MORSE-SMALE SURFACED DIFFEOMORPHISMS WITH ORIENTABLE HETEROCLINIC

A. MOROZOV and O. POCHINKA

ABSTRACT. In the present paper we consider preserving orientation Morse-Smale diffeomorphisms on surfaces. Using the methods of factorization and linearizing neighborhoods we prove that such diffeomorphisms have a finite number of orientable heteroclinic orbits.

1. INTRODUCTION

The Morse-Smale dynamics is a pattern of a regular structurally stable (rough) behaviour. The foundations of the theory of roughness of flows, laid down in the classic work by A. Andronov and L. Pontryagin [1], were developed by the associates of academician A. Andronov in the Gorky school of non-linear oscillations E. Leontovich and A. Mayer [12], [13]. Moreover, A. Meyer introduced the concept of roughness for a discrete dynamical system and obtained a complete topological classification of such systems (Morse-Smale systems) on the circle [14].

Since then, the theory of topological classification of Morse-Smale dynamic systems has gained wide popularity and has undergone intensive development (see, for example, review [8]). So the topological classification of structurally stable flows on surfaces is exhaustively described in the work by M. Peixoto [16]. On three-dimensional manifolds, the necessary and sufficient conditions of the topological conjugacy of Morse-Smale flows follows from the works by Ya. Umanskiy [20] and A. Prishlyak [18]. There are also multidimensional classification results, for example, the classification by S. Pilyugin [17] for Morse-Smale flows without heteroclinic intersections on the n-dimensional sphere. For three-dimensional Morse-Smale diffeomorphisms, the completed classification results were obtained only in 2018 by C. Bonatti, V. Grines, O. Pochinka [5] after twenty years of work by a team of Russian-French scientists, C. Bonatti, V. Grines, F. Laudenbach, V. Medvedev, E. Pecu, O. Pochinka. The researchers of the Nizhny Novgorod

school V. Grines, E. Gurevich, V. Medvedev, O. Pochinka also obtained a complete topological classification of Morse-Smale diffeomorphisms with saddles of co-dimension one on closed n-manifolds [9].

The classification of Morse-Smale cascades on surfaces was obtained in the work by C. Bonatti and R. Langevin [7], as part of the study of diffeomorphisms with zero-dimensional basic sets. Therefore, it uses a rather heavy apparatus of Markov partitions, due to the need to track the geometry of heteroclinic intersections that make up an infinite number of orbits. Given the obvious evolution of the theory of topological classification of dynamical systems, it would be natural to expect a separate and clear classification of two-dimensional Morse-Smale casacdes. Such attempts were made for some meaningful subclasses of surface diffeomorphisms, for example, in the works by A. Bezdenezhnykh, V. Grines, T. Mitryakova, O. Pochinka [4], [10], with the restriction imposed in the form of a finite number of heteroclinic orbits $(beh = 1)$.

However, the topology of heteroclinic intersections sometimes itself naturally imposes restrictions on the length of heteroclinic chains, which makes it possible to obtain a classification of such systems in combinatorial terms. In this paper, it is proven that Morse-Smale surfaced diffeomorphism with orientable heteroclinic intersections cannot have an infinite number of heteroclinic orbits. In the other words we state the following result.

Theorem 1. *If* $f : M^2 \to M^2$ *is a preserving orientation Morse-Smale diffeomorphism on surface with orientable heteroclinic then* $beh(f) = 1$.

This result was announced by V. Grines and A. Bezdenezhnyh here [3] and was proved as the chapter of the candidate's dissertation by A. Bezdenezhnyh [2]. That proof used a technique of the paper by A. Mayer [15] about the relation of the number of pairwise disjoint non-trivial recurrent trajectories with the genus of the ambient surface. In this paper we conduct an independent proof of this result using a factorization methods to compatible systems of neighborhoods.

2. General properties of Morse-Smale diffeomorphisms

Everywhere below M^n is a smooth closed connected orientable n-dimensional manifold $(n \geq 1)$ and $f : M^n \to M^n$ is a preserving orientation diffeomorphism.

Definition 2 (Morse-Smale diffeomorphism)**.** Diffeomorphism $f : M^n \to M^n$ is called *Morse-Smale diffeomorphism*, if

1. its non-wandering set Ω_f consists of a finite number of hyperbolic orbits;
2. the invariant manifolds W_p^s, W_q^u intersect transversally for any non-wandering points p, q.

It is well known that Morse-Smale diffeomorphisms have no cycles in the sense of the following definition.

Definition 3 (k-cycle)**.** A sequence of different periodic orbits $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_k$ that satisfies the condition

$$W^u_{\mathcal{O}_{i+1}} \cap W^s_{\mathcal{O}_i} \neq \emptyset, i \in \{0, \ldots, k\}, \mathcal{O}_{k+1} = \mathcal{O}_0,$$

is called k-cycle $(k \geq 1)$.

