

Local L^p -Brunn–Minkowski inequalities for $p < 1$ Alexander V. Kolesnikov¹ and Emanuel Milman²**Abstract**

The L^p -Brunn–Minkowski theory for $p \geq 1$, proposed by Firey and developed by Lutwak in the 90’s, replaces the Minkowski addition of convex sets by its L^p counterpart, in which the support functions are added in L^p -norm. Recently, Böröczky, Lutwak, Yang and Zhang have proposed to extend this theory further to encompass the range $p \in [0, 1)$. In particular, they conjectured an L^p -Brunn–Minkowski inequality for origin-symmetric convex bodies in that range, which constitutes a strengthening of the classical Brunn–Minkowski inequality. Our main result confirms this conjecture locally for all (smooth) origin-symmetric convex bodies in \mathbb{R}^n and $p \in [1 - \frac{c}{n^{3/2}}, 1)$. In addition, we confirm the local log-Brunn–Minkowski conjecture (the case $p = 0$) for small-enough C^2 -perturbations of the unit-ball of ℓ_q^n for $q \geq 2$, when the dimension n is sufficiently large, as well as for the cube, which we show is the conjectural extremal case. For unit-balls of ℓ_q^n with $q \in [1, 2)$, we confirm an analogous result for $p = c \in (0, 1)$, a universal constant. It turns out that the local version of these conjectures is equivalent to a minimization problem for a spectral-gap parameter associated with a certain differential operator, introduced by Hilbert (under different normalization) in his proof of the Brunn–Minkowski inequality. As applications, we obtain local uniqueness results in the even L^p -Minkowski problem, as well as improved stability estimates in the Brunn–Minkowski and anisotropic isoperimetric inequalities.

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Contents

1	Introduction	3
1.1	Previously Known Partial Results	5
1.2	Main Results	6
1.3	Spectral Interpretation via the Hilbert–Brunn–Minkowski operator	9
1.4	Method of Proof	10
1.5	Applications	11
2	Notation	13
3	Global vs. Local Formulations of the L^p-Brunn–Minkowski Conjecture	15
3.1	Standard Equivalent Global Formulations	15
3.2	Global vs. Local L^p -Brunn–Minkowski	17
4	Local L^p-Brunn–Minkowski Conjecture – Infinitesimal Formulation	21
4.1	Mixed Surface Area and Volume of C^2 functions	21
4.2	Properties of Mixed Surface Area and Volume	23
4.3	Second L^p -Minkowski Inequality	24
4.4	Comparison with classical $p = 1$ case	26
4.5	Infinitesimal Formulation On S^{n-1}	26
4.6	Infinitesimal Formulation On ∂K	27
5	Relation to Hilbert–Brunn–Minkowski Operator and Linear Equivariance	29
5.1	Hilbert–Brunn–Minkowski operator	30
5.2	Linear equivariance of the Hilbert–Brunn–Minkowski operator . . .	36
5.3	Spectral Minimization Problem and Potential Extremizers	39
6	Obtaining Estimates via the Reilly Formula	42
6.1	A sufficient condition for confirming the local p -BM inequality . .	44
6.2	General Estimate on $D(K)$	46
6.3	Examples	48
7	The second Steklov operator and $B_H(B_2^n)$	50
7.1	Second Steklov operator	50
7.2	Computing $B_H(B_2^n)$	52

8	Unconditional Convex Bodies and the Cube	52
8.1	Unconditional Convex Bodies	53
8.2	The Cube	54
9	Local log-Brunn–Minkowski via the Reilly Formula	56
9.1	Sufficient condition for verifying local log-Brunn–Minkowski	56
9.2	An alternative derivation via estimating $B_H(K)$	59
10	Continuity of B_H, B, D with application to B_q^n	61
10.1	Continuity of B_H, B, D in C -topology	61
10.2	The Cube	62
10.3	Unit-balls of ℓ_q^n	64
11	Local Uniqueness for Even L^p-Minkowski Problem	65
12	Stability Estimates for Brunn–Minkowski and Isoperimetric Inequalities	69
12.1	New stability estimates for origin-symmetric convex bodies with respect to variance	70
12.2	Improved stability estimates for all convex bodies with respect to asymmetry	75

1 Introduction

The celebrated Brunn–Minkowski inequality [60, 23] states that if K_0, K_1 are two convex sets in \mathbb{R}^n then:

$$V((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n}} \geq (1 - \lambda)V(K_0)^{\frac{1}{n}} + \lambda V(K_1)^{\frac{1}{n}} \quad \forall \lambda \in [0, 1]. \quad (1.1)$$

Here V denotes Lebesgue measure (volume) and

$$(1 - \lambda)K_0 + \lambda K_1 = \{(1 - \lambda)a + \lambda b ; a \in K_0, b \in K_1\}$$

denotes Minkowski addition (or interpolation). This inequality and its generalizations lie at the heart of the Brunn–Minkowski theory of convex sets, which is by now a classical object of study, having applications in a multitude of other fields [60, 24, 9, 23].

In the 60's, W. J. Firey [20] proposed an L^p extension ($p \in [1, \infty]$) of the Minkowski addition operation, the so-called L^p -Firey–Minkowski addition, or simply L^p -sum. To describe it, let h_K denote the support function of a convex body K (see Section 2 for definitions). If K_0, K_1 are convex compact sets containing the origin in their interior (“convex bodies”), their L^p interpolation, denoted $(1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1$, is defined when $p \geq 1$ and $\lambda \in [0, 1]$ as the convex body with support function:

$$h_{(1-\lambda) \cdot K_0 +_p \lambda \cdot K_1} = (1 - \lambda) \cdot h_{K_0} +_p \lambda \cdot h_{K_1} := ((1 - \lambda)h_{K_0}^p + \lambda h_{K_1}^p)^{\frac{1}{p}}. \quad (1.2)$$

The case $p = 1$ corresponds to the usual Minkowski sum. Note that $\lambda \cdot K = \lambda^{\frac{1}{p}} K$, and the implicit dependence of \cdot on p is suppressed. Firey established the following L^p -Brunn–Minkowski inequality when $p \geq 1$:

$$V((1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1)^{\frac{p}{n}} \geq (1 - \lambda)V(K_0)^{\frac{p}{n}} + \lambda V(K_1)^{\frac{p}{n}} \quad \forall \lambda \in [0, 1], \quad (1.3)$$

which turns out to be a consequence of the classical $p = 1$ case (1.1) by a simple application of Jensen’s inequality. The resulting L^p -Brunn–Minkowski theory was extensively developed by E. Lutwak [49, 50], leading to a rich theory with many parallels to the classical one (see also [28] and the references therein for further extensions to more general Orlicz norms).

Fairly recently, K. Böröczky, Lutwak, D. Yang and G. Zhang [7, 6] have proposed to extend the L^p -Brunn–Minkowski theory to the range $p \in [0, 1)$. To describe their extension, recall that the Aleksandrov body (or Wulff shape) associated to a positive (Wulff) function $w \in C(S^{n-1})$, is defined as the following convex body:

$$A[w] := \bigcap_{\theta \in S^{n-1}} \{x \in \mathbb{R}^n ; \langle x, \theta \rangle \leq w(\theta)\}.$$

In other words, $A[w]$ is the largest convex body K so that $h_K \leq w$. While the right-hand-side of (1.2) is no longer a support function in general when $p \in [0, 1)$, Böröczky–Lutwak–Yang–Zhang defined:

$$(1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1 := A[(1 - \lambda)h_{K_0}^p + \lambda h_{K_1}^p]^{\frac{1}{p}},$$

interpreting the case $p = 0$ in the limiting sense as:

$$(1 - \lambda) \cdot K_0 +_0 \lambda \cdot K_1 := A[h_{K_0}^{1-\lambda} h_{K_1}^\lambda].$$

This of course coincides with Firey’s definition when $p \geq 1$. With this notation, they proposed the following conjectural extension of (1.3) when $p \in [0, 1)$ and $n \geq 2$ (the case $n = 1$ is trivial):

Conjecture (L^p -Brunn–Minkowski Conjecture given $p \in [0, 1)$). *The L^p -Brunn–Minkowski inequality holds for all **origin-symmetric** convex bodies K_0, K_1 in \mathbb{R}^n :*

$$V((1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1) \geq \left((1 - \lambda)V(K_0)^{\frac{n}{p}} + \lambda V(K_1)^{\frac{n}{p}} \right)^{\frac{n}{p}} \quad \forall \lambda \in [0, 1]. \quad (1.4)$$

*In the case $p = 0$, called the *log-Brunn–Minkowski Conjecture*, (1.4) is interpreted in the limiting sense as:*

$$V((1 - \lambda) \cdot K_0 +_0 \lambda \cdot K_1) \geq V(K_0)^{1-\lambda} V(K_1)^\lambda \quad \forall \lambda \in [0, 1]. \quad (1.5)$$

The motivation for considering this question in [6] was an equivalence established by Böröczky–Lutwak–Yang–Zhang between the conjectured (1.5) for all origin-symmetric convex bodies, and the *uniqueness* question in the even log-Minkowski problem, the $p = 0$ analogue of the classical Minkowski problem (for the *existence* question, a novel necessary and sufficient condition was obtained in [7], see Section 11 or Subsection 1.5 below). The restriction to origin-symmetric convex bodies is an interesting feature of the above conjectures, which we will elucidate in this work (see Subsection 5.1); for now, let us just remark that the conjectures are known to be false for general convex bodies, as seen by selecting K_1 to be an infinitesimal translated version of an origin-symmetric convex body K_0 . As before, it is easy to check using Jensen’s inequality that the above conjectures become stronger as p decreases from 1 to 0, with the strongest case (implying the rest) being the $p = 0$ one. For general origin-symmetric convex bodies, there is no point to consider the L^p -Brunn–Minkowski conjecture for $p < 0$, since (1.4) is then easily seen to be false when $K_0 = \times_{i=1}^n [-a_i, a_i]$ and $K_1 = \times_{i=1}^n [-b_i, b_i]$ are two non-homothetic origin-symmetric cubes. However, other particular bodies K_0, K_1 may satisfy (1.4) with $p < 0$, explaining the convention of putting the exponent $\frac{n}{p}$ on the right-hand-side (when $p > 0$, this makes no difference).

Being a conjectural strengthening of the ubiquitous Brunn–Minkowski inequality, establishing the validity of the L^p -Brunn–Minkowski conjecture, for any p strictly smaller than 1, would be of fundamental importance to the Brunn–Minkowski theory and its numerous applications.

1.1 Previously Known Partial Results

The following partial results in regards to the log-Brunn–Minkowski conjecture have been previously obtained:

- Böröczky–Lutwak–Yang–Zhang [6] confirmed the conjecture in the plane \mathbb{R}^2 . See also Ma [52] for an alternative proof, and Xi–Leng [67] for an extension of this result which does not require origin-symmetry.
- C. Saroglou [58] verified the conjecture when $K_0, K_1 \subset \mathbb{R}^n$ are both simultaneously unconditional with respect to the same orthogonal basis, meaning that they are invariant under reflections with respect to the principle coordinate hyperplanes $\{x_i = 0\}$. In that case, a stronger version of the conjecture follows from a multiplicative version of the Prékopa–Leindler inequality on the positive orthant (e.g. [16, Proposition 10]).
- L. Rotem [57] observed that the conjecture for complex convex bodies $K_0, K_1 \subset \mathbb{C}^n$ follows from a more general theorem of D. Cordero–Erausquin [15].
- A. Colesanti, G. Livshyts and A. Marsiglietti [13] verified the conjecture locally for small-enough C^2 -perturbations of the Euclidean ball B_2^n (see below for more details).
- The log-Brunn–Minkowski conjecture has been shown by Saroglou [58, 59] to be intimately related to the generalized B -conjecture. Further connections to a conjecture of R. Gardner and A. Zvavitch [25] on a dimensional Brunn–Minkowski inequality for even log-concave measures were observed by Livshyts, Marsiglietti, P. Nayar and Zvavitch [47]. A surprising relation to a conjecture of S. Dar [17] was observed by Xi and Leng in [67].

1.2 Main Results

Our first main result in this work confirms the following *local* version of the L^p -Brunn–Minkowski conjecture for a certain range of p 's strictly smaller than 1. Let \mathcal{K}_+^2 denote the class of convex bodies with C^2 smooth boundary and strictly positive curvature, and let $\mathcal{K}_{+,e}^2$ denote its origin-symmetric members. We reserve the symbols c, C, C' etc.. to denote positive universal numeric constants, independent of any other parameter.

Theorem 1.1 (Local L^p -Brunn–Minkowski). *Let $n \geq 2$ and $p \in [1 - \frac{c}{n^{3/2}}, 1)$. Then for any $K_0, K_1 \in \mathcal{K}_{+,e}^2$ so that:*

$$(1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1 \in \mathcal{K}_{+,e}^2 \quad \forall \lambda \in [0, 1] , \quad (1.6)$$

the L^p -Brunn–Minkowski conjecture (1.4) for K_0, K_1 holds true:

$$V((1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1) \geq \left((1 - \lambda)V(K_0)^{\frac{n}{p}} + \lambda V(K_1)^{\frac{n}{p}} \right)^{\frac{p}{n}} \quad \forall \lambda \in [0, 1].$$

The condition (1.6) is deceptively appealing: while for $p \geq 1$ it is always automatically satisfied, this is not the case in general for $p \in [0, 1)$ (see Corollary 3.7). Consequently, we do not know how to conclude the validity of the (global) L^p -Brunn–Minkowski conjecture for all origin-symmetric K_0, K_1 and p in the above range. On the other hand, for every $K \in \mathcal{K}_{+,e}^2$, there exists a C^2 -neighborhood N_K so that for all $K_0, K_1 \in N_K$, (1.6) is satisfied, so we can confirm the L^p -Brunn–Minkowski conjecture locally. See also Section 3 for an equivalent local formulation in terms of the second variation $\frac{d^2}{(d\epsilon)^2}V(A[h_K +_p \epsilon \cdot f])^{\frac{p}{n}} \leq 0$ for appropriate test functions f .

Let us mention here that we conjecture the logical equivalence between the local and global formulations of the L^p -Brunn–Minkowski conjecture - see Conjecture 3.8 and the preceding discussion. This is an extremely interesting and tantalizing question, which leads to the study of the second variation of volume of an Aleksandrov body $\frac{d^2}{(d\epsilon)^2}V(A[h_K + \epsilon f])$, when no smoothness is assumed on K or f . While the first variation has been studied by Aleksandrov himself and is well understood (see [60, Lemma 6.5.3]), to the best of our knowledge, nothing concrete is known regarding the second variation when K is not assumed smooth, even when f is the difference of two support functions.

Our second type of results pertain to the verification of the local log-Brunn–Minkowski conjecture (case $p = 0$) for various classes of origin-symmetric convex bodies. For instance, we obtain the following result regarding B_q^n , the unit-ball of ℓ_q^n . Note that $B_q^n \notin \mathcal{K}_{+,e}^2$ unless $q = 2$. To emphasize the invariance of the conjecture under non-degenerate linear transformations GL_n , we state it explicitly.

Theorem 1.2 (Local log-Brunn–Minkowski for B_q^n). *For all $q \in [2, \infty)$, there exists $n_q \geq 2$ so that for all $n \geq n_q$, there exists a neighborhood $N_{B_q^n}^C$ of B_q^n in the Hausdorff topology, so that for any $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$, there exists a C^2 -neighborhood N_K of K in $\mathcal{K}_{+,e}^2$ so that for all $T \in GL_n$ and $K_1, K_0 \in T(N_K)$, the log-Brunn–Minkowski conjecture (1.5) for K_0, K_1 holds true:*

$$V((1 - \lambda) \cdot K_0 +_0 \lambda \cdot K_1) \geq V(K_0)^{1-\lambda} V(K_1)^\lambda \quad \forall \lambda \in [0, 1].$$

For more general statements, see Theorems 5.13, 9.1, 9.5, 9.6 and 10.4. The two extremal cases above are of particular interest. For the Euclidean ball B_2^n , Theorem 1.2 in fact holds with $n_2 = 2$, i.e. in all dimensions $n \geq 2$. In this particular case, Theorem 1.2 strengthens [13, Theorem 1.4] of Colesanti–Livshyts–Marsiglietti, where it was assumed that K_0, K_1, B_2^n are “co-linear”, i.e. that $K_0 = (1 - \epsilon) \cdot K_1 +_0 \epsilon \cdot B_2^n$ for some $\epsilon \in [0, 1]$; in addition, the linear invariance under

GL_n was not noted there (strictly speaking, the result in [13] does not yield a C^2 -neighborhood of B_2^n for which the latter statement holds, since the allowed C^2 -distance of K_1 from B_2^n depended on K_1 , but the latter caveat was very recently remedied by Colesanti and Livshyts in [12], concurrently to our work).

On the other extreme lies the unit-cube B_∞^n . In an appropriate sense, described in the next subsection, the cube turns out to be the conjectural extremal case for the log-Brunn–Minkowski inequality. The local extremality can be seen as follows (see Theorems 8.5 and 10.2 for a more streamlined formulation):

Theorem 1.3 (Extremal local log-Brunn–Minkowski for B_∞^n). *For any $\{K^i\}_{i \geq 1} \subset \mathcal{K}_{+,e}^2$ which approximate B_∞^n in the Hausdorff metric, there exist $\{p_i\}_{i \geq 1}$ converging to 0, with the following property: for all $i \geq 1$, there exists a C^2 -neighborhood N_{K^i} of K^i in $\mathcal{K}_{+,e}^2$, so that for all $T \in GL_n$ and $K_0^i, K_1^i \in T(N_{K^i})$, we have:*

$$V((1 - \lambda) \cdot K_0^i +_{p_i} \lambda \cdot K_1^i) \geq \left((1 - \lambda)V(K_0^i)^{\frac{p_i}{n}} + \lambda V(K_1^i)^{\frac{p_i}{n}} \right)^{\frac{n}{p_i}} \quad \forall \lambda \in [0, 1].$$

Moreover, if $K_i = B_{q_i}^n$ with $q_i \rightarrow \infty$, it is impossible to find $\{p_i\}$ with the above property, which instead of converging to zero, satisfy $\liminf p_i < 0$.

We do not know how to handle the range $q \in [1, 2)$ in Theorem 1.2. In particular, it would be very interesting to establish analogues of Theorem 1.2 or Theorem 1.3 (without the “moreover” part) for B_1^n . We can however obtain for any unconditional convex body, and in particular for B_1^n , a local *strengthening* of Saraoglu’s confirmation of the log-Brunn–Minkowski conjecture for unconditional convex bodies – see Theorem 8.3 and Corollary 10.3. We can also show (see Theorem 10.5):

Theorem 1.4 (Local L^c -Brunn–Minkowski for B_q^n). *There exists a universal constant $c \in (0, 1)$, so that for all $q \in [1, 2)$, there exists a neighborhood $N_{B_q^n}^C$ of B_q^n in the Hausdorff topology, so that for any $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$, there exists a C^2 -neighborhood N_K of K in $\mathcal{K}_{+,e}^2$ so that for all $T \in GL_n$ and $K_1, K_0 \in T(N_K)$, the p -Brunn–Minkowski conjecture (1.4) for K_0, K_1 holds true with $p = c$:*

$$V((1 - \lambda) \cdot K_0 +_c \lambda \cdot K_1) \geq \left((1 - \lambda)V(K_0)^{\frac{c}{n}} + \lambda V(K_1)^{\frac{c}{n}} \right)^{\frac{n}{c}} \quad \forall \lambda \in [0, 1].$$

An application of the above results to local uniqueness in the even L^p -Minkowski problem is presented in Section 11.

1.3 Spectral Interpretation via the Hilbert–Brunn–Minkowski operator

An additional contribution of this work lies in revealing the connection between the local L^p -Brunn–Minkowski conjecture and a spectral-gap property of a certain second-order elliptic operator L_K on S^{n-1} associated to any $K \in \mathcal{K}_+^2$. Modulo the different normalization we employ in our investigation, this operator was in fact considered by Hilbert in his proof of the classical Brunn–Minkowski inequality, and subsequently generalized by Aleksandrov in his second proof of the Aleksandrov–Fenchel inequality [5, pp. 108–110]. Consequently, we call L_K the Hilbert–Brunn–Minkowski operator. Our normalization has several advantages over the one employed by Hilbert (see Remark 5.6); for instance, it ensures an important equivariance property of the correspondence $K \mapsto L_K$ under linear transformations (see Theorem 5.8), which to the best of our knowledge was previously unnoted.

Let us denote by $\lambda_1(-L_K)$ the spectral-gap of $-L_K$ beyond the trivial 0 eigenvalue, and by $\lambda_{1,e}(-L_K)$ the *even* spectral-gap beyond 0 when restricting to *even* functions. By the previous comments, both are linear invariants of K . It was shown by Hilbert that (with our normalization) the Brunn–Minkowski inequality (1.1) is equivalent to the uniform spectral-gap estimate $\lambda_1(-L_K) \geq 1$ for all $K \in \mathcal{K}_+^2$; moreover, Hilbert showed that $\lambda_1(-L_K) = 1$, with the corresponding eigenspace being precisely the one spanned by (normalized) linear functions, generated by translations of K . A natural question is then:

is there a *uniform* spectral-gap of $-L_K$ for all $K \in \mathcal{K}_+^2$ *beyond* 1?

It turns out that the local L^p -Brunn–Minkowski conjecture for $K \in \mathcal{K}_{+,e}^2$ and $p \in [0, 1)$ is precisely equivalent to the conjecture that:

$$\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1} (> 1). \quad (1.7)$$

In other words, it is a “next eigenvalue” conjecture (similar in spirit to the B-conjecture for the Gaussian measure, established in [16]). This elucidates the requirement that K be origin-symmetric, since then even functions are automatically orthogonal to the odd (normalized) linear functions; moreover, this interpretation suggests a plausible extension of the conjecture to non-symmetric convex bodies (see Remark 5.5). Theorem 1.1 asserts that (1.7) holds with $p = 1 - \frac{c}{n^{3/2}}$, answering in the positive the question of whether there is a *uniform even* spectral-gap for $-L_K$ beyond 1.

It is easy to calculate that $\lambda_{1,e}(-L_{B_2^n}) = \frac{2n}{n-1}$ (corresponding to $p = -n$ in (1.7)), yielding Theorem 1.2 for $q = 2$ and $n_2 = 2$. A much greater challenge is to establish that:

$$\liminf_{\mathcal{K}_{+,e}^2 \ni K^i \rightarrow B_\infty^n \text{ in Hausdorff metric}} \lambda_{1,e}(-L_{K^i}) = \frac{n}{n-1},$$

corresponding to $p = 0$ in (1.7), and yielding Theorem 1.3. In other words, the local log-Brunn–Minkowski conjecture is equivalent to the conjecture that the cube B_∞^n is a minimizer of the linearly invariant spectral parameter $\lambda_{1,e}(-L_K)$ over all origin-symmetric convex bodies $K \in \mathcal{K}_{+,e}^2$. In our opinion, this provides the most convincing and natural reason for believing the validity of the local log-Brunn–Minkowski conjecture, and emphasizes its importance. We also conjecture that the Euclidean ball B_2^n is in fact a maximizer of $\lambda_{1,e}(-L_K)$.

1.4 Method of Proof

Our main tool for obtaining estimates on $\lambda_{1,e}(-L_K)$ is the Reilly formula, which is a well-known formula in Riemannian geometry obtained by integrating the Bochner–Lichnerowicz–Weitzenböck identity. In our previous work [39], we obtained a convenient weighted version of the Reilly formula, incorporating a general density. By applying the generalized Reilly formula with an appropriate log-convex (not log-concave!) density, we obtain a sufficient condition for establishing the local log-Brunn–Minkowski conjecture – see Theorem 9.1. Curiously, this condition resembles a dual log-convex form of the classical Brascamp–Lieb inequality [8]. Using the known estimates on the Poincaré constant of B_q^n , we are able to verify this condition when $q \in (2, \infty)$ and n is large enough, yielding Theorem 1.2.

To obtain our other results, we derive a different sufficient condition for establishing the local L^p -Brunn–Minkowski conjecture, in terms of a boundary Poincaré-type inequality for harmonic functions on K – see Theorem 6.1. In Section 7, we interpret this sufficient condition in terms of a first order pseudo-differential operator on ∂K , which we call the *second Steklov operator*, due to its analogy with the classical Steklov (or Dirichlet-to-Neumann) operator. We are able to precisely calculate the spectrum of this operator for B_2^n , and to calculate the sharp constant in the boundary Poincaré-type inequality for B_∞^n . After establishing the continuity of this boundary Poincaré-type constant $B(K)$ with respect to the Hausdorff metric, we are able to deduce Theorem 1.3.

To handle arbitrary origin-symmetric convex K , we obtain a general (rough) estimate on $B(K)$ in terms of the in and out radii of K , as well as the (usual) Poincaré

constant of K . Contrary to the linear invariance of the L^p -Brunn–Minkowski conjecture, our sufficient condition is no longer linearly invariant, and so we are required to apply it to a suitable linear image of K . By using the isotropic position, and the recent estimates of Y.-T. Lee and S. Vempala [45] on the Poincaré constant of K vis-à-vis the Kannan–Lovász–Simonovits conjecture [33], we are able to deduce Theorem 1.1.

1.5 Applications

We conclude this work by describing a couple of additional applications of our methods. It is by now a standard argument (see [49, 6]) to translate our local L^p -Brunn–Minkowski and log-Brunn–Minkowski inequalities into local uniqueness statements for the even L^p -Minkowski and log-Minkowski problems, respectively. The classical Minkowski problem (see [60, 51] and the references therein) asks for necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} , to guarantee the existence and uniqueness (up to translation) of a convex body $K \in \mathcal{K}$ so that its surface-area measure dS_K coincides with μ ; it was solved by Minkowski and Aleksandrov and has led to a rich theory pertaining to the regularity of the associated Monge–Ampère equation. In [49], Lutwak proposed to study the analogous L^p -Minkowski problem, where the role of the surface-area measure dS_K is replaced by the L^p -surface-area measure:

$$dS_{K,p} := h_K^{1-p} dS_K.$$

The case of *even* measures when $p > 1$ is well understood [49, 51], but the case $p < 1$ poses a much greater challenge for both existence and uniqueness questions, with the log-Minkowski problem (corresponding to the case $p = 0$) drawing the most attention (see Section 11 for more details). For the latter problem and for *even* measures μ , a novel necessary and sufficient *subspace concentration condition* ensuring the *existence* question was obtained by Böröczky–Lutwak–Yang–Zhang in [7], and the *uniqueness* question was settled in [6] in dimension $n = 2$; it remains open in full generality in dimension $n \geq 3$. In Theorem 11.2, we translate our results into new local uniqueness statements for the *even* L^p -Minkowski problem for appropriate values of $p < 1$; in particular, we establish local uniqueness for $p \in [1 - \frac{c}{n^{3/2}}, 1)$ in a C^2 -neighborhood of any $K \in \mathcal{K}_{+,e}^2$.

Our second application pertains to stability estimates in the Brunn–Minkowski and anisotropic isoperimetric inequalities. Recall that the sharp anisotropic isoperimetric inequality is the statement that:

$$P_L(K) \geq nV(L)^{\frac{1}{n}}V(K)^{\frac{n-1}{n}} \tag{1.8}$$

with equality when $K = L$, where $P_L(K) := \int_{S^{n-1}} h_L dS_K$ denotes the anisotropic perimeter of K with respect to the convex body L (note that when $L = B_2^n$ this boils down to the usual surface area of K). It is well known that the anisotropic isoperimetric inequality may be obtained by differentiating the Brunn–Minkowski inequality (and in fact, the two families of inequalities for all convex bodies K, L are equivalent). Since the local L^p -Brunn–Minkowski inequality for $p < 1$ is a strengthening of the classical one (in the class of origin-symmetric convex bodies), it is natural to expect that some stability results for isoperimetric inequalities could be derived from it, where an additional deficit term is added to the right-hand-side of (1.8), depending on some parameter measuring the proximity of K to L .

In Section 12, we obtain an interesting interpretation of the Hilbert–Brunn–Minkowski operator L_K as the operator controlling the deficit in Minkowski’s second inequality. As a consequence, we deduce new stability estimates for the (global) Brunn–Minkowski and anisotropic isoperimetric inequalities for origin-symmetric convex bodies, which depend on the variance of h_L/h_K with respect to the cone-measure dV_K . In addition, we derive a strengthening of the best known stability estimates in the Brunn–Minkowski and anisotropic isoperimetric inequalities for the class of all (not necessarily origin-symmetric) convex bodies with respect to the natural asymmetry parameter $\inf_{x_0 \in \mathbb{R}^n} V(K \triangle (x_0 + rL))$ with $V(rL) = V(K)$, which were previously due to Figalli, Maggi and Pratelli [19, 18] and Segal [61].

The rest of this work is organized as follows. In Section 2, we introduce some convenient notation. In Section 3, we establish a couple of standard equivalent versions of the global L^p -Brunn–Minkowski conjecture, introduce the local L^p -Brunn–Minkowski conjecture, and discuss the relation between the global and local versions. In Section 4 we derive various equivalent infinitesimal formulations of the local conjecture: in terms of a second L^p -Minkowski inequality involving mixed-volumes, in terms of a Poincaré-type inequality on S^{n-1} and finally in terms of an equivalent version on ∂K . In Section 5 we obtain an equivalent formulation involving the spectral-gap of the Hilbert–Brunn–Minkowski operator L_K , and establish its linear equivariance. In Section 6 we obtain a sufficient condition in terms of a boundary Poincaré-type inequality, establishing Theorem 1.1. In Section 7 we introduce the second Steklov operator, use it to rewrite our sufficient condition, and calculate its spectrum for B_2^n . In Section 8 we obtain sharp bounds on our boundary Poincaré-type inequality for unconditional convex bodies and calculate it precisely for the cube. In Section 9 we obtain another sufficient condition for establishing the log-Brunn–Minkowski conjecture, in preparation for Theorem 1.2. In Section 10, we establish the continuity of our boundary Poincaré-type inequality,

and use this to deduce Theorems 1.2, 1.3 and 1.4. The application to local uniqueness in the even L^p -Minkowski problem is presented in Section 11. The application pertaining to new and improved stability estimates for Minkowski's second inequality, the anisotropic isoperimetric inequality, and the Brunn–Minkowski inequality is presented in Section 12.

