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ON EMBEDDING OF MULTIDIMENSIONAL MORSE–SMALE DIFFEOMORPHISMS INTO TOPOLOGICAL FLOWS

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Abstract. J. Palis found necessary conditions for a Morse–Smale diffeomorphism on a closed n-dimensional manifold M^n to embed into a topological flow and proved that these conditions are also sufficient for $n = 2$. For the case $n = 3$ a possibility of wild embedding of closures of separatrices of saddles is an additional obstacle for Morse–Smale cascades to embed into topological flows. In this paper we show that there are no such obstructions for Morse–Smale diffeomorphisms without heteroclinic intersection given on the sphere S^n, $n \geqslant 4$, and Palis conditions again are sufficient for such diffeomorphisms.

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1. Introduction and Statements of Results

Let M^n be a smooth connected closed n-manifold. Recall that a C^m-*flow* ($m \geqslant 0$) on the manifold M^n is a (continuously depending on $t \in \mathbb{R}$) family of C^m-diffeomorphisms $X^t\colon M^n \to M^n$ that satisfies the following conditions:

1) $X^0(x) = x$ for any point $x \in M^n$;
2) $X^t(X^s(x)) = X^{t+s}(x)$ for any s, $t \in \mathbb{R}$, $x \in M^n$.

A C^0-flow is also called a *topological flow*. One says that a homeomorphism (diffeomorphism) $f\colon M^n \to M^n$ *embeds* into a C^m-flow on M^n if f is the time one map of this flow.

Obviously, if a homeomorphism embeds in a flow then it is isotopic to identity. For a homeomorphism of the line and a connected subset of the line this condition also is necessary (see [6], [8]). If an orientation preserving homeomorphism f of the circle satisfies either one of the three conditions: 1) f has a fixed point, 2) f has a dense orbit, or 3) f is periodic then it embeds in a flow (see [7]). Sufficient conditions of embedding in topological flow for a homeomorphisms of a compact two-dimensional disk and of the plane one can find in review [34]. An analytical, ε-closed to the identity diffeomorphism $f\colon M^n \to M^n$ can be approximated with accuracy $e^{-\frac{c}{\varepsilon}}$ by a diffeomorphism which embeds in an analytical flow, see [30].

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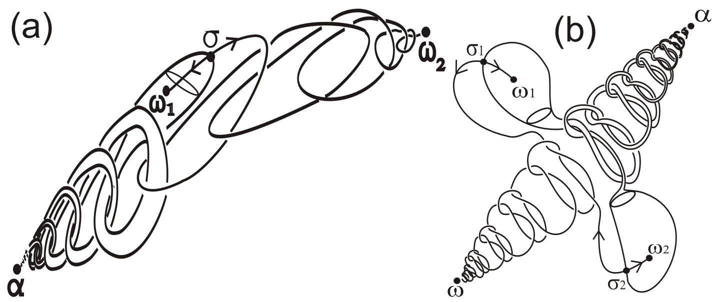

FIGURE 1. Phase portraits of Morse–Smale diffeomorphisms on S^3 which do not embed in topological flows

Due to [26] the set of C^r-diffeomorphisms ($r \geqslant 1$) which embed in C^1-flows, is a subset of the first category in $\mathrm{Diff}^r(M^n)$. As for embedding in C^0-flows, there exists an open (in $\mathrm{Diff}^r(M^n)$) set of diffeomorphisms embeddable in topological flows. This set contains neighborhoods of time one maps of Morse–Smale flows without periodic trajectories. According to [32] such flows exist on an arbitrary smooth manifold. According to [25], [27] Morse–Smale diffeomorphisms are structurally stable, so for any time-one map X^1 of Morse–Smale flow X^t there exist a neighborhood $U(X^1) \subset \mathrm{Diff}^r(M^n)$ such that any diffeomorphism $f \in U(X^1)$ is topologically conjugated with X^1 by means of some homeomorphism h, and, consequently, is embedding to a flow $h^{-1}X^th$.

Recall that a diffeomorphism $f\colon M^n \to M^n$ is called *a Morse–Smale diffeomorphism* if it satisfies the following conditions:

- the non-wandering set Ω_f is finite and consists of hyperbolic periodic points;
- for any two points p, $q \in \Omega_f$ the intersection of the stable manifold W^s_p of the point p and the unstable manifold W^u_q of the point q is transversal[1].

In [25] J. Palis established the following necessary conditions of the embedding of a Morse–Smale diffeomorphism $f\colon M^n \to M^n$ into a topological flow (we call them *Palis conditions*):

(1) the non-wandering set Ω_f coincides with the set of fixed points of f;
(2) the restriction of the diffeomorphism f to each invariant manifold of a fixed point $p \in \Omega_f$ preserves the orientation of the manifold;
(3) if for two distinct saddle points p, $q \in \Omega_f$ the intersection $W^s_p \cap W^u_q$ is not empty then it contains no compact connected components.

According to [25] these conditions are not only necessary but also sufficient for the case $n = 2$. For the case $n = 3$ a possibility of wild embedding of closures of separatrices of saddles is another obstruction for Morse–Smale cascades to embed in topological flows (phase portraits of such diffeomorphisms are shown on the Figure 1). In [13] examples of such cascades are described and a criteria for embedding of Morse–Smale 3-diffeomorphisms in topological flows is provided. In the present

[1]Definitions of stable and unstable manifolds and of transversality are given in Section 4; see also the book [16] for references.

paper we establish that the Palis conditions are sufficient for Morse–Smale diffeomorphisms on S^n, $n \geqslant 4$, such that for any distinct saddle points $p, q \in \Omega_f$ the intersection $W^s_p \cap W^u_q$ is empty.

Theorem 1. *Suppose that a Morse–Smale diffeomorphism* $f\colon S^n \to S^n$, $n \geqslant 4$, *satisfies the following conditions*:

(i) *the non-wandering set* Ω_f *of the diffeomorphism* f *coincides with the set of its fixed points*;

(ii) *the restriction of* f *to each invariant manifold of a fixed point* $p \in \Omega_f$ *preserves the orientation of the manifold*;

(iii) *for any distinct saddle points* p, q *the intersection* $W^s_p \cap W^u_p$ *is empty.*

Then f *embeds into a topological flow.*

2. Comments to Theorem 1

Due to [25] the conditions (i) and (ii) are necessary for embedding a Morse–Smale diffeomorphism into a flow. Our condition that the ambient manifold is the sphere S^n and the absence of heteroclinic intersections (condition (iii)) are not necessary but violation of each of them allows to construct examples of Morse–Smale diffeomorphisms which do not embed in topological flows. Below we describe such examples.

In [24] V. Medvedev and E. Zhuzhoma constructed a Morse–Smale diffeomorphism $f_0\colon M^4 \to M^4$ satisfying conditions (i)–(iii) on a projective-like manifold M^4 (different from S^4) whose non-wandering set consists of exactly three fixed points: a source, a sink and a saddle. Invariant manifolds of the saddle are two-dimensional and the closure of each of them is a wild sphere (see [24, Theorem 4, item 2]). Assume that f_0 embeds in a topological flow X^t_0. Then X^t_0 is a topological flow whose the non-wandering set consists of three equilibrium points with locally hyperbolic behavior. According to [35, Theorem 3] the closures of the invariant manifolds of the saddles are locally flat spheres. This is a contradiction because the closures of the invariant manifolds of the saddle singularities of X^t_0 and f_0 coincide. Thus, f_0 does not embed into a flow.

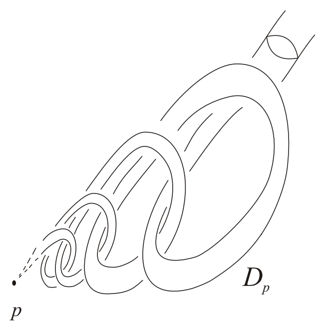

FIGURE 2. The disk $D_p \subset W^s_p$

In [23] T. Medvedev and O. Pochinka constructed an example of Morse–Smale diffeomorphism $f_1\colon S^4 \to S^4$ satisfying to the conditions (i)–(ii) of the Theorem 1. The non-wandering set of the diffeomorphism f_1 consists of two sources, two sinks, and two saddles p, q such that $\dim W^s_p = \dim W^u_q = 3$. The intersection $W^s_p \cap W^u_q$ is not empty and its closure in W^s_p is a wildly embedded open disk D_p (see Fig. 2). If $S^2 \subset W^s_p$ is a 2-sphere which bounds an open ball containing the point p then the intersection $S^2 \cap D_p$ contains at least three connected components. Assume that f_1 embeds into a topological flow X^t_1. Then due to [13] the restriction X^t_p of X^t_1 to $W^s_p \backslash p$ is topologically conjugated by means of a homeomorphism $h\colon W^s_p \backslash p \to \mathbb{S}^2 \times \mathbb{R}$ to a shift flow $\chi^t(s, r) = (s, r+t)$, $(s, r) \in \mathbb{S}^2 \times \mathbb{R}$. Let $\Sigma = h^{-1}(\mathbb{S}^2 \times \{0\})$. Then every trajectory of the flow X^t_p intersects the sphere Σ at a unique point. Since the disk D_p is invariant with respect to the flow X^t_p the intersection $D_p \cap \Sigma$ consists of a unique connected component and that is a contradiction. Thus, f_1 does not embed into a flow.

3. The Scheme of the Proof of Theorem 1

The proof of Theorem 1 is based on the technique developed for classification of Morse–Smale diffeomorphisms on orientable manifolds in a series of papers [1], [2], [3], [17], [10], [11], [12], [15]. The idea of the proof consists of the following.

In section 4 we introduce a notion of Morse–Smale homeomorphism on a topological n-manifold and define the subclass $G(S^n)$ of such homeomorphisms satisfying to conditions similar to (i)–(iii) of Theorem 1.

Let $f \in G(S^n)$. In [15, Theorem 1.3] it is shown that the dimension of the invariant manifolds of the fixed points of f can be only one of 0, 1, $n-1$ or n. Denote by Ω^i_f the set of all fixed points of f whose unstable manifolds have dimension $i \in \{0, 1, n-1, n\}$, and by m_f the number of all saddle points of f.

Represent the sphere S^n as the union of pairwise disjoint sets

$$A_f = \Big(\bigcup_{\sigma\in\Omega^1_f} W^u_\sigma\Big) \cup \Omega^0_f, \quad R_f = \Big(\bigcup_{\sigma\in\Omega^{n-1}_f} W^s_\sigma\Big) \cup \Omega^n_f, \quad V_f = S^n \setminus (A_f \cup R_f).$$

Similar to [18] one can prove that the sets A_f, R_f, V_f are connected, the set A_f is an attractor, R_f is a repeller[2] and V_f consists of wandering orbits of f moving from R_f to A_f.

Denote by $\widehat{V}_f = V_f/f$ the orbit space of the action of f on V_f and by $p_f\colon V_f \to \widehat{V}_f$ the natural projection. Let

$$\hat{L}^s_f = \bigcup_{\sigma\in\Omega^1_f} p_f(W^s_\sigma \setminus \sigma), \quad \hat{L}^u_f = \bigcup_{\sigma\in\Omega^{n-1}_f} p_f(W^u_\sigma \setminus \sigma).$$

Definition 3.1. The collection $S_f = (\widehat{V}_f, \hat{L}^s_f, \hat{L}^u_f)$ is called the *scheme of the homeomorphism* $f \in G(S^n)$.

[2]A set A is called an attractor of a homeomorphism $f\colon M^n \to M^n$ if there exists a closed neighborhood $U \subset M^n$ of the set A such that $f(U) \subset \operatorname{int} U$ and $A = \bigcap_{n\geqslant 0} f(U)$. A set R is called a repeller of a homeomorphism f if it is an attractor for the homeomorphism f^{-1}.

Definition 3.2. Schemes S_f and $S_{f'}$ of homeomorphisms $f,\, f' \in G(S^n)$ are called *equivalent* if there exists a homeomorphism $\hat\varphi\colon \widehat{V}_f \to \widehat{V}_{f'}$ such that $\hat\varphi(\hat L^s_f) = \hat L^s_{f'}$ and $\hat\varphi(\hat L^u_f) = \hat L^u_{f'}$.