The absence of cycles allows us to introduce the following partial order relation on the set of periodic orbits of the Morse-Smale diffeomorphism (it was first done by S. Smale [19]).

Definition 4 (Partial order relation, beh)**.** Let $\mathcal{O}_i, \mathcal{O}_j$ be periodic orbits of the Morse-Smale diffeomorphism $f : M^n \to M^n$. One says that the orbit $\mathcal{O}_i, \mathcal{O}_j$ are in a relation $\prec$ ($\mathcal{O}_i \prec \mathcal{O}_j$), if

$$W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset.$$

A sequence composed by different periodic orbits $\mathcal{O}_i = \mathcal{O}_{i_0}, \mathcal{O}_{i_1}, ..., \mathcal{O}_{i_k} = \mathcal{O}_j (k \geq 1)$ such that $\mathcal{O}_{i_0} \prec \mathcal{O}_{i_1} \prec \ldots \prec \mathcal{O}_{i_k}$ is called *chain of length k, connecting periodic orbits $\mathcal{O}_i$ and $\mathcal{O}_j$.* Number equal to the length of the maximum saddle chain of a Morse-Smale diffeomorphism $f : M^n \to M^n$ is denoted

$$beh(f).$$

For a subset P of the periodic orbits of f let us set

$$beh(\mathcal{O}_j|P) = \max_{\mathcal{O}_i \subset P} \{beh(\mathcal{O}_j|\mathcal{O}_i)\}.$$

Proposition 5 ([11], Theorem 2.1.1)**.** *Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism. Then*

1. $M^n = \bigcup\limits_{p \in \Omega_f} W^u_p$;
2. W^u_p *is a smooth submanifold of a manifold M^n diffeomorphic to $\mathbb{R}^{\dim W^u_p}$ for any periodic point $p \in \Omega_f$;*
3. $cl(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup\limits_{r \in \Omega_f : \ell^u_p \cap W^s_r \neq \emptyset} W^u_r$ *for any unstable separatrix ℓ^u_p (a connected component of $W^u_p \setminus p$) of periodic point $p \in \Omega_f$.*

A similar theorem holds for the stable manifolds of the diffeomorphism f.

3. Surfaced Morse-Smale diffeomorphisms

In this section we consider a class $MS(M^2)$ of orientation preserving Morse-Smale diffeomorphisms given on a closed orientable surface M^2.

3.1. Orientable heteroclinic. Following to [2], for the diffeomorphisms in this class we will introduce the concept of orientable heteroclinic as the following.

Definition 6 (Orientable heteroclinic). Let $f \in MS(M^2)$, σ_i, σ_j — saddle points of diffeomorphism f, such that $W_{\sigma_i}^s \cap W_{\sigma_j}^u \neq \emptyset$. For any heteroclinic point $x \in W_{\sigma_i}^s \cap W_{\sigma_j}^u$ we define an ordered pair of vectors $(\vec{v}_x^u, \vec{v}_x^s)$, where:

- $\vec{v}_x^u$ is the tangent vector to the unstable manifold of the point σ_j at the point x and directed from x to $f^{m^u}(x)$, where m^u is a period of the unstable separatrix containing x;
- $\vec{v}_x^s$ is the tangent vector to the stable manifold of the point σ_i at the point x and directed from x to $f^{m^s}(x)$, where m^s is a period of the stable separatrix containing x.

Heteroclinic intersection of diffeomorphism f is called *orientable*, if ordered pairs of vectors $(\vec{v}_x^u, \vec{v}_x^s)$ determines the same orientation of the ambient surface M^2 at every heteroclinic point x of the diffeomorphism f. Otherwise, the heteroclinic intersection is called *non-orientable* (see Fig. 1, Fig. 2).
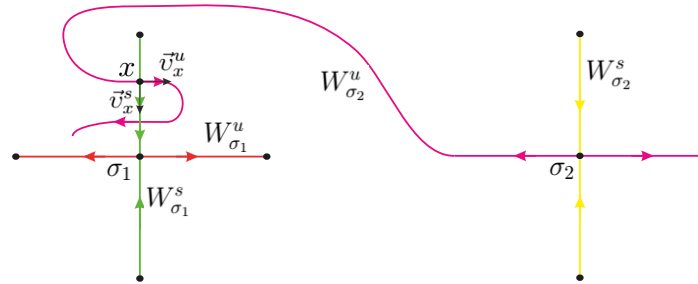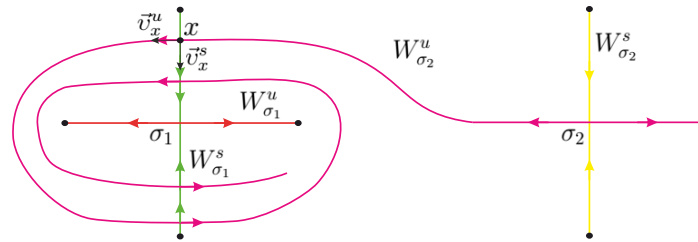