2 Notation

Given $t \in \mathbb{R}$ and $p > 0$, we denote:

$$t^p := \operatorname{sgn}(t) |t|^p.$$

If $f, g : X \rightarrow \mathbb{R}$, we employ the following notation:

$$f +_p g := (f^p + g^p)^{\frac{1}{p}}.$$

More generally, if $t, \alpha, \beta \in \mathbb{R}$, we write:

$$t \cdot g := t^{\frac{1}{p}} g \quad \text{and} \quad \alpha \cdot f +_p \beta \cdot g := (\alpha f^p + \beta g^p)^{\frac{1}{p}}, \quad (2.1)$$

suppressing the dependence of \cdot on p . When $p < 0$, we will specify $\frac{1}{p} f^p : X \rightarrow \mathbb{R}$, even though f itself may not be defined. Given $\frac{1}{p} f^p, \frac{1}{p} g^p : X \rightarrow \mathbb{R}$, we use (2.1) to define $\alpha \cdot f +_p \beta \cdot g$ only for $\alpha, \beta \in \mathbb{R}$ so that $\alpha f^p + \beta g^p$ is positive. The limiting case when $p = 0$ only makes sense when $f, g : X \rightarrow (0, \infty)$, and unless otherwise stated, is interpreted throughout this work as:

$$\frac{1}{p} f^p := \log f, \quad \alpha \cdot f +_p \beta \cdot g := f^\alpha g^\beta \quad \text{for } p = 0.$$

For instance, note that $\lim_{p \rightarrow 0} \frac{1}{p}(f^p - 1) = \log f$, but instead of using $\frac{1}{p}(f^p - 1)$ we use $\frac{1}{p} f^p$ in all statements pertaining to concavity or regularity, as the missing $-\frac{1}{p}$ makes no difference. In a few places where ambiguity may arise, we will use the full $\frac{1}{p}(f^p - 1)$ expression.

A convex body in \mathbb{R}^n is a convex, compact set with non-empty interior. We denote by \mathcal{K} the collection of convex bodies in \mathbb{R}^n having the origin in their interior. The support function $h_K : \mathbb{R}^n \rightarrow (0, \infty)$ of $K \in \mathcal{K}$ is defined as:

$$h_K(y) := \max_{x \in K} \langle y, x \rangle, \quad y \in \mathbb{R}^n.$$

It is easy to see that h_K is continuous and convex. Clearly, it is 1-homogeneous, so we will mostly consider its restriction to the Euclidean unit-sphere S^{n-1} . Conversely, a convex 1-homogeneous function $h : \mathbb{R}^n \rightarrow (0, \infty)$ is necessarily a support function of some $K \in \mathcal{K}$ (which is obtained as the polar body to $\{h \leq 1\}$). Given $f \in C(S^{n-1})$, we will denote:

$$K +_p \epsilon \cdot f := A[h_K +_p \epsilon \cdot f].$$

We will only consider $\epsilon \in \mathbb{R}$ so that $h_K +_p \epsilon \cdot f > 0$, ensuring that $K +_p \epsilon \cdot f \in \mathcal{K}$.

As usual, we denote by $C^k(M)$, $k = 0, 1, \dots, m$, the space of k -times continuously differentiable functions on a C^m -smooth differentiable manifold M , equipped with its natural C^k -norm. When $k = 0$, we simply write $C(M)$. It is known [60, Theorem 1.8.11] that convergence of elements of \mathcal{K} in the Hausdorff metric is equivalent to convergence of the corresponding support functions in the $C(S^{n-1})$ norm; we will refer to this as C -convergence for brevity. We denote by $C_{>0}^k(S^{n-1})$ the convex cone of positive functions in $C^k(S^{n-1})$.

It will be convenient to introduce the following notation for a function $h \in C^2(S^{n-1})$. Given a local orthonormal frame e_1, \dots, e_{n-1} on S^{n-1} , we use h_i and $h_{i,j}$ to denote $(\nabla_{S^{n-1}})_{e_i} h$ and $(\nabla_{S^{n-1}})_{e_i, e_j}^2 h$, respectively, where $\nabla_{S^{n-1}}$ is the covariant derivative on the sphere S^{n-1} with its canonical Riemannian metric δ . Extending h to a 1-homogeneous function on \mathbb{R}^n and denoting by $\nabla_{\mathbb{R}^n}$ the covariant derivative on Euclidean space \mathbb{R}^n , we define the symmetric 2-tensor D^2h on S^{n-1} as the restriction of $\nabla_{\mathbb{R}^n}^2 h$ onto TS^{n-1} ; in our local orthonormal frame, this reads as:

$$(D^2h)_{i,j} = (\nabla_{\mathbb{R}^n})_{e_i, e_j}^2 h = h_{i,j} + h\delta_{i,j}, \quad i, j = 1, \dots, n-1.$$

The latter identity follows since the second fundamental form of $S^{n-1} \subset \mathbb{R}^n$ satisfies $\text{II}_{S^{n-1}} = \delta$, and since in general:

$$\nabla_{S^{n-1}}^2 f(\theta) = \nabla_{\mathbb{R}^n}^2 f(\theta)|_{TS^{n-1}} - f_\theta \text{II}_{S^{n-1}}(\theta),$$

and $f_\theta = \langle \nabla_{\mathbb{R}^n} f, \theta \rangle = f$ for any 1-homogeneous function f . Note that $h \in C_{>0}^2(S^{n-1})$ is a support-function of $K \in \mathcal{K}$ if and only if $D^2h_K \geq 0$ as a $(n-1)$ by $(n-1)$ positive semi-definite tensor.

We denote by \mathcal{K}_+^m the subset of \mathcal{K} of convex bodies with C^m boundary and strictly positive curvature. By [60, pp. 106-107, 111], for $m \geq 2$, $K \in \mathcal{K}_+^m$ if and only if $h_K \in C^m(S^{n-1})$ and $D^2h_K > 0$ (as a $n-1$ by $n-1$ positive-definite tensor). It is well-known that \mathcal{K}_+^2 is dense in \mathcal{K} in the C -topology [60, p. 160]. We also denote by $C_h^2(S^{n-1}) := \{h_K ; K \in \mathcal{K}_+^2\} = \{h \in C_{>0}^2(S^{n-1}) ; D^2h > 0\}$,

the convex cone of strictly convex C^2 support functions. It is immediate to check that $C_h^2(S^{n-1})$ is open in $C^2(S^{n-1})$ (see e.g. [60, pp. 38, 111],[11], or simply use that the condition $D^2h_K > 0$ is open in $C^2(S^{n-1})$). This induces the natural C^2 -topology on \mathcal{K}_+^2 , where $K_i \rightarrow K$ in C^2 if and only if $h_{K_i} \rightarrow h_K$ in $C_h^2(S^{n-1})$.

We will always use S_e to denote the origin-symmetric (or even) members of a set S , e.g. \mathcal{K}_e and $\mathcal{K}_{+,e}^2$ denote subset of origin-symmetric bodies in \mathcal{K} and \mathcal{K}_+^2 , respectively, and $C_{h,e}^2(S^{n-1})$ and $C_e^2(S^{n-1})$ denote the subset of even functions in $C^2(S^{n-1})$ and $C_h^2(S^{n-1})$, respectively.

Finally, we abbreviate throughout this work the phrase “ L^p -Brunn–Minkowski” by p -BM, and “log-Brunn–Minkowski” by log-BM.

3 Global vs. Local Formulations of the L^p -Brunn–Minkowski Conjecture

3.1 Standard Equivalent Global Formulations

Lemma 3.1. *The following are equivalent for a given dimension $n \geq 2$ and $p < 1$ (with the usual interpretation when $p = 0$):*

(1) *For all $K, L \in \mathcal{K}_e$, the global p -BM conjecture is valid:*

$$V((1 - \lambda) \cdot K +_p \lambda \cdot L) \geq \left((1 - \lambda)V(K)^{\frac{n}{p}} + \lambda V(L)^{\frac{n}{p}} \right)^{\frac{p}{n}} \quad \forall \lambda \in [0, 1].$$

(2) *For all $K, L \in \mathcal{K}_e$, the following function is concave:*

$$[0, 1] \ni s \mapsto \frac{1}{p} V((1 - s) \cdot K +_p s \cdot L)^{\frac{n}{p}}.$$

(3) *For all $K, L \in \mathcal{K}_e$, the following function is concave:*

$$\mathbb{R}_+ \ni t \mapsto \frac{1}{p} V(K +_p t \cdot L)^{\frac{n}{p}}.$$

The derivation is completely standard when $p \geq 1$, since in that case the support and Wulff functions of $K +_p t \cdot L$ coincide, and hence:

$$(1 - \lambda) \cdot (K +_p t_1 \cdot L) +_p \lambda \cdot (K +_p t_2 \cdot L) = K +_p ((1 - \lambda)t_1 + \lambda t_2) \cdot L.$$

When $p < 1$, there is in general only a one-sided containment, which fortunately goes in the right direction for us.

Proof. Clearly (2) implies (1) by checking concavity at the three points $s = 0, \lambda, 1$. To show that (1) implies (2), let $s_0, s_1 \in [0, 1]$, set $s_\lambda = (1 - \lambda)s_0 + \lambda s_1$. When $p < 1$, we only have:

$$(1 - \lambda) \cdot ((1 - s_0) \cdot K +_p s_0 \cdot L) +_p \lambda \cdot ((1 - s_1) \cdot K +_p s_1 \cdot L) \subset (1 - s_\lambda) \cdot K +_p s_\lambda \cdot L.$$

Indeed, the support function of $(1 - s_i) \cdot K +_p s_i \cdot L$ on the left is not larger than the Wulff function $((1 - s_i)h_K^p + s_i h_L^p)^{1/p}$, and hence we have the above inequality for the corresponding Wulff functions, and hence for the associated Aleksandrov bodies. Applying $\frac{1}{p}V(\cdot)^{p/n}$ to both sides and invoking (1), the desired concavity (2) is established. An identical argument also shows that (1) implies (3).

To show that (3) implies (1), apply (3) to the bodies $K = (1 - \lambda) \cdot \bar{K}$ and $L = \frac{\lambda}{\beta} \cdot \bar{L}$ at the points $t = 0, \beta, 1$:

$$V((1 - \lambda) \cdot \bar{K} +_p \lambda \cdot \bar{L}) \geq \left((1 - \beta)V((1 - \lambda) \cdot \bar{K})^{\frac{p}{n}} + \beta V((1 - \lambda) \cdot \bar{K} +_p \frac{\lambda}{\beta} \cdot \bar{L})^{\frac{p}{n}} \right)^{\frac{n}{p}}.$$

Then letting $\beta \rightarrow 0$, it is easy to see by monotonicity and homogeneity that $\beta V((1 - \lambda) \cdot \bar{K} +_p \frac{\lambda}{\beta} \cdot \bar{L})^{p/n} \rightarrow \lambda V(\bar{L})^{p/n}$, and we obtain in the limit:

$$V((1 - \lambda) \cdot \bar{K} +_p \lambda \cdot \bar{L}) \geq \left((1 - \lambda)V(\bar{K})^{\frac{p}{n}} + \lambda V(\bar{L})^{\frac{p}{n}} \right)^{\frac{n}{p}}.$$

□

Remark 3.2. It is worth mentioning an alternative argument for showing that (3) implies (1): the concavity implies that the derivative at $t = 0$ is larger than the secant slope as $t \rightarrow \infty$, i.e.:

$$\frac{1}{n} V(K)^{\frac{p}{n}-1} \frac{d}{dt} \Big|_{t=0+} V(K +_p t \cdot L) \geq \lim_{t \rightarrow \infty} \frac{\frac{1}{p}V(K +_p t \cdot L)^{\frac{p}{n}} - \frac{1}{p}V(K)^{\frac{p}{n}}}{t} = \frac{1}{p}V(L)^{\frac{p}{n}}.$$

Equivalently, using the L^p -mixed-volume $V_p(K, L)$ introduced by Lutwak [49]:

$$V_p(K, L) := \frac{p}{n} \lim_{t \rightarrow 0+} \frac{V(K +_p t \cdot L) - V(K)}{t}, \quad (3.1)$$

we have:

$$\frac{1}{p} (V_p(K, L) - V(K)) \geq V(K) \frac{1}{p} \left(\left(\frac{V(L)}{V(K)} \right)^{\frac{p}{n}} - 1 \right), \quad (3.2)$$

with the case $p = 0$ interpreted in the limiting sense. The latter is precisely the first p -Minkowski inequality, which has been shown by Böröczky–Lutwak–Yang–Zhang in [6] when $p \geq 0$ to be equivalent to the global p -BM inequality (1). Their proof extends to $p < 0$ with appropriate modifications.

3.2 Global vs. Local L^p -Brunn–Minkowski

We now introduce the following local version of the global p -BM conjecture.

Conjecture 3.3 (Local L^p -Brunn–Minkowski conjecture). *Let $n \geq 2$ and $p \in [0, 1)$. For all $K \in \mathcal{K}_{+,e}^2$:*

$$\forall \frac{1}{p}f^p \in C_e^2(S^{n-1}) \quad \frac{d^2}{(d\epsilon)^2} \Big|_{\epsilon=0} \frac{1}{p}V(K +_p \epsilon \cdot f)^{\frac{p}{n}} \leq 0. \quad (3.3)$$

Recall that when $p = 0$, this is interpreted as:

$$\forall \text{ positive } f \in C_e^2(S^{n-1}) \quad \frac{d^2}{(d\epsilon)^2} \Big|_{\epsilon=0} \log V(K +_0 \epsilon \cdot f) \leq 0. \quad (3.4)$$

Whenever referring to (3.3) with $p = 0$, we will always interpret this as (3.4). It follows from the results of this work (see Theorem 10.2) that for a fixed $p < 0$, (3.3) cannot hold for all $K \in \mathcal{K}_{+,e}^2$, but nevertheless for a particular K , (3.3) may hold with $p = p_K < 0$.

The following is standard:

Lemma 3.4 (Global p -BM implies Local p -BM for given $K \in \mathcal{K}_{+,e}^2$). *Fix $p \in \mathbb{R}$ and $K \in \mathcal{K}_{+,e}^2$. Assume that the global p -BM conjecture holds for K , namely:*

$$[0, 1] \ni \lambda \mapsto \frac{1}{p}V((1 - \lambda) \cdot K +_p \lambda \cdot L)^{\frac{p}{n}} \text{ is concave } \quad \forall L \in N_K,$$

for some C^2 -neighborhood N_K of K in $\mathcal{K}_{+,e}^2$. Then the local p -BM conjecture (3.3) holds for K .

Proof. We know that $h_K \in C_{h,e}^2(S^{n-1})$ since $K \in \mathcal{K}_{+,e}^2$. Let $\frac{1}{p}f^p \in C_e^2(S^{n-1})$, and consider the Wulff function $w_\epsilon := h_K +_p \epsilon \cdot f = (h_K^p + \epsilon f^p)^{1/p}$, which is in $C_e^2(S^{n-1})$ for small enough $|\epsilon|$ since h_K is strictly positive. As w_ϵ is a C^2 perturbation of h_K and since $C_{h,e}^2(S^{n-1})$ is open in $C_e^2(S^{n-1})$ and locally convex, it follows that for small enough $\epsilon_0 > 0$, w_ϵ is the support function of a convex body $K_\epsilon = K +_p \epsilon \cdot f \in N_K$, for all $\epsilon \in [0, \epsilon_0]$.

Since $w_{\lambda\epsilon_0} = (1 - \lambda) \cdot h_K +_p \lambda \cdot w_{\epsilon_0}$ for all $\lambda \in [0, 1]$, it follows that $K_{\lambda\epsilon_0} = (1 - \lambda) \cdot K +_p \lambda \cdot K_{\epsilon_0}$. Our global assumption (with $L = K_{\epsilon_0}$) therefore implies that $[0, 1] \ni \lambda \mapsto \frac{1}{p}V(K_{\lambda\epsilon_0})^{p/n}$ is concave, and in particular, its second derivative at $\lambda = 0$ is non-positive, yielding (3.3). Note that the function $[-\epsilon_0, \epsilon_0] \ni \epsilon \mapsto V(K +_p \epsilon \cdot f)$ is indeed in C^2 , as witnessed by writing an explicit differential formula for the volume in terms of the support function (as we shall do in the next section). \square

We also have the following *conditional* converse:

Lemma 3.5 (Local p -BM implies Global p -BM assuming geodesic in $\mathcal{K}_{+,e}^2$). *Fix $p \in \mathbb{R}$ and $K_0, K_1 \in \mathcal{K}_{+,e}^2$. Assume that:*

$$\forall t \in [0, 1] \quad K_t := (1-t) \cdot K_0 +_p t \cdot K_1 \text{ is in } \mathcal{K}_{+,e}^2 \quad (3.5)$$

If the local p -BM conjecture (3.3) holds for K_t for all $t \in [0, 1]$, then the global p -BM conjecture holds between K_{t_0} and K_{t_1} for all $t_0, t_1 \in [0, 1]$:

$$V((1-\lambda) \cdot K_{t_0} + \lambda \cdot K_{t_1}) \geq \left((1-\lambda)V(K_{t_0})^{\frac{p}{n}} + \lambda V(K_{t_1})^{\frac{p}{n}} \right)^{\frac{n}{p}} \quad \forall \lambda \in [0, 1]. \quad (3.6)$$

This is not as obvious as it may seem. The proof crucially relies on a classical observation of Aleksandrov [60, Lemma 6.5.1]:

Lemma 3.6 (Aleksandrov). *If $K = A[w]$ for some $w \in C_{>0}(S^{n-1})$, then for any $\theta \in S^{n-1}$ so that $h_K(\theta) < w(\theta)$, ∂K has at least two normals at any contact point $x_0 \in \partial K \cap \{x \in \mathbb{R}^n; \langle x, \theta \rangle = h_K(\theta)\}$; in particular, ∂K is not C^1 smooth at x_0 .*

The usual application of this lemma is to deduce that the set of contact points corresponding to bad directions θ as above has zero $(n-1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} , since ∂K is (twice) differentiable \mathcal{H}^{n-1} almost-everywhere. We will only require the following:

Corollary 3.7. *Let $K_0, K_1 \in \mathcal{K}_+^2$, $p \in \mathbb{R}$ and $t \in [0, 1]$. Denote $w_t := (1-t) \cdot h_{K_0} +_p t \cdot h_{K_1} \in C^2(S^{n-1})$, and set $K_t := A[w_t]$. Then:*

$$K_t \in \mathcal{K}_+^2 \Leftrightarrow w_t = h_{K_t} \Leftrightarrow w_t \text{ is a support function.}$$

Proof. If w_t is a support function then it must coincide with h_{K_t} , and therefore $h_{K_t} \in C^2(S^{n-1})$, i.e. $K_t \in \mathcal{K}_+^2$. Conversely, if $K_t \in \mathcal{K}_+^2$, Aleksandrov's lemma implies that $w_t = h_{K_t}$ and so w_t is a support function. \square

Proof of Lemma 3.5. By the previous corollary, our assumption (3.5) is equivalent to:

$$h_{K_t} = (1-t) \cdot h_{K_0} +_p t \cdot h_{K_1} \quad \forall t \in [0, 1]. \quad (3.7)$$

The global concavity of $[0, 1] \ni \lambda \mapsto \frac{1}{p} V((1-\lambda) \cdot K_0 + \lambda \cdot K_1)^{\frac{p}{n}}$ now follows by testing its second derivative at a given $\lambda \in [0, 1]$, which must be non-positive by the local p -BM assumption (3.3) for the body $K = K_\lambda$ and the function $\frac{1}{p} f^p := \frac{1}{p} h_{K_1}^p - \frac{1}{p} h_{K_0}^p \in C_e^2(S^{n-1})$. The latter concavity is equivalent to the desired (3.6). \square

Note that when $p \geq 1$, (3.7) holds automatically since the Wulff function $(1-t) \cdot h_{K_0} +_p t \cdot h_{K_1}$ is a support function (being an L^p -combination of two support functions). Consequently, assumption (3.5) on the smoothness of the entire geodesic $t \mapsto K_t$ is satisfied when $p \geq 1$, and employing a standard approximation argument for general (non-smooth) end points K_0, K_1 , it is immediate to deduce the global p -BM formulation (1.4) from the local one (3.3).

However, this is definitely not the case in general when $p \in [0, 1)$. First, the semi-group property (3.7) will not hold in general when $p \in [0, 1)$, and one can only ensure that the left-hand-side is a subset of the right-hand one. This time, the inclusion goes in the unfavorable direction for us:

$$K_{\lambda, \epsilon} := K_\lambda +_p \epsilon \cdot f \subset K_{\lambda + \epsilon},$$

and so knowing that $\epsilon \mapsto \frac{1}{p} V(K_{\lambda, \epsilon})^{p/n}$ is concave at $\epsilon = 0$ for every $\lambda \in [0, 1]$ will not tell us much about the (local) concavity of $[0, 1] \ni \lambda \mapsto \frac{1}{p} V(K_\lambda)^{p/n}$. While it is possible to bypass this point, the main obstacle for deducing the global p -BM conjecture (1.4) from the local one (3.3) when $p \in [0, 1)$ is the violation of the smoothness assumption (3.5). By Aleksandrov's lemma, K_t will necessarily have a singular boundary whenever the Wulff function $(1-t) \cdot h_{K_0} +_p t \cdot h_{K_1}$ is no longer a support function. For such t 's, we will need to approximate K_t by bodies $K_t^i \in \mathcal{K}_{+, \epsilon}^2$, and establish a relation between:

$$\lim_{i \rightarrow \infty} \frac{d^2}{(d\epsilon)^2} \Big|_{\epsilon=0} V(K_t^i +_p \epsilon \cdot f) \quad \text{and} \quad \frac{d^2}{(d\epsilon)^2} \Big|_{\epsilon=0} V(K_t +_p \epsilon \cdot f).$$

This turns out to be an extremely difficult and tantalizing question, which boils down to the study of the second variation of the volume of the Aleksandrov body for non-smooth $K \in \mathcal{K}$:

$$\frac{d^2}{(d\epsilon)^2} \Big|_{\epsilon=0} V(A[h_K + \epsilon f]);$$

here f may be assumed to be the difference of two support functions. While the first variation of volume was studied by Aleksandrov himself and is well understood (see e.g. [60, Lemma 6.5.3]), to the best of our knowledge, the second variation is *terra incognita*.

Being unable to establish the equivalence between the global and local formulations, we state this as:

Conjecture 3.8. *Given $p \in [0, 1)$, the validity of the local p -BM conjecture (3.3) for all $K \in \mathcal{K}_{+, \epsilon}^2$ is logically equivalent to the validity of the global p -BM conjecture (1.4) for all $K_0, K_1 \in \mathcal{K}_e$.*

For future reference, we record the following:

Proposition 3.9. *Let $p \in \mathbb{R}$ and $K \in \mathcal{K}_{+,e}^2$. Then the following statements are equivalent:*

- (1) *There exists a C^2 -neighborhood N_K of K in $\mathcal{K}_{+,e}^2$ so that the local p -BM conjecture (3.3) holds for all $K' \in N_K$.*
- (2) *There exists a C^2 -neighborhood N'_K of K in $\mathcal{K}_{+,e}^2$ so that for all $K_0, K_1 \in N'_K$ and $t \in [0, 1]$, $K_t := (1-t) \cdot K_0 +_p t \cdot K_1 \in N'_K$, and the global p -BM conjecture (1.4) holds between K_{t_0}, K_{t_1} for all $t_0, t_1 \in [0, 1]$.*

Furthermore, given $p_0 \in \mathbb{R}$, the following statements are equivalent:

- (1') *For every $p > p_0$, (1) or (2) above hold.*
- (2') *The local p_0 -BM conjecture (3.3) holds for K .*

The equivalence between (1') and (2') follows from the results of the next sections, but at the risk of forward-referencing, we include this result in the present section as it fits more naturally here; the reader may wish to skip its proof in the first reading.

Proof. Lemma 3.4 verifies that (2) implies (1) with $N_K = N'_K$. To show that (1) implies (2), assume for simplicity that $p \neq 0$; the case $p = 0$ follows analogously. Denoting by $N_{h_K} = \{h_{K'} ; K' \in N_K\}$ the corresponding neighborhood of h_K in $C_{h,e}^2(S^{n-1})$, set $N_{h_K}^p := \{h^p ; h \in N_{h_K}\}$, which is an open subset of $C_e^2(S^{n-1})$. As $C_e^2(S^{n-1})$ is locally convex, we may find a convex neighborhood $N_{h_K}^{p'}$ of h_K^p in $N_{h_K}^p$. Setting $N'_{h_K} := \{h^{1/p'} ; h \in N_{h_K}^{p'}\}$, the latter is an open subset of $N_{h_K} \subset C_{h,e}^2(S^{n-1})$ containing h_K which is convex with respect to the $+_p$ operation: if $h_{K_0}, h_{K_1} \in N'_{h_K}$ then $w_\lambda := (1-\lambda) \cdot h_{K_0} +_p \lambda \cdot h_{K_1} \in N'_{h_K} \subset C_{h,e}^2(S^{n-1})$ for all $\lambda \in [0, 1]$. By Corollary 3.7, since w_λ is a support function, it must be that of $K_\lambda := (1-\lambda) \cdot K_0 +_p \lambda \cdot K_1$, and therefore $K_\lambda \in \mathcal{K}_{+,e}^2$. Consequently, defining $N'_K = \{K' \in \mathcal{K}_{+,e}^2 ; h_{K'} \in N'_{h_K}\}$, the latter is an open subset of $N_K \subset \mathcal{K}_{+,e}^2$ containing K which is convex with respect to the L^p -Minkowski operation. The smoothness assumption (3.5) is therefore satisfied between any $K_0, K_1 \in N'_K$, and so the assertion follows from Lemma 3.5.

To show that (1') implies (2'), apply the local p -BM conjecture (3.3) to K in any of the equivalent forms given in the next sections (e.g. Propositions 4.2, 4.4, 4.6 or 5.2), and take the limit as $p \rightarrow p_0$. The implication that (2') implies (1') follows from the equivalent spectral characterization of the local p -BM conjecture given in

Corollary 5.4, and the continuity of the spectrum of the Hilbert–Brunn–Minkowski operator $-L_K$ under C^2 perturbations asserted in Theorem 5.3 (4). \square

4 Local L^p -Brunn–Minkowski Conjecture – Infinitesimal Formulation

In this section we fix $K \in \mathcal{K}_{+,e}^2$, and derive equivalent infinitesimal formulations to the local p -Brunn–Minkowski conjecture for K .

4.1 Mixed Surface Area and Volume of C^2 functions

It was shown by Minkowski (e.g. [60, 5]) that when $\{K_i\}_{i=1}^m$ are convex bodies in \mathbb{R}^n , then the volume of their Minkowski sum is a polynomial in the scaling coefficients:

$$V\left(\sum_{i=1}^m t_i K_i\right) = \sum_{1 \leq i_1, \dots, i_n \leq m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, \dots, K_{i_n}) \quad \forall t_i \geq 0.$$

The coefficient $V(K_{i_1}, \dots, K_{i_n})$ is called the mixed volume of the n -tuple $(K_{i_1}, \dots, K_{i_n})$; it is uniquely defined by requiring in addition that it be invariant under permutation of its arguments. In this subsection, we extend the definition of mixed volume $V(h_1, \dots, h_n)$ to a n -tuple of functions in $C^2(S^{n-1})$, in a manner ensuring that:

$$V(h_{K_1}, \dots, h_{K_n}) = V(K_1, \dots, K_n) \quad \forall \{K_i\}_{i=1}^n \subset \mathcal{K}_+^2.$$

Recall our notation given a local orthonormal frame e_1, \dots, e_{n-1} on S^{n-1} :

$$(D^2 h)_{i,j} = (\nabla_{\mathbb{R}^n})_{e_i, e_j}^2 h = h_{i,j} + h \delta_{i,j}, \quad i, j = 1, \dots, n-1.$$

Let $D_m(A^1, \dots, A^m)$ denote the mixed discriminant (or mixed determinant) of an m -tuple (A^1, \dots, A^m) with $A^i \in \mathcal{M}_m$, the set of m by m matrices (over \mathbb{R}), namely:

$$D_m(A^1, \dots, A^m) := \frac{1}{m!} \sum_{\sigma, \tau \in S_m} (-1)^{\text{sgn}(\sigma) + \text{sgn}(\tau)} A_{\sigma(1), \tau(1)}^1 \cdots A_{\sigma(m), \tau(m)}^m,$$

where S_m denotes the permutation group on $\{1, \dots, m\}$. Recall that the mixed discriminant is simply the multi-linear polarization of the usual determinant functional \det on \mathcal{M}_m , and so in particular is invariant under permutation of its arguments and satisfies $D_m(A, \dots, A) = \det A$.