The next statement follows from paper [15, Theorem 1.2] (in fact, Theorem 1.2 was proven for Morse–Smale diffeomorphisms but the smoothness plays no role in the proof).

Statement 3.1. *Homeomorphisms $f,\, f' \in G(S^n)$ are topologically equivalent if and only if their schemes S_f, $S_{f'}$ are equivalent.*

The possibility of embedding of $f \in G(S^n)$ into a topological flow follows from triviality of the scheme in the following sense.

Let a^t be the flow on the set $\mathbb{S}^{n-1}\times\mathbb{R}$ defined by $a^t(x,\, s) = (x,\, s+t)$, $x \in S^{n-1}$, $s \in \mathbb{R}$ and let a be the time-one map of a^t. Let $\mathbb{Q}^n = \mathbb{S}^{n-1}\times\mathbb{S}^1$. Then the orbit space of the action a on $\mathbb{S}^{n-1}\times\mathbb{R}$ is $\mathbb{Q}^n$. Denote by $p_{\mathbb{Q}^n}\colon \mathbb{S}^{n-1}\times\mathbb{R} \to \mathbb{Q}^n$ the natural projection. Let $m \in \mathbb{N}$ and $c_1,\, \dots,\, c_m \subset \mathbb{S}^{n-1}$ be a collection of smooth pairwise disjoint $(n-2)$-spheres. Let $Q_i^{n-1} = \bigcup_{t\in\mathbb{R}} a^t(c_i)$, $\mathbb{L}_m = \bigcup_{i=1}^m Q_i^{n-1}$ and $\widehat{\mathbb{L}}_m = p_{\mathbb{Q}^n}(\mathbb{L}_m)$.

Definition 3.3. The scheme $S_f = (\widehat{V}_f,\, \hat L^s_f,\, \hat L^u_f)$ of a homeomorphism $f \in G(S^n)$ is called *trivial* if there exists a homeomorphism $\hat\psi\colon \widehat{V}_f \to \mathbb{Q}^n$ such that $\hat\psi(\hat L^s_f \cup \hat L^u_f) = \widehat{\mathbb{L}}_{m_f}$.

In Section 5 we prove the following key lemma.

Lemma 3.1. *If $f \in G(S^n)$ then its scheme S_f is trivial.*

In the section 6 we construct a topological flow X^t_f whose time one map belongs to the class $G(S^n)$ and has the scheme equivalent to S_f. According to Statement 3.1 there exists a homeomorphism $h\colon S^n \to S^n$ such that $f = hX^1_f h^{-1}$. Then the homeomorphism f embeds into the topological flow $Y^t_f = hX^t_f h^{-1}$.

4. Morse–Smale Homeomorphisms

This section contains some definitions and statements which were introduced and proved in [14].

4.1. Basic definitions. Recall that a linear automorphism $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is called hyperbolic if its matrix has no eigenvalues with absolute value equal one. In this case a space $\mathbb{R}^n$ have a unique decomposition into the direct sum of L-invariant subsets E^s, E^u such that $\|L|_{E^s}\| < 1$ and $\|L^{-1}|_{E^u}\| < 1$ in some norm $\|\cdot\|$ (see, for example, Propositions 2.9, 2.10 of Chapter 2 in [28]).

According to Proposition 5.4 of the book [28] any hyperbolic automorphism L is topologically conjugated with a linear map of the following form:

$$a_{\lambda,\mu,\nu}(x_1,\, x_2,\, \dots,\, x_\lambda,\, x_{\lambda+1},\, x_{\lambda+2},\, \dots,\, x_n) = \left(2\mu x_1,\, 2x_2,\, \dots,\, 2x_\lambda,\, \frac{1}{2}\nu x_{\lambda+1},\, \frac{1}{2}x_{\lambda+2},\, \dots,\, \frac{1}{2}x_n\right), \quad (1)$$

where $\lambda = \dim E^u \in \{0, 1, \ldots, n\}$, $\mu = -1$ ($\mu = 1$) if the restriction $L|_{E^u}$ reverses (preserves) an orientation of E^u, and $\nu = -1$ ($\nu = 1$) if the restriction $L|_{E^s}$ reverses (preserves) an orientation of E^s.

Put $\mathbb{E}^s_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1 = x_2 = \cdots = x_\lambda = 0\}$, $\mathbb{E}^u_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_{\lambda+1} = x_{\lambda+2} = \cdots = x_n = 0\}$ and denote by $P^s_x(P^u_y)$ a hyperplane that is parallel to the hyperplane $\mathbb{E}^s_\lambda$ ($\mathbb{E}^u_\lambda$) and contains a point $x \in \mathbb{E}^u_\lambda$ ($y \in \mathbb{E}^s_\lambda$). Unions $\mathcal{P}^s_\lambda = \{P^s_x\}_{x\in\mathbb{E}^u_\lambda}$, $\mathcal{P}^u_\lambda = \{P^u_y\}_{y\in\mathbb{E}^s_\lambda}$ form the $a_{\lambda,\mu,\nu}$-invariant foliation.

Suppose that M^n is an n-dimensional topological manifold, $f \colon M^n \to M^n$ is a homeomorphism and p is a fixed point of the homeomorphism f. We will call the point p *topologically hyperbolic point of index* λ_p, if there exists its neighborhood $U_p \subset M^n$, numbers $\lambda_p \in \{0, 1, \ldots, n\}$, $\mu_p, \nu_p \in \{+1, -1\}$, and a homeomorphism $h_p \colon U_p \to \mathbb{R}^n$ such that $h_p f|_{U_p} = a_{\lambda_p,\mu_p,\nu_p} h_p|_{U_p}$ when the left and right parts are defined. Call the sets $W^s_{p,\mathrm{loc}} = h_p^{-1}(E^s)$, $W^u_{p,\mathrm{loc}} = h_p^{-1}(E^u)$ the *local invariant manifolds* of the point p, and the sets $W^s_p = \bigcup_{i\in\mathbb{Z}} f^i(W^s_{p,\mathrm{loc}})$, $W^u_p = \bigcup_{i\in\mathbb{Z}} f^i(W^u_{p,\mathrm{loc}})$ the *stable and unstable invariant manifolds of the point* p.

It follows form the definition that $W^s_p = \{x \in M^n \colon \lim_{i\to+\infty} f^i(x) = p\}$, $W^u_p = \{x \in M^n \colon \lim_{i\to+\infty} f^{-i}(x) = p\}$ and $W^u_p \cap W^u_q = \varnothing$ ($W^s_p \cap W^s_q = \varnothing$) for any distinct hyperbolic points p, q. Moreover, there exists an injective continuous immersion $J \colon \mathbb{R}^{\lambda_p} \to M^n$ such that $W^u_p = J(\mathbb{R}^{\lambda_p})$[3].

A hyperbolic fixed point is called *the source* (*the sink*) if its index equals n (0), a hyperbolic fixed point p of index $0 < \lambda_p < n$ is called *the saddle point*.

A periodic point p of period m_p of a homeomorphism f is called a *topologically hyperbolic sink* (*source, saddle*) *periodic point* if it is the topologically hyperbolic (source, saddle) fixed point for the homeomorphism f^{m_p}. The stable and unstable manifolds of the periodic point p considered as the fixed point of the homeomorphism f^{m_p} are called the stable and unstable manifolds of the point p. Every connected component of the set $W^s_p \setminus p$ ($W^u_p \setminus p$) is called the *stable* (the *unstable*) *separatrix* and is denoted by l^s_p (l^u_p).

The linearizing homeomorphism $h_p \colon U_p \to \mathbb{R}^n$ induces a pair of transversal foliations $\mathcal{F}^s_p = h_p^{-1}(\mathcal{P}^s_{\lambda_p})$, $\mathcal{F}^u_p = h_p^{-1}(\mathcal{P}^u_{\lambda_p})$ on the set U_p. Every leaf of the foliation $\mathcal{F}^s_p$ ($\mathcal{F}^u_p$) is an open disk of dimension λ_p ($n - \lambda_p$). For any point $x \in U_p$ denote by $F^s_{p,x}$, $F^u_{p,x}$ the leaf of the foliation $\mathcal{F}^s_p$, $\mathcal{F}^u_p$, correspondingly, containing the point x.

The invariant manifolds of saddle periodic points of a homeomorphism f intersect *consistently transversely* if for any point saddle point p there exists a neighborhood U_p such that for any saddle points $q(r)$ such that $W^s_p \cap W^u_q \neq \varnothing$ ($W^u_p \cap W^s_r \neq \varnothing$) $F^u_{p,x} \subset W^u_q$ ($F^s_{p,y} \subset W^u_q$) for any points $x \in W^s_p$ ($y \in W^u_p$).

Notice that all saddle points of any Morse–Smale diffeomorphism have such neighborhoods due to [27, Theorem2.6]. In case of Morse–Smale homeomorphisms (not smooth) this property is necessary to require.

Definition 4.1. A homeomorphism $f \colon M^n \to M^n$ is called the *Morse–Smale homeomorphism* if it satisfies the next conditions:

[3]A map $J \colon \mathbb{R}^m \to M^n$ is called immersion if for any point $x \in \mathbb{R}^m$ there exists a neighborhood $U_x \in \mathbb{R}^m$ such that the restriction $J|_{U_x}$ of the map J on the set U_x is a homeomorphism.

(1) its non-wandering set Ω_f finite and any point $p \in \Omega_f$ is topologically hyperbolic;
(2) invariant manifolds of all saddle points intersect consistently transversely.

4.2. Properties of Morse–Smale homeomorphisms

Statement 4.1. *Let $f\colon M^n \to M^n$ be a Morse–Smale homeomorphism. Then*:

(1) $W^u_p \cap W^s_p = p$ *for any saddle point* $p \in \Omega_f$;
(2) *for any saddle points* $p,\, q,\, r \in \Omega_f$ *the conditions* $(W^s_p \setminus p) \cap (W^u_q \setminus q) \neq \varnothing$, $(W^s_q \setminus q) \cap (W^u_r \setminus r) \neq \varnothing$ *imply* $(W^s_p \setminus p) \cap (W^u_r \setminus r) \neq \varnothing$;
(3) *there are no sequence of distinct saddle points* $p_1,\, p_2,\, \dots,\, p_k \in \Omega_f$, $k > 1$, *such that* $(W^s_{p_i} \setminus p_i) \cap (W^u_{p_{i+1}} \setminus p_{i+1}) \neq \varnothing$ *for* $i \in \{1,\, \dots,\, k-1\}$ *and* $(W^s_{p_k} \setminus p_k) \cap (W^u_{p_1} \setminus p_1) \neq \varnothing$.

Statement 4.2. *Let $f\colon M^n \to M^n$ be a Morse–Smale homeomorphism. Then*:

1) $M^n = \bigcup\limits_{p \in \Omega_f} W^u_p$;
2) *for any point* $p \in \Omega_f$ *the manifold* W^u_p *is a topological submanifold of the manifold* M^n;
3) *for any point* $p \in \Omega_f$ *and any connected component* l^u_p *of the set* $W^u_p \setminus p$ *the following equality holds*: $\operatorname{cl} l^u_p \setminus (l^u_p \cup p) = \bigcup_{q \in \Omega_f\colon W^s_q \cap l^u_p \neq \varnothing} W^u_q$ [4].

Corollary 4.1. *If $f\colon M^n \to M^n$ is a Morse–Smale homeomorphism and $p \in \Omega_f$ is a saddle point such that $l^u_p \cap W^s_q = \varnothing$ for any saddle point $q \neq p$, then there exists a unique sink $\omega \in \Omega_f$ such that $\operatorname{cl} l^u_p = l^u_p \cup p \cup \omega$ and $\operatorname{cl} l^u_p$ is either a compact arc in case $\lambda_p = 1$ or a sphere of dimension λ_p in case $\lambda_p > 1$.*

For an arbitrary point $q \in \Omega_f$ and $\delta \in \{u,\, s\}$ put $V^\delta_q = W^\delta_q \setminus q$ and denote by $\widehat{V}^\delta_q = V^s_q / f$ the orbit space of the action of the homeomorphism f on the set V^δ_q. The following statement is proved in the book [17] (Proposition 2.1.5).