Fig. 1. Non-orientable heteroclinic intersection

Fig. 2. Orientable heteroclinic intersection

3.2. **Linearizing neighborhoods.** For $\nu \in \{-1, 1\}$ denote by $a_\nu : \mathbb{R}^2 \to \mathbb{R}^2$ diffeomorphism given by formula

$$a_\nu(x_1, x_2) = (\nu \cdot \frac{x_1}{2}, \nu \cdot 2x_2).$$

We call a_ν *canonical diffeomorphism*. Diffeomorphism a_ν has a single fixed saddle point at the origin O with stable manifold $W_O^s = Ox_1$ and unstable manifold $W_O^u = Ox_2$.

Let $\mathcal{N} = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x_1 x_2| \leq 1\}$. Notice that the set $\mathcal{N}$ is invariant with respect to the canonical diffeomorphism a_ν. Define in the neighborhood of $\mathcal{N}$ a pair of transversal foliations $\mathcal{F}^u$, $\mathcal{F}^s$ as follows:

$$\mathcal{F}^s = \bigcup_{c_2 \in Ox_2} \{(x_1, x_2) \in \mathcal{N} \ : \ x_2 = c_2\},$$

$$\mathcal{F}^u = \bigcup_{c_1 \in Ox_1} \{(x_1, x_2) \in \mathcal{N} \ : \ x_1 = c_1\}.$$

Notice that the canonical diffeomorphism a_ν which sends the leaves of foliation $\mathcal{F}^u$ ($\mathcal{F}^s$) to leaves of the same foliation.

Now let σ be a saddle point of a diffeomorphism $f \in MS(M^2)$. Suppose σ has a period m_σ and a type of orientation ν_σ, that is $\nu_\sigma = 1$ ($\nu_\sigma = -1$) if $f^{m_\sigma}|_{W_\sigma^u}$ preserves (changes) the orientation.

Definition 7 (Linearizing neighbourhood). An f^{m_σ}-invariant neighbourhood N_σ of saddle point $\sigma \in \Omega_f$ is called by *linearizing* if there is a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}$, conjugate diffeomorphism $f^{m_\sigma}|_{N_\sigma}$ with canonical diffeomorphism $a_{\nu_\sigma}|_{\mathcal{N}}$.

Foliations $\mathcal{F}^u$, $\mathcal{F}^s$ induced by means of the homeomorphism μ_σ^{-1}, f^{m_σ}-*invariant foliations* F_σ^u, F_σ^s on the linearizing neighboughood N_σ (see Fig. 3).

Neighbourhood $N_{\mathcal{O}_\sigma} = \bigcup_{k=0}^{m_\sigma - 1} f^k(N_\sigma)$, equipped with a mapping $\mu_{\mathcal{O}_\sigma}$, composed of homeomorphisms $\mu_\sigma f^{-k} : f^k(N_\sigma) \to \mathcal{N}$, $k = 0, \ldots, m_\sigma - 1$, called by *linearizing neighbourhood of orbit $\mathcal{O}_\sigma$*.

Definition 8 (The compatible system of neighbourhoods). An f-invariant collection N_f of linearizing neghbourhoods N_σ of all saddle point σ of diffeomorphism $f \in MS(M^2)$ is called *compatible* if the following properties are hold:

- if $W_{\sigma_1}^s \cap W_{\sigma_2}^u = \emptyset$ and $W_{\sigma_1}^u \cap W_{\sigma_2}^s = \emptyset$ for saddle points σ_1, σ_2 then $N_{\sigma_1} \cap N_{\sigma_2} = \emptyset$;
- if $W_{\sigma_1}^s \cap W_{\sigma_2}^u \neq \emptyset$ for different saddle points σ_1, σ_2 then