Definition. Given a tuple (h_1, \dots, h_{n-1}) of functions in $C^2(S^{n-1})$, define their “mixed surface area function” $S(h_1, \dots, h_{n-1}) \in C(S^{n-1})$ by:

$$\begin{aligned} S(h_1, \dots, h_{n-1})(\theta) &:= D_{n-1}(D^2h_1(\theta), \dots, D^2h_{n-1}(\theta)) \\ &= D_n(\nabla_{\mathbb{R}^n}^2 h_1(\theta), \dots, \nabla_{\mathbb{R}^n}^2 h_{n-1}(\theta), \theta \otimes \theta) \\ &= D_n(\nabla_{\mathbb{R}^n}^2 h_1(\theta), \dots, \nabla_{\mathbb{R}^n}^2 h_{n-1}(\theta), Id) \end{aligned}$$

It is easy to see that the former expression does not depend on the choice of local orthonormal frame. The latter two equalities follow easily by expanding D_n according to the last entry, since for any 1 homogeneous function h , θ is an eigenvector of the symmetric $\nabla_{\mathbb{R}^n}^2 h$ with eigenvalue zero, and hence $\nabla_{\mathbb{R}^n}^2 h = P_{\theta^\perp} \nabla_{\mathbb{R}^n}^2 h P_{\theta^\perp}$ where P_{θ^\perp} denotes orthogonal projection perpendicular to θ .

The surface-area measure of K , denoted dS_K , is the Borel measure on S^{n-1} obtained by pushing forward the $(n-1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} on ∂K via the Gauss map $\nu_{\partial K} : \partial K \rightarrow S^{n-1}$ [60, p. 115,207]. Here $\nu_{\partial K}(x)$ is the unit outer-normal to ∂K at x , which by convexity is well-defined and unique for \mathcal{H}^{n-1} -a.e. $x \in \partial K$.

Note that when $K \in \mathcal{K}_+^2$, $\nu_{\partial K} : \partial K \rightarrow S^{n-1}$ is in fact a C^1 diffeomorphism. Identifying between the tangent spaces $T_x \partial K$ and $T_{\nu_{\partial K}(x)} S^{n-1}$, we have [60, p. 107]:

$$d\nu_{\partial K}(x) = \Pi_{\partial K}(x) \quad \forall x \in \partial K,$$

where $\Pi_{\partial K}(x)$ denotes the second fundamental form of ∂K at x . The inverse of the Gauss map is the Weingarten map $\nabla_{\mathbb{R}^n} h_K : S^{n-1} \rightarrow \partial K$, and therefore [60, p. 108]:

$$D^2 h_K(\nu_{\partial K}(x)) = \Pi_{\partial K}^{-1}(x) \quad \forall x \in \partial K.$$

It follows by the change-of-variables formula that:

$$dS_K(\theta) = \det(D^2 h_K)(\theta) d\theta = S(h_K, \dots, h_K)(\theta) d\theta; \quad (4.1)$$

this explains the name “mixed surface-area function” for $S(h_1, \dots, h_n)$.

Recall that when $K_1, \dots, K_n \in \mathcal{K}_+^2$ then $h_{K_1}, \dots, h_{K_n} \in C^2(S^{n-1})$, and so their mixed volume may be expressed as [5, p. 64], [60, p. 115,275]:

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_n} S(h_{K_1}, \dots, h_{K_{n-1}}) d\theta; \quad (4.2)$$

this is just a multi-linear polarization of the usual formula:

$$V(K, \dots, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_K = \frac{1}{n} \int_{S^{n-1}} h_K S(h_K, \dots, h_K)(\theta) d\theta.$$

The fact that the expression on the right-hand-side of (4.2) is invariant under permutation of K_1, \dots, K_n is a nice exercise in integration by parts, which we shall reproduce below. Consequently, it is natural to give the following:

Definition. Given $h_1, \dots, h_n \in C^2(S^{n-1})$, define their “mixed-volume” as:

$$V(h_1, \dots, h_n) := \frac{1}{n} \int_{S^{n-1}} h_n S(h_1, \dots, h_{n-1}) d\theta.$$

Note that the mixed-volume is indeed multi-linear in its arguments.

4.2 Properties of Mixed Surface Area and Volume

Next, if $A^1, \dots, A^{n-1} \in \mathcal{M}_{n-1}$, a direct computation verifies:

$$D_{n-1}(A^1, \dots, A^{n-1}) = \sum_{i,j} A_{i,j}^1 Q^{i,j}(A^2, \dots, A^{n-1}), \quad (4.3)$$

$$Q^{i,j}(A^2, \dots, A^{n-1}) := \frac{(-1)^{i+j}}{n-1} D_{n-2}(M^{i,j}(A^2), \dots, M^{i,j}(A^{n-1})),$$

where $M^{i,j}(A)$ is the minor resulting after removing the i -th row and the j -th column from $A \in \mathcal{M}_{n-1}$. Consequently, when $A^2 = \dots = A^{n-1} = A \in GL_{n-1}$, we see that:

$$\begin{aligned} Q^{i,j}(A) &:= Q^{i,j}(A, \dots, A) = \frac{(-1)^{i+j}}{n-1} \det(M^{i,j}(A)) \\ &= \frac{1}{n-1} \text{adj}(A)^{i,j} = \frac{1}{n-1} \det(A) (A^{-1})^{i,j}. \end{aligned} \quad (4.4)$$

Clearly $Q^{i,j} = Q^{j,i}$ is symmetric and multi-linear in its arguments. Furthermore, the first of the following properties [1, Lemma 2-12] will be constantly used (the second is mentioned for completeness):

- For any $h_1, \dots, h_{n-1} \in C^3(S^{n-1})$ and $i = 1, \dots, n-1$, the (local) C^1 vector field on S^{n-1}

$$\sum_{j=1}^{n-1} Q^{i,j}(D^2 h_1, \dots, D^2 h_{n-1}) e_j$$

is divergence free.

- For any $K_1, \dots, K_{n-1} \in \mathcal{K}_+^2$, $\sum_{i,j} Q^{i,j}(D^2 h_{K_1}, \dots, D^2 h_{K_{n-1}}) e_i \otimes e_j$ is a positive definite (local) 2-tensor on S^{n-1} .

It is easy to see that the above two properties do not depend on the choice of local orthonormal frame. We will henceforth freely employ Einstein summation convention, summing over repeated indices. Our choice of using local orthonormal frames instead of local coordinates is in order to simplify notation, dispensing with the need to keep track of the covariance / contravariance of our various tensors. In local coordinates, we would apply the mixed discriminant D_m to 1-covariant 1-contravariant tensors A_j^i , and as suggested by the present notation, $Q^{i,j}$ would be a 2-contravariant tensor.

Lemma 4.1. *For any $h_1, \dots, h_n \in C^2(S^{n-1})$, the mixed volume $V(h_1, \dots, h_n)$ is invariant under permutation of its arguments.*

Proof. By approximation, we may assume $h_3, \dots, h_n \in C^3(S^{n-1})$; it is then enough to show that:

$$V(f, g, h_3 \dots, h_n) = V(g, f, h_3 \dots, h_n) \quad \forall f, g \in C^2(S^{n-1}),$$

since the last $n - 1$ arguments are invariant under permutation by definition of V and D . Abbreviating $Q^{i,j} = Q^{i,j}(D^2 h_3, \dots, D^2 h_n)$, integrating by parts, and using the divergence free property of $Q^{i,j}$, we have:

$$\begin{aligned} V(f, g, h_3 \dots, h_n) &= \frac{1}{n} \int_{S^{n-1}} f S(g, h_3, \dots, h_n) d\theta = \frac{1}{n} \int_{S^{n-1}} f (g_{i,j} + g \delta_{i,j}) Q^{i,j} d\theta \\ &= \frac{1}{n} \int_{S^{n-1}} (Q_i^i f g - (f Q^{i,j})_j g_i) d\theta = \frac{1}{n} \int_{S^{n-1}} (Q_i^i f g - Q^{i,j} f_j g_i) d\theta. \end{aligned}$$

The latter expression is symmetric in f, g (by symmetry of $Q^{i,j}$), and so the assertion is established. \square

4.3 Second L^p -Minkowski Inequality

We now fix $K \in \mathcal{K}_+^2$ and $p < 1$. Given $\frac{1}{p} f^p \in C^2(S^{n-1})$, the strict convexity of K and the positivity of h_K imply that for small enough $|\epsilon|$, the Wulff function $h_K +_p \epsilon \cdot f$ is a C^2 -smooth support function, and consequently:

$$h_{K+_p \epsilon \cdot f} = h_{A[h_K+_p \epsilon \cdot f]} = h_K +_p \epsilon \cdot f \quad \forall |\epsilon| \ll 1.$$

Denoting:

$$z := \begin{cases} \frac{1}{h_K^p} \frac{f^p}{p} & p \neq 0 \\ \log f & p = 0 \end{cases} \in C^2(S^{n-1}), \quad (4.5)$$

a second-order Taylor expansion immediately yields:

$$h_{K+p\epsilon \cdot f} = h_{K+\epsilon z h_K} + \frac{\epsilon^2}{2}(1-p)z^2 h_K + R(\epsilon), \quad \lim_{\epsilon \rightarrow 0} \frac{R_h(\epsilon)}{\epsilon^2} = 0 \text{ in } C^2(S^{n-1}). \quad (4.6)$$

Now denote:

$$J_p(\epsilon) := V(K +_p \epsilon \cdot f) = V(K +_p \epsilon \cdot f, \dots, K +_p \epsilon \cdot f),$$

apply the differential formula for mixed-volume (4.2), and expand using the above Taylor expansion. Inspecting the coefficients of ϵ and $\frac{\epsilon^2}{2}$ in the expansion of $J_p(\epsilon)$, and noting that the remainder term $R_{J_p}(\epsilon)$ satisfies:

$$|R_{J_p}(\epsilon)| \leq C_{h_K, z, p, n} \|R_h(\epsilon)\|_{C^2(S^{n-1})} \quad \text{and hence} \quad \lim_{\epsilon \rightarrow 0} \frac{R_{J_p}(\epsilon)}{\epsilon^2} = 0,$$

it follows by multi-linearity of mixed-volume and invariance under permutation of its arguments that:

$$J'_p(0) = nV(zh_K; 1), \quad J''_p(0) = nV((1-p)z^2 h_K; 1) + \binom{n}{2} 2V(zh_K; 2), \quad (4.7)$$

where we employ the abbreviation:

$$V(f; m) = V(\underbrace{f, \dots, f}_{m \text{ times}}, \underbrace{h_K, \dots, h_K}_{n-m \text{ times}}).$$

Proposition 4.2 (Second L^p -Minkowski Inequality). *Given $K \in \mathcal{K}_{+,e}^2$ and $p < 1$, the local p -BM conjecture (3.3) for K is equivalent to the assertion that:*

$$\forall z \in C_e^2(S^{n-1}) \quad \frac{1}{V(K)} V(zh_K; 1)^2 \geq \frac{n-1}{n-p} V(zh_K; 2) + \frac{1-p}{n-p} V(z^2 h_K; 1). \quad (4.8)$$

Proof. The local p -BM conjecture is the assertion that $(\frac{d}{d\epsilon})^2|_{\epsilon=0} \frac{1}{p} J_p(\epsilon)^{p/n} \leq 0$ for all $\frac{1}{p} f^p \in C_e^2(S^{n-1})$, or equivalently, that:

$$\frac{n-p}{n} J'_p(0)^2 \geq J_p(0) J''_p(0).$$

Plugging in the expressions for $J_p^{(m)}(0)$ obtained in (4.7), the equivalence with (4.8) immediately follows after noting that (4.5) gives a bijection between $\frac{1}{p} f^p \in C_e^2(S^{n-1})$ and $z \in C_e^2(S^{n-1})$. \square

Remark 4.3. Note that the validity of (4.8) remains invariant under $K \mapsto \lambda K$ (as all terms are n -homogeneous in K), $z \mapsto \lambda z$ (as all terms are quadratic in z) and $z \mapsto z + \lambda$ (this requires a quick check of both the linear term in λ and the quadratic one). Consequently, (4.8) is equivalent to:

$$\forall z \in C_e^2(S^{n-1}) \quad V(zh_K; 1) = 0 \quad \Rightarrow \quad -V(zh_K; 2) \geq \frac{1-p}{n-1} V(z^2h_K; 1).$$

4.4 Comparison with classical $p = 1$ case

Before proceeding, let us compare (4.8) with the classical case $p = 1$. Plugging in $p = 1$ in (4.8), the quadratic term in z^2 disappears, and denoting $w = zh_K$, we obtain:

$$\forall w \in C_e^2(S^{n-1}) \quad V(w; 1)^2 \geq V(w; 2)V(w; 0); \quad (4.9)$$

this is the classical Minkowski's second inequality, valid without any evenness assumption on K or w , which indeed is well-known to be equivalent to the local concavity of $\epsilon \mapsto V(K + \epsilon w)^{\frac{1}{n}}$, and hence to the global Brunn–Minkowski inequality.

In addition, let us check that the conjectured (4.8) for $p \in [0, 1)$ is indeed stronger than the classical case $p = 1$. To see this, first note that by (4.1), for all $w \in C^2(S^{n-1})$ (in fact, $C(S^{n-1})$ is enough for the first mixed volume):

$$V(w; 1) = V(w, h_K, \dots, h_K) = \frac{1}{n} \int_{S^{n-1}} w S(h_K, \dots, h_K) d\theta = \frac{1}{n} \int_{S^{n-1}} w dS_K. \quad (4.10)$$

In particular $V(K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_K$. Applying Cauchy–Schwarz, it follows that:

$$V(z^2h_K; 1) = \frac{1}{n} \int_{S^{n-1}} z^2 h_K dS_K \geq \frac{(\frac{1}{n} \int_{S^{n-1}} z h_K dS_K)^2}{\frac{1}{n} \int_{S^{n-1}} h_K dS_K} = \frac{V(zh_K; 1)^2}{V(K)}. \quad (4.11)$$

This means that when $p \in [0, 1)$, since $\frac{1-p}{n-p} > 0$, we can always make the inequality (4.8) weaker by replacing the last term by the one on the left-hand-side, and after rearranging terms and setting $w = zh_K$, we obtain (4.9) corresponding to the classical case $p = 1$.

4.5 Infinitesimal Formulation On S^{n-1}

Let us now obtain an explicit expression for the second mixed volume appearing in (4.8). Recall by (4.4) that:

$$Q_K^{i,j} := Q^{i,j}(D^2h_K, \dots, D^2h_K) = \frac{1}{n-1} \det(D^2h_K) ((D^2h_K)^{-1})^{i,j}. \quad (4.12)$$

Plugging this below assuming $K \in \mathcal{K}_+^3$, after integrating by parts in j and using the divergence-free property of $Q_K^{i,\cdot}$, and finally recalling (4.1), we obtain for any $w \in C^2(S^{n-1})$:

$$\begin{aligned}
V(w; 2) &= V(w, w, h_K, \dots, h_K) \\
&= \frac{1}{n} \int_{S^{n-1}} w S(w, h_K, \dots, h_K) d\theta = \frac{1}{n} \int_{S^{n-1}} w (w_{i,j} + w \delta_{i,j}) Q_K^{i,j} d\theta \\
&= \frac{1}{n} \left(\int_{S^{n-1}} (Q_K)_i^i w^2 d\theta - \int_{S^{n-1}} (Q_K^{i,j})_j w_i d\theta \right) \\
&= \frac{1}{n} \left(\int_{S^{n-1}} (Q_K)_i^i w^2 d\theta - \int_{S^{n-1}} Q_K^{i,j} w_j w_i d\theta \right) \\
&= \frac{1}{n(n-1)} \left(\int_{S^{n-1}} ((D^2 h_K)^{-1})_i^i w^2 dS_K - \int_{S^{n-1}} ((D^2 h_K)^{-1})^{i,j} w_i w_j dS_K \right).
\end{aligned} \tag{4.13}$$

Proposition 4.4 (Infinitesimal p -BM on S^{n-1}). *Given $K \in \mathcal{K}_{+,e}^2$ and $p < 1$, the local p -BM conjecture (3.3) for K is equivalent to the assertion that:*

$$\begin{aligned}
\forall w \in C_e^1(S^{n-1}) \quad & \int_{S^{n-1}} \langle (D^2 h_K)^{-1} \nabla_{S^{n-1}} w, \nabla_{S^{n-1}} w \rangle dS_K \geq \\
& \int_{S^{n-1}} \text{tr}((D^2 h_K)^{-1}) w^2 dS_K + (1-p) \int_{S^{n-1}} \frac{w^2}{h_K} dS_K - \frac{n-p}{n} \frac{1}{V(K)} \left(\int_{S^{n-1}} w dS_K \right)^2.
\end{aligned} \tag{4.14}$$

Proof. The equivalence for $w \in C_e^2(S^{n-1})$ and $K \in \mathcal{K}_+^3$ is immediate by setting $w = zh_K$ in (4.8) and using the expressions for the first and second mixed-volume derived in (4.10) and (4.13). When $w \in C_e^1(S^{n-1})$ and $K \in \mathcal{K}_+^2$, the equivalence follows by a standard approximation argument, utilizing the fact that only first derivatives of w and second derivatives of h_K appear in (4.13). \square

Remark 4.5. The case $p = 0$ (local log-BM) of Proposition 4.4 was previously derived by Colesanti–Livshyts–Marsiglietti in [13].

4.6 Infinitesimal Formulation On ∂K

By using the Weingarten map:

$$\nu_{\partial K}^{-1} : S^{n-1} \ni \theta \mapsto \nabla_{\mathbb{R}^n} h_K(\theta) \in \partial K,$$

we may transfer the infinitesimal formulation of the local p -BM conjecture obtained in the previous subsection from S^{n-1} to ∂K . Indeed, recall that $d\nu_{\partial K}(x) =$

$\Pi_{\partial K}(x)$, $d\nu_{\partial K}^{-1}(\theta) = D^2 h_K(\theta)$ and $D^2 h_K(\nu_{\partial K}(x)) = \Pi_{\partial K}^{-1}(x)$ (with the usual identification of tangent spaces), and that $\text{Jac}(\nu_{\partial K}^{-1})(\theta) = \det(D^2 h_K)(\theta) = \frac{dS_K(\theta)}{d\theta}$.

Denoting $\Psi(x) = w(\nu_{\partial K}(x))$, we see that:

$$\int_{S^{n-1}} w(\theta) dS_K(\theta) = \int_{\partial K} \Psi(x) dx,$$

$$\int_{S^{n-1}} \text{tr}((D^2 h_K)^{-1}) w^2(\theta) dS_K(\theta) = \int_{\partial K} \text{tr}(\Pi_{\partial K}) \Psi^2(x) dx,$$

and as $h_K(\nu_{\partial K}(x)) = \langle x, \nu_{\partial K}(x) \rangle$,

$$\int_{S^{n-1}} \frac{w^2(\theta)}{h_K(\theta)} dS_K(\theta) = \int_{\partial K} \frac{\Psi^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx.$$

Lastly, as $w(\theta) = \Psi(\nu_{\partial K}^{-1}(\theta))$ we have, setting $x = \nu_{\partial K}^{-1}(\theta)$:

$$\nabla_{S^{n-1}} w(\theta) = d\nu_{\partial K}^{-1}(\theta) \nabla_{\partial K} \Psi(x) = \Pi_{\partial K}^{-1}(x) \nabla_{\partial K} \Psi(x),$$

and therefore:

$$\begin{aligned} \int_{S^{n-1}} \langle (D^2 h_K)^{-1} \nabla_{S^{n-1}} w, \nabla_{S^{n-1}} w \rangle dS_K &= \int_{\partial K} \langle \Pi_{\partial K} \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi \rangle dx \\ &= \int_{\partial K} \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx. \end{aligned}$$

Plugging the above identities into Proposition 4.4, and using the fact that the Weingarten map is a C^1 diffeomorphism when $K \in \mathcal{K}_+^2$, we immediately obtain:

Proposition 4.6 (Infinitesimal p -BM on ∂K). *Given $K \in \mathcal{K}_{+,e}^2$ and $p < 1$, the local p -BM conjecture (3.3) for K is equivalent to the assertion that:*

$$\forall \Psi \in C_e^1(\partial K) \quad \int_{\partial K} \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx \geq \quad (4.15)$$

$$\int_{\partial K} H_{\partial K}(x) \Psi^2(x) dx + (1-p) \int_{\partial K} \frac{\Psi^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx - \frac{n-p}{n} \frac{1}{V(K)} \left(\int_{\partial K} \Psi(x) dx \right)^2,$$

where $H_{\partial K}(x) = \text{tr}(\Pi_{\partial K})(x)$ denotes the mean-curvature of ∂K at $x \in \partial K$.

Remark 4.7. Remark 4.3 translates into the fact that the validity of (4.14) is invariant under $w \mapsto w + \lambda h_K$, or equivalently, that (4.15) is invariant under $\Psi \mapsto \Psi + \lambda \langle x, \nu_{\partial K}(x) \rangle$. Consequently, (4.15) is equivalent to:

$$\forall \Psi \in C_e^1(\partial K) \quad \int_{\partial K} \Psi(x) dx = 0 \quad \Rightarrow \quad (4.16)$$

$$\int_{\partial K} \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx \geq \int_{\partial K} H_{\partial K}(x) \Psi^2(x) dx + (1-p) \int_{\partial K} \frac{\Psi^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx.$$

Remark 4.8. The classical case $p = 1$ of Proposition 4.6 was previously derived by Colesanti in [11]. See also [36] for extensions of the case $p = 1$ to the setting of weighted Riemannian manifolds satisfying the Curvature-Dimension condition $CD(0, N)$, [40] for an analogous statement involving Ehrhard's inequality for the Gaussian measure, and [14] for a version involving other intrinsic volumes.

5 Relation to Hilbert–Brunn–Minkowski Operator and Linear Equivariance

It is easy to show that the validity of the local p -BM conjecture (3.3) as a function of $K \in \mathcal{K}_{+,e}^2$ is invariant under linear transformations. More precisely, given $T \in GL_n$ and $\frac{1}{p}f^p \in C_e^2(S^{n-1})$, define $f_T : S^{n-1} \rightarrow \mathbb{R}$ by extending f as a 1-homogeneous function to \mathbb{R}^n and pushing it forward via T^{-t} , namely:

$$f_T(\theta) = T_*^{-t} f(\theta) = f(T^t \theta) = f\left(\frac{T^t \theta}{|T^t \theta|}\right) |T^t \theta|.$$

It is easy to see that $pf_T^p \in C_e^2(S^{n-1})$ and that:

$$T(K) +_p \epsilon \cdot f_T = T(K +_p \epsilon \cdot f). \quad (5.1)$$

Indeed, by definition, we have:

$$h_{T(K)}(\theta) = h_K(T^t \theta), \quad (5.2)$$

and so for small enough $|\epsilon| \ll 1$ and all $\theta \in S^{n-1}$:

$$\begin{aligned} h_{T(K) +_p \epsilon \cdot f_T}(\theta) &= \left(h_{T(K)}^p(\theta) + \epsilon f_T^p(\theta) \right)^{\frac{1}{p}} = \left(h_K^p\left(\frac{T^t \theta}{|T^t \theta|}\right) + \epsilon f^p\left(\frac{T^t \theta}{|T^t \theta|}\right) \right)^{\frac{1}{p}} |T^t \theta| \\ &= \left(h_K^p + \epsilon f^p \right)^{\frac{1}{p}} \left(\frac{T^t \theta}{|T^t \theta|} \right) |T^t \theta| = h_{K +_p \epsilon \cdot f}\left(\frac{T^t \theta}{|T^t \theta|}\right) |T^t \theta| = h_{K +_p \epsilon \cdot f}(T^t \theta) = h_{T(K +_p \epsilon \cdot f)}(\theta). \end{aligned}$$

In fact, it is equally easy to check that (5.1) remains valid for all $\epsilon \in \mathbb{R}$ as equality between Aleksandrov bodies, even when the support function of $K +_p \epsilon \cdot f$ does not coincide with the corresponding defining Wulff function.

In any case, since $\frac{1}{p}f^p \mapsto \frac{1}{p}f_T^p$ is clearly a bijection on $C_e^2(S^{n-1})$, the invariance under linear transformations $K \mapsto T(K)$ of the validity of (3.3) immediately follows. Consequently, the invariance under linear transformations of the validity

of the equivalent infinitesimal versions (4.14) on S^{n-1} and (4.15) on ∂K follows as well. This is not surprising, since this is just an infinitesimal manifestation of the (easily verifiable) fact that:

$$T(K +_p L) = T(K) +_p T(L) \quad \forall T \in GL_n,$$

which implies that the validity of the p -BM conjecture (1.4) for K, L is equivalent to that for $T(K), T(L)$.

However, one of our goals in this section is to establish a somewhat deeper linear equivariance of a certain second-order linear differential operator L_K associated to every $K \in \mathcal{K}_+^2$, which extends the above elementary observation. Modulo our different normalization, this operator was already been considered by Hilbert in his proof of the Brunn–Minkowski inequality, and generalized by Aleksandrov in his second proof of the Aleksandrov–Fenchel inequality (see [5, pp. 108–110]). Consequently, we call L_K the Hilbert–Brunn–Minkowski operator.

5.1 Hilbert–Brunn–Minkowski operator

Definition (Hilbert–Brunn–Minkowski operator). *Given $K \in \mathcal{K}_+^2$, the associated Hilbert–Brunn–Minkowski operator, denoted L_K , is the second-order linear differential operator on $C^2(S^{n-1})$ defined by:*

$$L_K := \tilde{L}_K - Id, \quad \tilde{L}_K z := \frac{S(zh_K, \overbrace{h_K, \dots, h_K}^{n-2 \text{ times}})}{S(\underbrace{h_K, \dots, h_K}_{n-1 \text{ times}})}.$$

Remark 5.1. Abbreviating $Q^{i,j} = Q^{i,j}(D^2 h_K)$ and $h = h_K$, observe that:

$$\begin{aligned} & S(zh, h, \dots, h) - zS(h, h, \dots, h) \\ &= Q^{i,j} (((zh)_{i,j} + zh\delta_{i,j}) - z(h_{i,j} + h\delta_{i,j})) \\ &= Q^{i,j} (z_i h_j + h_i z_j + h z_{i,j}) = Q^{i,j} \frac{1}{h} (h^2 z_i)_j, \end{aligned} \quad (5.3)$$

where the last transition follows by the symmetry of $Q^{i,j}$. Recalling (4.12), we obtain:

$$L_K z = \frac{1}{n-1} ((D^2 h)^{-1})^{i,j} (z_i h_j + h_i z_j + h z_{i,j}) = \frac{1}{n-1} \frac{((D^2 h)^{-1})^{i,j}}{h} (h^2 z_i)_j. \quad (5.4)$$

In particular, we see that L_K has no zeroth order term.

Definition (Cone Measure). *The cone measure of K , denoted dV_K , is the Borel measure on S^{n-1} defined by:*

$$dV_K := \frac{1}{n} h_K dS_K.$$

It is easy to check that for any Borel set $A \subset S^{n-1}$, $V_K(A)$ equals the volume of the cone in K generated by $\nu_K^{-1}(A)$ with vertex at the origin. In particular, $V_K(S^{n-1}) = V(K)$. When $K \in \mathcal{K}_+^2$, we have by (4.1):

$$dV_K = \frac{1}{n} h_K S(h_K, \dots, h_K) d\theta.$$

Fixing $K \in \mathcal{K}_+^2$, and recalling the terms appearing in the second p -Minkowski inequality (Proposition 4.2), we rewrite:

$$\begin{aligned} V(zh_K; 1) &= \frac{1}{n} \int_{S^{n-1}} zh_K dS_K = \int_{S^{n-1}} z dV_K \\ V(z^2h_K; 1) &= \frac{1}{n} \int_{S^{n-1}} z^2 h_K dS_K = \int_{S^{n-1}} z^2 dV_K \\ V(zh_K; 2) &= \frac{1}{n} \int_{S^{n-1}} zh_K S(zh_K, h_K, \dots, h_K) d\theta = \int_{S^{n-1}} (\tilde{L}_K z) z dV_K. \end{aligned} \quad (5.5)$$

Plugging these expressions into Proposition 4.2, and applying Remark 4.3, we obtain:

Proposition 5.2. *Given $K \in \mathcal{K}_{+,e}^2$ and $p < 1$, the local p -BM conjecture (3.3) for K is equivalent to the assertion that:*

$$\forall z \in C_e^2(S^{n-1}) \quad \int_{S^{n-1}} z dV_K = 0 \quad \Rightarrow \quad \int_{S^{n-1}} (-L_K z) z dV_K \geq \frac{n-p}{n-1} \int_{S^{n-1}} z^2 dV_K.$$

The latter formulation has a clear spectral flavor. Let us make this more precise.

Theorem 5.3. *Let $K \in \mathcal{K}_+^2$.*

- (1) *The operator $L_K : C^2(S^{n-1}) \rightarrow C(S^{n-1})$ is symmetric on $L^2(dV_K)$.*
- (2) *The operator L_K is elliptic and hence admits a unique self-adjoint extension in $L^2(dV_K)$ with domain $\text{Dom}(L_K) = H^2(S^{n-1})$, which we continue to denote by L_K . Its spectrum $\sigma(L_K) \subset \mathbb{R}$ is discrete, consisting of a countable sequence of eigenvalues of finite multiplicity tending (in absolute value) to ∞ .*

(3) The Dirichlet form associated to L_K is given by:

$$\int_{S^{n-1}} (-L_K z) z dV_K = \frac{1}{n-1} \int_{S^{n-1}} h_K ((D^2 h_K)^{-1})^{i,j} z_i z_j dV_K. \quad (5.6)$$

In particular, $-L_K$ is positive semi-definite and therefore $\sigma(-L_K) \subset \mathbb{R}_+$. We denote its eigenvalues (arranged in non-decreasing order and repeated according to multiplicity) by $\{\lambda_m\}_{m \geq 0}$.