Statement 4.3. *The space $\widehat{V}^u_q$ is homeomorphic to $\mathbb{S}^{\lambda_q - 1} \times \mathbb{S}^1$ and the space $\widehat{V}^s_q$ is homeomorphic to $\mathbb{S}^{n - \lambda_q - 1} \times \mathbb{S}^1$.*

Remark that $\mathbb{S}^0 \times \mathbb{S}^1$ means a union of two disjoint closed curves.

Proposition 4.1. *Suppose $f\colon M^n \to M^n$ is a Morse–Smale homeomorphism, $n \geqslant 4$, and $\sigma \in \Omega_f$ is a saddle point of index $(n-1)$ such that $l^u_\sigma \cap W^s_q = \varnothing$ for any saddle point $q \neq p$. Then the sphere $\operatorname{cl} l^u_\sigma$ is bicollared.*

Proof. Let $\omega \in \Omega^0_f$ be a sink point such that $l^u_\sigma \subset W^s_\omega$. Due to Corollary 4.1 and the item 2 of Statement 4.2 the set $\operatorname{cl} l^u_\sigma = l^u_\sigma \cup \omega$ is an $(n-1)$-sphere which is locally flat embedded in M^n at all its points except possibly one point ω. According to [5], [20] an $(n-1)$-sphere in a manifold M^n of dimension $n \geqslant 4$ is either locally flat or have more than countable set of points of wildness. Therefore the sphere $\operatorname{cl} l^u_\sigma$ is locally flat at point ω. According to [4] a locally flat sphere is bicollared. $\square$

By $G(S^n)$ we denoted a class of Morse–Smale homeomorphism on the sphere S^n such that any $f \in G(S^n)$ satisfy the following conditions:

[4] Here $\operatorname{cl} l^u_p$ means the closure of the set l^u_p.

(i) Ω_f consists of fixed points;
(ii) $W^s_p \cap W^u_q = \varnothing$ for any distinct saddle points $p,\, q \in \Omega_f$;
(iii) the restriction of a homeomorphism f on every invariant manifolds of an arbitrary fixed point $p \in \Omega_f$ preserves its orientation.

Proposition 4.2. *If $f \in G(S^n)$, then any saddle fixed point has index* 1 *and* $(n-1)$.

Proof. Suppose that, on the contrary, there exists a point $\sigma \in \Omega_f$ of index $j \in (1,\, n-1)$. According to Corollary 4.1 the closures $\operatorname{cl} W^u_\sigma$, $\operatorname{cl} W^s_\sigma$ of the stable and unstable manifolds of the point σ are spheres of dimensions j and $n-j$ correspondingly. Due to item 1 of Statements 4.1, the spheres $S^j = \operatorname{cl} W^u_\sigma$, $S^{n-j} = \operatorname{cl} W^s_\sigma$ intersect at a single point σ. Therefore their intersection index equals either 1 or -1 (depending on the choice of orientations of the spheres S^j, S^{n-j} and S^n). Since homology groups $H_j(S^n)$, $H_{n-j}(S^n)$ are trivial it follows that there is a sphere $\tilde{S}^j$ homological to the sphere S^j and having the empty intersection with the sphere S^{n-j}. Then the intersection number of the spheres S^j, S^{n-j} must be equal to zero as the intersection number is the homology invariant (see, for example, [31], § 69). This contradiction proves the statement. □

4.3. Canonical manifolds connected with saddle fixed points of a homeomorphism $f \in G(S^n)$. It follows from Statement 4.2 that for each saddle point of a homeomorphism $f \in G(S^n)$ there exists a neighborhood where f is topologically conjugated either with the map $a_1 \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $a_1(x_1,\, x_2,\, \dots,\, x_n) = (2x_1,\, \frac{1}{2}x_2,\, \dots,\, \frac{1}{2}x_n)$ or with the map a_1^{-1}. In this section we describe canonical manifolds defined by the action of the map a_1 and prove Proposition 4.3 allowing to define similar canonical manifolds for the homeomorphism $f \in G(S^n)$.

Put $\mathbb{U}_\tau = \{(x_1,\, \dots,\, x_n) \in \mathbb{R}^n \colon x_1^2(x_2^2 + \dots + x_n^2) \leqslant \tau^2\}$, $\tau \in (0,\, 1]$, $\mathbb{U} = \mathbb{U}_1$; $\mathbb{U}_0 = \{(x_1,\, \dots,\, x_n) \in \mathbb{R}^n \colon x_1 = 0\}$, $\mathbb{N}^s = \mathbb{U} \setminus Ox_1$, $\mathbb{N}^u = \mathbb{U} \setminus \mathbb{U}_0$, $\widehat{\mathbb{N}}^s = \mathbb{N}^s / a_1$, $\widehat{\mathbb{N}}^u = \mathbb{N}^u / a_1$. Denote by $p_s : \mathbb{N}^s \to \widehat{\mathbb{N}}^s$, $p_u : \mathbb{N}^u \to \widehat{\mathbb{N}}^u$ the natural projections and put $\widehat{V}^s = p_s(\mathbb{U}_0)$.

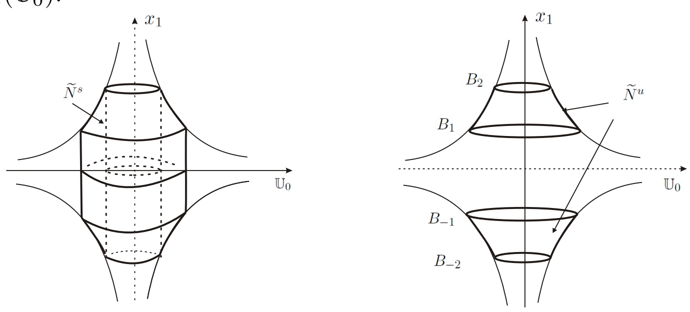

FIGURE 3. Fundamental domains $\widetilde{N}^s$, $\widetilde{N}^u$ of the action of the homeomorphism a_1 on the sets $\mathbb{N}^s$, $\mathbb{N}^u$

The following statement is proved in [12] (Propositions 2.2, 2.3).

Statement 4.4. *The space $\widehat{\mathbb{N}}^s$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1,\, 1]$, the space $\widehat{\mathbb{N}}^u$ consists of two connected components each of which is homeomorphic to the direct product $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Recall that *an annulus* of dimension n is a manifold homeomorphic to $\mathbb{S}^{n-1} \times [0, 1]$.

On the Figure 3 we present the neighborhoods $\mathbb{N}^s$, $\mathbb{N}^u$ and the fundamental domains $\widetilde{N}^s = \{(x_1, \dots, x_n) \in \mathbb{N}^s \colon \frac{1}{4} \leqslant x_2^2 + \dots + x_n^2 \leqslant 1\}$, $\widetilde{N}^u = \{(x_1, \dots, x_n) \in \mathbb{N}^u \colon |x_1| \in [1, 2]\}$ of the action of the diffeomorphism a_1[5]. Put $\mathcal{C} = \{\{(x_1, \dots, x_n) \in \mathbb{R}^n \colon \frac{1}{4} \leqslant x_2^2 + \dots + x_n^2 \leqslant 1\}$. The set $\mathbb{N}^s$ is the union of the hyperplanes $\mathcal{L}_t = \{(x_1, \dots, x_n) \in N^s \colon x_1^2(x_2^2 + \dots + x_n^2) = t^2\}$, $t \in [-1, 1]$. Then the fundamental domain $\widetilde{N}^s$ is the union of the pairs of annuli $\mathcal{K}_t = \mathcal{L}_t \cap \mathcal{C}$, $t \in [-1, 1]$ and the space $\widehat{\mathbb{N}}^s$ can be obtained from $\widetilde{N}^s$ by gluing the connected components of the boundary of each annulus by means of the diffeomorphism a_1. The set $\widetilde{N}^u$ consists of two connected components each of which is homeomorphic to the direct product $\mathbb{B}^{n-1} \times [0, 1]$. The space $\widehat{\mathbb{N}}^u$ is obtained from $\widetilde{N}^u$ by gluing the disk $B_1 = \{(x_1, \dots, x_n) \in \mathbb{N}^u \colon x_1 = 1\}$ to the disk $B_2 = \{(x_1, \dots, x_n) \in \mathbb{N}^u \colon x_1 = 2\}$ and the disk $B_{-1} = \{(x_1, \dots, x_n) \in \mathbb{N}^u \colon x_1 = -1\}$ to the disk $B_{-2} = \{(x_1, \dots, x_n) \in \mathbb{N}^u \colon x_1 = -2\}$ by means of the diffeomorphism a_1.

Proposition 4.3. *Suppose $f \in G(S^n)$; then there exists a set of pairwise disjoint neighborhoods $\{N_\sigma\}_{\sigma \in \Omega_f^1 \cup \Omega_f^{n-1}}$ such that for any neighborhood N_σ there exists a homeomorphism $\chi_\sigma \colon N_\sigma \to \mathbb{U}$ such that $\chi_\sigma f|_{N_\sigma} = a_1 \chi_\sigma|_{N_\sigma}$ whenever $\lambda_\sigma = 1$ and $\chi_\sigma f|_{N_\sigma} = a_1^{-1} \chi_\sigma|_{N_\sigma}$ whenever $\lambda_\sigma = n - 1$.*

Proof. Put $V^\delta_{\Omega_f^i} = \bigcup_{q \in \Omega_f^i} V_q^\delta$, $\widehat{V}^\delta_{\Omega_f^i} = \bigcup_{q \in \Omega_f^i} \widehat{V}_q^\delta$, $i \in \{0, 1, n-1, n\}$, $\delta \in \{s, u\}$ and denote by $p^\delta_{\Omega_f^i} \colon V^\delta_{\Omega_f^i} \to \widehat{V}^\delta_{\Omega_f^i}$ the natural projection such that $p^\delta_{\Omega_f^i}|_{V_q^\delta} = p_q^\delta|_{V_q^\delta}$ for any point $q \in \Omega_f$.

Put $\Sigma_f = \Omega_f^1 \cup \Omega_f^{n-1}$, $\widehat{L}^u_{\Sigma_f} = p^s_{\Omega_f^0}(V^u_{\Omega_f^1} \cup V^u_{\Omega_f^{n-1}})$.

The set $\widehat{L}^u_{\Sigma_f}$ consists of finite number of compact topological submanifolds. Then there is a set of pairwise disjoint compact neighborhoods $\{\widehat{K}^u_\sigma, \sigma \in \Sigma_f\}$ of these manifolds in $\widehat{V}^s_{\Omega^0}$. For every point $\sigma \in \Sigma_f$ put $K^u_\sigma = (p^s_{\Omega_f^0})^{-1}(\widehat{K}^u_\sigma)$ and $\widetilde{N}_\sigma = K^u_\sigma \cup W^s_\sigma$.

Let $U_\sigma \subset \widetilde{N}_\sigma$ be a neighborhood of the point σ such that a homeomorphism $g_\sigma \colon U_\sigma \to \mathbb{R}^n$ satisfying the condition $g_\sigma f|_{U_\sigma} = a_{\lambda_\sigma} g_\sigma|_{U_\sigma}$ is defined.

Recall that we denoted by $\mathbb{U}_\tau$ the set $\{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_1^2(x_2^2 + \dots + x_n^2) \leqslant \tau^2\}$, $\tau \in (0, 1]$. Put $u_\tau = \{(x_1, \dots, x_n) \in \mathbb{U}_\tau \colon x_2^2 + \dots + x_n^2 \leqslant 1, |x_1| \leqslant 2\tau\}$, $D^u_\tau = \{(x_1, \dots, x_n) \in \mathbb{U}_\tau \colon \tau < |x_1| \leqslant 2\tau\}$, $D^s_\tau = \{(x_1, \dots, x_n) \in \mathbb{U}_\tau \colon \frac{1}{4} \leqslant x_2^2 + \dots + x_n^2 \leqslant 1\}$, $\tilde{u}_\tau = g_\sigma^{-1}(u_\tau)$, $\widetilde{D}^\delta_\tau = g_\sigma^{-1}(D^\delta_\tau)$, $\delta \in \{s, u\}$, and $N_\tau = \bigcup_{i \in \mathbb{Z}} f^i(\widetilde{u}_\tau)$.