$$(F_{\sigma_1, x}^u \cap N_{\sigma_2}) \subset F_{\sigma_2, x}^u, \ \ (F_{\sigma_2, x}^s \cap N_{\sigma_1}) \subset F_{\sigma_1, x}^s,$$
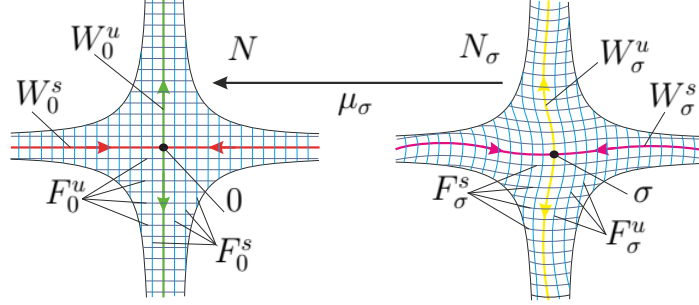
for $x \in (N_{\sigma_1} \cap N_{\sigma_2})$ (see Fig. 4).
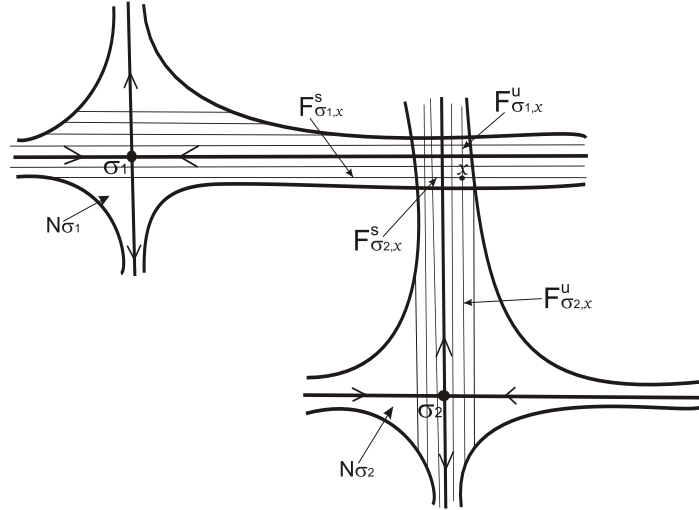
Fig. 3. A linearizing neighborhood

Fig. 4. A compatible system of neighborhoods

Proposition 9 ([6], Theorem 1)**.** *For every diffeomorphism $f \in MS(M^2)$ there is a compatible system of neighbourhoods.*

3.3. **Factorization method.** Let $f \in MS(M^2)$. Denote by $\Sigma_0, \Sigma, \Sigma_{beh(f)}$ the set of all sinks, saddles, sources of f, accordingly. We decompose the set Σ on the subsets $\Sigma_1, \ldots, \Sigma_{beh(f)-1}$ inductively, as follows: define Σ_i as the set of all saddle points of the diffeomorphism f, such that $beh(\mathcal{O}_\sigma|\Sigma_{i-1}) = 1$ for every orbit $\mathcal{O}_\sigma$, $\sigma \in \Sigma_i$.

For every $i \in \{0, \ldots, beh(f) - 1\}$ define

$$\mathcal{A}_i = \bigcup_{j=0}^{i} W_{\Sigma_j}^u, \ \mathcal{V}_i = \bigcup_{j=0}^{i} W_{\Sigma_j}^s \setminus \mathcal{A}_i.$$

It follows from [11] that $\mathcal{A}_i$ is an attractor of the diffeomorphism f and f acts discontinuously on the $\mathcal{V}_i$. Let $\hat{\mathcal{V}}_i = \mathcal{V}_i/f$ and denote by

$$p_i : \mathcal{V}_i \to \hat{\mathcal{V}}_i$$

the natural projection.

Let $\hat{V}$ be a connected component of $\hat{\mathcal{V}}_i$ and $V = p_i^{-1}(\hat{V})$. Denote by m_V the number of connected components in V. Set $p_V = p_i|_V$.

Proposition 10 ([11] Proposition 2.1.5). *The space $\hat{V}$ is diffeomorphic to two-dimensional torus, natural projection $p_V : V \to \hat{V}$ is a covering map inducing an epimorphism $\eta_V : \pi_1(\hat{V}) \to m_V\mathbb{Z}$ (here $m_V\mathbb{Z}$ is the group of integers multiple to m_V) with the following property. Let $[\hat{c}] \in \pi_1(\hat{V})$. Any lift c of the loop $\hat{c}$ joints a point $x \in V$ with the point $f^n(x)$, where $n \in \mathbb{Z}$ does not depend on the choice of the lift. Then $\eta_V([\hat{c}]) = n$.*

For an unstable separatrix γ^u of a saddle point σ denote by m_{γ^u} its *period*, that is, the smallest natural number μ, such that $f^\mu(\gamma^u) = \gamma^u$. Let us suppose, that an unstable saddle separatrix γ^u belongs to V. Let $\hat{\gamma}^u = p_V(\gamma^u)$ and denote by $j_{\hat{\gamma}^u} : \hat{\gamma}^u \to \hat{V}$ the inclusion map.

