time for unsteady vortex motions. Due to a nonlinearity of the left parts of Eqs. (51) and (52) and to the presence of trigonometric functions there the time spectrum of the function P_s will be wide, though discrete. Taking into account that the choice of the functions G and F is sufficiently arbitrary it can be concluded that Eqs. (51) and (52) define a wide class of surface pressure distributions. This gives ground to admit an existence of natural fluid flows of Ptolemaic type in the ocean.

6. Conclusion

The presented model is based on a number of idealizations and approximations. But it allows us to calculate both the flow parameters of fluid flow in the gyre and the trajectories of liquid particles explicitly. It is formulated in the spherical coordinates and overcomes the frame of β-approximation. We focused here on the large-scale subtropical gyres. However the developed model is applicable to smaller-scale eddies as well. It allows one to take into account the effect of displacement of the vortex center [see formula (42)]. But in this case the boundary condition at the gyre's edge should be formulated, which was not required in our investigation. The study is based on the Lagrangian approach. Unlike the Eulerian description [4] it helps to clarify unsteady features of the flow within the gyre. In general, the proposed model gives an explanation of the inner mechanism of large-scale circulation in the gyres.

Acknowledgements. The publication was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2018-2019 (Grant № 18-01-0006) and by the Russian Academic Excellence Project "5-100". The author has the pleasure to thank Prof. A. Constantin for stimulating discussions.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Verkley, T.M.: The construction of barotropic modons on a sphere. J. Atmos. Sci. **41**, 2492–2504 (1984)
[2] Paldor, N.: Shallow Water Waves on the Rotating Earth. Springer, Berlin (2015)
[3] Constantin, A., Johnson, R.S.: Steady large-scale ocean flows in spherical coordinates. Oceanography **31**, 42–50 (2018)
[4] Constantin, A., Johnson, R.S.: Large gyres as a shallow-water asymptotic solution of Euler's equation in spherical coordinates. Proc. R. Soc. A **473**, 20170063 (2017)
[5] Constantin, A., Johnson, R.S.: Large-scale oceanic currents as shallow-water asymptotic solutions of the Navier–Stokes equation in rotating spherical coordinates. Deep Sea Res. Part II Top. Stud. Oceanogr. **160**, 32–40 (2019)

[6] Constantin, A., Johnson, R.S.: Ekman-type solutions for shallow-water flows on a rotating sphere: a new perspective on a classical problem. Phys. Fluids **31**, 021401 (2019)
[7] Pedlosky, J.: The stability of currents in the atmosphere and the ocean: part I. J. Atmos. Sci. **27**, 15–30 (1964)
[8] Philander, S.G.H.: Instability of zonal equatorial currents. J. Geophys. Res. **81**(21), 3725–3735 (1976)
[9] Pedlosky, J.: Geophysical Fluid Dynamics. Springer, Berlin (1987)
[10] Kamenkovich, I.V., Pedlosky, J.: Radiating instability of nonzonal ocean currents. J. Phys. Oceanogr. **26**, 622–643 (1996)
[11] Walker, A., Pedlosky, J.: Instability of meridional baroclinic currents. J. Phys. Oceanogr. **32**, 1075–1093 (2002)
[12] Pedlosky, J., Thomson, J.: Baroclinic instability of time-dependent currents. J. Fluid Mech. **490**, 189–215 (2003)
[13] Hristova, H.G., Pedlosky, J., Spall, M.A.: Radiating instability of a meridional boundary current. J. Phys. Oceanogr. **38**, 2294–2307 (2008)
[14] Cushman-Roisin, B.: Trajectories in Gulf Stream meanders. J. Geophys. Res. **98**(C2), 2543–2554 (1993)
[15] Yuan, G.-C., Pratt, L.J., Jones, C.K.R.T.: Barrier destruction and Lagrangian predictability at depth in a meandering jet. Dyn. Atmos. Oceans **35**, 41–61 (2002)
[16] Gill, A.E., Green, J.S.A., Simmons, A.I.: Energy partition in the large-scale ocean circulation and the production of mid-ocean eddies. Deep Sea Res. **21**, 499–528 (1974)
[17] Robinson, A.R., McWilliams, J.C.: The baroclinic instability of the open ocean. J. Phys. Oceanogr. **4**, 281–294 (1974)
[18] Robinson, A.R.: Eddies in Marine Science. Springer, Berlin (1983)
[19] Spall, M.A.: Cooling spirals and recirculation in the subtropical gyre. J. Phys. Oceanogr. **22**, 564–571 (1991)
[20] Spall, M.A.: Generation of strong mesoscale eddies by weak ocean gyres. J. Mar. Res. **58**, 97–116 (2000)
[21] Manucharyan, G.E., Spall, M.A., Thompson, A.F.: A theory of the wind-driven Beaufort gyre variability. J. Phys. Oceanogr. **46**, 3263–3278 (2016)
[22] Bennett, A.: Lagrangian Fluid Dynamics. Cambridge University Press, Cambridge (2006)
[23] Abrashkin, A.A., Zen'kovich, D.A., Yakubovich, E.I.: Matrix formulation of hydrodynamics and extension of Ptolemaic flows to three-dimensional motions. Radiophys. Quantum Electron. **39**, 518–526 (1996)
[24] Zakharov, V.E., Kuznetsov, E.A.: Hamiltonian formalism for nonlinear waves. Phys. Usp. **40**, 1087–1116 (1997)
[25] Kuznetsov, E.A.: Vortex line representation for the hydrodynamic type equations. J. Nonlinear Math. Phys. **13**(1), 64–80 (2006)
[26] Frisch, U., Villone, B.: Cauchy's almost forgotten Lagrangian formulation of the Euler equation for 3D incompressible flow. Eur. Phys. J. H **39**, 325–351 (2014)
[27] Besse, N., Frisch, U.: Geometric formulation of the Cauchy invariants for incompressible Euler flow in flat and curved spaces. J. Fluid Mech. **825**, 412–478 (2017)
[28] Lamb, G.: Hydrodynamics, 6th edn. Cambridge University Press, Cambridge (1932)
[29] Viudez, A., Dritschel, D.G.: Vertical velocity in mesoscale geophysical flows. J. Fluid Mech. **483**, 199–223 (2015)
[30] Herbei, R., McKeague, I., Speer, K.G.: Gyres and jets: inversion of tracer data for ocean circulation structure. J. Phys. Oceanogr. **39**, 1180–1202 (2009)
[31] Kochin, N.E., Kibel, I.A., Roze, N.V.: Theoretical Hydromechanics. Interscience, New York (1964)
[32] Constantin, A.: Nonlinear water waves with applications to wave–current interactions and tsunamis. In: CBMS-NSF Conference Series in Applied Mathematics, vol. 81. SIAM, Philadelphia, PA (2011)
[33] Abrashkin, A.A., Yakubovich, E.I.: Planar rotational flows of an ideal fluid. Sov. Phys. Dokl. **29**, 370–371 (1984)
[34] Aleman, A., Constantin, A.: Harmonic maps and ideal fluid flows. Arch. Ration. Mech. Anal. **204**, 479–513 (2012)
[35] Guimbard, D., Leblanc, S.: Local stability of the Abrashkin–Yakubovich family of vortices. J. Fluid Mech. **567**, 91–110 (2006)

Anatoly Abrashkin
National Research University
Higher School of Economics
25/12 Bol'shaya Pecherskaya str.
Nizhny Novgorod
Russia 603155
e-mail: aabrashkin@hse.ru

(accepted: March 25, 2019)