Let us show that there is a number $\tau_1 > 0$ such that for any $i \in \mathbb{N}$ the intersection $f^i(\widetilde{D}^u_{\tau_1}) \cap \tilde{u}_{\tau_1}$ is empty. Suppose $\sigma \in \Omega_f^{n-1}$ (the argument for the case $\sigma \in \Omega_f^1$ is similar). By the Statement 4.2, the set $\bigcup_{i \in \mathbb{N}} f^i(\widetilde{D}^u_\tau)$ lies in the stable manifold of a unique sink point ω. Since the homeomorphism f is locally conjugated with the linear compression a_0 in a neighborhood of the point ω, we have that there exists

[5] A fundamental domain of the action of a group G on a set X is a closed set $D_G \subset X$ containing a subset $\tilde{D}_G$ with the following properties: 1) $\operatorname{cl} \tilde{D}_G = D_G$; 2) $g(\tilde{D}_G) \cap \tilde{D}_G = \varnothing$ for any $g \in G$ distinct from the neutral element; 3) $\bigcup_{g \in G} g(\tilde{D}_G) = X$.

a ball $B^n \subset W^s_\omega \setminus U_\sigma$ such that $\omega \subset B^n$ and $f(B^n) \subset \operatorname{int} B^n$. Since $\widetilde{D}^u_\tau$ is compact, there is $i^* > 0$ such that $f^i(\widetilde{D}^u_\tau) \cap U_\sigma \subset B^n$ for all $i > i^*$. Hence the set of numbers i_j such that $f^{i_j}(\widetilde{D}^u_\tau) \cap \tilde{u}_\tau \neq \varnothing$ is finite. Then one can choose $\tau_1 \in (0, \tau)$ such that $\tilde{u}_{\tau_1} \cap f^i(\widetilde{D}^u_\tau)) = \varnothing$ and therefore $\tilde{u}_{\tau_1} \cap f^i(\widetilde{D}^u_{\tau_1})) = \varnothing$ for any $i \in \mathbb{N}$. Similarly one can show that there exists a number $\tau_2 \in (0, \tau_1]$ such that for any $i \in \mathbb{N}$ the intersection of $f^{-i}(\widetilde{D}^s_{\tau_2}) \cap \tilde{u}_{\tau_2}$ is empty.

Suppose $\lambda_\sigma = 1$, put $N_\sigma = \bigcup_{i \in \mathbb{Z}} f^i(\tilde{u}_{\tau_2})$, and define a homeomorphism ${\chi^*}_\sigma \colon N_\sigma \to U_{\tau_2}$ by the following: $\chi^*_\sigma(x) = g_\sigma(x)$ whenever $x \in \tilde{u}_{\tau_2}$, and $\chi^*_\sigma(x) = a^{-k}_{\lambda_\sigma}(g_\sigma(f^k(x)))$ whenever $x \in N_\sigma \setminus (\tilde{u}_{\tau_2})$, where $k \in \mathbb{Z}$ is such that $f^k(x) \in \tilde{u}_{\tau_2}$. The homeomorphism ${\chi^*}_\sigma$ conjugates the homeomorphism $f|_{N_\sigma}$ with the linear diffeomorphism $a_1|_{\mathbb{U}_{\tau_2}}$. Since the homeomorphism $a_1|_{\mathbb{U}_{\tau_2}}$ is topologically conjugated with $a_1|_{\mathbb{U}}$ by means of the diffeomorphism $g(x_1, \ldots, x_n) = \left(\frac{x_1}{\sqrt{\tau_2}}, \ldots, \frac{x_n}{\sqrt{\tau_2}}\right)$, we see that the superposition $\chi_\sigma = g\chi^*_\sigma \colon N_\sigma \to \mathbb{U}$ topologically conjugates $f|_{N_\sigma}$ with $a_1|_{\mathbb{U}}$. A homeomorphism χ_σ for the case $\lambda_\sigma = n - 1$ can be constructed in the same way. $\square$

Put $N^u_\sigma = N_\sigma \setminus W^s_\sigma$, $N_{\tau,\sigma} = \chi^{-1}_\sigma(\mathbb{U}_\tau)$, $N^s_\sigma = N_\sigma \setminus W^u_\sigma$, $\widehat{N}^s_\sigma = N^s_\sigma / f$, $\widehat{N}^u_\sigma = N^u / f$.

5. Triviality of the Scheme of the Homeomorphism $f \in G(S^n)$

This section is devoted to the proof of Lemma 3.1. In subsections 5.1-5.3 we establish some axillary results.

5.1. Introduction results on the embedding of closed curves and their tubular neighborhoods in a manifold M^n. Further we denote by M^n a topological manifold possibly with non-empty boundary.

Recall that a manifold $N^k \subset M^n$ of dimension k without boundary is *locally flat in a point* $x \in N^k$ if there exists a neighborhood $U(x) \subset M^n$ of the point x and a homeomorphism $\varphi \colon U(x) \to \mathbb{R}^n$ such that $\varphi(N^k \cap U(x)) = \mathbb{R}^k$, where $\mathbb{R}^k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_{k+1} = x_{k+2} = \cdots = x_n = 0\}$.

A manifold N^k is *locally flat* in M^n or *the submanifold* of the manifold M^n if it is locally flat at each its point.

If the condition of local flatness fails in a point $x \in N^k$ then the manifold N^k is called *wild* and the point x is called *the point of wildness.*

A topological space X is called *m-connected* (for $m > 0$) if it is non-empty, path-connected and its first m homotopy groups $\pi_i(X)$, $i \in \{1, \ldots, m\}$, are trivial. The requirements of being non-empty and path-connected can be interpreted as (-1)-connected and 0-connected correspondingly.

A topological space P generated by points of a simplicial complex K with the topology induced from $\mathbb{R}^n$ is called *the polyhedron.* The complex K is called *the partition* or *the triangulation* of the polyhedron P.

A map $h \colon P \to Q$ of polyhedra is called *piecewise linear* if there exists partitions K, L of polyhedra P, Q correspondingly such that h moves each simplex of the complex K into a simplex of the complex L (see for example [29]).

A polyhedron P is called *piecewise linear manifold* of dimension n with boundary if it is a topological manifold with boundary and for any point $x \in \operatorname{int} P$ ($y \in \partial P$)

there is a neighborhood U_x (U_y) and a piecewise linear homeomorphism $h_x\colon U_x \to \mathbb{R}^n$ ($h_y\colon U_y \to \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \subset \mathbb{R}^n\colon x_1 \geqslant 0\}$).

The following important statement follows from Theorem 4 of [19].

Statement 5.1. *Suppose that N^k, M^n are compact piecewise linear manifolds of dimension k, n correspondingly, N^k is the manifold without boundary, M^n possibly has a non-empty boundary, $\tilde{e}$, $e\colon N^k \to \operatorname{int} M^n$ are homotopic piecewise linear embeddings, and the following conditions hold*:

(1) $n - k \geqslant 3$;
(2) N^k *is* $(2k - n + 1)$*-connected;*
(3) M^n *is* $(2k - n + 2)$*-connected.*

Then there exists a family of piecewise linear homeomorphisms $h_t\colon M^n \to M^n$, $t \in [0, 1]$, such that $h_0 = \mathrm{id}$, $h_1\tilde{e} = e$, $h_t|_{\partial M^n} = \mathrm{id}$ for any $t \in [0, 1]$.

We will say that a topological submanifold $N^k \subset M^n$ of the manifold M^n is *an essential* if a homomorphism $e_{\gamma_*}\colon \pi_1(N^k) \to \pi_1(M^n)$ induced by an embedding $e_{N^k}\colon N^k \to M^n$ is the isomorphism. We will call an essential manifold β homeomorphic to the circle $\mathbb{S}^1$ *the essential knot*.

Let $\beta \in M^n$ be an essential knot and $h\colon \mathbb{B}^{n-1} \times \mathbb{S}^1 \to M^n$ be a topological embedding such that $h(\{O\} \times \mathbb{S}^1) = \beta$. Call the image $N_\beta = h(\mathbb{B}^{n-1} \times \mathbb{S}^1)$ the *tubular neighborhood* of the knot β.

Proposition 5.1. *Suppose that $\mathbb{P}^{n-1}$ is $\mathbb{S}^{n-1}$ or $\mathbb{B}^{n-1}$, $\beta_1, \dots, \beta_k \subset \operatorname{int} \mathbb{P}^{n-1} \times \mathbb{S}^1$ are essential knots, and $x_1, \dots, x_k \subset \operatorname{int} \mathbb{P}^{n-1}$ are arbitrary points. Then there is a homeomorphism $h\colon \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ such that $h(\bigcup_{i=1}^k \beta_i) = \bigcup_{i=1}^k \{x_i\} \times \mathbb{S}^1$ and $h|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = \mathrm{id}$.*

Proof. Put $b_i = \{x_i\} \times \mathbb{S}^1$, $i \in \{1, \dots, k\}$. Choose pairwise disjoint neighborhoods $U_1, \dots, U_k$ of knots $\beta_1, \dots, \beta_k$ in $\operatorname{int} \mathbb{P}^{n-1} \times \mathbb{S}^1$. It follows from Theorem 1.1 of the paper [9] that there exists a homeomorphism $g\colon \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ that is identity outside the set $\bigcup_{i=1}^k U_i$ and such that for any $i \in \{1, \dots, k\}$ the set $g(\beta_i)$ is a subpolyhedron.

By assumption, piecewise linear embeddings $\tilde{e}\colon \mathbb{S}^1 \times \mathbb{Z}_k \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, $e\colon \mathbb{S}^1 \times \mathbb{Z}_k \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ such that $\tilde{e}(\mathbb{S}^1 \times \mathbb{Z}_k) = \bigcup_{i=1}^k g(\beta_i)$, $e(\mathbb{S}^1 \times \mathbb{Z}_k) = \bigcup_{i=1}^k b_i$ are homotopic. By Statement 5.1, there exists a family of piecewise linear homeomorphisms $h_t\colon \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, $t \in [0, 1]$, such that $h_0 = \mathrm{id}$, $h_1\tilde{e} = e$, $h_t|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = \mathrm{id}$ for any $t \in [0, 1]$. Then h_1 is the desired homeomorphism. □

The following Statement 5.2 is proved in the paper [12] (see Lemma 2.1).

Statement 5.2. *Let $h\colon \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \operatorname{int} \mathbb{B}^{n-1} \times \mathbb{S}^1$ be a topological embedding such that $h(\{O\} \times \mathbb{S}^1) = \{O\} \times \mathbb{S}^1$. Then the manifold $\mathbb{B}^{n-1} \times \mathbb{S}^1 \setminus \operatorname{int} h(\mathbb{B}^{n-1} \times \mathbb{S}^1)$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]$.*

Proposition 5.2. *Suppose that Y is a topological manifold with boundary, X is a closed component of its boundary, Y_1 is a manifold homeomorphic to $X \times [0, 1]$, and $Y \cap Y_1 = X$. Then a manifold $Y \cup Y_1$ is homeomorphic to Y. Moreover, if the manifold Y is homeomorphic to the direct product $X \times [0, 1]$ then there exists a homeomorphism $h : X \times [0, 1] \to Y \cup Y_1$ such that $h(X \times \{\frac{1}{2}\}) = X$.*

Proof. By [4, Theorem 2], there exists a topological embedding $h_0\colon X\times[0,\,1]\to Y$ such that $h_0(X\times\{1\})=X$. Put $Y_0=h_0(X\times[0,\,1])$. Let $h_1\colon X\times[0,\,1]\to Y_1$ be a homeomorphism such that $h_1(X\times\{0\})=X=h_0(X\times\{1\})$.