Proposition 11 ([11] proposition 2.1.3). *The set $\hat{\gamma}^u$ is a circle smoothly embedded in $\hat{V}$, such that $\eta_V(j_{\hat{\gamma}^u *}(\pi_1(\hat{\gamma}^u))) = m_{\gamma^u}\mathbb{Z}$.*
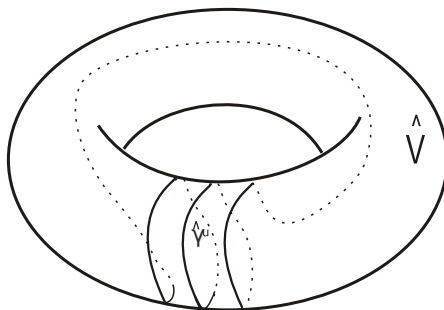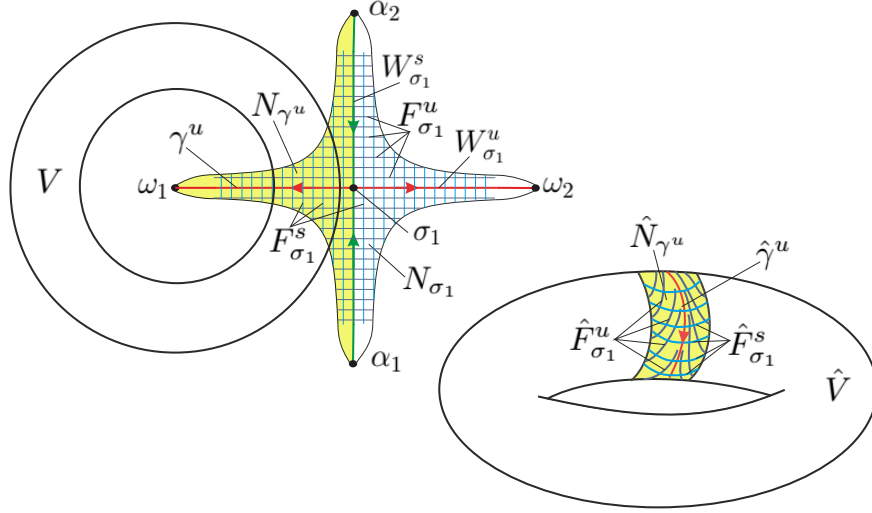

Fig. 5. Projection of an unstable saddle separatrix to $\hat{V}$

On figure 5 the torus $\hat{V}$ is represented with projection $\hat{\gamma}^u$ of separatrix γ^u such that $\frac{m_{\gamma^u}}{m_\omega} = 3$.

Let N_{γ^u} be a connected component of the set $N_\sigma \setminus W_\sigma^s$ containing γ^u and $\hat{N}_{\gamma^u}^u = p_V(N_{\gamma^u})$.

Proposition 12 ([11] Proposition 2.1.3). *The set $\hat{N}_{\gamma^u}$ is an annulus smoothly embedded in $\hat{V}$.*

Fig. 6. Projection of compatible foliations to $\hat{V}$

Similar statements can be formulated for the stable saddle separatrix when it belongs to V.

On Figure 6 a part of the phase portrait of a diffeomorphism $f \in MS(M^2)$ is represented. Here α_1, α_2 are source points, ω_1, ω_2 are sink points, σ_1 is a saddle point and N_{σ_1} is a linearizing neighbourhood of it. All non-wandering point are supposed to be fixed and $V = W^s_{\omega_1} \setminus \omega_1$ — the punctured basin of the sink ω_1. Then on the space orbit $\hat{V}$ we can see annulus $\hat{N}_{\gamma^u} = p_V(N_{\gamma^u})$ as the projection of a part of N_{σ_1} containing the unstable separatrix γ^u. Also we can see projections of both foliations in N_{σ_1}:

$$\hat{F}^u_{\sigma_1} = p_V(F^u_{\sigma_1}), \ \ \hat{F}^s_{\sigma_1} = p_V(F^s_{\sigma_1}).$$

4. Proof of the main result (Theorem 1)

Let $f \in MS(M^2)$ and N_f be its compatible system of linearizing neghbourhoods N_σ of all saddle point σ. In the denotations of previous section for $i = 1, \ldots, beh(f) - 1$ let

$$\mathcal{W}^s_i = \bigcup_{\sigma \in \Sigma_i} W^s_\sigma, \ \mathcal{W}^u_i = \bigcup_{\sigma \in \Sigma_i} W^u_\sigma,$$

$$\mathcal{N}_i = \bigcup_{\sigma \in \Sigma_i} N_\sigma, \ \mathcal{F}^s_i = \bigcup_{\sigma \in \Sigma_i} F^s_\sigma, \ \mathcal{F}^u_i = \bigcup_{\sigma \in \Sigma_i} F^u_\sigma.$$

Let us prove, that from the conditions of the orientability heteroclinic intersections of f follows that $beh(f) = 1$.