- (4) If $\{K_i\} \subset \mathcal{K}_+^2$ and $K_i \rightarrow K$ in C^2 then $\lim_{i \rightarrow \infty} \lambda_m(-L_{K_i}) = \lambda_m(-L_K)$ for all $m \geq 0$.
- (5) 0 is an eigenvalue of $-L_K$ with multiplicity one corresponding to the one-dimensional subspace of constant functions $E_0 := \text{span}(\mathbf{1})$.
- (6) $-L_K|_{\mathbf{1}^\perp} \geq \text{Id}|_{\mathbf{1}^\perp}$ (as positive semi-definite operators on $L^2(dV_K)$).
- (7) 1 is an eigenvalue of $-L_K$ with multiplicity precisely n corresponding to the n -dimensional subspace E_1^K spanned by the (renormalized) linear functions:

$$\ell_v^K(\theta) = \frac{\langle \theta, v \rangle}{h_K(\theta)}, \quad v \in \mathbb{R}^n.$$

Proof. (1) If $z_1, z_2 \in C^2(S^{n-1})$ then:

$$\begin{aligned} \int_{S^{n-1}} (L_K z_1) z_2 dV_K &= \frac{1}{n} \int_{S^{n-1}} z_1 h_K S(z_2 h_K, h_K, \dots, h_K) d\theta \\ &= V(z_1 h_K, z_2 h_K, h_K, \dots, h_K). \end{aligned}$$

By Lemma 4.1, the mixed volume is invariant under permutations, so the right-hand-side is symmetric in z_1, z_2 , and hence so is the left-hand-side, as asserted.

- (2) The ellipticity of L_K follows since by (5.4), its (leading) second-order term is given by:

$$\frac{h_K}{n-1} ((D^2 h_K)^{-1})^{i,j} z_{i,j}.$$

Since $K \in \mathcal{K}_+^2$, there exist $a, b > 0$ so that $a\delta \leq D^2 h_K \leq b\delta$ on S^{n-1} , where δ denotes the standard metric on S^{n-1} , and the (uniform) ellipticity follows.

As for essential self-adjointness, it is well known using elliptic regularity theory [65, Section 8.2], [64] that a symmetric second-order elliptic operator with continuous coefficients on a compact closed manifold M has a unique self-adjoint extension from $C^2(M)$ to the Sobolev space $H^2(M)$. Its resolvent is necessarily compact, and hence its spectrum is discrete.

- (3) By density, it is enough to verify the representation (5.6) for $z \in C^2(S^{n-1})$. Abbreviating as usual $Q^{i,j} = Q^{i,j}(D^2h_K)$ and $h = h_K$, it follows by (5.3) and the divergence free property of $Q^{i,\cdot}$ when $K \in \mathcal{K}_+^3$ that:

$$\int_{S^{n-1}} (-L_K z) z dV_K = -\frac{1}{n} \int_{S^{n-1}} z Q^{i,j} (h^2 z_i)_j d\theta = \frac{1}{n} \int_{S^{n-1}} Q^{i,j} h^2 z_i z_j d\theta.$$

The case of a general $K \in \mathcal{K}_+^2$ follows by approximation. Recalling (4.12), the representation (5.6) immediately follows. The positive semi-definiteness of $-L_K$ follows since D^2h is positive definite; note that this also follows by (5) and (6) as well.

- (4) The continuity of the eigenvalues as a function of the coefficients of a family of uniformly elliptic operators on a compact manifold is classical (e.g. [30, Theorem 2.3.3]). Indeed, the C^2 convergence of K_i to K ensures that the coefficients of $-L_{K_i}$ converge in $C(S^{n-1})$ to those of $-L_K$; using the uniform ellipticity (as $D^2h_{K_i} \geq \frac{1}{2}D^2h_K \geq c\delta$ for some $c > 0$ and large enough i), one shows pointwise convergence of the corresponding compact resolvent operators, from whence norm convergence of the resolvent operators is deduced, yielding the convergence of eigenvalues.
- (5) Clearly $L_K \mathbf{1} = 0$ as it has no zeroth order term (or since clearly $\tilde{L}_K \mathbf{1} = \mathbf{1}$). The standard fact that the multiplicity of the 0 eigenvalue is precisely one follows since L_K is second-order elliptic with no zeroth order term and since the sphere is a connected compact manifold. Note that this also follows from (6).
- (6) The spectral-gap estimate $-L_K|_{\mathbf{1}^\perp} \geq Id|_{\mathbf{1}^\perp}$ is a deep fact which is equivalent to the (local, and hence global) Brunn–Minkowski inequality. Under a different normalization, this equivalence was first noted by Hilbert (see [5, pp. 108–109] and Remark 5.6), who obtained a direct proof of the former spectral-gap estimate by employing the method of continuity, thereby obtaining a novel proof of the Brunn–Minkowski inequality. To see the equivalence, simply apply Proposition 5.2 in the classical case $p = 1$, and note that the Brunn–Minkowski inequality holds without any symmetry assumptions on K, L , so that the evenness assumption on the test function z is unnecessary in this case.
- (7) It is immediate to check that ℓ_v^K is indeed an eigenfunction of $-L_K$ with eigenvalue 1, since:

$$\nabla_{\mathbb{R}^n}^2 \langle \theta, v \rangle = 0 \quad \text{and hence} \quad \tilde{L}_K \ell_v^K = \frac{S(\langle \theta, v \rangle, h_K, \dots, h_K)}{S(h_K, h_K, \dots, h_K)} = 0.$$

Consequently, the multiplicity of the eigenvalue 1 is at least n (the dimension of linear functionals on \mathbb{R}^n). The fact that there are no other eigenfunctions with eigenvalue 1, and hence that the corresponding multiplicity is *precisely* n , was established by Hilbert (see [5, p. 110] for an alternative argument) in his proof of the spectral-gap estimate (6), and in fact constitutes the crux of Hilbert's argument. □

Theorem 5.3, which modulo our different normalization is essentially due to Hilbert (see Remark 5.6 below), interprets the Brunn–Minkowski inequality as a uniform spectral-gap statement (beyond the trivial $\lambda_0 = 0$ eigenvalue corresponding to $E_0 = \text{span}(\mathbf{1})$):

$$\lambda_1(-L_K) := \min \sigma(-L_K|_{\mathbf{1}^\perp}) \geq 1 \quad \forall K \in \mathcal{K}_+^2.$$

Moreover, it provides the *additional* information that $\lambda_1(-L_K) = 1$ with corresponding eigenspace E_1^K of dimension precisely n . Consequently, the next eigenvalue $\lambda_{n+1}(-L_K)$, which is obtained by restricting $-L_K$ to the (invariant) subspace perpendicular to $E_1^K + E_0$, satisfies:

$$\lambda_{n+1}(-L_K) := \min \sigma(-L_K|_{(E_1^K)^\perp \cap \mathbf{1}^\perp}) > 1 \quad \forall K \in \mathcal{K}_+^2.$$

A naturally arising question, which perhaps could have been asked by Hilbert himself (had he been using our normalization), is whether the above *next* eigenvalue gap *beyond* 1 is actually *uniform* over all $K \in \mathcal{K}_+^2$. The most convenient way to obtain a necessary condition for this to hold, is to assume that K is origin-symmetric, and so the ℓ_v^K eigenfunctions will all be odd, as the ratio of linear (odd) functions and an even one. If we only consider test-functions $z \in H^2(S^{n-1})$ which arise from perturbations of K by another origin-symmetric body L , they will always be even, and hence constitute an invariant subspace E_{even} for $-L_K$, which is in addition automatically perpendicular to E_1^K , and hence:

$$\lambda_{1,e}(-L_K) := \min \sigma(-L_K|_{E_{\text{even}} \cap \mathbf{1}^\perp}) \geq \min \sigma(-L_K|_{(E_1^K)^\perp \cap \mathbf{1}^\perp}) = \lambda_{n+1}(-L_K).$$

Proposition 5.2 thus translates into an interpretation of the local p -BM conjecture as a question on the *even* spectral-gap of $-L_K$ beyond 1:

Corollary 5.4. *Given $K \in \mathcal{K}_{+,e}^2$ and $p < 1$, the local p -BM conjecture (3.3) for K is equivalent to the following even spectral-gap estimate for $-L_K$ beyond 1:*

$$\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}.$$

This gives a concrete spectral reason for the restriction to origin-symmetric convex bodies in the p -BM conjecture when $p \in [0, 1)$, and explains why the conjecture fails for general convex bodies K – without being perpendicular to E_1^K , the spectral-gap beyond 0 is precisely 1 and never better (as seen by the test functions ℓ_v^K , corresponding to translations of K).

Remark 5.5. The above discussion suggests a plausible extension of the local p -BM conjecture which does not require that $K \in \mathcal{K}_+^2$ be origin-symmetric. In spectral terms it naturally reads as:

$$\lambda_{n+1}(-L_K) \geq \frac{n-p}{n-1},$$

or equivalently:

$$\begin{aligned} \forall z \in C^2(S^{n-1}) \quad \int_{S^{n-1}} zh_K dS_K = 0 \quad \text{and} \quad \int_{S^{n-1}} \vec{\theta} z(\theta) dS_K(\theta) = \vec{0} \quad \Rightarrow \quad (5.7) \\ \int_{S^{n-1}} (-L_K z) z dV_K \geq \frac{n-p}{n-1} \int_{S^{n-1}} z^2 dV_K. \end{aligned}$$

In terms of local concavity as in (3.3), recalling the derivation in Subsection 4.3, this reads as:

$$\frac{d^2}{(d\epsilon)^2} \Big|_{\epsilon=0} \frac{1}{p} V(A[h_K(1 + \epsilon z)^{\frac{1}{p}}])^{\frac{p}{n}} \leq 0 \quad \text{for all } z \in C^2(S^{n-1}) \text{ satisfying (5.7).}$$

However, we do not know how to translate the local condition (5.7) on $z = \frac{1}{p} \frac{h_L^p - h_K^p}{h_K^p}$ into a global requirement on K, L which would guarantee the local condition along the p -Minkowski interpolation between K and L . In this regard, we note that it was shown in [67] that for all convex bodies K, L in the plane, there exist translations of K, L (“dilation positions”) for which the log-BM conjectured inequality holds true.

Remark 5.6. Hilbert originally considered [5, pp. 108-109] the operator:

$$H_K w := S(w, h_K, \dots, h_K),$$

which is elliptic and essentially self adjoint on $L^2(d\theta)$ with respect to the Lebesgue measure $d\theta$ on S^{n-1} . However, this operator is not well suited for our investigation. Indeed, setting as usual $w = zh_K$ in Remark 4.3, the local p -BM conjecture for $K \in \mathcal{K}_{+,e}^2$ is then equivalent to:

$$\forall w \in C_e^2(S^{n-1}) \quad \int w dS_K = 0 \quad \Rightarrow \quad \int_{S^{n-1}} (-H_K w) w d\theta \geq \frac{1-p}{n-1} \int_{S^{n-1}} \frac{w^2}{h_K} dS_K,$$

which does not have a nice spectral interpretation when $p \neq 1$. Furthermore, the correspondence $K \mapsto H_K$ does not possess the useful equivariance property under linear transformations we shall establish for L_K in the next subsection, and it is not even invariant under homothety, so there is no chance of obtaining uniform estimates for H_K valid for all convex bodies K .

The normalization we employ in our definition of L_K may be uniquely characterized (up to scaling) as defining a second-order differential operator with no zeroth order term, which is essentially self-adjoint on $L^2(\mu_K)$, and so that the conjectured second p -Minkowski inequality (4.8) may be equivalently rewritten as a spectral-gap inequality on the subspace of $L^2(\mu_K)$ of even functions perpendicular to the constant ones. The unique (up to scaling) measure μ_K satisfying these requirements turns out to be the cone measure dV_K (for all values of p , not just $p = 0$!), amounting further evidence to the intimate relation between the p -BM conjecture and the cone measure.

5.2 Linear equivariance of the Hilbert–Brunn–Minkowski operator

In this subsection, we establish an important equivariance property of the Hilbert–Brunn–Minkowski operators $L_{T(K)}$ under linear transformations $T \in GL_n$.

Denote by $T^{(0)}$ the following “0-homogeneous” linear change of variables:

$$T^{(0)} : S^{n-1} \ni \theta \mapsto \frac{T^{-t}\theta}{|T^{-t}\theta|} \in S^{n-1}.$$

We denote by $T_*^{(0)}z$ the push-forward of a (Lebesgue) measurable function $z \in \mathcal{L}(S^{n-1})$ via $T^{(0)}$, i.e. an application of a linear change of variables when treating z as a 0-homogeneous function on \mathbb{R}^n :

$$T_*^{(0)} : \mathcal{L}(S^{n-1}) \ni z(\theta) \mapsto z((T^{(0)})^{-1}\theta) = z(T^t\theta) \in \mathcal{L}(S^{n-1}).$$

We now state the two main results of this subsection:

Lemma 5.7. $T^{(0)}$ pushes forward dV_K onto $\frac{1}{|\det(T)|}dV_{T(K)}$. In particular, $T_*^{(0)}$ is an isometry from $L^2(d\tilde{V}_K)$ to $L^2(d\tilde{V}_{T(K)})$, where $d\tilde{V}_Q := dV_Q/V(Q)$ is the normalized cone probability measure.

Theorem 5.8. For any $K \in \mathcal{K}_+^2$ and $T \in GL_n$, the following diagram commutes:

$$\begin{array}{ccc} L_K : L^2(dV_K) \supset H^2(S^{n-1}) & \rightarrow & L^2(dV_K) \\ & \begin{array}{ccc} T_*^{(0)} \downarrow & & T_*^{(0)} \downarrow \end{array} & \\ L_{T(K)} : L^2(dV_{T(K)}) \supset H^2(S^{n-1}) & \rightarrow & L^2(dV_{T(K)}). \end{array}$$

In particular, it follows by the previous lemma that L_K and $L_{T(K)}$ are conjugates via an isometry of Hilbert spaces, and therefore have the same spectrum:

$$\sigma(-L_K) = \sigma(-L_{T(K)}).$$

The proof involves several calculations. We will constantly use the linear contravariance of the support function (5.2).

Lemma 5.9.

$$\left| \text{Jac } T^{(0)}(\theta) \right| = \frac{1}{|\det(T)| |T^{-t}\theta|^n}.$$

Proof. Complete θ to an orthonormal basis $\theta, e_1, \dots, e_{n-1}$ of \mathbb{R}^n . Denote by $V(\{v_i\}_{i=1}^{n-1})$ the volume of the $n-1$ -dimensional parallelepiped spanned by v_1, \dots, v_{n-1} . Now calculate:

$$\left| \text{Jac } T^{(0)}(\theta) \right| = V \left(\left\{ \frac{P_{(T^{-t}\theta)^\perp} T^{-t} e_i}{|T^{-t}\theta|} \right\}_{i=1}^{n-1} \right) = \frac{1}{|T^{-t}\theta|^{n-1}} V \left(\left\{ P_{(T^{-t}\theta)^\perp} T^{-t} e_i \right\}_{i=1}^{n-1} \right).$$

On the other hand, by expanding the determinant of T^{-t} (volume of the parallelepiped spanned by the vectors $T^{-t}\theta, T^{-t}e_1, \dots, T^{-t}e_{n-1}$):

$$|\det(T^{-t})| = V \left(\left\{ P_{(T^{-t}\theta)^\perp} T^{-t} e_i \right\}_{i=1}^{n-1} \right) |T^{-t}\theta|,$$

and so the assertion follows. \square

Lemma 5.10. Let h_1, \dots, h_{n-1} denote 1-homogeneous C^2 functions on \mathbb{R}^n . Then:

$$S(h_1 \circ T^t, \dots, h_{n-1} \circ T^t)(T^{(0)}\theta) = \det(T)^2 |T^{-t}\theta|^{n+1} S(h_1, \dots, h_{n-1})(\theta).$$

Proof. Let us abbreviate $\nabla^2 = \nabla_{\mathbb{R}^n}^2$. Since $\nabla^2(g \circ T^t)(x) = T(\nabla^2 g)(T^t x)T^t$, we calculate, using the fact that $\nabla^2 g$ is -1 -homogeneous if g is 1-homogeneous and the multi-linearity of the mixed discriminant D_n :

$$\begin{aligned} & S(h_1 \circ T^t, \dots, h_{n-1} \circ T^t)(T^{(0)}\theta) \\ &= D_n(\nabla_{\mathbb{R}^n}^2(h_1 \circ T^t)(T^{(0)}\theta), \dots, \nabla_{\mathbb{R}^n}^2(h_{n-1} \circ T^t)(T^{(0)}\theta), T^{(0)}\theta \otimes T^{(0)}\theta) \\ &= |T^{-t}\theta|^{n-3} D_n(T(\nabla^2 h_1)(\theta)T^t, \dots, T(\nabla^2 h_{n-1})(\theta)T^t, T^{-t}\theta \otimes T^{-t}\theta) \\ &= \det(T)^2 |T^{-t}\theta|^{n-3} D_n(\nabla^2 h_1(\theta), \dots, \nabla^2 h_{n-1}(\theta), T^{-2t}\theta \otimes T^{-2t}\theta). \end{aligned}$$

Recalling that $\nabla^2 h_1(\theta) = P_{\theta^\perp} \nabla^2 h_1(\theta) P_{\theta^\perp}$, we proceed by expanding the mixed discriminant in the last entry:

$$\begin{aligned} &= \det(T)^2 |T^{-t}\theta|^{n-3} \langle T^{-2t}\theta, \theta \rangle^2 D_{n-1}(P_{\theta^\perp} \nabla^2 h_1(\theta) P_{\theta^\perp}, \dots, P_{\theta^\perp} \nabla^2 h_{n-1}(\theta) P_{\theta^\perp}) \\ &= \det(T)^2 |T^{-t}\theta|^{n+1} S(h_1, \dots, h_{n-1})(\theta). \end{aligned}$$

□

Proof of Lemma 5.7. Recall that the surface area measure dS_K and the cone measure dV_K are defined as the following measures on S^{n-1} :

$$dS_K = S(h_K, \dots, h_K) d\theta, \quad dV_K = \frac{1}{n} h_K dS_K.$$

Lemma 5.10 implies that:

$$\begin{aligned} dS_{T(K)}(T^{(0)}\theta) &= S(h_K \circ T^t, \dots, h_K \circ T^t)(T^{(0)}\theta) d\theta \\ &= \det(T)^2 |T^{-t}\theta|^{n+1} S(h_K, \dots, h_K)(\theta) d\theta, \end{aligned}$$

and so together with Lemma 5.9:

$$\begin{aligned} dV_{T(K)}(T^{(0)}\theta) &= \frac{1}{n} h_{T(K)}(T^{(0)}\theta) dS_{T(K)}(T^{(0)}\theta) \\ &= \det(T)^2 |T^{-t}\theta|^{n+1} \frac{1}{n} h_K(T^t T^{(0)}\theta) S(h_K, \dots, h_K)(\theta) d\theta \\ &= \det(T)^2 |T^{-t}\theta|^n \frac{1}{n} h_K(\theta) S(h_K, \dots, h_K)(\theta) d\theta = \frac{1}{|\text{Jac } T^{(0)}(\theta)|} |\det(T)| dV_K(\theta), \end{aligned}$$

confirming that $T^{(0)}$ pushes forward dV_K onto $\frac{1}{|\det(T)|} dV_{T(K)}$. □

Proof of Theorem 5.8. We would like to prove that:

$$(T_*^{(0)})^{-1} \circ L_{T(K)} \circ T_*^{(0)} = L_K,$$

or equivalently, that for all $z \in H^2(S^{n-1})$:

$$L_{T(K)}(T_*^{(0)} z)(T^{(0)}\theta) = (L_K z)(\theta). \quad (5.8)$$

By density, it is enough to establish this for $z \in C^2(S^{n-1})$. Recall that in this case:

$$\tilde{L}_K(z) = \frac{S(zh_K, h_K, \dots, h_K)}{S(h_K, \dots, h_K)},$$

so that:

$$\tilde{L}_{T(K)}(T_*^{(0)} z)(T^{(0)} \theta) = \frac{S(T_*^{(0)} z \cdot h_{T(K)}, h_{T(K)}, \dots, h_{T(K)})(T^{(0)} \theta)}{S(h_{T(K)}, \dots, h_{T(K)})(T^{(0)} \theta)}.$$

We think of z as 0-homogeneous ($T_*^{(0)} z(x) = z(T^t x)$) and of course h_K is 1-homogeneous and satisfies $h_{T(K)}(x) = h_K(T^t x)$, and so $w := z h_K$ is 1-homogeneous and satisfies:

$$T_*^{(0)} z \cdot h_{T(K)}(x) = z(T^t x) h(T^t x) = w(T^t x).$$

By Lemma 5.10:

$$\begin{aligned} & S(T_*^{(0)} z \cdot h_{T(K)}, h_{T(K)}, \dots, h_{T(K)})(T^{(0)} \theta) \\ &= S(w \circ T^t, h_K \circ T^t, \dots, h_K \circ T^t)(T^{(0)} \theta) \\ &= \det(T)^2 |T^{-t} \theta|^{n+1} S(w, h_K, \dots, h_K)(\theta), \end{aligned}$$

and in particular:

$$S(h_{T(K)}, \dots, h_{T(K)})(T^{(0)} \theta) = \det(T)^2 |T^{-t} \theta|^{n+1} S(h_K, h_K, \dots, h_K)(\theta).$$

Taking the quotient of the latter two expressions, we verify:

$$\tilde{L}_{T(K)}(T_*^{(0)} z)(T^{(0)} \theta) = (\tilde{L}_K z)(\theta).$$

Since $L = \tilde{L} - \text{Id}$, (5.8) is established. \square

5.3 Spectral Minimization Problem and Potential Extremizers

We now restrict our discussion to origin-symmetric $K \in \mathcal{K}_{+,e}^2$. Recall that:

$$\lambda_{1,e}(-L_K) := \min \sigma(-L_K|_{E_{\text{even}} \cap \mathbf{1}^{\perp, L^2(dV_K)}}),$$

and that Theorem 5.3 implies that:

$$\lambda_{1,e}(-L_K) > 1 \quad \forall K \in \mathcal{K}_{+,e}^2. \quad (5.9)$$

In addition, since the isometry $T_*^{(0)} : L^2(d\tilde{V}_K) \rightarrow L^2(d\tilde{V}_{T(K)})$ clearly maps $E_{\text{even}} \cap \mathbf{1}^{\perp, L^2(dV_K)}$ onto $E_{\text{even}} \cap \mathbf{1}^{\perp, L^2(dV_{T(K)})}$, it follows by Theorem 5.8 that:

$$\lambda_{1,e}(-L_{T(K)}) = \lambda_{1,e}(-L_K) \quad \forall K \in \mathcal{K}_{+,e}^2 \quad \forall T \in GL_n.$$

Corollary 5.4 therefore translates into:

Corollary 5.11. *Given $p < 1$, the validity of the local p -BM conjecture (3.3) for all $K \in \mathcal{K}_{+,e}^2$ is equivalent to the validity of the following lower bound for the minimization problem over the linearly invariant spectral parameter $\lambda_{1,e}(-L_K)$:*

$$\inf_{K \in \mathcal{K}_{+,e}^2/GL_n} \lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}. \quad (5.10)$$

Observe that by F. John's Theorem [24, Theorem 4.2.12], \mathcal{K}_e/GL_n is a compact set of equivalence classes of origin-symmetric convex bodies (with respect to the natural Hausdorff topology \mathcal{C}), the so-called Banach-Mazur compactum. Also note that by Theorem 5.3 (4), $\mathcal{K}_{+,e}^2 \ni K \mapsto \lambda_{1,e}(-L_K)$ is continuous in the C^2 topology. Unfortunately, $\mathcal{K}_{+,e}^2/GL_n$ isn't closed in either of these topologies, and is only a dense subset of the Banach-Mazur compactum. In particular, we do not know how to show from general functional-analytic arguments that the infimum in (5.10) is strictly greater than 1, even though we have the individual estimate (5.9). However, we will see in the next section that we can verify the validity of (4.15) without resorting to compactness arguments for a concrete range of $p < 1$, which translates into the following:

Theorem 5.12. *There exists a constant $c > 0$ so that:*

$$\inf_{K \in \mathcal{K}_{+,e}^2/GL_n} \lambda_{1,e}(-L_K) \geq \frac{n-p_n}{n-1} > 1 \quad \text{where} \quad p_n := 1 - \frac{c}{n^{3/2}}.$$

This provides a positive answer to the qualitative question of whether there is a uniform even spectral-gap for $-L_K$ beyond 1, and so the only remaining question is the quantitative one – how big is it? By Corollary 5.11, the (local) log-BM conjecture ($p = 0$ case) predicts it should be $\frac{n}{n-1}$, an a-priori mysterious quantity. Better insight is gained by inspecting several natural candidates K for being a minimizer in (5.10). As with essentially all minimization problems over linearly invariant parameters in Convexity Theory, there are three immediate suspects:

- $K = B_2^n$, the Euclidean unit-ball. Recalling (5.4), we immediately see that $L_{B_2^n} = \frac{1}{n-1} \Delta_{S^{n-1}}$, where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on S^{n-1} . The spectral decomposition of $\Delta_{S^{n-1}}$ is classical [66, 10], with k -th distinct eigenvalue ($k \geq 0$) equal to $k(k+n-2)$, corresponding to the eigenspace of spherical harmonics of degree k . As expected, spherical harmonics of degree 0 are constant functions, of degree 1 are linear functions ℓ_v , and of degree 2 are homogeneous quadratic harmonic polynomials (which are in particular even). It follows that for $-L_{B_2^n}$:

$$\lambda_0 = 0, \quad \lambda_1 = \dots = \lambda_n = \frac{n-1}{n-1} = 1, \quad \lambda_{1,e} = \lambda_{n+1} = \frac{2n}{n-1},$$

and we see that we get a much better even spectral-gap (corresponding to $p = -n$ in (5.10)) than the conjectured lower bound $\frac{n}{n-1}$. So B_2^n is not a minimizer for (5.10).

Applying Proposition 3.9 (with $p_0 = -n$ and $p = 0$) and the invariance under linear transformations, we obtain:

Theorem 5.13. $\lambda_{1,e}(-L_{B_2^n}) = \frac{2n}{n-1}$; equivalently, the local $(-n)$ -BM inequality (3.3) holds for B_2^n . In particular, there exists a C^2 -neighborhood $N_{B_2^n}$ of B_2^n in $\mathcal{K}_{+,e}^2$ so that for all $T \in GL_n$, for all $K_1, K_0 \in T(N_{B_2^n})$, the local log-BM conjecture (3.4) holds for K_0 and

$$V((1-\lambda) \cdot K_0 + \lambda \cdot K_1) \geq V(K_0)^{1-\lambda} V(K_1)^\lambda \quad \forall \lambda \in [0, 1].$$

This confirms Theorem 1.2 for $q = 2$ for all $n \geq 2$.

- $K = B_\infty^n = [-1, 1]^n$, the unit cube. Note that B_∞^n is not smooth, so $L_{B_\infty^n}$ is not well-defined. However, defining for any $K \in \mathcal{K}_e$:

$$\lambda_{1,e}(K) := \liminf_{\mathcal{K}_{+,e}^2 \ni K_i \rightarrow K \text{ in } C} \lambda_{1,e}(-L_{K_i}), \quad (5.11)$$

we obtain a lower semi-continuous function on the Banach–Mazur compactum \mathcal{K}_e/GL_n , which must attain a minimum. By Theorem 5.12, this minimum is strictly greater than 1. We will verify in Theorem 10.2 that:

$$\lambda_{1,e}(B_\infty^n) = \frac{n}{n-1}.$$

Consequently, we have the following natural interpretation:

Proposition 5.14. *The validity of the local log-BM conjecture (3.4) for all $K \in \mathcal{K}_{+,e}^2$ is equivalent to the validity of the conjecture that the cube $K = B_\infty^n$ is a minimizer of the linearly invariant even spectral-gap $\lambda_{1,e}(K)$:*

$$\min_{K \in \mathcal{K}_e/GL_n} \lambda_{1,e}(K) = \lambda_{1,e}(B_\infty^n).$$

In our opinion, the latter conjecture is extremely natural, and constitutes the best justification for believing that the local log-BM conjecture is true.

- $K = B_1^n$, the unit-ball of ℓ_1^n . This might be the only natural potential counter-example to the log-BM conjecture, and we presently do not know

how to verify the conjecture for it. As before, B_1^n is not smooth so $L_{B_1^n}$ is not well-defined. However, we will verify in Corollary 10.3 that:

$$\lambda_{1,\text{uncond}}(K) \geq \frac{n}{n-1} \quad \forall K \in \mathcal{K}_{\text{uncond}}.$$

Here $\mathcal{K}_{\text{uncond}}$, $\mathcal{K}_{+,\text{uncond}}^2$ and E_{uncond} denote the unconditional elements of \mathcal{K} , \mathcal{K}_+^2 and $H^2(S^{n-1})$, respectively, meaning that they are invariant under reflections with respect to the coordinate hyperplanes; for $K \in \mathcal{K}_{+,\text{uncond}}^2$ we define:

$$\lambda_{1,\text{uncond}}(-L_K) := \min \sigma(-L_K|_{E_{\text{uncond}} \cap \mathbf{1}^\perp}),$$

and for $K \in \mathcal{K}_{\text{uncond}}$ we set:

$$\lambda_{1,\text{uncond}}(K) := \liminf_{\mathcal{K}_{+,\text{uncond}}^2 \ni K_i \rightarrow K \text{ in } C} \lambda_{1,\text{uncond}}(-L_{K_i}). \quad (5.12)$$

In particular, we have $\lambda_{1,\text{uncond}}(B_1^n) \geq \frac{n}{n-1}$, which is a good sign.

While the p -BM conjecture pertains to the minimization problem (5.10), it also makes sense to consider the corresponding *maximization* problem. In view of the above examples, we make the following:

Conjecture 5.15.

$$\max_{K \in \mathcal{K}_{+,e}^2/GL_n} \lambda_{1,e}(-L_K) = \frac{2n}{n-1},$$

with equality for origin-symmetric ellipsoids $K = T(B_2^n)$.