Define homeomorphisms $g\colon X\times[0,\,1]\to X\times[0,\,1]$, $\tilde h_1\colon X\times[0,\,1]\to Y_1$, $h\colon X\times[0,\,1]\to Y_0\cup Y_1$ by $g(x,\,t)=(h_1^{-1}(h_0(x,\,1)),\,t)$, $\tilde h_1=h_1g$,

$$h(x,\,t)=\begin{cases}h_0(x,\,2t), & t\in[0,\,\frac12];\\ \tilde h_1(x,\,2t-1), & t\in(\frac12;\,1],\end{cases}$$

and define a homeomoprhism $H\colon Y\cup Y_1\to Y$ by

$$H(x)=\begin{cases}h_0(h^{-1}(x)), & x\in Y_0\cup Y_1;\\ x, & x\in Y\setminus Y_0.\end{cases}$$

To prove the second item of the statement it is enough to put $Y=Y_0$. Then the homeomorphism $h\colon X\times[0,\,1]\to Y\cup Y_1$ defined above is the desired one. $\square$

Proposition 5.3. *Suppose that $\mathbb{P}^{n-1}$ is either the ball $\mathbb{B}^{n-1}$ or the sphere $\mathbb{S}^{n-1}$, $\beta_1,\,\ldots,\,\beta_k\subset\operatorname{int}\mathbb{P}^{n-1}\times\mathbb{S}^1$ are essential knots, $N_{\beta_1},\,\ldots,\,N_{\beta_k}\subset\mathbb{P}^{n-1}\times\mathbb{S}^1$ are their pairwise disjoint neighborhoods, $D_1^{n-1},\,\ldots,\,D_k^{n-1}\subset\mathbb{P}^{n-1}$ are pairwise disjoint disks, and $x_1,\,\ldots,\,x_k$ are inner points of the disks $D_1^{n-1},\,\ldots,\,D_k^{n-1}$ correspondingly. Then there exist a homeomorphism $h\colon\mathbb{P}^{n-1}\times\mathbb{S}^1\to\mathbb{P}^{n-1}\times\mathbb{S}^1$ such that $h(\beta_i)=\{x_i\}\times\mathbb{S}^1$, $h(N_{\beta_i})=D_i^{n-1}\times\mathbb{S}^1$, $i\in\{1,\,\ldots,\,k\}$, and $h|_{\partial\mathbb{P}^{n-1}\times\mathbb{S}^1}=\mathrm{id}$.*

Proof. By Proposition 5.1, there exists a homeomorphism $h_0\colon\mathbb{P}^{n-1}\times\mathbb{S}^1\to\mathbb{P}^{n-1}\times\mathbb{S}^1$ such that $h_0(\beta_i)=\{x_i\}\times\mathbb{S}^1$, $h_0|_{\partial\mathbb{P}^{n-1}\times\mathbb{S}^1}=\mathrm{id}$. Put $\tilde N_i=h_0(N_{\beta_i})$. By [4], there exist topological embeddings $e_i\colon\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1]\to\operatorname{int}\mathbb{P}^{n-1}\times\mathbb{S}^1$ such that $e_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times\{1\})=\partial\tilde N_{\beta_i}$, $e_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1])\cap e_j(\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1])=\varnothing$ for $i\neq j$, $i,j\in\{1,\,\ldots,\,k\}$. Put $U_i=e_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1])\cup\tilde N_i$.

Suppose that $D_{0,1}^{n-1},\,\ldots,\,D_{0,k}^{n-1},\,D_{1,1}^{n-1},\,\ldots,\,D_{1,k}^{n-1}\subset\mathbb{P}^{n-1}$ are disks such that $x_i\subset\operatorname{int}D_{j,i}^{n-1}$, $D_{j,i}^{n-1}\subset\operatorname{int}D_i^{n-1}$, $j\in\{0,\,1\}$, $D_{0,i}^{n-1}\subset\operatorname{int}D_{1,i}^{n-1}$, and $D_{1,i}^{n-1}\times\mathbb{S}^1\subset\operatorname{int}\tilde N_i$.

By Proposition 5.2, every set $\tilde N_i\setminus(\operatorname{int}D_{1,i}^{n-1}\times\mathbb{S}^1)$, $(D_{1,i}^{n-1}\setminus\operatorname{int}D_{0,1}^{n-1})\times\mathbb{S}^1$ is homeomorphic to the direct product $\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1]$. By Proposition 5.2, there exists a homeomorphism $g_i\colon\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1]\to U_i\setminus\operatorname{int}D_{0,i}^{n-1}\times\mathbb{S}^1$ such that $g_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times\{t_1\})=\partial\tilde N_i$, $g_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times\{t_2\})=\partial D_{1,i}^{n-1}\times\mathbb{S}^1$ for some $t_1,\,t_2\subset(0,\,1)$. Let $\xi\colon[0,\,1]\to[0,\,1]$ be a homeomorphism that is identity on the ends of the interval $[0,\,1]$ and such that $\xi(t_1)=t_2$. Define a homeomorphism $\tilde g_i\colon\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1]\to\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,\,1]$ by $\tilde g_i(x,\,t)=(x,\,\xi(t))$.

Define a homeomorphism $h_i\colon\mathbb{P}^{n-1}\times\mathbb{S}^1\to\mathbb{P}^{n-1}\times\mathbb{S}^1$ by

$$h_i(x)=\begin{cases}g_i(\tilde g_i(g_i^{-1}(x))), & x\in U_i\setminus\operatorname{int}D_{0,i}^{n-1}\times\mathbb{S}^1;\\ x, & x\in(\mathbb{P}^{n-1}\times\mathbb{S}^1\setminus U_i).\end{cases}$$

The superposition $\eta=h_k\cdots h_1h_0$ maps every knot β_i into the knot $\{x_i\}\times\mathbb{S}^1$, the neighborhood N_{β_i} into the set $D_{1,i}^{n-1}\times\mathbb{S}^1$, and keeps the set $\partial\mathbb{P}^{n-1}\times\mathbb{S}^1$ fixed. Construct a homeomorphism $\Theta\colon\mathbb{P}^{n-1}\times\mathbb{S}^1\to\mathbb{P}^{n-1}\times\mathbb{S}^1$ that be identity on the set

$\partial\mathbb{P}^{n-1}\times\mathbb{S}^1$ and on the knots $\{x_1\}\times\mathbb{S}^1, \dots, \{x_k\}\times\mathbb{S}^1$ and move the set $D^{n-1}_{1,i}\times\mathbb{S}^1$ into the set $D^{n-1}_i\times\mathbb{S}^1$ for every $i\in\{1,\dots,k\}$. It follows from the Annulus Theorem[6] that the set $D^{n-1}_i\setminus\operatorname{int}D^{n-1}_{1,i}$ is homeomorphic to the annulus $\mathbb{S}^{n-2}\times[0,1]$. Then apply the construction similar to one described above to define a homeomorphism $\theta\colon\mathbb{P}^{n-1}\to\mathbb{P}^{n-1}$ such that $\theta(x_i)=x_i$, $\theta(D^{n-1}_i)=D^{n-1}_{1,i}$, $\theta|_{\partial\mathbb{P}^{n-1}}=\mathrm{id}$. Put $\Theta(x,t)=(\theta^{-1}(x),t)$, $x\in\mathbb{P}^{n-1}$, $t\in\mathbb{S}^1$. Then $h=\Theta\eta$ is the desired homeomorphism. □

Corollary 5.1. *If $N\subset\mathbb{S}^{n-1}\times\mathbb{S}^1$ is a tubular neighborhood of an essential knot than the manifold $(\mathbb{S}^{n-1}\times\mathbb{S}^1)\setminus\operatorname{int}N$ is homeomorphic to the direct product $\mathbb{B}^{n-1}\times\mathbb{S}^1$.*

5.2. A surgery of the manifold $\mathbb{S}^{n-1}\times\mathbb{S}^1$ along an essential submanifold homeomorphic to $\mathbb{S}^{n-2}\times\mathbb{S}^1$. Recall that we put $\mathbb{Q}^n=\mathbb{S}^{n-1}\times\mathbb{S}^1$. Suppose that $N\subset\mathbb{Q}^n$ is an essential submanifold homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$, $T=\partial N$, and $e_T\colon\mathbb{S}^{n-2}\times\mathbb{S}^1\times[-1;1]\to\mathbb{Q}^n$ is a topological embedding such that $e_T(\mathbb{S}^{n-2}\times\mathbb{S}^1\times\{0\})=T$. Put $K=e_T(\mathbb{S}^{n-2}\times\mathbb{S}^1\times[-1;1])$ and denote by N_+, N_- connected components of the set $\mathbb{Q}^n\setminus\operatorname{int}K$. It follows from Propositions 5.3, 5.2 that the manifolds N_+, N_- are homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$. Let N'_+, N'_- manifolds homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$. Denote by $\psi_\delta:\partial N_\delta\to\partial N'_\delta$ an arbitrary homeomorphism reversing the natural orientation, by Q_δ a manifold obtained by gluing the manifolds N_δ and N'_δ by means of homeomorphism ψ_δ, and by $\pi_\delta:(N_\delta\cup N'_\delta)\to Q_\delta$ the natural projection, $\delta\in\{+,-\}$.

We will say that the manifolds Q_+, Q_- are obtained from $\mathbb{Q}^n$ *by the surgery along the submanifold T.*

Note that $\mathbb{S}^{n-2}\times\mathbb{S}^1$ is the boundary of $\mathbb{B}^{n-1}\times\mathbb{S}^1$. By [22, Theorem 2], the following statement holds.

Statement 5.3. *Let $\psi\colon\mathbb{S}^{n-2}\times\mathbb{S}^1\to\mathbb{S}^{n-2}\times\mathbb{S}^1$ be an arbitrary homeomorphism. Then there exists a homeomorphism $\Psi\colon\mathbb{B}^{n-1}\times\mathbb{S}^1\to\mathbb{B}^{n-1}\times\mathbb{S}^1$ such that $\Psi|_{\mathbb{S}^{n-2}\times\mathbb{S}^1}=\psi|_{\mathbb{S}^{n-2}\times\mathbb{S}^1}$.*

Proposition 5.4. *The manifolds Q_+, Q_- are homeomorphic to $\mathbb{Q}^n$.*

Proof. Let $D^{n-1}\subset\mathbb{S}^{n-1}$ be an arbitrary disk, $\mathbb{N}_\delta=D^{n-1}\times\mathbb{S}^1$ and $h_\delta:\pi_\delta(N_\delta)\to\mathbb{N}_\delta$ be an arbitrary homeomorphism. Put $\tilde\psi_\delta=h_\delta\pi_\delta\psi_\delta\pi_\delta^{-1}h_\delta^{-1}|_{\partial\mathbb{N}_\delta}$. Due to Proposition 5.3 a homeomorphism $\tilde\psi_\delta$ can extend up to a homeomorphism $h'_\delta\colon\pi_\delta(N'_\delta)\to\mathbb{Q}^n\setminus\operatorname{int}\mathbb{N}_\delta$. Then a map $H_\delta\colon Q_\delta\to\mathbb{Q}^n$ defined by $H_\delta(x)=h_\delta(x)$ whenever $x\in\pi_\delta(N_\delta)$ and $H_\delta(x)=h'_\delta(x)$ whenever $x\in\pi_\delta(N'_\delta)$ is the desired homeomorphism. □

5.3. A surgery of manifolds homeomorphic to $\mathbb{S}^{n-1}\times\mathbb{S}^1$ along essential knots. Let $Q^n_1,\dots,Q^n_{k+1}$ be manifolds homeomorphic to $\mathbb{Q}^n$. Denote by $\beta_1,\dots,\beta_{2k}\subset\bigcup_{i=1}^{k+1}Q^n_i$ essential knots such that for any $j\in\{1,\dots,k\}$ knots

[6]The Annulus Theorem states that the closure of an open domain on the sphere S^{n+1} bounded by two disjoint locally flat spheres S^n_1, S^n_2 is homeomorphic to the annulus $\mathbb{S}^n\times[0,1]$. In dimension 2 it was proved by Rado in 1924, in dimension 3 — by Moise in 1952, in dimension 4 — by Quinn in 1982, and in dimension 5 and greater — by Kirby in 1969.