Suppose the opposite: $beh(f) > 1$. Then there is a chain of saddle points $\sigma_1 \prec \sigma_2 \prec \sigma_3$ (see Figure 7), such that $\sigma_i \in \Sigma_i$ and (see Fig. 7)

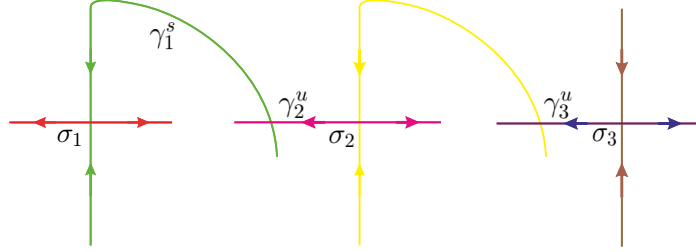$$W^u_{\sigma_{i+1}} \cap W^s_{\sigma_i} \neq \emptyset, i = 1, 2.$$

Fig. 7. The chain of the periodic points $\sigma_1 \prec \sigma_2 \prec \sigma_3$ of the diffeomorphism f.

Then these saddle points have separatrix $\gamma^s_1, \gamma^u_2, \gamma^u_3$ such that $\gamma^s_1 \cap \gamma^u_2 \neq \emptyset$ and $\gamma^u_3 \cap W^s_{\sigma_2} \neq \emptyset$. Let $\hat{\gamma}^s_1 = p_1(\gamma^s_1)$, $\hat{\gamma}^u_2 = p_1(\gamma^u_2)$. By the construction, all points of the separatrix γ^s_1, γ^u_2 belong to space $\mathcal{V}_1$. Due to Proposition 11, each set $\hat{\gamma}^s_1, \hat{\gamma}^u_2$ is a circle on a disjoint union of the tori $\hat{\mathcal{V}}_1$. Since the intersection $\hat{\gamma}^s_1 \cap \hat{\gamma}^u_2$ is not empty then these circles belong to the same torus (denote it by $\hat{V}$). Also we will see annuli $\hat{N}_1 = p_1(N_{\gamma^s_1})$ and $\hat{N}^1_2 = p_1(N_{\gamma^u_2})$ on the torus $\hat{V}$.

Every point of the separatrix γ^u_3, except the finite number of heteroclinic orbits $\gamma^u_3 \cap \mathcal{W}^s_2$, belongs to the space $\mathcal{V}_1$. Then the set $\hat{\gamma}^u_3 = p_1(\gamma^u_3)$ consists of a finite number of non-compact arcs. As $\gamma^u_3 \cap W^s_{\sigma_2} \neq \emptyset$ then, due to Proposition 5, $\gamma^u_3 \cap N_{\gamma^u_2} \neq \emptyset$. Thus there is a connected component $\hat{\ell}$ of the set $\hat{\gamma}^u_3$ such that $\hat{\ell} \cap \hat{N}^1_2 \neq \emptyset$.

Let $\hat{\Gamma}^u_2 = \hat{V} \cap p_1(\mathcal{W}^u_2)$ and $\hat{\mathcal{N}}_2 = p_1(\mathcal{N}_2)$. Due to Proposition 11 the set $\hat{\Gamma}^u_2$ is a disjoint union of homotopically non-trivial circles on the torus $\hat{V}$ and $\hat{\gamma}^u_2$ is one of them. From condition of the orientability of heteroclinic intersections, it follows that the circle $\hat{\gamma}^s_1$ intersects each of the circles of the set $\hat{\Gamma}^u_2$ and the index of the intersection at each point of such the intersection is the same.

As the curve $\hat{\ell}$ belongs $\hat{V}$ then there is annuli $\hat{N}^2_2$ (possible $\hat{N}^2_2 = \hat{N}^1_2$) which is a connected components of the set $\hat{N}_2$ and such that $(\hat{\ell} \cap \hat{\mathcal{N}}_2) \subset (\hat{N}^1_2 \cup \hat{N}^2_2)$. From the compatible system of neighborhoods follows, that each connected component of the intersection $\hat{\ell} \cap (\hat{N}^1_2 \cup \hat{N}^2_2)$ is a leaf of the foliation $\hat{\mathcal{F}}^u_2 = p_1(\mathcal{F}^u_2)$ (see Fig. 8).

Let $\hat{x}_1 = \hat{\ell} \cap \partial \hat{N}^1_2$ and $\hat{x}_2 = \hat{\ell} \cap \partial \hat{N}^2_2$. Due to the orientability of the heteroclinic intersection the points $\hat{x}_1, \hat{x}_2$ belong to different connected components of the set $\partial(\hat{N}^1_2 \cup \hat{N}^2_2)$. Moreover, according to compatibility of
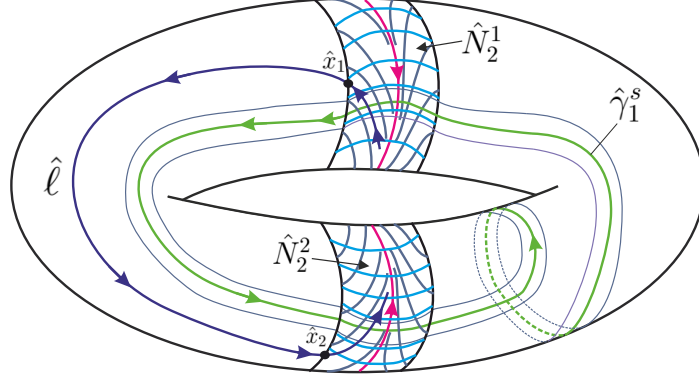
Fig. 8. Orbit space $\hat{V}$