6 Obtaining Estimates via the Reilly Formula

We are finally ready to prove our main results in this work. These are based on an integral formula obtained by twice integrating-by-parts the Bochner–Lichnerowicz–Weitzenböck identity, which in the Riemannian setting is due to Reilly [56]. In the description below, we specialize the Reilly formula to our Euclidean setting (see [39, Theorem 1.1] for a proof of a more general version, which holds on weighted Riemannian manifolds, and involves an additional curvature term).

We denote by ∇ the Euclidean connection, and by Δ the Euclidean Laplacian. We denote by $\|\nabla^2 u\|$ the Hilbert-Schmidt norm of the Euclidean Hessian $\nabla^2 u$. If $\Omega \subset \mathbb{R}^n$ is a compact set with Lipschitz boundary, we denote by $\mathcal{S}_0(\Omega)$ the class of functions u on Ω which are in $C^2(\text{int}(\Omega)) \cap C^1(\Omega)$. We shall henceforth assume that

$\partial\Omega$ is C^2 smooth with outer normal $\nu = \nu_{\partial\Omega}$, and denote by $\mathcal{S}_N(\Omega)$ the elements in $\mathcal{S}_0(\Omega)$ which satisfy $u_\nu := \langle \nabla u, \nu \rangle \in C^1(\partial\Omega)$. Let $\mu = \exp(-V(x))dx$ denote a measure on Ω with $V \in C^2(\Omega)$, and denote $\mu_{\partial\Omega} = \exp(-V(x))d\mathcal{H}^{n-1}|_{\partial\Omega}(x)$. Introduce the following weighted Laplacian, defined by:

$$L_\mu u := \Delta u - \langle \nabla V, \nabla u \rangle \quad \forall u \in C^2(\text{int}(\Omega));$$

it satisfies the following weighted integration-by-parts property (see [39, Remark 2.2]):

$$\int_{\Omega} L_\mu u \, d\mu = \int_{\partial\Omega} u_\nu d\mu_{\partial\Omega} \quad \forall u \in \mathcal{S}_0(\Omega). \quad (6.1)$$

As usual, we denote by $\text{II}_{\partial\Omega}$ the second fundamental form of $\partial\Omega \subset \mathbb{R}^n$, and define its generalized mean curvature at $x \in \partial\Omega$ as:

$$H_{\partial\Omega, \mu} := \text{tr}(\text{II}_{\partial\Omega}) - \langle \nabla V, \nu \rangle.$$

Finally, $\nabla_{\partial\Omega}$ denotes the induced connection on $\partial\Omega$.

Theorem (Generalized Reilly Formula). *For any function $u \in \mathcal{S}_N(\Omega)$:*

$$\begin{aligned} \int_{\Omega} (L_\mu u)^2 d\mu &= \int_{\Omega} \|\nabla^2 u\|^2 d\mu + \int_{\Omega} \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu + \\ &\int_{\partial\Omega} H_{\partial\Omega, \mu} u_\nu^2 d\mu_{\partial\Omega} + \int_{\partial\Omega} \langle \text{II}_{\partial\Omega} \nabla_{\partial\Omega} u, \nabla_{\partial\Omega} u \rangle d\mu_{\partial\Omega} - 2 \int_{\partial\Omega} \langle \nabla_{\partial\Omega} u_\nu, \nabla_{\partial\Omega} u \rangle d\mu_{\partial\Omega}. \end{aligned} \quad (6.2)$$

We will also use the following classical existence and regularity results for linear elliptic PDEs (e.g. [26, Chapter 8], [46, Chapter 5], [41, Chapter 3]):

Theorem. *Let $f \in C^\alpha(\text{int}(\Omega))$ for some $\alpha \in (0, 1)$, let $\Psi \in C^1(\partial\Omega)$, and assume that:*

$$\int_{\Omega} f d\mu = \int_{\partial\Omega} \Psi d\mu_{\partial\Omega}. \quad (6.3)$$

Then there exists a function $u \in C_{loc}^{2, \alpha}(\text{int}(\Omega)) \cap C^{1, \beta}(\Omega)$ for all $\beta \in (0, 1)$, which solves the following Poisson equation with Neumann boundary conditions:

$$L_\mu u = f \text{ in } \text{int}(\Omega), \quad u_\nu = \Psi \text{ on } \partial\Omega. \quad (6.4)$$

Moreover, u is unique up to an additive constant.

Note that in particular, the function u above is in $\mathcal{S}_N(\Omega)$. By (6.1), the compatibility condition (6.3) is also a necessary condition for solving (6.4).

6.1 A sufficient condition for confirming the local p -BM inequality

We now derive a sufficient condition for confirming the local p -BM inequality (3.3) in its equivalent infinitesimal form (4.16) on ∂K . Our motivation comes from our previous work [36], where we obtained a new proof of the (local, and hence global) Brunn–Minkowski inequality ($p = 1$ case), by verifying (4.16) directly (for all test functions $\Psi \in C^1(\partial K)$, without any evenness assumption). In fact, our proof in [36] applies to a general weighted Riemannian manifold satisfying the Curvature-Dimension condition $\text{CD}(0, N)$, yielding a novel interpretation of Minkowski addition in the Riemannian setting.

Given $K \in \mathcal{K}_{+,e}^2$, let μ denote the Lebesgue measure dx on $\Omega = K$ (corresponding to $V = 0$, $L_\mu = \Delta$ and $H_{\partial K, \mu} = H_{\partial K}$ above). Given $\Psi \in C_e^1(\partial K)$ with $\int_{\partial K} \Psi dx = 0$, the classical L^2 -method consists of solving for $u \in \mathcal{S}_N(K)$ the Laplace equation:

$$\Delta u = 0 \text{ in } \text{int}(K), \quad u_\nu = \Psi \text{ on } \partial K,$$

(which clearly satisfies the necessary and sufficient compatibility condition (6.3)). The origin-symmetry of K and evenness of Ψ guarantee that $u(-x)$ is also a solution, and so by uniqueness of the solution it follows that u is necessarily even; we denote by $\mathcal{S}_{N,e}(K)$ the even elements of $\mathcal{S}_N(K)$ (and similarly for $\mathcal{S}_{0,e}(K)$). We see that the above procedure yields a bijection between $\Psi \in C_e^1(\partial K)$ and harmonic $u \in \mathcal{S}_{N,e}(K)$, characterized by the property that $u_\nu = \Psi$.

Now, applying the Reilly formula (6.2) to u , using that $\Delta u = 0$, and plugging in the resulting expression for $\int_{\partial K} H_{\partial K} u_\nu^2 dx$ into (4.16), we obtain:

Theorem 6.1. *Given $K \in \mathcal{K}_{+,e}^2$ and $p < 1$, the local p -BM conjecture (3.3) for K is equivalent to the assertion that:*

$$\begin{aligned} \forall u \in \mathcal{S}_{N,e}(K) \quad \Delta u = 0 \text{ in } \text{int}(K) \quad \Rightarrow \\ \int_K \|\nabla^2 u\|^2 dx \geq (1-p) \int_{\partial K} \frac{u_\nu^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx - R_K(u), \end{aligned}$$

where:

$$\begin{aligned} R_K(u) := & \int_{\partial K} \langle II_{\partial K} \nabla_{\partial K} u, \nabla_{\partial K} u \rangle dx \\ & + \int_{\partial K} \langle II_{\partial K}^{-1} \nabla_{\partial K} u_\nu, \nabla_{\partial K} u_\nu \rangle dx - 2 \int_{\partial K} \langle \nabla_{\partial K} u_\nu, \nabla_{\partial K} u \rangle dx. \end{aligned}$$

In particular, as $R_K(u) \geq 0$ by Cauchy–Schwarz (since $II_{\partial K} > 0$), a sufficient condition for the local p -BM conjecture to hold for K is that:

$$\forall u \in \mathcal{S}_{0,e}(K) \quad \Delta u = 0 \text{ in } \text{int}(K) \quad \Rightarrow \quad \int_K \|\nabla^2 u\|^2 dx \geq (1-p) \int_{\partial K} \frac{u_\nu^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx.$$

Remark 6.2. For future reference, we mention the following alternative expression for $R_K(u)$ when $u \in C^2(K)$:

$$R_K(u) = \int_{\partial K} \langle \Pi_{\partial K}^{-1} P_{T_{\partial K}} [\nabla^2 u \cdot \nu], P_{T_{\partial K}} [\nabla^2 u \cdot \nu] \rangle dx,$$

where $P_{T_{\partial K}}$ denotes projection to the tangent space to ∂K . Indeed, this follows by plugging above:

$$\nabla_{\partial K} u_\nu = \nabla_{\partial K} \langle \nabla u, \nu \rangle = \Pi_{\partial K} \nabla_{\partial K} u + P_{T_{\partial K}} [\nabla^2 u \cdot \nu].$$

The sufficient condition of Theorem 6.1 naturally leads us to the following:

Definition ($B_H(K)$ and $B(K)$). *Given $K \in \mathcal{K}_e$, let $B_H(K)$ denote the best constant B in the following boundary Poincaré-type inequality for harmonic functions:*

$$\forall u \in \mathcal{S}_{0,e}(K) \quad \Delta u = 0 \text{ in } \text{int}(K) \Rightarrow \int_{\partial K} \frac{u_\nu^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx \leq B \int_K \|\nabla^2 u\|^2 dx.$$

Without the requirement that $\Delta u = 0$, the above inequality is called a boundary Poincaré-type inequality, and the best constant B above is denoted by $B(K)$.

Note that all expressions above are well-defined without any smoothness or strict convexity assumptions on ∂K , since $\nu_{\partial K}(x)$ exists for \mathcal{H}^{n-1} -a.e. $x \in \partial K$. We also take this opportunity to introduce:

Definition ($D(K)$). *Given $K \in \mathcal{K}$, let $D(K)$ denote the best constant in the following absolute boundary Poincaré-type inequality:*

$$\forall u \in \mathcal{S}_0(K) \quad \int_K \vec{\nabla} u \, dx = \vec{0} \Rightarrow \int_{\partial K} \frac{|\nabla u(x)|^2}{\langle x, \nu_{\partial K}(x) \rangle} dx \leq D(K) \int_K \|\nabla^2 u\|^2 dx.$$

Note that the evenness assumption on u from the former definitions has been replaced by a balancing condition in the latter one, and that the u_ν^2 term has been replaced by (the possibly larger) $|\nabla u|^2$ one. Since ∇u is odd for any even function u , and hence integrates to zero on any origin-symmetric K , it immediately follows that:

$$B_H(K) \leq B(K) \leq D(K) \quad \forall K \in \mathcal{K}_e.$$

We will soon see that $D(K) < \infty$ for all $K \in \mathcal{K}$, so the above definitions are non-trivial. It is easy to see that the constants $B_H(K)$, $B(K)$ and $D(K)$ are invariant under homothety $K \mapsto \lambda K$. However, they are no longer invariant under general linear transformations as in Section 5. This can be seen from the following:

Example 6.3. For a 2-dimensional cube $K = [-a, a] \times [-b, b]$, $B_H(K) \geq \frac{1}{6} \left(\frac{b^2}{a^2} + \frac{a^2}{b^2} \right)$, as witnessed by the even harmonic function $u = x_1 x_2$. Consequently, $B_H(K), B(K), D(K) \rightarrow \infty$ as the aspect-ratio of K grows to infinity, demonstrating the absence of invariance under GL_n .

Consequently, while the validity of the first condition of Theorem 6.1 is invariant under GL_n (being equivalent to the local p -BM conjecture), the validity of the second sufficient condition is not, and requires putting K in a “good position”, i.e. a suitable linear image. This is due to application of the Cauchy–Schwarz inequality when transitioning from the first formulation to the second, which identifies between tangent and cotangent spaces, and thus destroys the natural covariance–contravariance enjoyed by the first formulation. We summarize all of the relevant information we have obtained thus far in the following:

Theorem 6.4. Given $K \in \mathcal{K}_{+,e}^2$, assume that $B_H(T_0(K)) \leq \frac{1}{1-p}$ for some $p < 1$ and $T_0 \in GL_n$. Then the local p -BM conjecture (3.3) holds for $T(K)$ for all $T \in GL_n$.

In the notation of Section 5, we equivalently have:

Theorem 6.5. For all $K \in \mathcal{K}_{+,e}^2$, $\lambda_{1,e}(-L_K) \geq 1 + \sup_{T \in GL_n} \frac{1}{(n-1)B_H(T(K))}$.

Proof. This follows immediate from Corollary 5.4, which asserts that $\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}$ if and only if the local p -BM conjecture (3.3) holds for K , and the sufficient condition for its validity for $T(K)$ given by Theorem 6.4. \square

6.2 General Estimate on $D(K)$

Unfortunately, getting a handle on $B_H(K)$ or $B(K)$ directly is quite a formidable task, and it is easier to upper bound the larger $D(K)$ constant. Recall that the Poincaré constant $C_{Poin}(K)$ is defined as the best constant in the following inequality:

$$\forall f \in C^1(\text{int}(K)) \quad \int_K f(x) dx = 0 \quad \Rightarrow \quad \int_K f^2(x) dx \leq C_{Poin}^2(K) \int_K |\nabla f|^2 dx.$$

Equivalently, $1/C_{Poin}^2(K)$ is the first positive eigenvalue of the Neumann Laplacian on K .

Theorem 6.6. Let $K \in \mathcal{K}$ and assume that $rB_2^n \subset K \subset RB_2^n$. Then:

$$D(K) \leq \frac{1}{r^2} (C_{Poin}^2(K)n + 2C_{Poin}(K)R).$$

Proof. Let $u \in \mathcal{S}_0(K)$ be such that $\int_K \vec{\nabla} u \, dx = \vec{0}$. Applying Cauchy-Schwarz, we have for any $\lambda > 0$:

$$\operatorname{div}(|\nabla u|^2 x) = n |\nabla u|^2 + 2 \langle \nabla^2 u \cdot \nabla u, x \rangle \leq n |\nabla u|^2 + \lambda \|\nabla^2 u\|^2 + \frac{1}{\lambda} |\nabla u|^2 |x|^2.$$

Using the assumption that $\langle x, \nu_{\partial K}(x) \rangle = h_K(\nu_{\partial K}(x)) \geq r$, integrating by parts, and finally that $K \subset RB_2^n$, we obtain:

$$\begin{aligned} \int_{\partial K} \frac{|\nabla u|^2}{\langle x, \nu_{\partial K}(x) \rangle} dx &\leq \frac{1}{r^2} \int_{\partial K} |\nabla u|^2 \langle x, \nu_{\partial K}(x) \rangle dx = \frac{1}{r^2} \int_K \operatorname{div}(|\nabla u|^2 x) dx \\ &\leq \left(\frac{n}{r^2} + \frac{R^2}{r^2} \frac{1}{\lambda} \right) \int_K |\nabla u|^2 dx + \frac{\lambda}{r^2} \int_K \|\nabla^2 u\|^2 dx. \end{aligned}$$

Since $\int_K u_i dx = 0$ for all $i = 1, \dots, n$, by applying the Poincaré inequality to u_i and summing the resulting inequalities, we obtain:

$$\int_K |\nabla u|^2 dx \leq C_{Poin}^2(K) \int_K \|\nabla^2 u\|^2 dx.$$

It follows that for all $\lambda > 0$:

$$\int_{\partial K} \frac{|\nabla u|^2}{\langle x, \nu_{\partial K}(x) \rangle} dx \leq \left(\left(\frac{n}{r^2} + \frac{R^2}{r^2} \frac{1}{\lambda} \right) C_{Poin}^2(K) + \frac{\lambda}{r^2} \right) \int_K \|\nabla^2 u\|_{HS}^2 dx.$$

Using the optimal $\lambda = C_{Poin}(K)R$ and recalling the definition of $D(K)$, the assertion is established. \square

Corollary 6.7. *For all $K \in \mathcal{K}$, $D(K) < \infty$.*

Proof. Any $K \in \mathcal{K}$ satisfies $rB_2 \subset K \subset RB_2^n$ with some $r, R > 0$. By a well-known theorem of Payne–Weinberger [55], for any $K \in \mathcal{K}$ we have $C_{Poin}(K) \leq \frac{D}{\pi}$ where D is the diameter of K . As $D \leq 2R$, the assertion follows by Theorem 6.6. \square

In order to apply Theorem 6.4, we would like to apply our estimate for $D(K)$ in a good position of $K \in \mathcal{K}$. Recall that the isotropic position is defined as an affine image of $K \in \mathcal{K}$ having barycenter at the origin and for which:

$$\int_K \langle x, \theta \rangle^2 dx = |\theta|^2 \quad \forall \theta \in S^{n-1}.$$

It is well known [53] that such a position always exists and is unique up to orthogonal transformations. In this position, we have the following (sharp) estimates on the in and out radii of K [33]:

$$\sqrt{\frac{n+2}{n}} B_2^n \subset K \subset \sqrt{(n+2)n} B_2^n.$$

As for the Poincaré constant in isotropic position, a bold conjecture of Kannan, Lovász and Simonovits [33] predicts that $C_{Poin}(K) \leq C$ for some universal numeric constant $C > 1$, independent of the dimension n . The current best known estimate on the KLS conjecture has recently been improved by Lee and Vempala [45], who showed that:

$$C_{Poin}(K) \leq C' \sqrt[4]{n} \tag{6.5}$$

for all isotropic $K \in \mathcal{K}$. We immediately deduce from Theorem 6.6 the following:

Corollary 6.8. *There exists a universal numeric constant $C > 1$, independent of the dimension n , so that for all $K \in \mathcal{K}$ in isotropic position, $D(K) \leq Cn^{3/2}$. Assuming a positive answer to the KLS conjecture, the latter estimate may be improved to $D(K) \leq Cn$.*

Since any $K \in \mathcal{K}_e$ has a linear isotropic image and satisfies $B_H(K) \leq D(K)$, Theorem 6.4 in conjunction with Corollary 6.8 and Lemma 3.5 immediately yield Theorem 1.1, which we re-state as follows:

Theorem 6.9. *The local p -BM conjecture (3.3) holds for all $K \in \mathcal{K}_{+,e}^2$ and $p \in [1 - \frac{1}{Cn^{3/2}}, 1)$. Equivalently, for all p in this range, for all $K_0, K_1 \in \mathcal{K}_{+,e}^2$ so that $(1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1 \in \mathcal{K}_{+,e}^2$ for all $\lambda \in [0, 1]$, we have:*

$$V((1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1) \geq \left((1 - \lambda)V(K_0)^{\frac{p}{n}} + \lambda V(K_1)^{\frac{p}{n}} \right)^{\frac{n}{p}} \quad \forall \lambda \in [0, 1].$$

6.3 Examples

Of course, using the known estimates on $C_{Poin}(K)$ which improve over the general (6.5) for various classes of convex bodies K (see e.g. [34, 3, 32, 37, 38]), one may obtain an improved estimate for p above. It will be instructive in this work to concentrate on the unit-balls of ℓ_q^n , denoted B_q^n .

Theorem 6.10 (Sodin, Latała–Wojtaszczyk). *For all $q \in [1, \infty]$, $C_{Poin}(B_q^n)$ is of the order of $n^{-\frac{1}{q}}$.*

Proof. It was shown by S. Sodin [62] for $q \in [1, 2]$ and by R. Latała and J. Wojtaszczyk [42] for $q \in [2, \infty]$ that if $\lambda_q^n B_q^n$ has volume 1 then $\lambda_q^n C_{Poin}(B_q^n) = C_{Poin}(\lambda_q^n B_q^n)$ is of the order of 1. An easy and well-known computation verifies that λ_q^n is of the order of $n^{1/q}$, yielding the claim. \square

Lemma 6.11. *For any $K \in \mathcal{K}_e$, $D(K) \geq B(K) \geq 1$.*

Proof. Testing the even function $u(x) = \frac{|x|^2}{2}$, note that:

$$\int_{\partial K} \frac{u_\nu^2}{\langle x, \nu_{\partial K}(x) \rangle} dx = \int_{\partial K} \langle x, \nu_{\partial K}(x) \rangle dx = \int_K \operatorname{div}(x) dx = nV(K) = \int_K \|\nabla^2 u\|^2 dx.$$

It follows by definition that $D(K) \geq B(K) \geq 1$. \square

Lemma 6.12. *For any $q \in [1, 2]$, $1 \leq D(B_q^n) \leq C$. For any $q \in [2, \infty]$, $D(B_q^n) \leq Cn^{1-\frac{2}{q}}$.*

Proof. The lower estimate is given by the previous lemma. The upper bound follows from the general estimate of Theorem 6.6 in combination with Theorem 6.10 and the obvious estimates $rB_2^n \subset B_q^n \subset RB_2^n$ with $r = n^{\min(0, \frac{1}{2}-\frac{1}{q})}$ and $R = n^{\max(0, \frac{1}{2}-\frac{1}{q})}$. \square

For the cube B_∞^n , we can obtain rather tightly matching lower and upper estimates.

Lemma 6.13. $\frac{1}{3}n \leq D(B_\infty^n) \leq \frac{4}{\pi^2}n + \frac{4}{\pi}\sqrt{n}$.

Proof. The upper bound follows from the general estimate of Theorem 6.6, using $B_2^n \subset B_\infty^n \subset \sqrt{n}B_2^n$ and $C_{Poin}^2(B_\infty^n) = \frac{4}{\pi^2}$ [43]. For the lower bound, consider the function $u(x) = x_1^2/2 \in \mathcal{S}_{0,e}(B_\infty^n)$, for which $u_1(x) = x_1$ and $u_i(x) = 0$ for all $i = 2, \dots, n$. Calculating the contribution on each boundary facet, we have:

$$\int_{\partial B_\infty^n} \frac{|\nabla u|^2}{\langle x, \nu \rangle} dx = 2 \cdot 2^{n-1} + 2(n-1)2^{n-2} \int_{-1}^1 x_1^2 dx_1,$$

and clearly:

$$\int_{B_\infty^n} \|\nabla^2 u\|^2 dx = 2^n.$$

Taking the quotient of these two expressions, we see that $D(B_\infty^n) \geq 1 + (n-1)/3$. \square

Remark 6.14. Is not hard to improve the constant $\frac{1}{3}$ in Lemma 6.13 to $\frac{3}{8}$ by using a function $u(x) = u(x_1)$ so that $u_1(x_1) = x_1/\epsilon \vee -1 \wedge +1$ for an appropriate $\epsilon > 0$. In addition, we see that the conjectural estimate $D(K) \leq Cn$ for isotropic K , which by Corollary 6.8 would follow from a positive answer to the KLS conjecture, is best possible (up to the value of the constant C).

The above examples demonstrate that the general estimate given by Theorem 6.6 is in fact fairly accurate in a variety of situations, and so in order to make further progress on the local p -BM conjecture, it is best to work with the $B_H(K)$ or $B(K)$ constants. Indeed, we will see in the next sections that $B_H(B_2^n) = \frac{2}{n+2} < 1$ and $B_H(B_\infty^n) = 1$, which are better by an order of n from the corresponding values of $D(B_2^n), D(B_\infty^n)$ estimated above. Furthermore, we will see that when $q \in (2, \infty)$, for $n \geq n_q$ large enough, $B_H(B_q^n) < 1$. In view of these results and examples, we make the following:

Conjecture 6.15. *For all $K \in \mathcal{K}_e$, there exists $T_0 \in GL_n$ so that $B_H(T_0(K)) \leq 1$.*

By Theorem 6.4, a positive answer to the latter conjecture will imply a positive answer to the local log-BM conjecture.

7 The second Steklov operator and $B_H(B_2^n)$

In this section, we obtain an operator-theoretic interpretation of the inequality:

$$\forall u \in \mathcal{S}_{0,e}(K) \quad \Delta u = 0 \text{ in int}(K) \quad \Rightarrow \quad \int_{\partial K} \frac{u_\nu^2}{\langle x, \nu_{\partial K}(x) \rangle} dx \leq B_H(K) \int_K \|\nabla^2 u\|^2 dx, \quad (7.1)$$

which we will use for calculating $B_H(B_2^n)$. It is related to the classical Steklov (or Dirichlet-to-Neumann) 1st order elliptic pseudo-differential operator S [2, 27].

7.1 Second Steklov operator

Let us assume for simplicity that ∂K is C^∞ smooth, and denote by $C_0^\infty(\partial K)$ the subspace of smooth functions integrating to zero on ∂K . The Neumann-to-Dirichlet operator D , which is the inverse of S on $C_0^\infty(\partial K)$, is the linear operator defined by:

$$D : C_0^\infty(\partial K) \ni \Psi \mapsto u|_{\partial K} \in C^\infty(\partial K),$$

where $u = u_\Psi \in C^\infty(K)$ solves:

$$\Delta u = 0 \text{ in } K, \quad u_\nu = \Psi \text{ on } \partial K.$$

In fact, D may be extended to a compact operator [2] on:

$$L_0^2(\partial K) := \left\{ \Psi \in L^2(dx|_{\partial K}); \int_{\partial K} \Psi dx = 0 \right\},$$

(and moreover to the Sobolev space $H_0^{-1/2}(\partial K)$, but we will not require this here).

Note that D is self-adjoint and positive semi-definite on $L_0^2(\partial K)$, since for all $\Psi, \Phi \in C_0^\infty(\partial K)$, denoting $v = u_\Psi$ and $w = u_\Phi$, we have (integrating by parts and using that $\Delta w = 0$):

$$\int_{\partial K} (D\Psi)\Phi dx = \int_{\partial K} v w_\nu dx = \int_K \operatorname{div}(v \nabla w) dx = \int_K \langle \nabla v, \nabla w \rangle dx.$$

By analogy, we introduce the second Steklov operator S_2 , by requiring that:

$$\int_{\partial K} (S_2\Psi)\Phi dx = \int_K \langle \nabla^2 v, \nabla^2 w \rangle dx. \quad (7.2)$$

Indeed, on $C_0^\infty(\partial K)$, $S_2\Psi$ has the following explicit description:

$$S_2\Psi := -\Delta_{\partial K}(D\Psi) - D(\Delta_{\partial K}\Psi) - H_{\partial K}\Psi + D\nabla_{\partial K} \cdot \Pi_{\partial K} \nabla_{\partial K} D\Psi,$$

where of course $\nabla_{\partial K} \cdot$ denotes the divergence operator on ∂K . To see this, denote $v = u_\Psi$ (so that $v_\nu = \Psi$), integrate by parts on ∂K , use the self-adjointness of D , and finally apply the Reilly formula (6.2), to obtain:

$$\begin{aligned} \int_{\partial K} (S_2\Psi)\Psi dx &= - \int_{\partial K} \langle \Delta_{\partial K} v, v_\nu \rangle dx - \int_{\partial K} \langle D\Delta_{\partial K} v_\nu, v_\nu \rangle - \int_{\partial K} H_{\partial K} v_\nu^2 dx \\ &+ \int_{\partial K} (D\nabla_{\partial K} \cdot \Pi_{\partial K} \nabla_{\partial K} v) v_\nu dx = 2 \int_{\partial K} \langle \nabla_{\partial K} v, \nabla_{\partial K} v_\nu \rangle - \int_{\partial K} H_{\partial K} v_\nu^2 dx \\ &- \int_{\partial K} \langle \Pi_{\partial K} \nabla_{\partial K} v, \nabla_{\partial K} v \rangle dx = \int_K \|\nabla^2 v\|^2 dx, \end{aligned}$$

and so (7.2) follows by polarization. In particular, (7.2) implies that S_2 is symmetric and positive semi-definite on $L_0^2(\partial K)$, and hence admits a Friedrichs self-adjoint extension. Note that as S is of order 1, D is of order -1 , and hence S_2 is also of order 1, like S , explaining our nomenclature.

Recalling (7.1), we see that $B_H(K)$ for $K \in \mathcal{K}_e^\infty$ is the best constant in the following inequality for the second Steklov operator:

$$\forall \Psi \in C_{0,e}^\infty(\partial K) \quad \int_{\partial K} (S_2\Psi)\Psi dx \geq \frac{1}{B_H(K)} \int_{\partial K} \frac{\Psi^2}{\langle x, \nu_{\partial K}(x) \rangle} dx.$$

In this sense, we can think of the sufficient condition of Theorem 6.1 as a 1st order relaxation (via the second Steklov operator) of the original 2nd order spectral problem (for the Hilbert–Brunn–Minkowski operator).

7.2 Computing $B_H(B_2^n)$

When $K = B_2^n$, $B_H(B_2^n)$ is the best constant in:

$$\forall \Psi \in C_{0,e}^\infty(S^{n-1}) \quad \int_{S^{n-1}} (S_2 \Psi) \Psi d\theta \geq \frac{1}{B_H(B_2^n)} \int_{S^{n-1}} \Psi^2 d\theta,$$

and so $B_H(B_2^n)$ is the reciprocal of the first eigenvalue of S_2 corresponding to an even eigenfunction in $C_0^\infty(S^{n-1})$. As $H_{S^{n-1}} \equiv n-1$ and $\Pi_{S^{n-1}} = \delta_{S^{n-1}}$, we see that:

$$S_2 = -\Delta_{S^{n-1}} D - D \Delta_{S^{n-1}} - (n-1)\text{Id} + D \Delta_{S^{n-1}} D. \quad (7.3)$$

As both operators $\Delta_{S^{n-1}}$ and D clearly intertwine the natural action of $SO(n)$ on $L_0^2(\partial K)$, so does S_2 . It follows by Schur's lemma [66] that the eigenspaces of S_2 are given by H_k , the subspace of degree k spherical harmonics on S^{n-1} , for $k \geq 1$ ($k=0$ is excluded since we are in $C_0^\infty(S^{n-1})$). For $h \in H_k$ it is well known [66, 10] that $-\Delta_{S^{n-1}} h = k(k+n-2)h$. In addition, h is already the restriction to S^{n-1} of a harmonic homogeneous polynomial of degree k on B_2^n , which we continue to denote by h ; it follows by Euler's identity that $h_\nu = kh$, and so by definition $Dh = \frac{1}{k}h$. Consequently, (7.3) yields a complete description of the spectral decomposition of S_2 :

$$S_2|_{H_k} = \left(2\frac{k(k+n-2)}{k} - (n-1) - \frac{k(k+n-2)}{k^2} \right) \text{Id}|_{H_k}.$$

It follows that the first even eigenfunction of S_2 lies in H_2 (quadratic harmonic polynomials), with corresponding eigenvalue $1 + \frac{n}{2}$. We thus obtain:

Theorem 7.1. $B_H(B_2^n) = \frac{2}{n+2} < 1$.