β_{2j-1}, β_{2j} belongs to distinct manifolds from the union $\bigcup_{i=1}^{k+1} Q_i^n$ and every manifold Q_i^n contains at least one knot from the set $\beta_1, \ldots, \beta_{2k}$. Let $N_{\beta_1}, \ldots, N_{\beta_{2k}}$ be tubular neighborhoods of the knots $\beta_1, \ldots, \beta_{2k}$ correspondingly.

Let $K_1, \ldots, K_k$ be manifolds homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1; 1]$. For every $j \in \{1, \ldots, k\}$ denote by $T_j \subset K_j$ a manifold homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$ that cuts K_j into two connected components whose closures are homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0; 1]$, and by $\psi_j \colon \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j$ an arbitrary reversing the natural orientation homeomorphism.

Glue manifolds $\widetilde{Q} = \bigcup_{i=1}^{k+1} Q_i^n \setminus \bigcup_{\nu=1}^{2k} \operatorname{int} N_\nu$ and $K = \bigcup_{j=1}^{k} K_j$ by means of the homeomorphisms $\psi_1, \ldots, \psi_k$, denote by Q the obtained manifold and by $\pi \colon \widetilde{Q} \cup K \to Q$ the natural projection. We will say that the manifold Q is obtained from $Q_1^n, \ldots, Q_{k+1}^n$ by *the surgery along knots* $\beta_1, \ldots, \beta_{2k}$ and call every pair β_{2j-1}, β_{2j} *the binding pair*, $j \in \{1, 2, \ldots, k\}$.

Proposition 5.5. *The manifold Q is homeomorphic to $\mathbb{Q}^n$ and every manifold $\pi(T_j)$ cuts Q into two connected components whose closures are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Proof. Prove the proposition by induction on k. Consider the case $k = 1$. Due to Propositions 5.3, 5.2 manifolds $\widetilde{N}_1 = Q_1^n \setminus \operatorname{int} N_1$, $\widetilde{N}_2 = Q_2^n \setminus \operatorname{int} N_2$, $\widetilde{N}_1 \cup_{\psi_1|_{\partial N_1}} K_1$ are homeomorphic to the direct product $\mathbb{B}^{n-1} \times \mathbb{S}^1$. By definition, the manifold T_1 cuts the manifold K_1 into two connected components whose closures are homeomorphic to $\mathbb{Q}^{n-1} \times [0, 1]$. It follows from Proposition 5.2 that T_1 cuts $\widetilde{N}_1 \cup_{\psi_1|_{\partial N_1}} K_1$ into two connected components such that the closure of one of which, denote it by N, is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$ and the closure of another is homeomorphic to $\mathbb{Q}^{n-1} \times [0, 1]$. Suppose that $D_0^{n-1} \subset \mathbb{S}^{n-1}$ is an arbitrary disk, $N_0 = D_0^{n-1} \times \mathbb{S}^1$ and $h_0 \colon \pi(\widetilde{N}_1 \cup K_1) \to N_0$ is an arbitrary homeomorphism. Put $\tilde{\psi}_1 = h_0 \pi \psi_1^{-1} \pi^{-1} h_0^{-1}|_{\partial N_0}$. In virtue of Proposition 5.3 a homeomorphism $\tilde{\psi}$ can be extended up to a homeomorphism $h_1 \colon \pi(\widetilde{N}_2) \to \mathbb{Q}^n \setminus \operatorname{int} N_0$. Then the map $h \colon Q \to \mathbb{Q}^n$ defined by $h(x) = h_0(x)$ for $x \in \pi(\widetilde{N}_1 \cup K_1)$ and $h(x) = h_1(x)$ for $x \in \pi(\widetilde{N}_2)$ is the desired homeomorphism. The manifold $\pi(T_1)$ cuts Q into two connected components such that the closure of one of them is $\pi(N)$, which is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. By Corollary 5.1, the closure of another connected component is also homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

Suppose that the statement is true for all $\lambda = k$ and show that it is true also for $\lambda = k + 1$. Since $2k \geqslant k + 1$ we have that there exists at least one manifold among the manifolds $Q_1^n, \ldots, Q_{\lambda+1}^n$, say $Q_{\lambda+1}^n$, containing exactly one knot from the set $\beta_1, \ldots, \beta_{2k}$ (if every of that manifolds would contain no less than two knots, then the total number of all knots be no less than $2k + 2$). Let $\beta_{2\lambda} \subset Q_{\lambda+1}^n$, $\beta_{2\lambda-1} \subset Q_i^n$, $i \in \{1, \ldots, \lambda\}$, be a binding pair. By the induction hypothesis and Corollary 5.1, the manifold Q_λ obtained by the surgery of manifolds $Q_1^n, \ldots, Q_\lambda^n$ along knots $\beta_1, \ldots, \beta_{2\lambda-2}$ is homeomorphic to $\mathbb{Q}^n$; the projection of every manifold (T_j) cuts Q_λ into two connected components such that the closure of each of which is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$; and the projection of the knot $\beta_{2\lambda-1}$ is the essential knot. Now apply the surgery to manifolds Q_λ, $Q_{\lambda+1}^n$ along knots $\pi(\beta_{2\lambda-1})$, $\beta_{2\lambda}$ and use the first step arguments to obtain the desired statement. $\square$

5.4. Proof of Lemma 3.1. Step 1. *Proof of the fact that the manifold $\widehat{V}_f$ is homeomorphic to $\mathbb{Q}^n$ and every connected component $\mathcal{Q}^{n-1}$ of the set $\hat{L}^u_f \cup \hat{L}^s_f$ cuts $\widehat{V}_f$ into two connected components whose closures are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Put $k_i = |\Omega^i_f|$, $i \in \{0, 1, n-1, n\}$. Due to Statement 4.2 and the fact that the closure of every separatrix of dimension $(n-1)$ cuts the ambient sphere S^n into two connected components one gets $k_0 = k_1 + 1$, $k_n = k_{n-1} + 1$.

Denote by $\beta_1, \ldots, \beta_{2k_1}$ the essential knots in the set $\widehat{V} = \bigcup_{\omega \in \Omega^0_f} \widehat{V}^s_\omega$ which are projections (by means of $p_{\widehat{V}}$) of all one-dimension unstable separatrices of the diffeomorphism f. Without loss of generality assume that knots β_{2j-1}, β_{2j} are the projection of the separatrices of the same saddle point $\sigma_j \in \Omega^1_f$, $j \in \{1, \ldots, k_1\}$.

It follows from Statement 4.2 that every manifold $\widehat{V}^s_\omega$ contains at least one knot from the set $\beta_1, \ldots, \beta_{2k_1}$. Since stable and unstable manifolds of different saddle points do not intersect we have that for any $j \in \{1, \ldots, k_1\}$ knots β_{2j-1}, β_{2j} belong to distinct connected components of $\widehat{V}$. Indeed, if one suppose that $\beta_{2j-1}, \beta_{2j} \subset \widehat{V}^s_\omega$ for some j, ω, then the set $\operatorname{cl} W^u_{\sigma_j} = W^u_{\sigma_j} \cup \omega$ is homeomorphic to the circle. Since $\operatorname{cl} W^s_{\sigma_j}$ divides the sphere S^n into two parts and intersect the circle $\operatorname{cl} W^u_{\sigma_j}$ at the point σ_j we have that there exists at least one point in $\operatorname{cl} W^s_{\sigma_j} \cap \operatorname{cl} W^u_{\sigma_j}$ different from σ_j. This fact contradicts to the item 1 of Statement 4.1.

Let N_{σ_j}, $\chi_{\sigma_j} \colon N_{\sigma_j} \to \mathbb{U}$ be the neighborhood of the point σ_j and the homeomorphism defined in Proposition 4.3. Further we use denotations of Sections 4.2, 4.3. Denote by N_{2j-1}, N_{2j} the connected components of the set $\widehat{N}^u_{\sigma_j}$ containing knots β_{2j-1}, β_{2j} correspondingly. Let $\psi \colon \partial\widehat{\mathbb{N}}^u \to \partial\widehat{\mathbb{N}}^s$ be a homeomorphism such that $\psi p_u|_{\partial\mathbb{U}} = p_s|_{\partial\mathbb{U}}$. Put $K_j = \widehat{N}^s_{\sigma_j}$, $T_j = \widehat{V}^s_{\sigma_j}$ and define homeomorphisms $\varphi_{u,j} \colon N_{2j-1} \cup N_{2j} \to \widehat{\mathbb{N}}^u$, $\varphi_{s,j} \colon K_j \to \widehat{\mathbb{N}}^s$, $\psi_j \colon \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j$ by

$$\begin{aligned} \varphi_{u,j} &= p_u \chi_{\sigma_j} p^{-1}_{\widehat{V}_f}|_{N_{2j-1}\cup N_{2j}}, \\ \varphi_{s,j} &= p_s \chi_{\sigma_j} p^{-1}_{\widehat{V}_f}|_{K_j}, \\ \psi_j &= \varphi^{-1}_{s,j} \psi \varphi_{u,j}|_{\partial N_{2j-1}\cup\partial N_{2j}}, \end{aligned}$$

and denote by

$$\Psi \colon \bigcup_{j=1}^{k_1} (\partial N_{2j-1} \cup \partial N_{2j}) \to \bigcup_{j=1}^{k_1} K_j$$

the homeomorphism such that

$$\Psi|_{\partial N_{2j-1}\cup\partial N_{2j}} = \psi_j|_{\partial N_{2j-1}\cup\partial N_{2j}}.$$

Since

$$V_f = \Big(\bigcup_{\omega\in\Omega^0_f} V^s_\omega \setminus \Big(\bigcup_{\sigma\in\Omega^1_f} V^u_\sigma\Big)\Big) \cup \Big(\bigcup_{\sigma\in\Omega^1_f} V^s_\sigma\Big) = \Big(V_f \setminus \Big(\bigcup_{\sigma\in\Omega^1_f} N^u_\sigma\Big)\Big) \cup \Big(\bigcup_{\sigma\in\Omega^1_f} N^s_\sigma\Big)$$

it follows that

$$\widehat{V}_f = \Big(\widehat{V}_f \setminus \Big(\bigcup_{\sigma\in\Omega^1_f} \widehat{N}^u_\sigma\Big)\Big) \cup_\Psi \Big(\bigcup_{\sigma\in\Omega^1_f} \widehat{N}^s_\sigma\Big) = \Big(\widehat{V}_f \setminus \Big(\bigcup_{j=1}^{2k_1} N_j\Big)\Big) \cup_\Psi \Big(\bigcup_{j=1}^{k_1} K_j\Big).$$

So, the manifold $\widehat{V}_f$ is obtained from $\bigcup_{\omega\in\Omega_f^0}\widehat{V}_\omega^s$ by the surgery along knots $\beta_1, \dots, \beta_{2k_1}$. Due to Proposition 5.5, the manifold $\widehat{V}_f$ is homeomorphic to $\mathbb{Q}^n$ and every connected component of the set $\hat{L}_f^s$ cuts the set $\widehat{V}_f$ into two connected components such that the closure of each of which is homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$.

On the other hand,

$$V_f=\Big(\bigcup_{\alpha\in\Omega_f^n}V_\alpha^u\setminus\Big(\bigcup_{\sigma\in\Omega_f^{n-1}}V_\sigma^s\Big)\Big)\cup\Big(\bigcup_{\sigma\in\Omega_f^{n-1}}V_\sigma^u\Big)=\Big(V_f\setminus\Big(\bigcup_{\sigma\in\Omega_f^{n-1}}N_\sigma^s\Big)\Big)\cup\Big(\bigcup_{\sigma\in\Omega_f^{n-1}}N_\sigma^u\Big).$$

Similar to previous arguments one can conclude that the set $\widehat{V}_f$ is obtained from $\bigcup_{\alpha\in\Omega_f^n}\widehat{V}_\alpha^u$ by the surgery along the projections of all one-dimensional stable separatrices of the saddle points of the diffeomorphism f. In virtue of Proposition 5.5 every connected component of the set $\hat{L}_f^u$ cuts the set $\widehat{V}_f$ into two connected components such that the closure of each of which is homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$.