system of linearizing neighborhoods, the arc $\hat{\ell}$ is a union of three arcs with the pairwise disjoint interiors

$$\hat{\ell} = \hat{F}^u_{1,\hat{\ell}} \cup \hat{\ell}_{\hat{x}_1,\hat{x}_2} \cup \hat{F}^u_{2,\hat{\ell}},$$

where $\hat{F}^u_{1,\hat{\ell}}, \hat{F}^u_{2,\hat{\ell}}$ are different leaves of foliation $\hat{\mathcal{F}}^u_2$.

It follows from the definition of the compatible system of linearizing neighborhoods that every connected component of the intersection $\hat{\gamma}^s_1 \cap (\hat{N}^1_2 \cup \hat{N}^2_2)$ is a leaf of the foliation $\hat{\mathcal{F}}^s_2 = p_1(\mathcal{F}^s_2)$. Choose one connected component $\hat{F}^s_{1,\hat{\ell}} (\hat{F}^s_{2,\hat{\ell}})$ in the intersection $\hat{\gamma}^s_1 \cap \hat{N}^1_2$ ($\hat{\gamma}^s_1 \cap \hat{N}^2_2$), possible $\hat{F}^s_{1,\hat{\ell}} = \hat{F}^s_{2,\hat{\ell}}$ if $\hat{N}^1_2 = \hat{N}^2_2$. Let $\hat{y}_1 = \hat{F}^s_{1,\hat{\ell}} \cap \hat{\Gamma}_2$ and $\hat{y}_2 = \hat{F}^s_{2,\hat{\ell}} \cap \hat{\Gamma}_2$. Also choose one pair of the points $\hat{z}_1, \hat{z}_2$ from each of the countable set $\hat{F}^s_{1,\hat{\ell}} \cap \hat{F}^u_{1,\hat{\ell}}$, $\hat{F}^s_{2,\hat{\ell}} \cap \hat{F}^u_{2,\hat{\ell}}$ (see Fig. 9).

According to the orientability of the heteroclinic intersection, the pairs of the vectors $(\vec{v}^u_{\hat{y}_1}, \vec{v}^s_{\hat{y}_1})$ and $(\vec{v}^u_{\hat{y}_2}, \vec{v}^s_{\hat{y}_2})$ define the same orientation of the torus $\hat{V}$. Also, the pairs of the vectors $(\vec{v}^u_{\hat{y}_1}, \vec{v}^s_{\hat{y}_1})$, $(\vec{v}^u_{\hat{z}_1}, \vec{v}^s_{\hat{z}_1})$ and $(\vec{v}^u_{\hat{y}_2}, \vec{v}^s_{\hat{y}_2})$, $(\vec{v}^u_{\hat{z}_2}, \vec{v}^s_{\hat{z}_2})$ also need to be consistently oriented. However, this contradicts with any orientation on the curve $\hat{\ell}$.
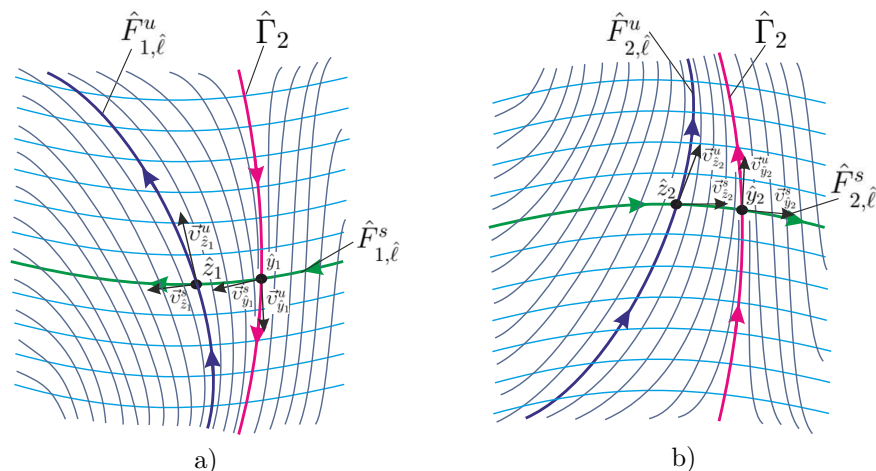
Fig. 9. Intersection of separatrices in the space orbit $\hat{V}$