By Theorem 6.4, this corresponds to sufficient condition for confirming the local p -BM conjecture with $p = -\frac{n}{2}$. Note that this is worse by a factor of 2 than the equivalent characterization from Subsection 5.3 using $\lambda_{1,e}(-L_{B_2^n}) = \frac{2n}{n-1}$, which corresponds to $p = -n$. This means that the Cauchy-Schwarz inequality we have employed in Theorem 6.1 is indeed wasteful for B_2^n , but still we obtain a good enough condition to reaffirm the local log-BM conjecture (case $p=0$) for B_2^n (and its C^2 -perturbations), as $B_H(B_2^n) < 1$.

8 Unconditional Convex Bodies and the Cube

It is also a challenging task to compute $B(K)$ even for some concrete convex bodies. In this section, we precisely compute the variant $B_{\text{uncond}}(K)$, when only testing

unconditional functions on an unconditional convex body K . In the case of the cube B_∞^n , we also manage to precisely compute $B(B_\infty^n)$ and consequently $B_H(B_\infty^n)$, in precise agreement with the worst-possible predicted value by the local log-BM conjecture.

8.1 Unconditional Convex Bodies

Let $\mathcal{K}_{\text{uncond}}$ denote the class of unconditional convex bodies, namely convex bodies which are invariant under reflections with respect to the coordinate hyperplanes $\{x_i = 0\}$, $i = 1, \dots, n$. We denote by $u \in \mathcal{S}_{0, \text{uncond}}(K)$ the elements of $\mathcal{S}_0(K)$ which are invariant under the aforementioned reflections.

Definition ($B_{\text{uncond}}(K)$). *Given $K \in \mathcal{K}_{\text{uncond}}$, let $B_{\text{uncond}}(K)$ denote the best constant in the following boundary Poincaré-type inequality for unconditional functions:*

$$\forall u \in \mathcal{S}_{0, \text{uncond}}(K) \quad \int_{\partial K} \frac{u_\nu^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx \leq B_{\text{uncond}}(K) \int_K \|\nabla^2 u\|^2 dx.$$

Observe that $B_{\text{uncond}}(K) \geq 1$, by testing the unconditional function $u(x) = \frac{|x|^2}{2}$ as in Lemma 6.11. Note that when $K \in \mathcal{K}_{\text{uncond}}$, it is easy to see that $\nu_i x_i \geq 0$ for all $i = 1, \dots, n$ and \mathcal{H}^{n-1} -a.e. $x \in \partial K$. The Cauchy–Schwarz inequality immediately yields:

Lemma 8.1. *For all $K \in \mathcal{K}_{\text{uncond}}$ and $u \in C^1(K)$, we have for \mathcal{H}^{n-1} -a.e. $x \in \partial K$:*

$$\frac{u_\nu^2}{\langle x, \nu \rangle} = \frac{(\sum_{i=1}^n u_i \nu_i)^2}{\sum_{i=1}^n x_i \nu_i} \leq \sum_{i=1}^n \frac{u_i^2}{x_i} \nu_i.$$

We also have the following lemma, inspired by the method in our previous work [37]:

Lemma 8.2. *For any $K \in \mathcal{K}$, $u \in \mathcal{S}_0(K)$ and $i = 1, \dots, n$ so that $u_i \equiv 0$ on $K \cap \{x_i = 0\}$, we have:*

$$\int_{\partial K} \frac{u_i^2}{x_i} \nu_i = \int_K \left(2 \frac{u_i}{x_i} u_{ii} - \frac{u_i^2}{x_i^2} \right) dx \leq \int_K u_{ii}^2 dx.$$

Proof. The first identity follows by integration-by-parts on $K_+ := K \cap \{x_i \geq 0\}$ and $K_- := K \cap \{x_i \leq 0\}$ separately. The assumption that $u_i \equiv 0$ on $K \cap \{x_i = 0\}$

and $u \in \mathcal{S}_0(K)$ are crucial here, to ensure that $\lim_{x \rightarrow x^0} \frac{u_i(x)}{x_i} = u_{ii}(x^0)$ if $x^0 \in \text{int}(K) \cap \{x_i = 0\}$. Defining:

$$\xi(x) := \begin{cases} \frac{u_i^2}{x_i} e_i & K \cap \{x_i > 0\} \\ 0 & K \cap \{x_i = 0\} \end{cases},$$

it follows that the vector field ξ is in $C^1(\text{int}(K_+) \cap C(K_+))$, and hence integrating by parts:

$$\int_{\partial K \cap \{x_i \geq 0\}} \langle \xi, \nu_{\partial K} \rangle dx = \int_{\partial K_+} \langle \xi, \nu_{\partial K_+} \rangle dx = \int_{K_+} \text{div}(\xi) dx.$$

Repeating the argument for K_- and summing, the first equality follows. The second inequality follows by applying the Cauchy–Schwarz (Geometric–Arithmetic mean) inequality $2ab \leq a^2 + b^2$. \square

Applying the previous two lemmas and summing over $i = 1, \dots, n$, we deduce:

Theorem 8.3. *Let $K \in \mathcal{K}_{\text{uncond}}$. Then for all $u \in \mathcal{S}_0(K)$ such that $u_i \equiv 0$ on $K \cap \{x_i = 0\}$ for all $i = 1, \dots, n$, we have:*

$$\int_{\partial K} \frac{u_\nu^2}{\langle x, \nu \rangle} dx \leq \int_K \sum_{i=1}^n u_{ii}^2 dx \leq \int_K \|\nabla^2 u\|^2 dx. \quad (8.1)$$

In particular, this holds for all $u \in \mathcal{S}_{0,\text{uncond}}(K)$, and therefore $B_{\text{uncond}}(K) = 1$.

Remark 8.4. Note that the latter theorem does not follow from Saroglou’s global affirmation of the log-BM conjecture for unconditional convex bodies [58]. When $K, L \in \mathcal{K}_{\text{uncond}}$, all relevant test functions Ψ on ∂K (and thus the harmonic u on K) for the local log-BM conjecture will indeed be unconditional. However, Theorem 8.3 confirms the *stronger* sufficient condition given in Theorem 6.4, and moreover, *without the requirement that u be harmonic*.

8.2 The Cube

In the case of the cube, we can use the off-diagonal elements of D^2u to control the non-unconditional part of a general test function u :

Theorem 8.5. $B_H(B_\infty^n) = B(B_\infty^n) = 1$.

Proof. To show that $B(B_\infty^n) \leq 1$, we need to show for all $u \in \mathcal{S}_{0,e}(B_\infty^n)$ that:

$$\int_{\partial B_\infty^n} u_\nu^2 dx \leq \int_{B_\infty^n} \|\nabla^2 u\|^2 dx.$$

To this end, it is enough to establish for all $i = 1, \dots, n$ that:

$$\int_{\partial B_\infty^n} u_i^2 |\nu_i| dx \leq \int_{B_\infty^n} |\nabla u_i|^2 dx. \quad (8.2)$$

Without loss of generality, we may assume that $i = 1$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, set $y = (x_2, \dots, x_n)$, and write $u = u^+ + u^-$, where u^+ is even w.r.t. both x_1 and y ($u^+(-x_1, y) = u^+(x_1, -y) = u^+(x_1, y)$) and u^- is odd w.r.t. both x_1 and y ($u^-(-x_1, y) = u^-(x_1, -y) = -u^-(x_1, y)$), namely:

$$u^+(x) := \frac{1}{2}(u(x_1, y) + u(x_1, -y)), \quad u^-(x) := \frac{1}{2}(u(x_1, y) - u(x_1, -y))$$

(recall that u was assumed even). It is enough to verify (8.2) for u^+ and u^- separately, since it is easy to see that the behavior under reflections and the unconditionality of the cube guarantee that $\int_{\partial B_\infty^n} u_1^+ u_1^- |\nu_1| dx = 0$ and $\int_{B_\infty^n} \langle \nabla u_1^+, \nabla u_1^- \rangle dx = 0$.

Note that u_1^+ is odd w.r.t. x_1 and hence $u_1^+ \equiv 0$ on $B_\infty^n \cap \{x_1 = 0\}$. It follows by Lemma 8.2 that:

$$\int_{\partial B_\infty^n} (u_1^+)^2 |\nu_1| dx = \int_{\partial B_\infty^n} \frac{(u_1^+)^2}{x_1} \nu_1 dx \leq \int_{B_\infty^n} (u_{11}^+)^2 dx \leq \int_{B_\infty^n} |\nabla u_1^+|^2 dx.$$

As for u_1^- , which is even w.r.t. x_1 , write:

$$(u_1^-)^2(1, y) = \int_0^1 \frac{\partial}{\partial x_1} (x_1 (u_1^-)^2(x_1, y)) dx_1 = \int_0^1 (2x_1 u_{11}^-(x_1, y) u_1^-(x_1, y) + (u_1^-)^2(x_1, y)) dx_1.$$

Using the evenness of the above integrand in x_1 , we obtain:

$$\begin{aligned} \int_{\partial B_\infty^n} (u_1^-)^2 |\nu_1| dx &= \int_{B_\infty^{n-1}} \int_{-1}^1 (2x_1 u_{11}^-(x_1, y) u_1^-(x_1, y) + (u_1^-)^2(x_1, y)) dx_1 dy \\ &\leq \int_{B_\infty^n} (x_1^2 (u_{11}^-)^2(x) + 2(u_1^-)^2(x)) dx, \end{aligned} \quad (8.3)$$

where the last inequality follows by completing the square. The first term on the right is trivially controlled by:

$$\int_{B_\infty^n} x_1^2 (u_{11}^-)^2(x) dx \leq \int_{B_\infty^n} (u_{11}^-)^2(x) dx.$$

For the second term, note the $u_1^-(x_1, y)$ is odd w.r.t. y , and hence integrates to zero on each $(n-1)$ -dimensional slice $B_t := B_\infty^n \cap \{x_1 = t\}$. Applying the Poincaré inequality on B_t , and recalling the well known fact [43] that $C_{Poin}^2(B_\infty^k) = \frac{4}{\pi^2}$ for any $k \geq 1$, it follows that:

$$2 \int_{B_\infty^n} (u_1^-)^2(x) dx = 2 \int_{-1}^1 \int_{B_{x_1}} (u_1^-)^2(x_1, y) dy dx_1 \leq \frac{8}{\pi^2} \int_{-1}^1 \int_{B_{x_1}} |\nabla_y u_1^-|^2 dy dx_1.$$

Since $\frac{8}{\pi^2} < 1$, combining the contributions of the above two terms to (8.3), we obtain:

$$\int_{\partial B_\infty^n} (u_1^-)^2 |\nu_1| dx \leq \int_{B_\infty^n} |\nabla u_1^-|^2 dx,$$

as required.

This concludes the proof that $B(B_\infty^n) \leq 1$. Consequently, we also have $B_H(B_\infty^n) \leq 1$. It remains to note that the constant 1 is sharp in both cases, as witnessed by the even harmonic function $u(x) = \frac{x_1^2}{2} - \frac{x_2^2}{2}$, and therefore $B_H(B_\infty^n) = B(B_\infty^n) = 1$. \square

Remark 8.6. Note that the value $B_H(B_\infty^n) = 1$ is in precise accordance with the threshold required in Theorem 6.4 for confirming the local log-BM conjecture in the case of smooth bodies in $\mathcal{K}_{+,e}^2$. In this formal sense, the cube can be thought as satisfying the local log-BM conjecture. We will give this a more rigorous sense in Section 10.

9 Local log-Brunn–Minkowski via the Reilly Formula

In this section, we apply the generalized Reilly formula to a measure on K with log-convex (not log-concave!) density, specifically constructed for verifying the local log-BM conjecture for certain classes of convex bodies.

9.1 Sufficient condition for verifying local log-Brunn–Minkowski

Recall by Proposition 4.6 and Remark 4.7 that the validity of the local log-BM conjecture (3.4) for $K \in \mathcal{K}_{+,e}^2$ is equivalent to the validity of the following assertion:

$$\begin{aligned} \forall \Psi \in C_e^1(\partial K) \quad \int_{\partial K} \Psi(x) dx = 0 \quad \Rightarrow \\ \int_{\partial K} \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx \geq \int_{\partial K} H_{\partial K}(x) \Psi^2(x) dx + \int_{\partial K} \frac{\Psi^2(x)}{\langle x, \nu_{\partial K}(x) \rangle} dx. \end{aligned} \quad (9.1)$$

Given $K \in \mathcal{K}_e$, the associated norm $\|\cdot\|_K$ is defined by $\|x\|_K = \min \{t > 0; x \in tK\}$. Let $w : [0, 1] \rightarrow \mathbb{R}$ denote a C^2 function with $w'(0) = 0$ and $w'(1) = 1$.

$$W(x) := w(\|x\|_K) , \quad \mu := \exp(W(x))dx|_K.$$

Note that w cannot be concave, and typically will be chosen to be convex, so that μ is log-convex (and not log-concave). Assuming $K \in \mathcal{K}_{+,e}^2$ and abbreviating $\|x\| = \|x\|_K$ and $\nu = \nu_{\partial K}$, observe that on ∂K :

$$\langle \nabla W, \nu \rangle = w'(1) \langle \nabla \|x\|, \nu \rangle = |\nabla \|x\|| = \frac{\|x\|}{\langle x, \nu \rangle} = \frac{1}{\langle x, \nu \rangle},$$

and hence:

$$H_{\partial K, \mu} = H_{\partial K} + \langle \nabla W, \nu \rangle = H_{\partial K} + \frac{1}{\langle x, \nu \rangle}.$$

Also note that:

$$d\mu_{\partial K} = e^{w(1)} d\mathcal{H}^{n-1}|_{\partial K}.$$

Given $\Psi \in C_e^1(\partial K)$ with $\int_{\partial K} \Psi dx = 0$, since also $\int_{\partial K} \Psi d\mu_{\partial K} = 0$, we may solve for $u \in \mathcal{S}_{N,e}(K)$ the Laplace equation:

$$L_\mu u = 0 \text{ in } \text{int}(K) , \quad u_\nu = \Psi \text{ on } \partial K. \quad (9.2)$$

Applying the Reilly formula to u on K equipped with the measure μ , we have:

$$\begin{aligned} 0 &= \int_K (L_\mu u)^2 d\mu = \int_K \|\nabla^2 u\|^2 d\mu - \int_K \langle \nabla^2 W \nabla u, \nabla u \rangle d\mu + \\ &e^{w(1)} \left(\int_{\partial K} \left(H_{\partial K} + \frac{1}{\langle x, \nu \rangle} \right) u_\nu^2 dx + \int_{\partial K} \langle \Pi_{\partial K} \nabla_{\partial K} u, \nabla_{\partial K} u \rangle dx \right. \\ &\quad \left. - 2 \int_{\partial K} \langle \nabla_{\partial K} u_\nu, \nabla_{\partial K} u \rangle dx \right) . \end{aligned}$$

Using $\Pi_{\partial K} > 0$ and applying the Cauchy–Schwarz inequality as in Theorem 6.1, we deduce (recalling that $u_\nu = \Psi$):

$$\begin{aligned} \int_K \langle \nabla^2 W \nabla u, \nabla u \rangle e^W dx &\geq \int_K \|\nabla^2 u\|^2 e^W dx \\ + e^{w(1)} \left(\int_{\partial K} \left(H_{\partial K} + \frac{1}{\langle x, \nu \rangle} \right) \Psi^2 dx - \int_{\partial K} \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx \right) . \end{aligned} \quad (9.3)$$

Comparing this with our desired inequality (9.1), we deduce:

Theorem 9.1. Let $w : [0, 1] \rightarrow \mathbb{R}$ denote a C^2 function with $w'(0) = 0$ and $w'(1) = 1$. Given $K \in \mathcal{K}_{+,e}^2$ denote $W(x) = w(\|x\|_K)$, and assume that:

$$\begin{aligned} \forall u \in \mathcal{S}_{N,e}(K) \quad \Delta u + \langle \nabla W, \nabla u \rangle = 0 \text{ in } \text{int}(K) \quad \Rightarrow \\ \int_K \langle \nabla^2 W \nabla u, \nabla u \rangle e^W dx \leq \int_K \|\nabla^2 u\|^2 e^W dx. \end{aligned}$$

Then the local log-BM conjecture (3.4) holds for $T(K)$ for all $T \in GL_n$.

As usual, the application of the Cauchy–Schwarz inequality destroyed the linear invariance of the validity of the above sufficient condition, in contrast with the invariance of the local log-BM conjecture.

Remark 9.2. Observe that when w is convex, the sufficient condition in Theorem 9.1 is some sort of dual log-convex formulation of the classical Brascamp–Lieb inequality [8] (which in itself is known to be equivalent to the Prékopa–Leindler, and hence Brunn–Minkowski, inequality).

We can also obtain the following version of Theorem 9.1 for perturbations:

Theorem 9.3. With the same assumptions as in Theorem 9.1, assume in addition the existence of $\epsilon > 0$ so that:

$$\begin{aligned} \forall u \in \mathcal{S}_{N,e}(K) \quad \Delta u + \langle \nabla W, \nabla u \rangle = 0 \text{ in } \text{int}(K) \quad \Rightarrow \\ \int_K \langle \nabla^2 W \nabla u, \nabla u \rangle e^W dx \leq (1 - \epsilon) \int_K \|\nabla^2 u\|^2 e^W dx. \end{aligned} \quad (9.4)$$

Then there exists a C^2 neighborhood N_K of K in $\mathcal{K}_{+,e}^2$, so that the local log-BM conjecture (3.4) holds for $T(K')$ for all $K' \in N_K$ and $T \in GL_n$. Equivalently, for all $T \in GL_n$ and $K_1, K_0 \in T(N_K)$:

$$V((1 - \lambda) \cdot K_0 +_0 \lambda \cdot K_1) \geq V(K_0)^{1-\lambda} V(K_1)^\lambda \quad \forall \lambda \in [0, 1].$$

Proof. Plugging (9.4) into (9.3), we obtain for all $\Psi \in C_e^1(\partial K)$ with $\int_{\partial K} \Psi dx = 0$:

$$\int_{\partial K} \left(H_{\partial K} + \frac{1}{\langle x, \nu \rangle} \right) \Psi^2 dx - \int_{\partial K} \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx \leq -\delta \int_K \|\nabla^2 u\|^2 dx \quad (9.5)$$

with $\delta = \epsilon e^{\min w - w(1)}$, where u solves (9.2). By definition:

$$\int_K \|\nabla^2 u\|^2 dx \geq \frac{1}{B(K)} \int_{\partial K} \frac{u_\nu^2}{\langle x, \nu \rangle} dx \quad \forall u \in \mathcal{S}_{0,e}(K),$$

and by Theorem 6.6, $B(K) \leq D(K) < \infty$. Consequently, we deduce from (9.5) that:

$$\int_K \langle \Pi_{\partial K}^{-1} \nabla_{\partial K} \Psi, \nabla_{\partial K} \Psi \rangle dx \geq \int_{\partial K} H_{\partial K} \Psi^2 dx + \left(1 + \frac{\delta}{D(K)}\right) \int_{\partial K} \frac{\Psi^2}{\langle x, \nu \rangle} dx.$$

In other words, (4.16) holds with $p_K := -\frac{\delta}{D(K)}$, and so the local p_K -BM conjecture (3.3) holds for K . The assertion then follows by Proposition 3.9 (with $p_0 = p_K < 0$ and $p = 0$) and the invariance under linear images. \square

Corollary 9.4. *The assumption and hence conclusion of Theorem 9.3 hold if:*

$$Q_{K,w} := \max_{x \in K} \|\nabla^2 W(x)\|_{op} e^{\max w - \min w} C_{Poin}^2(K) < 1.$$

Proof. For any $u \in \mathcal{S}_{N,e}(B_q^n)$:

$$\int_K \langle \nabla^2 W \nabla u, \nabla u \rangle e^W dx \leq \max_{x \in K} \|\nabla^2 W(x)\|_{op} e^{\max w} \int_K \sum_{i=1}^n u_i^2(x) dx.$$

Since u is even, u_i is odd, and hence integrates to zero on K . Applying the Poincaré inequality on K for each u_i and summing, we obtain:

$$\int_K \langle \nabla^2 W \nabla u, \nabla u \rangle e^W dx \leq \max_{x \in K} \|\nabla^2 W(x)\|_{op} e^{\max w - \min w} C_{Poin}^2(K) \int_K \|\nabla^2 u\|^2 e^W dx.$$

The assertion follows from Theorem 9.3. \square

9.2 An alternative derivation via estimating $B_H(K)$

The approach of the previous subsection has the advantage of uncovering a certain duality between the sufficient condition of Theorem 9.1 and the Brascamp–Lieb inequality (see Remark 9.2). In this subsection, we provide an alternative simpler derivation of an estimate very similar to that of Corollary 9.4, which is devoid of the former insight. On the other hand, it has the advantage of providing an upper estimate on $B_H(K)$, so that even when the latter is strictly larger than 1, Theorem 6.4 may be used to deduce the local p -BM conjecture for K for some $p \in (0, 1)$. In addition, we do not need to assume that $K \in \mathcal{K}_+^2$.

Theorem 9.5. *Let $w : [0, 1] \rightarrow \mathbb{R}$ denote a C^2 function with $w'(0) = 0$ and $\max_{t \in [0,1]} |w'(t)| = w'(1) = 1$. Given $K \in \mathcal{K}_e$ so that $\|\cdot\|_K \in C^2(S^{n-1})$, denote $W(x) = w(\|x\|_K)$, and assume that $K \supset rB_2^n$. Then:*

$$B_H(K) \leq \frac{C_{Poin}(K)}{r} + C_{Poin}^2(K) \max_{x \in K} \|\nabla^2 W(x)\|_{op}.$$

Proof. Let $u \in \mathcal{S}_{0,e}(K)$ be harmonic in $\text{int}(K)$. As usual:

$$\nabla W(x) = w'(\|x\|)\nabla \|x\| = w'(\|x\|)\frac{\|x\|}{\langle x, \nu(x/\|x\|) \rangle} \nu(x/\|x\|) \quad \forall x \in K,$$

so $\nabla W|_{\partial K} = \frac{1}{\langle x, \nu \rangle} \nu$ and $|\nabla W| \leq \frac{\max_{t \in [0,1]} |w'(t)|}{\max_{\nu \in S^{n-1}} h_K(\nu)} \leq \frac{1}{r}$ on K . Integrating by parts and utilizing the harmonicity of u :

$$\begin{aligned} \int_{\partial K} \frac{u_\nu^2}{\langle x, \nu \rangle} dx &= \int_{\partial K} u_\nu \langle \nabla u, \nabla W \rangle dx = \int_K \text{div}(\nabla u \langle \nabla u, \nabla W \rangle) dx \\ &= \int_K (\langle \nabla^2 u \nabla u, \nabla W \rangle + \langle \nabla^2 W \nabla u, \nabla u \rangle) dx. \end{aligned}$$

Applying Cauchy–Schwarz and the usual Poincaré inequality on each u_i , we have for any $\lambda > 0$:

$$\begin{aligned} &\leq \frac{\lambda}{2} \int_K \|\nabla^2 u\|^2 dx + \frac{1}{2\lambda} \int_K |\nabla u|^2 |\nabla W|^2 dx + \max_{x \in K} \|\nabla^2 W(x)\|_{op} \int_K |\nabla u|^2 dx \\ &\leq \frac{\lambda}{2} \int_K \|\nabla^2 u\|^2 dx + \left(\frac{1}{2\lambda r^2} + \max_{x \in K} \|\nabla^2 W(x)\|_{op} \right) \int_K |\nabla u|^2 dx \\ &\leq \left(\frac{\lambda}{2} + \left(\frac{1}{2\lambda r^2} + \max_{x \in K} \|\nabla^2 W(x)\|_{op} \right) C_{Poin}^2(K) \right) \int_K \|\nabla^2 u\|^2 dx. \end{aligned}$$

Setting $\lambda = \frac{C_{Poin}(K)}{r}$, the assertion follows. \square

It is particularly convenient to apply Theorem 9.5 to B_q^n , the unit-balls of ℓ_q^n , for $q \in (2, \infty)$.

Theorem 9.6. *For all $q \in (2, \infty)$, $B_H(B_q^n) \leq C(n^{-1/q} + qn^{-2/q})$.*

Proof. Set $w(t) = \frac{1}{q}t^q$ and $W(x) = w(\|x\|_{\ell_q^n}) = \frac{1}{q} \sum_{i=1}^n |x_i|^q$. Observe that $\nabla^2 W = (q-1)\text{diag}(|x_i|^{q-2})$, and hence $\max_{x \in B_q^n} \|\nabla^2 W(x)\|_{op} = q-1$ whenever $q \geq 2$. It remains to recall that $C_{Poin}(B_q^n)$ is of the order of $n^{-1/q}$ by Theorem 6.10 and that $B_2^n \subset B_q^n$ when $q \geq 2$, and so Theorem 9.5 yields the assertion. \square

10 Continuity of B_H, B, D with application to B_q^n

10.1 Continuity of B_H, B, D in C -topology

Proposition 10.1. *Let $\{K_i\} \subset \mathcal{K}_e$. If $K_i \rightarrow K$ in the C -topology then $B(K_i) \rightarrow B(K)$, $B_H(K_i) \rightarrow B_H(K)$ and $D(K_i) \rightarrow D(K)$.*

Proof. As this is not a cardinal point in this work, let us only sketch the proof, as providing all details would be tedious.

It is easy to see that the mappings $\mathcal{K}_e \ni K \mapsto B_H(K), B(K), D(K)$ are lower semi-continuous in the C topology, being the suprema of a continuous family of functionals (parametrized by u). For instance, for B_H, B and a fixed u , the functional is:

$$\mathcal{K}_e \ni K \mapsto \frac{\int_{\partial K} \frac{u_\nu^2}{\langle x, \nu_{\partial K}(x) \rangle} dx}{\int_K \|\nabla^2 u\|^2 dx},$$

which are continuous in C since the vector valued measure $\frac{1}{\langle x, \nu_{\partial K}(x) \rangle} \nu_{\partial K} \mathcal{H}^{n-1}|_{\partial K}$ weakly converges under C convergence of convex bodies (for a more general statement regarding generalized curvature measures, also known as support measures, see [60, Theorem 4.2.1]).

The harder part is to show the upper semi-continuity. To see this for $D(K)$, for instance, let $\{K_i\}$ denote a sequence on which $\limsup_{K_i \rightarrow K} D(K_i)$ is attained. Since $K \mapsto D(K)$ is invariant under homothety, we may assume w.l.o.g. that $rB_2^n \subset K_1 \subset K_2 \subset \dots \subset K \subset RB_2^n$. Denote by u^i the test function for which:

$$\int_{K_i} \|\nabla^2 u^i\|^2 dx = 1 \text{ and } \int_{\partial K_i} \frac{|\nabla u^i|^2}{\langle x, \nu_{\partial K_i}(x) \rangle} dx \geq D(K_i) - \frac{1}{i}.$$

As $\{u^i\}_{i \geq j}$ are bounded in $H^2(K_j)$, they have a weakly convergent subsequence in $H^2(K_j)$, and by a diagonalization argument, we may extract a subsequence (which we continue to denote $\{u^i\}$) weakly converging to $u \in H^2(K)$, so that:

$$\lim_{i \rightarrow \infty} \int_{K_i} \langle \nabla^2(u^i - u), \varphi \rangle dx = 0$$

for any smooth 2-tensor φ on K . The weak convergence implies that $\int_K \|\nabla^2 u\| dx \leq 1$. By compactness [4, Corollary 7.4] of the trace embedding of Sobolev space on Lipschitz domains with upper Ahlfors measures (such as $\mu_K := \frac{1}{\langle x, \nu_{\partial K} \rangle} d\mathcal{H}^{n-1}|_{\partial K}$ in our setting), which in fact holds with a uniform constant for all K_i (as they are uniformly Lipschitz and μ_K is uniformly upper Ahlfors, owing to convexity and

r, R being uniform), the weak convergence in H^2 implies strong convergence in the trace H^1 norm:

$$\lim_{i \rightarrow \infty} \int_{\partial K_i} |\nabla u^i - \nabla u|^2 d\mu_{K_i} = 0.$$

By weak convergence of μ_{K_i} to μ_K (as for the lower semi-continuity direction), there exists for any $\epsilon > 0$ a large enough i_ϵ , so that for all $i \geq i_\epsilon$:

$$\begin{aligned} D(K) &\geq \frac{\int_{\partial K} \frac{|\nabla u|^2}{\langle x, \nu_{\partial K}(x) \rangle} dx}{\int_K \|\nabla^2 u\|^2 dx} \geq \int_{\partial K} |\nabla u|^2 d\mu_K \geq \int_{\partial K_i} |\nabla u|^2 d\mu_{K_i} - \epsilon \\ &\geq \int_{\partial K_i} |\nabla u^i|^2 d\mu_{K_i} - \int_{\partial K_i} |\nabla u^i - \nabla u|^2 d\mu_{K_i} - \epsilon \\ &\geq D(K_i) - \frac{1}{i} - \int_{\partial K_i} |\nabla u^i - \nabla u|^2 d\mu_{K_i} - \epsilon. \end{aligned}$$

Taking the limit as $i \rightarrow \infty$ and $\epsilon \rightarrow 0+$, the upper semi-continuity of D follows. Note that the limiting u is not guaranteed to be in $\mathcal{S}_{N,e}(K)$, only in $H^2(K)$, but can be approximated in $H^2(K)$ by functions in $\mathcal{S}_{N,e}(K)$, and by the trace embedding theorem, also in $H^1(d\mu_K)$, and hence the above lower bound on $D(K)$ is legitimate.