Step 2. *Proof of the fact that there is a set $\widehat{\mathbb{L}}_{m_f}\subset\mathbb{Q}^n$ and a homeomorphism $\hat{\varphi}\colon\widehat{V}_f\to\mathbb{Q}^n$ such that $\hat{\varphi}(\hat{L}_f^s\cup\hat{L}_f^u)=\widehat{\mathbb{L}}_{m_f}$.*

Denote by $\mathcal{Q}_1^{n-1}, \dots, \mathcal{Q}_{k_1+k_{n-1}}^{n-1}$ all elements of the set $\hat{L}_f^s\cup\hat{L}_f^u$ and suppose that $\mathcal{Q}_1^{n-1}$ is an element such that all elements of the set $\hat{L}_f^s\cup\hat{L}_f^u\setminus\mathcal{Q}_1^{n-1}$ are contained exactly in one of the connected components of the manifold $\widehat{V}_f\setminus\mathcal{Q}_1^{n-1}$. Denote by N_1 the closure of this connected component. By Step 1, N_1 is homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$. By Proposition 5.3, there exists a disk $D_1^{n-1}\subset\mathbb{S}^{n-1}$ and a homeomorphism $\psi_0\colon\hat{V}_f\to\mathbb{Q}^n$ such that $\psi_0(N_1)=D_1^{n-1}\times\mathbb{S}^1$. If $k_1+k_{n-1}=1$ then the proof is complete and $\hat{\varphi}=\psi_0$, $\widehat{\mathbb{L}}_{m_f}=\partial D_1^{n-1}\times\mathbb{S}^1$.

Suppose that $k_1+k_{n-1}>1$. Denote the images of $\mathcal{Q}_1^{n-1}, \dots, \mathcal{Q}_{k_1+k_{n-1}}^{n-1}$ under the homeomorphism ψ_0 by the same symbols as their originals. For $i\in\{2, \dots, k_1+k_{n-1}\}$ denote by N_i the connected component of the set $\mathbb{Q}^n\setminus\mathcal{Q}_i^{n-1}$ contained in the set $D_1^{n-1}\times\mathbb{S}^1$. Without loss of generality suppose that the numeration of the sets $\mathcal{Q}_1^{n-1}, \dots, \mathcal{Q}_{k_1+k_{n-1}}^{n-1}$ is chosen in such a way that there exist a number $l_1\in[2, k_1+k_{n-1}]$ and pairwise disjoint sets $N_2, \dots, N_{l_1}$ such that $\bigcup_{i=2}^{l_1}N_i=\bigcup_{i=2}^{k_1+k_{n-1}}N_i$. Choose in the interior of the disk D_1^{n-1} arbitrary pairwise disjoint disks $D_2^{n-1}, \dots, D_{l_1}^{n-2}$. Due to Proposition 5.3 there exists a homeomorphism $\psi_1\colon\mathbb{Q}^n\to\mathbb{Q}^n$ such that $\psi_1|_{\mathbb{Q}^n\setminus\operatorname{int}D_1^{n-1}\times\mathbb{S}^1}=\mathrm{id}$, $\psi_1(N_i)=D_i^{n-1}\times\mathbb{S}^1$, $i\in\{2, \dots, l_1\}$. If $l_1=k_1+k_{n-1}$ then the proof is complete and $\hat{\varphi}=\psi_1\psi_0$, $\widehat{\mathbb{L}}_{m_f}=\bigcup_{i=1}^{l_1}\partial D_i^{n-1}\times\mathbb{S}^1$.

Suppose that $l_1<k_1+k_{n-1}$. Denote the images of $\mathcal{Q}_1^{n-1}, \dots, \mathcal{Q}_{k_1+k_{n-1}}^{n-1}$ and $N_1, \dots, N_{k_1+k_{n-1}}$ under the homeomorphism ψ_1 by the same symbols as their originals. Put $\mathcal{N}=\bigcup_{i=l_1+1}^{k_1+k_{n-1}}N_i$.

If for fixed $i\in\{2, \dots, l_1\}$ the set N_i has non-empty intersection with the set $\mathcal{N}$, then denote by l_i, $\tilde{k}_i$, $l_i\leqslant\tilde{k}_i$, the positive numbers such that $N_{i,1}, \dots, N_{i,\tilde{k}_i}$ are all elements from $N_i\cap\mathcal{N}$ and $N_{i,1}, \dots, N_{i,l_i}$ are pairwise disjoint elements from $N_i\cap\mathcal{N}$ such that $\bigcup_{j=1}^{l_i}N_{i,j}=\bigcup_{j=2}^{\tilde{k}_i}N_{i,j}$. Choose in the interior of the every disk D_i^{n-1} pairwise disjoint disks $D_{i,1}^{n-1}, \dots D_{i,l_i}^{n-1}$. It follows from Proposition 5.3 that there exists a homeomorphism $\psi_i\colon \mathbb{Q}^n \to \mathbb{Q}^n$ such that $\psi_i|_{\mathbb{Q}^n\setminus \operatorname{int} N_i} = \mathrm{id}$, $\psi_i(N_{i,j}) = D_{i,j}^{n-1}\times\mathbb{S}^1$, $j\in\{1, \dots, l_i\}$, $i\in\{2, \dots, l_1\}$. If $N_i\cap\mathcal{N}=\varnothing$, put $\psi_i=\mathrm{id}$.

If $l_i=\tilde{k}_i$ for any $i\in\{2, \dots, l_1\}$ such that the numbers l_i, $\tilde{k}_i$ are defined, then the proof is complete and $\hat{\varphi} = \psi_{l_1}\psi_{l_1-1}\cdots\psi_1$, $\widehat{\mathbb{L}}_{m_f} = \bigcup_{i=1}^{l_1}\bigcup_{j=1}^{l_i}\partial D_{i,j}^{n-1}\times\mathbb{S}^1$. Otherwise, continue the process and after finite number of steps get the desired set $\widehat{\mathbb{L}}_{m_f}$ and the desired homeomorphism $\hat{\varphi}$ as a superposition of all constructed homeomorphisms.

6. Embedding of Diffeomorphisms from the Class $G(M^n)$ into Topological Flows

6.1. Free and properly discontinuous action of a group of maps. In this section we collect axillary facts on properties of the transformation group $\{g^n,\, n\in\mathbb{Z}\}$, which is an infinite cyclic group acting freely and properly discontinuously on a topological (in general, non-compact) manifold X and generated by a homeomorphism $g\colon X\to X$[7].

Denote by X/g the orbit space of the action of the group $\{g^n,\, n\in\mathbb{Z}\}$ and by $p_{X/g}\colon X\to X/g$ the natural projection. In virtue of [33] (Theorem 3.5.7 and Proposition 3.6.7) the natural projection $p_{X/g}\colon X\to X/g$ is a covering map and the space X/g is a manifold.

Denote by $\eta_{X/g}\colon \pi_1(X/g)\to\mathbb{Z}$ a homomorphism defined in the following way. Let $\hat{c}\subset X/g$ be a loop non-homotopic to zero in X/g and $[\hat{c}]\in\pi_1(X/g)$ be a homotopy class of $\hat{c}$. Choose an arbitrary point $\hat{x}\in\hat{c}$, denote by $p_{X/g}^{-1}(\hat{x})$ the complete inverse image of $\hat{x}$, and fix a point $\tilde{x}\in p_{X/g}^{-1}(\hat{x})$. As $p_{X/g}$ is the covering map then there is a unique path $\tilde{c}(t)$ beginning at the point $\tilde{x}$ ($\tilde{c}(0)=\tilde{x}$) and covering the loop c (such that $p_{X/g}(\tilde{c}(t))=\hat{c}$). Then there exists an element $n\in\mathbb{Z}$ such that $\tilde{c}(1)=f^n(\tilde{x})$. Put $\eta_{X/g}([\hat{c}])=n$. It follows from [21] (ch. 18) that the homomorphism $\eta_{X/g}$ is an epimorphism.

The next statement 6.1 can be found in [21] (Theorem 5.5) and [3] (Propositions 1.2.3, 1.2.4).

Statement 6.1. *Suppose that X, Y are connected topological manifolds and $g\colon X\to X$, $h\colon Y\to Y$ are homeomorphisms such that groups $\{g^n,\, n\in\mathbb{Z}\}$, $\{h^n,\, n\in\mathbb{Z}\}$ acts freely and properly discontinuously on X, Y correspondingly. Then*:

1) *if $\varphi\colon X\to Y$ is a homeomorphism such that $h=\varphi g\varphi^{-1}$ and $\varphi_*\colon\pi_1(X/g)\to\pi_1(Y/h)$ is the induced homomorphism, then a map $\widehat{\varphi}\colon X/g\to Y/h$ defined by $\widehat{\varphi}=p_{Y/h}\varphi p_{X/g}^{-1}$ is a homeomorphism and $\eta_{X/g}=\eta_{Y/h}\varphi_*$;*

[7] A group $\mathcal{G}$ *acts* on the manifold X if there is a map $\zeta\colon\mathcal{G}\times X\to X$ with the following properties:

1) $\zeta(e, x)=x$ for all $x\in X$, where e is the identity element of the group $\mathcal{G}$;

2) $\zeta(g, \zeta(h, x))=\zeta(gh, x)$ for all $x\in X$ and $g, h\in\mathcal{G}$.

A group $\mathcal{G}$ acts *freely* on a manifold X if for any different $g, h\in X$ and for any point $x\in X$ an inequality $\zeta(g, x)\neq\zeta(h, x)$ holds.

A group $\mathcal{G}$ acts *properly discontinuously* on the manifold X if for every compact subset $K\subset X$ the set of elements $g\in\mathcal{G}$ such that $\zeta(g, K)\cap K\neq\varnothing$ is finite.

2) *if* $\widehat{\varphi}\colon X/g \to Y/h$ *is a homeomorphism such that* $\eta_{X/g} = \eta_{Y/h}\varphi_*$ *and* $\hat{x} \in X/g$, $\tilde{x} \in p^{-1}_{X/g}(x)$, $y = \widehat{\varphi}(x)$, $\tilde{y} \in p^{-1}{}_{Y/h}(y)$, *then there exists a unique homeomorphism* $\varphi\colon X \to Y$ *such that* $h = \varphi g \varphi^{-1}$ *and* $\varphi(\tilde{x}) = \tilde{y}$.

6.2. Proof of Theorem 1. Suppose that a Morse–Smale diffeomorphism $f\colon S^n \to S^n$ has no heteroclinic intersection and satisfy Palis conditions. To prove the theorem it is enough to construct a topological flow X^t_f such that its time one map X^1_f belongs to the class $G(S^n)$ and the scheme $S_{X^1_f}$ is equivalent to the scheme S_f (see Section 3).

It follows from Lemma 3.1 and Proposition 6.1 that the restriction of the diffeomorphism f on V_f is embedding in topological flow, conjugated with the flow $a^t(x, s) = (x, s+t)$, $x \in \mathbb{S}^{n-1}$, $s \in \mathbb{R}$. Below we describe a modification of the flow $a^(x, s)$ and its ambient manifolds $\mathbb{S}^{n-1} \times \mathbb{R}$ to obtain the desire flows X^t_f.

Step 1. *Adding saddle points of indices* 1.