REFERENCES

1. A.A. Andronov, L.S. Pontryagin. Rough systems. *Report, The Academy of Sciences USSR.* **14:5** (1937), 247–250.
2. A.N. Bezdenezhnyh. Dissertation: Topological classification of Morse-Smale diffeomorphisms with orientable heteroclinic set on two-dimensional manifolds. *The University of N.I. Lobachevskiy, Gor'kij.* (1985).
3. A.N. Bezdenezhnykh and V.Z. Grines. Diffeomorphisms with orientable heteroclinic sets on two-dimensional manifolds. *Methods of the quantitative theory of differential equations, Gorkii State University, Gorkii.* (1985), 139-152.
4. A.N. Bezdenezhnyh, V.Z. Grines. Dynamic properties and topological classification of gradient-like diffeomorphisms on two-dimensional manifolds I (II). *KTUD Methods, editor E.A. Leontovich-Andronova. Gor'kij.* (1985), 22–38;(1987), 24–32. *Selected translations. Selecta Math. Soviet.* **11** (1992), No. 1, 1–11.
5. Ch. Bonatti, V.Z. Grines, O.V. Pochinka. Topological classification of Morse-Smale diffeomorphisms on 3-manifolds. *Duke Mathematical Journal.* (2019).

6. Ch. Bonatti, V. Grines, O. Pochinka, S. Van Strien. A complete topological classification of Morse-Smale diffeomorphisms on surfaces: a kind of kneading theory in dimension two. *Cornell University Library.* (2017).

7. Ch. Bonatti, R. Langevin. Diffeomorphismes de Smale des surfaces. *Aste'risque socie'te' mathe'matique de France.* (1998), No. 250.

8. V.Z. Grines, E.Ya. Gurevich, E.V. Zhuzhoma, O.V. Pochinka. Classification of Morse-Smale systems and topological structure of bearing manifolds. *Russian Mathematical Surveys.* (2019), No. 74 issue 1(445), 41-116.

9. V.Z. Grines, E.Ya. Gurevich, O.V. Pochinka, V.S. Medvedev. On embedding Morse-Smale diffeomorphisms on 3-manifolds in a topological flow. *Sbornik: Mathematics.* **203:12**, (2012), 81-104.

10. V.Z. Grines, T.M. Mitryakova, O.V.Pochinka. New topological invariants of non-gradient-like diffeomorphisms on orientable surfaces. *Journal of the middle Volga mathematical society. Saransk.* **7:1**, (2005), 123-129.

11. V. Grines V., T. Medvedev, O. Pochinka, Dynamical Systems on 2- and 3-Manifolds. *Switzerland, Springer International Publishing.* (2016).

12. E.A. Leontovich, A.G. Maier. On trajectories determining the qualitative structure of partition of a sphere into trajectories.*Report, The Academy of Sciences USSR.* **14:5**, (1937), 251-257.

13. E.A. Leontovich, A.G. Maier. About the scheme determining the topological structure of partition into trajectories.*Report, The Academy of Sciences USSR.* **103:4**, (1955), 557-560.

14. A.G. Maier. Rough transformation of a circle into a circle.*Scientific notes of the Gorky state University.* **12**, (1939), 215-229.

15. A.G. Maier. On trajectories on orientable surfaces. *Sbornik: Mathematics.* **12(54):1** (1943), 7184 .

16. M.M. Peixoto. Structural stability on two-dimensional manifolds. *Topology* **1**, (1962), 101-120; *a further remark. Topology.* **2**, (1963), 179-180.

17. S. Yu. Pilyugin. Phase diagrams defining Morse-Smale systems without periodic trajectories on spheres. *Differential equations.* **14:2**, (1978), 245-254.

18. A.O. Prishlyak. A complete topological invariant of Morse-Smale flows and tube decompositions of three-dimensional manifolds. *Fundamental and applied mathematics.*, **11:4**, (2005), 185-196.

19. S. Smale.Diferentiable dynamical systems. *Bull. Amer. Math. Soc.* **73(6)**, (1967), 747–817.

20. Ya.L. Umanskij. Necessary and sufficient conditions for the topological equivalence of three-dimensional dynamical Morse-Smale systems with a finite number of special trajectories. *Sbornik: Mathematics.* **181:2**, (1990), 212–239.

Authors' addresses:
   A. Morozov
   National Research University Higher School of Economics, Nizhnij Novgorod, Russia.
   ORCID: http://orcid.org/0000-0003-3125-1825
   E-mail: andreifrostnn@gmail.com

   O. Pochinka
   National Research University Higher School of Economics, Nizhnij Novgorod, Russia.
   ORCID: http://orcid.org/0000-0002-6587-5305
   E-mail: olga-pochinka@yandex.ru