The proof is identical for $B(K)$. For $B_H(K)$, one just has to note that the limiting u will be harmonic as the weak H^2 limit of the harmonic u^i . \square

10.2 The Cube

We can now extend Theorem 8.5 to a result on the even spectral-gap $\lambda_{1,e}(B_\infty^n)$ of the formal Hilbert–Brunn–Minkowski operator associated to B_∞^n . Recalling the notation from Section 5, and in particular the definition (5.11):

$$\lambda_{1,e}(B_\infty^n) := \liminf_{\mathcal{K}_{+,e}^2 \ni K \rightarrow B_\infty^n \text{ in } C} \lambda_{1,e}(-L_K),$$

we have:

Theorem 10.2. $\lambda_{1,e}(B_\infty^n) = \frac{n}{n-1}$.

Proof. By Proposition 10.1 and Theorem 8.5, we have:

$$\lim_{\mathcal{K}_{+,e}^2 \ni K_i \rightarrow B_\infty^n \text{ in } C} B_H(K_i) = B(B_\infty^n) = 1.$$

By Theorem 6.5, we know that $\lambda_{1,e}(-L_K) \geq 1 + \frac{1}{(n-1)B_H(K)}$ for any $K \in \mathcal{K}_{+,e}^2$. Consequently:

$$\liminf_{\mathcal{K}_{+,e}^2 \ni K \rightarrow B_\infty^n \text{ in } C} \lambda_{1,e}(-L_K) \geq \frac{n}{n-1}.$$

To see that we actually have equality in the above inequality, it is enough to test the specific sequence $\{B_q^n\} \subset \mathcal{K}_{+,e}^2$ which converges to B_∞^n in C as $q \rightarrow \infty$. Moreover, it is enough to show that the inequality $R_{B_q^n}(u^0) \geq 0$ we employed in Theorem 6.1, when transitioning from the equivalent condition for the local log-BM conjecture to the sufficient one, is not wasteful for the extremal even harmonic function $u^0 := \frac{x_1^2}{2} - \frac{x_2^2}{2}$ for B_∞^n . Using Remark 6.2, we need to show:

$$R_{B_q^n}(u^0) = \int_{\partial B_q^n} \left\langle \Pi_{\partial B_q^n}^{-1} P_{T_{\partial B_q^n}} [\nabla^2 u^0 \cdot \nu], P_{T_{\partial B_q^n}} [\nabla^2 u^0 \cdot \nu] \right\rangle dx \rightarrow 0 \quad \text{as } q \rightarrow \infty. \quad (10.1)$$

A computation verifies that on the positive orthant:

$$\nu(x) = \frac{\{x_i^{q-1}\}_{i=1}^n}{\sqrt{\sum_{i=1}^n x_i^{2(q-1)}}}, \quad \Pi_{\partial B_q^n} = \Lambda^{1/2} U \Lambda^{1/2} |_{T_{\partial B_q^n}},$$

where:

$$\Lambda(x) = \frac{q-1}{\sqrt{\sum_{i=1}^n x_i^{2(q-1)}}} \text{diag}(x_i^{q-2}), \quad U = \langle \Lambda \nu, \nu \rangle \langle \Lambda^{-1} \nu, \nu \rangle \hat{e}_2 \otimes \hat{e}_2 + \sum_{k>2}^n \hat{e}_k \otimes \hat{e}_k,$$

and $\{\hat{e}_k\}_{k=1}^n$ is an orthonormal frame with:

$$\hat{e}_i = \frac{\tilde{e}_i}{|\tilde{e}_i|}, \quad \tilde{e}_1 = \Lambda^{1/2} \nu, \quad \tilde{e}_2 = \Lambda^{-1/2} \nu - \frac{1}{\langle \Lambda \nu, \nu \rangle} \Lambda^{1/2} \nu.$$

Consequently:

$$\Pi_{\partial B_q^n}^{-1} = (\Lambda^{-\frac{1}{2}} \tilde{U} \Lambda^{-\frac{1}{2}}) |_{T_{\partial K}} \quad \text{with} \quad \tilde{U} = \frac{1}{\langle \Lambda \nu, \nu \rangle \langle \Lambda^{-1} \nu, \nu \rangle} \hat{e}_2 \otimes \hat{e}_2 + \sum_{k>2}^n \hat{e}_k \otimes \hat{e}_k.$$

It follows that the integrand in (10.1) is bounded by:

$$\begin{aligned} \left\langle \Pi_{\partial B_q^n}^{-1} P_{T_{\partial K}} [\nabla^2 u^0 \cdot \nu], P_{T_{\partial K}} [\nabla^2 u^0 \cdot \nu] \right\rangle \leq \\ \max \left(1, \frac{1}{\langle \Lambda \nu, \nu \rangle \langle \Lambda^{-1} \nu, \nu \rangle} \right) \left| \Lambda^{-\frac{1}{2}} P_{T_{\partial K}} [\nabla^2 u^0 \cdot \nu] \right|^2. \end{aligned}$$

It is easy to check that:

$$P_{T_{\partial K}}[\nabla^2 u^0 \cdot \nu] = \frac{\{a_i(x)x_i^{q-1}\}_{i=1}^n}{\sqrt{\sum_{i=1}^n x_i^{2(q-1)}}}, \quad |a_i(x)| \leq 2.$$

Plugging in the above expression for Λ and applying Hölder's inequality (using $\sum_{i=1}^n x_i^q = 1$), a straightforward calculation verifies that the integrand goes to zero uniformly in x as $q \rightarrow \infty$, and the claim is established. \square

In view of Corollary 5.4 and Proposition 3.9, Theorem 1.3 is a reformulation of Theorem 10.2. Similarly, recalling definition (5.12), we extend Theorem 8.3 to non-smooth $K \in \mathcal{K}_{\text{uncond}}$:

Corollary 10.3. *For all $K \in \mathcal{K}_{\text{uncond}}$, $\lambda_{1,\text{uncond}}(K) \geq \frac{n}{n-1}$.*

10.3 Unit-balls of ℓ_q^n

Observe that $B_q^n \notin \mathcal{K}_+^2$ whenever $q \neq 2$. Consequently, we will use Proposition 10.1 to obtain a neighborhood $N_{B_q^n}^C$ of B_q^n in the C_e -topology, so that the results of the previous sections may be applied to its dense subset $N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$. This yields Theorems 1.2 and 1.4 from the Introduction, which we restate here as follows:

Theorem 10.4. *For all $q \in (2, \infty)$, there exists $n_q \geq 2$ so that for all $n \geq n_q$, there exists a neighborhood $N_{B_q^n}^C$ of B_q^n in the C_e -topology, so that the local log-BM conjecture (3.4) holds for $T(K)$ for all $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$ and $T \in GL_n$. In addition, for any $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$, there exists a C^2 -neighborhood N_K of K in $\mathcal{K}_{+,e}^2$ so that for all $T \in GL_n$ and $K_1, K_0 \in T(N_K)$:*

$$V((1-\lambda) \cdot K_0 +_0 \lambda \cdot K_1) \geq V(K_0)^{1-\lambda} V(K_1)^\lambda \quad \forall \lambda \in [0, 1].$$

Proof. Recall that by Theorem 9.6, $B_H(B_q^n) \leq C(n^{-1/q} + qn^{-2/q})$. Setting $n_q = \exp(\frac{q}{2} \log(C'q))$, it follows that $B_H(B_q^n) \leq \frac{1}{2}$ for all $n \geq n_q$. By Proposition 10.1, there exists a neighborhood $N_{B_q^n}^C$ of B_q^n in the C_e -topology so that $B_H(K) \leq \frac{3}{4}$ for all $K \in N_{B_q^n}^C$. Consequently, Theorem 6.4 implies that the local p -BM conjecture (3.3) holds with $p = -\frac{1}{3}$ for $T(K)$ for all $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$ and $T \in GL_n$, implying in particular the first assertion. The second assertion follows by invoking Proposition 3.9. \square

Theorem 10.5. *There exists a universal constant $c \in (0, 1)$ so that for all $q \in [1, 2)$, there exists a neighborhood $N_{B_q^n}^C$ of B_q^n in the C_e -topology, so that the p -BM conjecture (3.3) holds with $p = c$ for $T(K)$ for all $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$ and $T \in GL_n$. In addition, for any $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$, there exists a C^2 -neighborhood N_K of K in $\mathcal{K}_{+,e}^2$ so that for all $T \in GL_n$ and $K_1, K_0 \in T(N_K)$:*

$$V((1 - \lambda) \cdot K_0 +_c \lambda \cdot K_1) \geq \left((1 - \lambda)V(K_0)^{\frac{c}{n}} + \lambda V(K_1)^{\frac{c}{n}} \right)^{\frac{n}{c}} \quad \forall \lambda \in [0, 1].$$

Proof. By Lemma 6.12, there exists a universal constant $C > 1$ so that for all $q \in [1, 2)$, $B_H(B_q^n) \leq D(B_q^n) \leq C$. By Proposition 10.1, there exists a neighborhood $N_{B_q^n}^C$ of B_q^n in the C_e -topology so that $B_H(K) \leq 2C$ for all $K \in N_{B_q^n}^C$. Consequently, Theorem 6.4 implies that the local p -BM conjecture (3.3) holds with $p = 1 - \frac{1}{2C}$ for $T(K)$ for all $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$ and $T \in GL_n$, implying in particular the first assertion. Setting $c = 1 - \frac{1}{3C}$, the second assertion follows by invoking Proposition 3.9. \square

11 Local Uniqueness for Even L^p -Minkowski Problem

In this section, we describe an application (which is by now well-understood and standard – see [49, 6]) of our local p -BM and log-BM inequalities to local uniqueness statements for the even L^p -Minkowski and log-Minkowski problems.

The classical Minkowski problem (see [60, 51] and the references therein) asks for necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} , to guarantee the existence and uniqueness (up to translation) of a convex body $K \in \mathcal{K}$ so that its surface-area measure dS_K coincides with μ . It was shown by Minkowski for polytopes and by Aleksandrov for general convex bodies, that a necessary and sufficient condition is to require that the centroid of μ is at the origin and that its support is not contained in a great subsphere. In [49], Lutwak proposed to study the analogous L^p -Minkowski problem, where the role of the surface-area measure dS_K is replaced by the L^p -surface-area measure:

$$dS_{K,p} := h_K^{1-p} dS_K.$$

For *even* measures, Lutwak showed in [49] that Minkowski's condition is again necessary and sufficient for existence and uniqueness (no translations required now) in the case $1 < p \neq n$ (see also Lutwak–Yang–Zhang [51] for the case $p = n$).

The same question may be extended to the range $p < 1$. Of particular interest is the the log-Minkowski problem, which pertains to the cone-measure dV_K (corresponding to the case $p = 0$). For *even* measures μ , a novel necessary and sufficient *subspace concentration condition* ensuring the *existence* question was obtained in [7], and the *uniqueness* question was settled in [6] in dimension $n = 2$; it remains open in full generality in dimension $n \geq 3$ (see also [22] for the planar uniqueness question for smooth convex bodies with strictly positive curvature, and [63] for existence and uniqueness in the even planar problem when μ is assumed discrete). Various other partial results pertaining to the uniqueness question are known (see e.g. [31, 52, 67, 12] and the references therein). Without assuming evenness of μ , both existence and uniqueness problems are more delicate, and there is a huge body of works on this topic which we do not attempt to survey here. Instead, let us mention the known intimate relation between the uniqueness question and the p -BM inequality with its equality conditions.

Recall the definition (3.1) of the L^p -mixed-volume $V_p(K, L)$, introduced by Lutwak in [49]. It was shown in [6] (for $p \in (0, 1)$, but the proof extends to all $p < 1$) that:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p dS_{K,p}.$$

Proposition 11.1. *Let $K_0, K_1 \in \mathcal{K}_e$ and $p < 1$. Then each of the following statements implies the subsequent one:*

- (1) *The function $[0, 1] \ni \lambda \mapsto g_p(\lambda) := \frac{1}{p} V((1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1)^{\frac{p}{n}}$ is concave, and it is affine if and only if K_0 and K_1 are dilates.*
- (2) *The first L^p -Minkowski inequality (3.2) holds for the pair $K, L = K_0, K_1$ and for the pair $K, L = K_1, K_0$, with equality in either of these cases if and only if K_0 and K_1 are dilates.*
- (3) *$dS_{K_0,p} = dS_{K_1,p}$ implies $K_0 = K_1$.*

Slightly more is required for the converse implications to hold, as worked out in [6], but we do not require this here.

Proof. It was shown in [6] (for $p \in [0, 1)$, but the proof extends to all $p < 1$) that:

$$\begin{aligned} \left. \frac{d}{d\lambda} \right|_{\lambda=0+} V((1 - \lambda) \cdot K_0 +_p \lambda \cdot K_1) &= \frac{1}{p} \int_{S^{n-1}} h_{K_0}^{1-p} (h_{K_1}^p - h_{K_0}^p) dS_{K_0} \\ &= \frac{n}{p} (V_p(K_0, K_1) - V(K_0)). \end{aligned}$$

Consequently, the chain rule yields:

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0+} g_p(\lambda) = V(K_0)^{\frac{p}{n}-1} \frac{1}{p} (V_p(K_0, K_1) - V(K_0)),$$

with the case $p = 0$ understood in the limiting sense. The concavity in statement (1) implies $\left. \frac{d}{d\lambda} \right|_{\lambda=0+} g_p(\lambda) \geq g_p(1) - g_p(0)$, which is precisely (3.2). Reversing the roles of K_0, K_1 by the symmetry of (1), (3.2) also holds in that case. Clearly we have equality in (3.2) if K_0 and K_1 are dilates. Conversely, equality in (3.2) translates to $\left. \frac{d}{d\lambda} \right|_{\lambda=0+} g_p(\lambda) = g_p(1) - g_p(0)$, and since g_p is assumed concave, it follows that it must be affine, and so the equality conditions in (1) imply those in (2).

If $dS_{K_0,p} = dS_{K_1,p}$, then for any $Q \in \mathcal{K}$:

$$V_p(K_0, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q^p dS_{K_0,p} = \frac{1}{n} \int_{S^{n-1}} h_Q^p dS_{K_1,p} = V_p(K_1, Q).$$

Assuming for simplicity that $p > 0$ (but an identical argument holds for general p), we have by (3.2) for both $i = 0, 1$ that:

$$V(K_i) = V_p(K_i, K_i) = V_p(K_{1-i}, K_i) \geq V(K_{1-i})^{1-\frac{p}{n}} V(K_i)^{\frac{p}{n}}.$$

It follows that $V(K_0) = V(K_1)$ and hence we have equality in (3.2) for the pair K_{1-i}, K_i . The equality conditions in (2) therefore imply that $K_0 = K_1$. \square

Definition. Given $p < 1$ and $K \in \mathcal{K}_{+,e}^2$, we will say that the even L^p -Minkowski problem has a locally unique solution in a neighborhood of K if there exists a C^2 -neighborhood $N_{K,p}$ of K in $\mathcal{K}_{+,e}^2$, so that for all $T \in GL_n$, for all $K_0, K_1 \in T(N_{K,p})$, if $dS_{K_0,p} = dS_{K_1,p}$ then $K_0 = K_1$.

Theorem 11.2. Assume that the local p_0 -BM conjecture (3.3) holds for $K \in \mathcal{K}_{+,e}^2$ and some $p_0 < 1$. Then for any $p \in (p_0, 1)$, the even L^p -Minkowski problem has a locally unique solution in a neighborhood $N_{K,p}$ of K .

Proof. Given $p \in (p_0, 1)$, denote $p_1 = \frac{p+p_0}{2} \in (p_0, p)$. Proposition 3.9 ensures the existence of a neighborhood $N_{K,p}$ so that for all $T \in GL_n$ and $K_0, K_1 \in T(N_{K,p})$, K_0 satisfies the local p_1 -BM inequality (3.3), and in addition, the function $g_p(\lambda)$ appearing in Proposition 11.1 (1) is concave. It remains to establish the equality conditions in Proposition 11.1 (1) to deduce the local uniqueness statement in (3). Assume that $K_0, K_1 \in T(N_{K,p})$ are such that the function $g_p(\lambda)$ is affine. It follows by the argument in the proof of Lemmas 3.4 and 3.5 that equality holds in the local p -BM inequality (3.3) for the body K_0 and $\frac{1}{p} f_0^p = \frac{1}{p} h_{K_1}^p - \frac{1}{p} h_{K_0}^p \in$

$C_e(S^{n-1})$. Recalling the equivalent formulations of the local p -BM equality and p_1 -BM inequality derived in Section 4, and denoting:

$$z_0 := \begin{cases} \frac{1}{h_{K_0}^p} \frac{f_0^p}{p} = \frac{1}{p} \left(\left(\frac{h_{K_1}}{h_{K_0}} \right)^p - 1 \right) & p \neq 0 \\ \log f_0 = \log \frac{h_{K_1}}{h_{K_0}} & p = 0 \end{cases}$$

as in (4.5), we deduce (say, using the formulation of (4.8)):

$$\begin{aligned} \frac{1}{V(K_0)} V(z_0 h_{K_0}; 1)^2 &= \frac{n-1}{n-p} V(z_0 h_{K_0}; 2) + \frac{1-p}{n-p} V(z_0^2 h_{K_0}; 1), \\ \frac{1}{V(K_0)} V(z_0 h_{K_0}; 1)^2 &\geq \frac{n-1}{n-p_1} V(z_0 h_{K_0}; 2) + \frac{1-p_1}{n-p_1} V(z_0^2 h_{K_0}; 1). \end{aligned}$$

Equating the $V(z_0 h_{K_0}; 2)$ terms above and using that $p_1 < p < 1 \leq n$, it follows that:

$$V(z_0^2 h_{K_0}; 1) \leq \frac{1}{V(K_0)} V(z_0 h_{K_0}; 1)^2.$$

On the other hand, the reverse inequality is always satisfied by Cauchy–Schwarz (4.11). By the equality conditions of Cauchy–Schwarz, it follows that z_0 must be a constant dS_{K_0} -a.e. on S^{n-1} . Using the fact that dS_{K_0} and the Lebesgue measure are equivalent since $K_0 \in \mathcal{K}_+^2$, and as support functions are continuous, it follows that $h_{K_1} = Ch_{K_0}$ identically on S^{n-1} for some $C > 0$, and the equality case in Proposition 11.1 (1) is established. \square

It is now immediate to translate the results of this work into the following:

Theorem 11.3. *The even L^p -Minkowski problem has a locally unique solution in a neighborhood of K for all $p \in (p_K, 1)$, in the following cases:*

- (1) For any $K \in \mathcal{K}_{+,e}^2$ and $p_K = 1 - \frac{c}{n^{3/2}}$.
- (2) For $K = B_2^n$ and $p_K = -n$.
- (3) For any $\epsilon > 0$, for all $K \in N_{B_\infty^n}^{C,\epsilon} \cap \mathcal{K}_{+,e}^2$ and $p_K = \epsilon$, where $N_{B_\infty^n}^{C,\epsilon}$ is an appropriate C -neighborhood of B_∞^n (depending on ϵ).
- (4) If $q \in (2, \infty)$, for all $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$ and $p_K = 1 - \frac{c}{n^{-1/q} + qn^{-2/q}}$, where $N_{B_q^n}^C$ is an appropriate C_e -neighborhood of B_q^n .
- (5) If $q \in [1, 2)$, for all $K \in N_{B_q^n}^C \cap \mathcal{K}_{+,e}^2$ and $p_K = c \in (0, 1)$, where $N_{B_q^n}^C$ is an appropriate C_e -neighborhood of B_q^n .

Proof. (1) follows from Theorem 6.9. (2) follows from Theorem 5.13. (3) follows from Theorem 10.2 and Corollary 5.4. (4) follows from Theorem 10.4. (5) follows from Theorem 10.5. \square

Case (2) should be compared with a result of Colesanti and Livshyts [12], who considered local uniqueness for the log-Minkowski problem (the case $p = 0$), and showed the existence of a C^2 -neighborhood $N_{B_2^n}$ of B_2^n , so that for all $K \in N_{B_2^n}$, if $dV_K = dV_{B_2^n}$ then necessarily $K = B_2^n$.

12 Stability Estimates for Brunn–Minkowski and Isoperimetric Inequalities

It is well-known that by differentiating the Brunn–Minkowski inequality, one can obtain isoperimetric inequalities. Recall that the sharp anisotropic isoperimetric inequality is the statement that:

$$P_L(K) \geq nV(L)^{\frac{1}{n}}V(K)^{\frac{n-1}{n}}, \quad (12.1)$$

with equality when $K = L$, where:

$$\begin{aligned} P_L(K) &:= \lim_{\epsilon \rightarrow 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon} = nV(L, K, \dots, K) \\ &= \int_{S^{n-1}} h_L dS_K = \int_{\partial K} h_L(\nu_{\partial K}) dx, \end{aligned} \quad (12.2)$$

is the anisotropic perimeter of K with respect to the convex body L . Note that in the isotropic case, when $L = B_2^n$, $P_{B_2^n}(K)$ boils down to the usual surface area $S(K)$ of K . While the Brunn–Minkowski and isoperimetric inequalities hold for all Borel sets K , we restrict (as usual in this work) our discussion to convex bodies only.

Since the (local) p -BM inequality for $p < 1$ is a strengthening of the classical (local) BM inequality (in the class of origin-symmetric convex bodies), it is natural to expect that some stability results for isoperimetric inequalities could be derived from it. In the first part of this section, we rewrite the equivalent formulations for the local p -BM inequality obtained in Section 4 in a manner which is more suitable for obtaining stability estimates. In particular, we obtain an interesting interpretation of the Hilbert–Brunn–Minkowski operator L_K as the operator controlling the deficit in Minkowski’s second inequality, and as a consequence, deduce new stability estimates for the anisotropic isoperimetric inequality and the (global) Brunn–Minkowski inequality for origin-symmetric convex bodies.

In the second part of this section, we show how to derive a strengthening of the best known stability estimates in the (global) Brunn–Minkowski and anisotropic isoperimetric inequalities for the class of all (not necessarily origin-symmetric) convex bodies.

12.1 New stability estimates for origin-symmetric convex bodies with respect to variance

Throughout this subsection, the body K is assumed to be a convex body in $\mathcal{K}_{+,e}^2$. Recall that L_K denotes the Hilbert–Brunn–Minkowski operator on $L^2(dV_K)$ associated to K defined in Section 5, and that the local p -BM inequality for K is equivalent by Proposition 5.2 to the inequality:

$$\int_{S^{n-1}} (-L_K z) z dV_K \geq \frac{n-p}{n-1} \text{Var}_{dV_K}(z) \quad \forall z \in C_e^2(S^{n-1}), \quad (12.3)$$

where:

$$\text{Var}_{dV_K}(z) := \int_{S^{n-1}} z^2 dV_K - \frac{1}{V(K)} \left(\int_{S^{n-1}} z dV_K \right)^2.$$

Also recall that one of our main results in this work is verifying the validity of (12.3) for $p = 1 - cn^{-3/2}$.

Theorem 12.1 (Stability estimate for Minkowski’s second inequality). *Assume that $K \in \mathcal{K}_{+,e}^2$ satisfies the local p -BM inequality (12.3) with $p \leq 1$. Then for all $L \in \mathcal{K}_{+,e}^2$:*

$$\begin{aligned} \frac{V(L, K, \dots, K)^2}{V(K)} &\geq V(L, L, K, \dots, K) + \frac{1-p}{n-p} R_K(L) \\ &\geq V(L, L, K, \dots, K) + \frac{1-p}{n-1} \text{Var}_{dV_K} \left(\frac{h_L}{h_K} \right), \end{aligned}$$

where:

$$\begin{aligned} R_K(L) &:= \int_{S^{n-1}} \left(-L_K \frac{h_L}{h_K} \right) \frac{h_L}{h_K} dV_K \\ &= \frac{1}{n(n-1)} \int_{S^{n-1}} \langle (D^2 h_K)^{-1} \xi, \xi \rangle dS_K \end{aligned} \quad (12.4)$$

$$= \frac{1}{n(n-1)} \int_{\partial K} \langle H_{\partial K} \zeta, \zeta \rangle dx, \quad (12.5)$$

and:

$$\begin{aligned}\xi &:= h_K \nabla_{S^{n-1}} \frac{h_L}{h_K} = \nabla_{S^{n-1}} h_L - h_L \nabla_{S^{n-1}} \log h_K, \\ \zeta &:= \xi(\nu_{\partial K}) = \nabla_{S^{n-1}} h_L(\nu_{\partial K}) - h_L(\nu_{\partial K}) \frac{x - \langle x, \nu_{\partial K} \rangle \nu_{\partial K}}{\langle x, \nu_{\partial K} \rangle}.\end{aligned}$$

In particular, $R_K(L) = 0$ if and only if $L = CK$ for some constant $C > 0$.

Remark 12.2. Note that in the isotropic case (when $L = B_2^n$), $h_L(\theta) \equiv 1$ and $\nabla_{S^{n-1}} h_L = 0$, so the above expressions for ξ and ζ simplify considerably. In addition, it will be clear from the proof that L need not be a convex body, and that any $w \in C_e^2(S^{n-1})$ will work equally well.

Proof of Theorem 12.1. Recall that by (4.8), the local p -BM inequality for K is equivalent to:

$$\forall z \in C_e^2(S^{n-1}) \quad \frac{1}{V(K)} V(zh_K; 1)^2 \geq \frac{n-1}{n-p} V(zh_K; 2) + \frac{1-p}{n-p} V(z^2h_K; 1),$$

where as usual $V(w; m)$ denotes the mixed volume of w repeated m times and h_K repeated $n - m$ times. Applying this to $z := \frac{h_L}{h_K}$, we deduce:

$$\frac{V(L, K, \dots, K)^2}{V(K)} \geq V(L, L, K, \dots, K) + \frac{1-p}{n-p} (V(z^2h_K; 1) - V(zh_K; 2)).$$

On the other hand, recall by (5.5) that $L_K = \tilde{L}_K - \text{Id}$ satisfies

$$R_K(L) = \int_{S^{n-1}} (-L_K z) z dV_K = \int_{S^{n-1}} z^2 dV_K - \int_{S^{n-1}} (\tilde{L}_K z) z dV_K = V(z^2h_K; 1) - V(zh_K; 2).$$

This verifies the first asserted inequality. The second asserted inequality follows immediately by (12.3). Note that $R_K(L) = 0$ if and only if $h_L = Ch_K$ since 0 is an eigenvalue of $-L_K$ with corresponding eigenspace spanned by the constant functions.

It remains to verify the alternative expressions derived for $R_K(L)$. Recall that by (5.6):

$$\begin{aligned}\int_{S^{n-1}} (-L_K z) z dV_K &= \frac{1}{n-1} \int_{S^{n-1}} h_K ((D^2 h_K)^{-1})^{i,j} z_i z_j dV_K \\ &= \frac{1}{n(n-1)} \int_{S^{n-1}} ((D^2 h_K)^{-1})^{i,j} (h_K z_i) (h_K z_j) dS_K,\end{aligned}$$

which yields the expression in (12.4). The one in (12.5) follows by the change-of-variables $\theta = \nu_{\partial K}(x)$, as in Subsection 4.6. Finally, recall that the inverse of the latter mapping is the Weingarten map $x = \nabla_{\mathbb{R}^n} h_K(\theta)$, and since $\nabla_{S^{n-1}} h_K(\theta) = \nabla_{\mathbb{R}^n} h_K(\theta) - \langle \nabla_{\mathbb{R}^n} h_K, \theta \rangle \theta$, we obtain the last expression for ζ . \square

To deduce an isoperimetric inequality from Theorem 12.1, we employ the following well-known consequence of the Aleksandrov–Fenchel inequality (or simply Minkowski’s inequality for mixed-volumes) [60, p. 334]:

$$V(L, L, K, \dots, K) \geq V(L)^{\frac{2}{n}} V(K)^{\frac{n-2}{n}}.$$

It is also possible to employ the stronger inequality:

$$V(L, L, K, \dots, K) \geq V(L)^{\frac{1}{n-1}} V(L, K, \dots, K)^{\frac{n-2}{n-1}},$$

but this would yield a slightly less aesthetically pleasing expression below, and so we leave this possibility to the interested reader. Together with (12.2), we immediately obtain:

Corollary 12.3 (Stability estimate for anisotropic isoperimetric inequality). *Assume that $K \in \mathcal{K}_{+,e}^2$ satisfies the local p -BM inequality (12.3) with $p \leq 1$. Then for all $L \in \mathcal{K}_{+,e}^2$:*

$$\begin{aligned} P_L^2(K) &\geq \left(nV(L)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}} \right)^2 + n^2 \frac{1-p}{n-p} V(K) R_K(L) \\ &\geq \left(nV(L)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}} \right)^2 + n^2 \frac{1-p}{n-1} V(K) \text{Var}_{dV_K} \left(\frac{h_L}{h_K} \right). \end{aligned}$$

Note that whenever $p < 1$, the above inequalities constitute a strict improvement over the classical sharp anisotropic isoperimetric inequality (12.1); the latter is recovered by applying Corollary 12.3 with $p = 1$ (which is always valid by the classical Brunn–Minkowski inequality, regardless of origin-symmetry). Consequently, the terms on the right-hand-sides above may be viewed as some type of isoperimetric deficits. The term involving $\text{Var}_{dV_K}(h_L/h_K)$ is a very intuitive way of quantifying how different L is from K . The term involving $R_K(L)$ is less intuitive, but nevertheless may be sometimes estimated from below by a simple geometric expression.