It follows from Lemma 3.1 and Proposition 6.1 that there exists a homeomorphism $\psi_f\colon V_f \to \mathbb{S}^{n-1} \times \mathbb{R}$ such that:

1) $f|_{V_f} = \psi^{-1}_f a \psi_f$, where a is the time one map of the flow $a^t(x, s) = (x, s+t)$, $x \in S^{n-1}$, $s \in \mathbb{R}$;

2) for $(n-1)$-dimensional separatrix l_σ of an arbitrary saddle point $\sigma \in \Omega_f$ there exists a sphere $S^{n-2}_\sigma \subset \mathbb{S}^{n-1}$ such that $\psi_f(l_\sigma) = \bigcup_{t\in\mathbb{R}} a^t(S^{n-2}_\sigma)$.

Recall that we denote by L^s_f and L^u_f the union of all $(n-1)$-dimensional stable and unstable separatrices of the diffeomorphism f correspondingly. Put $\mathbb{L}^s = \psi_f(L^s_f)$, $\mathbb{L}^u = \psi_f(L^u_f)$. Then $\mathbb{L}^\delta$ is the union of pairwise disjoint cylinders $\tilde{Q}^\delta_1 \cup \dots \cup \tilde{Q}^\delta_{k^\delta}$, $\delta \in \{s, u\}$. Denote by $N(\mathbb{L}^\delta) = N(\tilde{Q}^\delta_1) \cup \dots \cup N(\tilde{Q}^\delta_{k^\delta})$ the set of their pairwise disjoint closed tubular neighborhoods such that $N(\tilde{Q}^\delta_i) = K^\delta_i \times \mathbb{R}$, where $K^\delta_i \subset \mathbb{S}^{n-1}$ is an annulus of dimension $(n-1)$, $i = 1, \dots, k^\delta$.

Define a flow a^t_1 on the set $\mathbb{U} = \{(x_1, \dots, x_n) \in \mathbb{R}^n\colon x^2_1(x^2_2 + \dots + x^2_n) \leqslant 1\}$ by

$$a^t_1(x_1, x_2, \dots, x_n) = (2^t x_1, 2^{-t}x_2, \dots, 2^{-t}x_n).$$

It follows from Statements 4.4, 6.1 that there exists a homeomorphism $\chi^s_i\colon N(\tilde{Q}^s_i) \to \mathbb{N}^s$ such that $a^1_1|_{\mathbb{N}^s} = \chi^s_i a^1 (\chi^s_i)^{-1}|_{\mathbb{N}^s}$. Denote by $\chi^s\colon N(\mathbb{L}^s) \to \mathbb{U} \times \mathbb{Z}_{k^s}$ a homeomorphism such that $\chi^s|_{N(\tilde{Q}^s_i)} = \chi^s_i$ for any $i \in \{1, \dots, k^s\}$. Put $\mathbb{Q}^s = (\mathbb{S}^{n-1} \times \mathbb{R}) \cup_{\chi^s} (\mathbb{U} \times \mathbb{Z}_{k^s})$. A topological space $\mathbb{Q}^s$ is a connected oriented n-manifold without boundary.

Denote by $\pi_s\colon (\mathbb{S}^{n-1} \times \mathbb{R}) \cup (\mathbb{U} \times \mathbb{Z}_{k^s}) \to Q^s$ a natural projection. Put $\pi_{s,1} = \pi_s|_{\mathbb{S}^{n-1}\times\mathbb{R}}$, $\pi_{s,2} = \pi_s|_{\mathbb{U}\times\mathbb{Z}_{k^s}}$. Define a flow $\tilde{Y}^t_s$ on the manifold Q^s by

$$\tilde{Y}^t_s(x) = \begin{cases} \pi_{s,1}(a^t(\pi^{-1}_{s,1}(x))), & x \in \pi_{s,1}(\mathbb{S}^{n-1} \times \mathbb{R}); \\ \pi_{s,2}(a^t_1(\pi^{-1}_{s,2}(x))), & x \in \pi_{s,2}(\mathbb{U} \times \{i\}),\ i \in \mathbb{Z}_{k^s} \end{cases}.$$

By construction the non-wandering set of the flow $\tilde{Y}^t_s$ consists of k^s equilibria such that the flow $\tilde{Y}^t_s$ is locally topologically conjugated with the flow a^t_1 at the neighborhood of each equilibrium.

Step 2. *Adding saddle points of indices* $(n-1)$.

Denote the images of the sets $\mathbb{L}^u$, $N(\mathbb{L}^u)$ by means of the projection π_s by the same symbols as their originals. Due to Statements 4.4, 6.1 there exists a homeomorphism $\chi_i^u\colon N(\tilde{Q}_i^u)\to\mathbb{N}^u$ such that $a_1^{-1}|_{\mathbb{N}^u}=\chi_i^u\tilde{Y}_s^1(\chi_i^u)^{-1}$, $i=1,\,\dots,\,k^u$. Denote by $\chi^u\colon N(\mathbb{L}^u)\to\mathbb{U}\times\mathbb{Z}_{k^u}$ the homeomorphism such that $\chi^u|_{N(\tilde{Q}_i^u)}=\chi_i^u|_{N(\tilde{Q}_i^u)}$ for any $i=1,\,\dots,\,k^u$. Put $\mathbb{Q}^u=\mathbb{Q}^s\cup_{\chi^u}(\mathbb{U}\times\mathbb{Z}_{k^u})$. A topological space $\mathbb{Q}^u$ is a connected oriented n-manifold without boundary.

Denote by $\pi_u\colon\mathbb{Q}^s\cup(\mathbb{U}\times\mathbb{Z}_{k^u})\to\mathbb{Q}^u$ the natural projection. Put $\pi_{u,1}=\pi_u|_{\mathbb{Q}^s}$, $\pi_{u,2}=p_u|_{\mathbb{U}\times\mathbb{Z}_{l^u}}$. Define a flow $\tilde{Y}_u^t$ on the manifold $\mathbb{Q}^u$ by

$$\tilde{Y}_u^t(x)=\begin{cases}\pi_{u,1}(\tilde{Y}_s^t(\pi_{u,1}^{-1}(x))), & x\in\pi_{u,1}(\mathbb{Q}^s);\\ \pi_{u,2}(a_1^{-t}(\pi_{u,2}^{-1}(x))), & x\in\pi_{u,2}(\mathbb{U}\times\{i\}),\ i\in\mathbb{Z}_{k^u}\end{cases}.$$

The non-wandering set $\Omega_{\tilde{Y}_u^t}$ of the flow $\tilde{Y}_u^t$ consists of k^s equilibria such that the flow $\tilde{Y}_u^t$ is locally topological conjugated with the flow a_1^t in each of their neighborhoods and k^u equilibria such that the flow $\tilde{Y}_u^t$ is locally topologically conjugated with the flow a_1^{-t} in each of their neighborhoods.

Step 3. *Adding sinks.*

Put $R^s=\mathbb{Q}^u\setminus W^s_{\Omega_{\tilde{Y}_u^t}}$, denote by $\rho_1^s,\,\dots,\,\rho_{n^s}^s$ connected components of the set R^s and put $\hat{\rho}_i^s=\rho_i^s/_{\tilde{Y}_u^1}$. A union of the orbit spaces $\bigcup_{i=1}^{n^s}\hat{\rho}_i^s$ is obtained from the manifold $\widehat{V}_a$ by a sequence of the surgeries along essential submanifolds of codimension 1. In virtue of Proposition 5.4 for any $i\in\{1,\,\dots,\,n^s\}$ the manifold $\hat{\rho}_i^s$ is homeomorphic to $\mathbb{S}^{n-1}\times\mathbb{S}^1$, the manifold ρ_i^s is homeomorphic to $\mathbb{S}^{n-1}\times\mathbb{R}$ and the flow $\tilde{Y}_u^t|_{\rho_i^s}$ is topologically conjugated with the flow $a^t|_{\mathbb{R}^n\setminus O}$ by means of a homeomorphism ν_i^s. Denote by $\nu^s\colon R^s\to(\mathbb{R}^n\setminus\{0\})\times\mathbb{Z}_{n^s}$ the homeomorphism consisting of the homeomorphisms $\nu_1^s,\,\dots,\,\nu_{n^s}^s$. Put $M^s=\mathbb{Q}^u\cup_{\nu^s}(\mathbb{R}^n\times\mathbb{Z}_{n^s})$. Then M^s is a connected oriented n-manifold without boundary.

Put $\bar{M}^s=\mathbb{Q}^u\cup(\mathbb{R}^n\times\mathbb{Z}_{n^s})$ and denote by $q_s\colon\bar{M}^s\to M^s$ the natural projection. Put $q_{s,1}=q_s|_{Q^u}$, $q_{s,2}=q_s|_{\mathbb{R}^n\times\mathbb{Z}_{n^s}}$. Define a flow $\tilde{X}_s^t$ on the manifold M^s by

$$\tilde{X}_s^t(x)=\begin{cases}q_{s,1}(\tilde{Y}_u^t(q_{s,1}^{-1}(x))), & x\in q_{s,1}(Q^u);\\ q_{s,2}(a^t(q_{s,2}^{-1}(x))), & x\in q_{s,2}(\mathbb{R}^n\times\{i\}),\ i\in\mathbb{Z}_{n^s}\end{cases}.$$

By construction the non-wandering set of the time one map of the flow $\tilde{X}_s^t$ consists of k^s saddle topologically hyperbolic fixed points of index 1, k^u saddle topologically hyperbolic fixed points of index $(n-1)$ and n^s sink topologically hyperbolic fixed points.

Step 4. *Adding sources.*

Put $R^u=M^s\setminus W^u_{\Omega_{\tilde{X}_s^t}}$ and denote by $\rho_1^u,\,\dots,\,\rho_{n^u}^u$ connected components of the set R^u. Similar to Step 3 one can prove that every component ρ_i^u is homeomorphic to $\mathbb{S}^{n-1}\times\mathbb{R}$ and the flow $\tilde{X}_s^t|_{\rho_i^u}$ is conjugated with the flow $a^{-t}|_{\mathbb{R}^n\setminus\{O\}}$ by a homeomorphism μ_i^u. Denote by $\mu^u\colon R^u\to(\mathbb{R}^n\setminus\{O\})\times\mathbb{Z}_{n^u}$ a homeomorphism consisting of the homeomorphisms $\mu_1^u,\,\dots,\,\mu_{n^u}^u$. Put $M^u=M^s\cup_{\mu^u}(\mathbb{R}^n\times\mathbb{Z}_{n^u})$. M^u is a connected closed oriented n-manifold.

Put $\bar{M}^u = M^s \cup (\mathbb{R}^n \times \mathbb{Z}_{n^u})$, denote by $q_u \colon \bar{M}^u \to M^u$ the natural projection, and put $q_{u,1} = q_u|_{M^s}$, $q_{u,2} = q_u|_{\mathbb{R}^n \times \mathbb{Z}_{n^u}}$. Define a flow $\tilde{X}_u^t$ on the manifold M^u by

$$\tilde{X}_u^t(x) = \begin{cases} q_{u,1}(\tilde{X}_s^t(q_{u,1}^{-1}(x))), & x \in q_{u,1}(M^s); \\ q_{u,2}(a_0^{-t}(q_{u,2}^{-1}(x))), & x \in q_{u,2}(\mathbb{R}^n \times \{i\}),\ i \in \mathbb{Z}_{n^u} \end{cases}.$$

By construction the non-wandering set of the time one map of the flow $\tilde{X}_u^t$ consists of k^s saddle topologically hyperbolic fixed points of index 1, k^u saddle topologically hyperbolic fixed points of index $(n-1)$, n^s sink and n^u source topologically hyperbolic fixed points.

Step 5. Put $\tilde{f} = \tilde{X}_u^1$. By construction $\tilde{f}$ is a Morse–Smale homeomorphism on the manifold M^u and its restriction $\tilde{f}|_{V_{\tilde{f}}}$ is topologically conjugated with the diffeomorphism $f|_{V_f}$ by a homeomorphism mapping the $(n-1)$-dimensional separatrices of the diffeomorphism $\tilde{f}$ to the $(n-1)$-dimensional separatrices of the diffeomorphism f and preserving their stability. Due to Statement 3.1 homeomorphisms $\tilde{f}$ and f are topologically conjugated. Hence $M^u = S^n$ and $X^t = \tilde{X}_u^t$ is the desired flow.

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