Let us demonstrate this for $n = 2$. Denote by r and R the in and out radii of K with respect to L , respectively, namely the best constants so that $rL \subset K \subset RL$, or equivalently:

$$r := \min_{\theta \in S^{n-1}} \frac{h_K(\theta)}{h_L(\theta)}, \quad R := \max_{\theta \in S^{n-1}} \frac{h_K(\theta)}{h_L(\theta)}.$$

Lemma 12.4. For all $K, L \in \mathcal{K}_{+,e}^2$ in \mathbb{R}^2 , we have:

$$R_K(L) \geq 4 \max \left(\frac{\left(\frac{1}{r} - \frac{1}{R}\right)^2}{V(K^\circ)}, \frac{\log^2\left(\frac{R}{r}\right)}{V(L^\circ)} \right),$$

where Q° denotes the polar-body to Q , i.e. the unit-ball for the norm h_Q .

Proof. When $n = 2$, since S^{n-1} is one dimensional and $dS_K = \det(D^2 h_K)$, we see from (12.4) that:

$$R_K(L) = \frac{1}{2} \int_{S^1} (h_K(h_L/h_K)')^2 d\theta.$$

To obtain the first estimate above, we applying Cauchy–Schwarz in the following form:

$$R_K(L) \geq \frac{1}{2} \frac{(\int_{S^1} |(h_L/h_K)'| d\theta)^2}{\int_{S^1} 1/h_K^2 d\theta}.$$

By origin-symmetry, it is clear that the function h_L/h_K changes its value from $1/r$ to $1/R$ to $1/r$ in half a revolution of S^1 , and similarly for the second half, and so:

$$\int_{S^1} |(h_L/h_K)'| d\theta \geq 4 \left(\frac{1}{r} - \frac{1}{R} \right).$$

On the other hand, since $h_K(\theta) = \|\theta\|_{K^\circ}$, it follows by integration in polar coordinates that:

$$\int_{S^1} 1/h_K^2 d\theta = 2V(K^\circ),$$

yielding the first estimate. The second estimate is established similarly by applying Cauchy–Schwarz in the following form:

$$R_K(L) = \frac{1}{2} \int_{S^1} (h_L \log(h_L/h_K)')^2 d\theta \geq \frac{1}{2} \frac{(\int_{S^1} |\log(h_L/h_K)'| d\theta)^2}{\int_{S^1} 1/h_L^2 d\theta}.$$

It is clear that other variations in this spirit are also possible. \square

Plugging the latter estimate on $R_K(L)$ into Corollary 12.3, we deduce:

Corollary 12.5. For all $K, L \in \mathcal{K}_{+,e}^2$ in the plane, so that K satisfies (12.3) for some $p \leq 0$, we have:

$$\frac{P_L(K)^2}{V(K)} \geq 4V(L) + \frac{2(1-p)}{2-p} \max \left(\frac{8}{V(K^\circ)} \left(\frac{1}{r} - \frac{1}{R} \right)^2, \frac{8}{V(L^\circ)} \log^2\left(\frac{R}{r}\right) \right)$$

Recall that by the result of Böröczky–Lutwak–Yang–Zhang [6], (12.3) always holds for any $K \in \mathcal{K}_{+,e}^2$ in the plane with $p = 0$, and so by approximation, the above estimate clearly holds for all origin-symmetric K, L in the plane with $p = 0$.

Remark 12.6. The above isoperimetric-deficit estimates should be compared with the classical estimate of Bonnesen (see [60, p. 324]):

$$\frac{P_L(K)^2}{V(K)} \geq 4V(L) + \frac{V(L)^2}{V(K)}(R - r)^2.$$

Note that in the isotropic case, when $L = B_2^n$, Bonnesen showed that the constant $V(L)^2 = \pi^2$ improves to 4π . It is not hard to check that Bonnesen’s numeric constant is better than the one in our estimates in the isotropic case when K is itself close to a Euclidean ball. On the other hand, in the non-isotropic case, it is not clear to us whether the estimates are comparable. In addition, note that by Theorem 5.13 and Proposition 3.9 we know that when K is ϵ -close to a Euclidean ball in C^2 then (12.3) in fact holds with $p \rightarrow -2$ as $\epsilon \rightarrow 0$, and so our constant 8 improves to $\frac{2(1-p)}{2-p}8 \rightarrow 12$, which is already better than Bonnesen’s anisotropic constant π^2 (but worse than the isotropic constant 4π).

To conclude this subsection, we use the well-known equivalence between the family of all anisotropic isoperimetric inequalities for all (say origin-symmetric) convex bodies K, L and (global) Brunn–Minkowski inequalities for such bodies, to obtain stability estimates for the Brunn–Minkowski inequality.

Corollary 12.7 (Stability estimate for Brunn–Minkowski inequality). *Let $K, L \in \mathcal{K}_{+,e}^2$ so that $K + L$ satisfies (12.3) for some $p \leq 1$. Then:*

$$\frac{V(K + L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}} - 1 \geq 2 \frac{1 - p}{n - p} \frac{R_{K+L}(K)}{V(K + L)} \geq 2 \frac{1 - p}{n - 1} \frac{\text{Var}_{dV_{K+L}}\left(\frac{h_K}{h_K + h_L}\right)}{V(K + L)}.$$

Remark 12.8. Note that both expressions on the right-hand-side above are symmetric in K and L since $\frac{h_L}{h_K + h_L} = 1 - \frac{h_K}{h_K + h_L}$.

Proof of Corollary 12.7. By Corollary 12.3 applied to the pair $K + L$ and Q for $Q \in \mathcal{K}_{+,e}^2$, we have:

$$P_Q(K + L) \geq nV(Q)^{\frac{1}{n}}V(K + L)^{\frac{n-1}{n}} \sqrt{1 + B_{K+L}(Q) \left(\frac{V(K + L)}{V(Q)}\right)^{\frac{2}{n}}}$$

where we denote for brevity $B_{K+L}(Q) := \frac{1-p}{n-p} \frac{R_{K+L}(Q)}{V(K+L)}$. Applying the above inequality to $Q = K$, $Q = L$, and summing, since $P_K(K+L) + P_L(K+L) = P_{K+L}(K+L) = nV(K+L)$, we deduce:

$$\begin{aligned} \frac{V(K+L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}} &\geq \alpha \sqrt{1 + B_{K+L}(K) \left(\frac{V(K+L)}{V(K)} \right)^{\frac{2}{n}}} \\ &\quad + (1-\alpha) \sqrt{1 + B_{K+L}(L) \left(\frac{V(K+L)}{V(L)} \right)^{\frac{2}{n}}}, \end{aligned}$$

with:

$$\alpha := \frac{V(K)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}}.$$

Denote the expression on the left-hand-side above by M , and recall that $B_{K+L}(K) = B_{K+L}(L)$. By convexity of the function $x \mapsto \sqrt{1+x^2}$, we conclude that:

$$\begin{aligned} M &\geq \sqrt{1 + B_{K+L}(K) \left(\alpha \left(\frac{V(K+L)}{V(K)} \right)^{\frac{1}{n}} + (1-\alpha) \left(\frac{V(K+L)}{V(L)} \right)^{\frac{1}{n}} \right)^2} \\ &= \sqrt{1 + 4B_{K+L}(K)M^2}. \end{aligned}$$

Consequently, we see that $4B_{K+L}(K) < 1$ and:

$$M \geq \frac{1}{\sqrt{1 - 4B_{K+L}(K)}} \geq 1 + 2B_{K+L}(K),$$

verifying the first asserted inequality. As usual, the second inequality follows immediately by (12.3). \square

12.2 Improved stability estimates for all convex bodies with respect to asymmetry

In this subsection, K and L are only assumed to be two convex bodies in \mathbb{R}^n (no origin-symmetry is required). The anisotropic isoperimetric deficit of K with respect to L is defined as:

$$\delta(K, L) := \frac{P_L(K)}{nV(L)^{\frac{1}{n}}V(K)^{\frac{n-1}{n}}} - 1.$$

The Brunn–Minkowski deficit of K and L is defined as:

$$\beta(K, L) := \frac{V(K + L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}} - 1.$$

The relative asymmetry of K with respect to L is defined as:

$$A(K, L) := \inf_{x_0 \in \mathbb{R}^n} \frac{V(K \triangle (x_0 + rL))}{V(K)},$$

where $r > 0$ is such that $V(rL) = V(K)$. The scaling factor of K and L is then defined as:

$$\sigma(K, L) := \max\left(r, \frac{1}{r}\right) = \max\left(\frac{V(K)^{\frac{1}{n}}}{V(L)^{\frac{1}{n}}}, \frac{V(L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\right).$$

The classical anisotropic isoperimetric inequality and Brunn–Minkowski inequality state that $\delta(K, L) \geq 0$ and $\beta(K, L) \geq 0$, respectively. It is also known that equality in both cases occurs if and only if $K = x_0 + rL$ for some $r > 0$ and $x_0 \in \mathbb{R}^n$, i.e. when $A(K, L) = 0$. A natural question is then whether one can obtain stable versions of these inequalities, lower bounding the deficits $\delta(K, L)$ and $\beta(K, L)$ by some function of $A(K, L)$.

In a groundbreaking work, it was shown by N. Fusco, F. Maggi and A. Pratelli [21] in the isotropic case (when $L = B_2^n$), that for all Borel sets K in \mathbb{R}^n of finite perimeter:

$$\delta(K, L) \geq \frac{1}{C(n)^2} A(K, L)^2, \tag{12.6}$$

for some constant $C(n)$ depending on the dimension n . Various other estimates on $\delta(K, L)$ as a function of $A(K, L)$ had been previously known (see [60, p. 326], [21] and the references therein), but that was the first time the best-possible quadratic dependence on $A(K, L)$ was obtained. The approach in [21] was based on a careful analysis of symmetrization of K , and thus confined to the isotropic case $L = B_2^n$. In another major milestone [19], it was shown by A. Figalli, Maggi and Pratelli that (12.6) holds for all convex bodies L and sets of finite perimeter K with $C(n)$ of the order of $n^{8.5}$. The approach in [19] was based on M. Gromov’s optimal-transport proof of the anisotropic isoperimetric inequality [54], which was itself inspired by Knöthe’s proof of the Brunn–Minkowski inequality [35]. As already mentioned, it is well-known that the family of anisotropic isoperimetric inequalities for all convex K, L is equivalent to the Brunn–Minkowski inequality for all convex

K, L ; consequently, Figalli–Maggi–Pratelli also obtained in [19] the following stable version of the Brunn–Minkowski inequality for convex bodies K, L :

$$\beta(K, L) \geq \frac{1}{C(n)^2} \frac{A(K, L)^2}{\sigma(K, L)}. \quad (12.7)$$

Soon after, Figalli–Maggi–Pratelli gave a second simpler proof of (12.7) (and hence of (12.6)) in [18] for all convex bodies K, L , which however yielded an exponential dependence of $C(n)$ on n . By carefully refining their argument, A. Segal was able in [61] to improve their estimate (in both (12.6) and (12.7) for all convex bodies K, L) to $C(n)$ of the order of $n^{3.5}$ (improving further to n^3 in (12.6) when K is origin-symmetric, and in (12.7) when both K and L are origin-symmetric). An improvement over Segal’s estimates by a factor of \sqrt{n} was claimed by Harutyunyan in [29], but unfortunately the proof contains a mistake (formula (3.18) in [29] is off by a factor of \sqrt{n}), a correction of which precisely cancels out the claimed improvement. To the best of our knowledge, prior to our own results, the best known estimate on $C(n)$ was due to Segal.

In this subsection, we improve the best-known dependence on $C(n)$ further in both (12.6) and (12.7) for all convex bodies (regardless of origin-symmetry) as follows:

Theorem 12.9. *The stable versions of the anisotropic isoperimetric inequality (12.6) and Brunn–Minkowski inequality (12.7) hold with $C(n) \leq Cn^{\frac{11}{4}}$ for all convex bodies K, L in \mathbb{R}^n .*

All the proofs in [18, 61, 29] employ the following so-called “Poincaré-type trace inequality” proved by Figalli–Maggi–Pratelli in [18]. We prefer referring to it as a “Cheeger-type boundary inequality”, since it involves L^1 rather than L^2 norms.

Lemma 12.10 (Figalli–Maggi–Pratelli). *Let K be a convex body in \mathbb{R}^n with:*

$$r(K)B_2^n \subset K \subset R(K)B_2^n.$$

Then:

$$\inf_{c \in \mathbb{R}} \int_{\partial K} |f(x) - c| dx \leq \frac{\sqrt{2}}{\log 2} \frac{nR(K)}{r(K)} \int_K |\nabla f| dx \quad \forall f \in C^1(K).$$

Our improved estimates in Theorem 12.9 are based on the following improvement of Lemma 12.10. We denote by $C_{Che}(K)$ the (reciprocal of the) Cheeger constant of K , namely the best constant in the following inequality:

$$\int_K |f(x) - m_f| dx \leq C_{Che}(K) \int_K |\nabla f| dx \quad \forall f \in C^1(K),$$

where m_f denotes a median value of the law of f under the Lebesgue measure on K .

Lemma 12.11. *With the same assumptions as in Lemma 12.10:*

$$\int_{\partial K} |f(x) - m_f| dx \leq \frac{nC_{Che}(K) + R(K)}{r(K)} \int_K |\nabla f| dx \quad \forall f \in C^1(K).$$

Remark 12.12. By a result of Kannan–Lovász–Simonovits [33] (see also Lovász–Simonovits [48] for a slightly worse numeric constant), it is known that $C_{Che}(K) \leq \frac{2}{\log 2} R(K)$ for any convex body K , and so the estimate in Lemma 12.11 is always better than the one in Lemma 12.10, up to numeric constants. Our improvement is based on the fact that $C_{Che}(K)$ can be significantly smaller than $R(K)$ when an appropriate affine transformation is applied to K .

Our proof of Lemma 12.11 is based on the same idea as in the proof of Theorem 6.6.

Proof of Lemma 12.11. Using convexity and integrating-by-parts:

$$\begin{aligned} \int_{\partial K} |f - m_f| dx &\leq \frac{1}{r(K)} \int_{\partial K} |f - m_f| \langle x, \nu_{\partial K} \rangle dx \\ &= \frac{1}{r(K)} \int_K \operatorname{div}(|f - m_f| x) dx \\ &= \frac{1}{r(K)} \left(n \int_K |f - m_f| dx + \int_K \langle \nabla |f - m_f|, x \rangle dx \right) \\ &\leq \frac{nC_{Che}(K) + R(K)}{r(K)} \int_K |\nabla f| dx. \end{aligned}$$

Note that used that $|\nabla |f - m_f|| = |\nabla(f - m_f)| = |\nabla f|$ on $\{f \neq m_f\}$ and the well-known $\int_{\{f=c\}} |\nabla |f - c|| dx = 0$, which is valid for all (say) locally Lipschitz functions f . \square

The improvement over the result of [61] is now achieved by substituting the application of Lemma 12.10 by that of Lemma 12.11. Observe that the desired estimates (12.6) and (12.7) are both invariant under a simultaneous application of a linear transformation to both K and L , as well as individually translating K or L (note that $P_L(K)$ is indeed invariant under translation of L). Consequently, we may always assume that K is in a desirable affine position. In [18, 61, 29], the authors employed John’s position, which ensures that $R(K)/r(K) \leq n$ for general convex bodies, and $R(K)/r(K) \leq \sqrt{n}$ for origin-symmetric ones. To obtain our

improvement, we apply an affine transformation to K so that K is in isotropic position, which was already defined in Subsection 6.2. As explained there, it is guaranteed that when K is in isotropic position then:

$$r(K) \geq \sqrt{\frac{n+2}{n}}, \quad R(K) \leq \sqrt{(n+2)n}. \quad (12.8)$$

As for the (reciprocal of the) Cheeger constant $C_{Che}(K)$, it is known by results of Maz'ya, Cheeger, Buser and Ledoux (see [44]) that for all convex bodies K , $C_{Che}(K)$ is equivalent (up to numeric constants, independent of the dimension) to the Poincaré constant $C_{Poin}(K)$, defined in Subsection 6.2. Recall from that subsection that the conjecture of Kannan, Lovász and Simonovits [33] predicts that if K is convex and in isotropic position then $C_{Che}(K) \leq C$ for some universal numeric constant $C > 0$, independent of the dimension n ; also recall that the presently best-known estimate for the KLS conjecture is due to Lee and Vempala [45], who showed that:

$$C_{Che}(K) \leq C\sqrt[4]{n} \quad (12.9)$$

for all isotropic convex bodies K in \mathbb{R}^n . This results in a strict improvement in Lemma 12.11 over the estimates in [18, 61, 29], which employed Lemma 12.10 – an improvement by an order of $n^{\frac{3}{4}}$ for general convex bodies, and by an order of $n^{\frac{1}{4}}$ for origin-symmetric ones – thereby yielding Theorem 12.9. Note that confirmation of the KLS conjecture would further improve our estimate on $C(n)$ down to an order of $n^{\frac{10}{4}} = n^{2.5}$.

While all other arguments remain unchanged, and the reader can easily reconstruct all the remaining details by following Segal's proof from [61], for the reader's convenience, we sketch the most essential steps in the proof.

Sketch of proof of Theorem 12.9. By scaling, one can assume that K, L have equal volumes. Since $A(K, L) \leq 2$, we may assume that $\delta(K, L) \leq 1$ (say), otherwise there is nothing to prove. Let

$$T(x) = \nabla\Phi(x)$$

be the Brenier optimal-transport map pushing forward the Lebesgue measure on K onto the Lebesgue measure on L , with $\Phi : K \rightarrow \mathbb{R}$ convex. The classical argument of Knöthe and Gromov (see [61, p. 385]) then yields:

$$V(L)\delta(K, L) \geq \int_K \left(\frac{\text{tr } dT(x)}{n} - (\det dT(x))^{1/n} \right) dx.$$

The usual arithmetic-geometric means (AM-GM) inequality applied to the (positive) eigenvalues of $dT(x) = \nabla^2\Phi(x) \geq 0$ already shows that the right-hand-side is non-negative, and the goal is to obtain a lower bound depending on $A(K, L)$.

Applying a stable version of the AM-GM inequality and Cauchy–Schwarz, Segal obtains (see [61, formula (10)]):

$$2nV(L)\sqrt{\delta(K, L)} \geq \int_K \|dT(x) - \text{Id}\| dx, \quad (12.10)$$

where as usual $\|\cdot\|$ denotes the Hilbert-Schmidt norm. By translating L we may always assume that the median of T_i satisfies $m_{T_i} = 0$ for each $i = 1, \dots, n$. Now applying Lemma 12.11 instead of Lemma 12.10 with $f(x) = T_i(x)$ to each coordinate $i = 1, \dots, n$, and passing back and forth between the ℓ_2^n to the ℓ_1^n norm (the latter step was not accounted for in [29]), we obtain:

$$\begin{aligned} \int_K \|dT(x) - \text{Id}\| dx &\geq \frac{1}{\sqrt{n}} \int_K \sum_{i=1}^n |\nabla T_i(x) - e_i| dx \\ &\geq \frac{1}{\sqrt{n}C_{Che, \partial}(K)} \int_{\partial K} \sum_{i=1}^n |T_i(x) - x_i| dx \geq \frac{1}{\sqrt{n}C_{Che, \partial}(K)} \int_{\partial K} |T(x) - x| dx, \end{aligned} \quad (12.11)$$

where:

$$C_{Che, \partial}(K) := \frac{nC_{Che}(K) + R(K)}{r(K)}.$$

While the passage between the ℓ_2^n to the ℓ_1^n norms seems wasteful, we have not been able to make this step more efficient.

Now, a beautiful geometric argument of Figalli–Maggi–Pratelli (see [61, p. 387]) verifies that (recall $V(K) = V(L)$):

$$\frac{1}{2}V(K)A(K, L) \leq \frac{1}{2}V(K \triangle L) = V(K \setminus L) \leq \int_{\partial K} |T(x) - x| dx. \quad (12.12)$$

Combining (12.10), (12.11) and (12.12), we deduce:

$$A(K, L) \leq 4n^{\frac{3}{2}}C_{Che, \partial}(K)\sqrt{\delta(K, L)}.$$

As already explained above, it remains to translate K and apply a simultaneous linear transformation to both K and L , so that K is in isotropic position. The estimates (12.8) and (12.9) then yield:

$$C_{Che, \partial}(K) \leq Cn^{\frac{5}{4}},$$

concluding the proof of Theorem 12.9 for the isoperimetric deficit (12.6). The estimate on the Brunn–Minkowski deficit (12.7) follows by the standard equivalence between the family of anisotropic isoperimetric inequalities for all convex bodies K, L and the Brunn–Minkowski inequality for all such pairs, as in the previous subsection (see [19, p. 203]). \square

References

- [1] B. Andrews. Entropy estimates for evolving hypersurfaces. *Comm. Anal. Geom.*, 2(1):53–64, 1994.
- [2] W. Arendt and R. Mazzeo. Friedlander’s eigenvalue inequalities and the Dirichlet-to-Neumann semigroup. *Commun. Pure Appl. Anal.*, 11(6):2201–2212, 2012.
- [3] F. Barthe and D. Cordero-Erausquin. Invariances in variance estimates. *Proc. Lond. Math. Soc. (3)*, 106(1):33–64, 2013.
- [4] M. Biegert. On traces of Sobolev functions on the boundary of extension domains. *Proc. Amer. Math. Soc.*, 137(12):4169–4176, 2009.
- [5] T. Bonnesen and W. Fenchel. *Theory of convex bodies*. BCS Associates, Moscow, ID, 1987. Translated from the German and edited by L. Boron, C. Christenson and B. Smith.
- [6] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The log-Brunn-Minkowski inequality. *Adv. Math.*, 231(3-4):1974–1997, 2012.
- [7] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. *J. Amer. Math. Soc.*, 26(3):831–852, 2013.
- [8] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Func. Anal.*, 22(4):366–389, 1976.
- [9] Yu. D. Burago and V. A. Zalgaller. *Geometric inequalities*, volume 285 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988.
- [10] I. Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1984.
- [11] A. Colesanti. From the Brunn-Minkowski inequality to a class of Poincaré-type inequalities. *Commun. Contemp. Math.*, 10(5):765–772, 2008.
- [12] A. Colesanti and G. V. Livshyts. Uniqueness of a smooth convex body with a uniform cone volume measure in the neighborhood of a ball. Manuscript, 2017.

- [13] A. Colesanti, G. V. Livshyts, and A. Marsiglietti. On the stability of Brunn-Minkowski type inequalities. *J. Funct. Anal.*, 273(3):1120–1139, 2017.
- [14] A. Colesanti and E. Saorín Gómez. Functional inequalities derived from the Brunn-Minkowski inequalities for quermassintegrals. *J. Convex Anal.*, 17(1):35–49, 2010.
- [15] D. Cordero-Erausquin. Santaló’s inequality on \mathbb{C}^n by complex interpolation. *C. R. Math. Acad. Sci. Paris*, 334(9):767–772, 2002.
- [16] D. Cordero-Erausquin, M. Fradelizi, and B. Maurey. The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems. *J. Funct. Anal.*, 214(2):410–427, 2004.
- [17] S. Dar. A Brunn-Minkowski-type inequality. *Geom. Dedicata*, 77(1):1–9, 1999.
- [18] A. Figalli, F. Maggi, and A. Pratelli. A refined Brunn-Minkowski inequality for convex sets. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(6):2511–2519, 2009.
- [19] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.
- [20] W. J. Firey. p -means of convex bodies. *Math. Scand.*, 10:17–24, 1962.
- [21] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.
- [22] M. E. Gage. Evolving plane curves by curvature in relative geometries. *Duke Math. J.*, 72(2):441–466, 1993.
- [23] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)*, 39(3):355–405, 2002.
- [24] R. J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2006.
- [25] R. J. Gardner and A. Zvavitch. Gaussian Brunn-Minkowski inequalities. *Trans. Amer. Math. Soc.*, 362(10):5333–5353, 2010.
- [26] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [27] A. Girouard and I. Polterovich. Spectral geometry of the Steklov problem (survey article). *J. Spectr. Theory*, 7(2):321–359, 2017.
- [28] C. Haberl, E. Lutwak, D. Yang, and G. Zhang. The even Orlicz–Minkowski problem. *Adv. Math.*, 224(6):2485–2510, 2010.
- [29] D. Harutyunyan. Quantitative anisotropic isoperimetric and Brunn-Minkowski inequalities for convex sets with improved defect estimates. to appear in ESAIM: Control, Optimisation and Calculus of Variations,

- arxiv.org/abs/1604.04302, 2017.
- [30] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
 - [31] Y. Huang, J. Liu, and L. Xu. On the uniqueness of L_p -Minkowski problems: the constant p -curvature case in \mathbb{R}^3 . *Adv. Math.*, 281:906–927, 2015.
 - [32] N. Huet. Spectral gap for some invariant log-concave probability measures. *Mathematika*, 57(1):51–62, 2011.
 - [33] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.*, 13(3-4):541–559, 1995.
 - [34] B. Klartag. A Berry-Esseen type inequality for convex bodies with an unconditional basis. *Probab. Theory Related Fields*, 45(1):1–33, 2009.
 - [35] H. Knothe. Contributions to the theory of convex bodies. *Michigan Math. J.*, 4:39–52, 1957.
 - [36] A. V. Kolesnikov and E. Milman. Poincaré and Brunn–Minkowski inequalities on the boundary of weighted Riemannian manifolds. to appear in *Amer. J. Math.*, arxiv.org/abs/1711.08825, 2014.
 - [37] A. V. Kolesnikov and E. Milman. Remarks on the KLS conjecture and Hardy-type inequalities. In *Geometric aspects of functional analysis, Israel Seminar 2011-2013*, volume 2116 of *Lecture Notes in Math.*, pages 273–292. Springer, Berlin, 2014.
 - [38] A. V. Kolesnikov and E. Milman. The KLS isoperimetric conjecture for generalized Orlicz balls. to appear in *Ann. Probab.*, arxiv.org/abs/1610.06336, 2016.
 - [39] A. V. Kolesnikov and E. Milman. Brascamp-Lieb-type inequalities on weighted Riemannian manifolds with boundary. *J. Geom. Anal.*, 27(2):1680–1702, 2017.
 - [40] A. V. Kolesnikov and E. Milman. Sharp Poincaré-type inequality for the Gaussian measure on the boundary of convex sets. In *Geometric aspects of functional analysis*, volume 2169 of *Lecture Notes in Math.*, pages 221–234. Springer, Cham, 2017.
 - [41] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.
 - [42] R. Latała and J. O. Wojtaszczyk. On the infimum convolution inequality. *Studia Math.*, 189(2):147–187, 2008.
 - [43] M. Ledoux. *The concentration of measure phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Provi-

- dence, RI, 2001.
- [44] M. Ledoux. Spectral gap, logarithmic Sobolev constant, and geometric bounds. In *Surveys in differential geometry. Vol. IX*, pages 219–240. Int. Press, Somerville, MA, 2004.
 - [45] Y. T. Lee and S. Vempala. Eldan’s stochastic localization and the KLS hyperplane conjecture: An improved lower bound for expansion. arxiv.org/abs/1612.01507, to appear in FOCS 2017, 2016.
 - [46] G. M. Lieberman. *Oblique derivative problems for elliptic equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.
 - [47] G. Livshyts, A. Marsiglietti, P. Nayar, and A. Zvavitch. On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities. *Trans. Amer. Math. Soc.*, 369(12):8725–8742, 2017.
 - [48] L. Lovász and M. Simonovits. Random walks in a convex body and an improved volume algorithm. *Random Structures Algorithms*, 4(4):359–412, 1993.
 - [49] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.*, 38(1):131–150, 1993.
 - [50] E. Lutwak. The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas. *Adv. Math.*, 118(2):244–294, 1996.
 - [51] E. Lutwak, D. Yang, and G. Zhang. On the L_p -Minkowski problem. *Trans. Amer. Math. Soc.*, 356(11):4359–4370, 2004.
 - [52] L. Ma. A new proof of the log-Brunn-Minkowski inequality. *Geom. Dedicata*, 177:75–82, 2015.
 - [53] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n -dimensional space. In *Geometric Aspects of Functional Analysis*, volume 1376 of *Lecture Notes in Mathematics*, pages 64–104. Springer-Verlag, 1987-1988.
 - [54] V. D. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.
 - [55] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292, 1960.
 - [56] R. C. Reilly. Applications of the Hessian operator in a Riemannian manifold. *Indiana Univ. Math. J.*, 26(3):459–472, 1977.
 - [57] L. Rotem. A letter: The log-Brunn-Minkowski inequality for complex bodies. Unpublished, arxiv.org/abs/1412.5321, 2014.
 - [58] C. Saroglou. Remarks on the conjectured log-Brunn-Minkowski inequality. *Geom. Dedicata*, 177:353–365, 2015.

- [59] C. Saroglou. More on logarithmic sums of convex bodies. *Mathematika*, 62(3):818–841, 2016.
- [60] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.
- [61] A. Segal. Remark on stability of Brunn-Minkowski and isoperimetric inequalities for convex bodies. In *Geometric aspects of functional analysis*, volume 2050 of *Lecture Notes in Math.*, pages 381–391. Springer, Heidelberg, 2012.
- [62] S. Sodin. An isoperimetric inequality on the ℓ_p balls. *Ann. Inst. H. Poincaré Probab. Statist.*, 44(2):362–373, 2008.
- [63] A. Stancu. The discrete planar L_0 -Minkowski problem. *Adv. Math.*, 167(1):160–174, 2002.
- [64] R. S. Strichartz. Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal.*, 52(1):48–79, 1983.
- [65] M. E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.
- [66] N. Ja. Vilenkin. *Special functions and the theory of group representations*. Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, R. I., 1968.
- [67] D. Xi and G. Leng. Dar’s conjecture and the log-Brunn-Minkowski inequality. *J. Differential Geom.*, 103(1):145–189, 2016.