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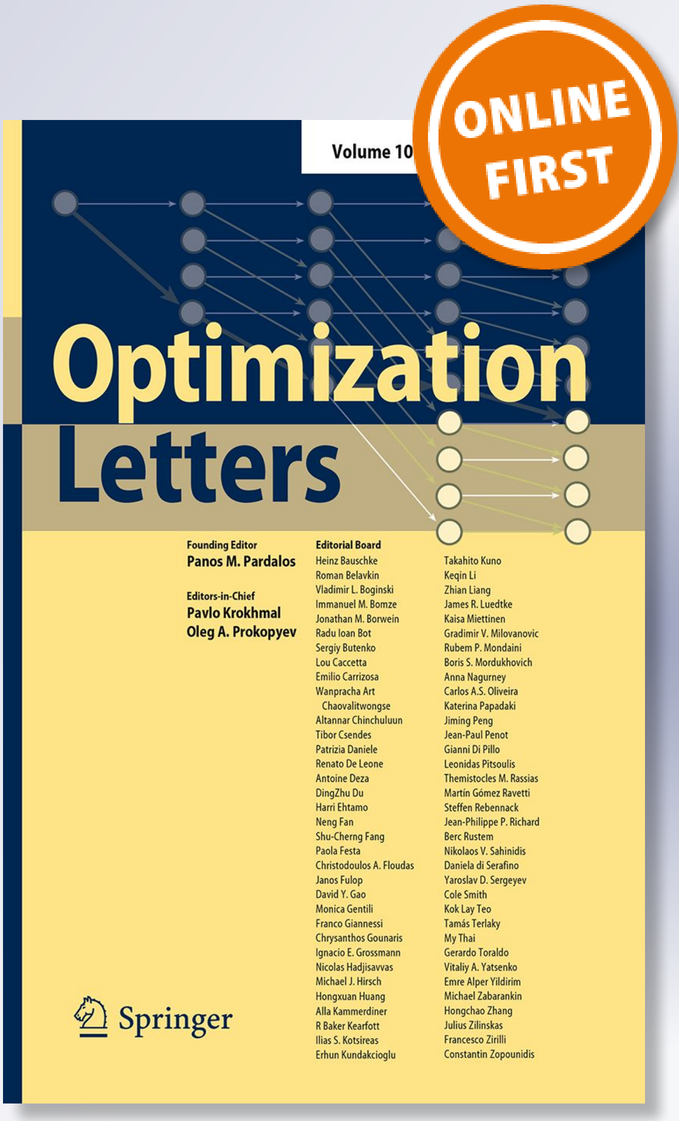


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ORIGINAL PAPER

# A polynomial-time algorithm of finding a minimum $k$-path vertex cover and a maximum $k$-path packing in some graphs

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**Abstract**
For a graph $G$ and a positive integer $k$, a subset $C$ of vertices of $G$ is called a $k$-path vertex cover if $C$ intersects all paths of $k$ vertices in $G$. The cardinality of a minimum $k$-path vertex cover is denoted by $\beta_{P_k}(G)$. For a graph $G$ and a positive integer $k$, a subset $M$ of pairwise vertex-disjoint paths of $k$ vertices in $G$ is called a $k$-path packing. The cardinality of a maximum $k$-path packing is denoted by $\mu_{P_k}(G)$. In this paper, we describe some graphs, having equal values of $\beta_{P_k}$ and $\mu_{P_k}$, for $k \geq 5$, and present polynomial-time algorithms of finding a minimum $k$-path vertex cover and a maximum $k$-path packing in such graphs.

**Keywords** $k$-path vertex cover · $k$-path packing · Computational complexity

## 1 Introduction

By default, all graphs in this paper are finite, undirected, without loops and multiple edges. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph $G$, respectively. We call a *$k$-path* a path of $k$ vertices and use $P_k$ to denote it.

For a positive integer $k$, a set of pairwise vertex-disjoint $k$-paths of a graph $G$ is called a *$k$-path packing* of $G$. The *$k$-path packing problem* is to find a maximum

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$k$-path packing in a graph. For a positive integer $k$, a set of vertices of a graph $G$, which intersects all $k$-paths of $G$, is called a *$k$-path vertex cover* of $G$. The *$k$-path vertex cover problem* is to find a minimum $k$-path vertex cover in a given graph. For a given graph $H$, the $H$-packing problem can be defined in a similar way. The $k$-path vertex cover problem can be motivated by a problem, related to security protocols in wireless sensor networks (see, for example [2,10,12,14]) or the problem of installing cameras on roads [11].

A lot of papers on packing problems are devoted to algorithmic aspects (see [4,6, 7,13]). It is known that the matching problem (i.e., the 2-path packing problem) can be solved in polynomial time [3], but the $H$-packing problem is NP-complete for any graph $H$, having a connected component on three or more vertices [5].

It seems perspective to find new polynomially solvable cases for the $k$-path packing and $k$-path vertex cover problems. Several results are known for $k = 3$ [1], $k = 4$ [9], and for the general case [8].

The aim of this paper is to describe a family of graph classes, on which the $k$-path packing and $k$-path vertex cover problems for $k \geq 5$ have polynomial-time algorithms. Namely, we consider some graphs, which hereditarily meet the equality of $\beta_{P_k}$ and $\mu_{P_k}$. This property admits us to present a polynomial-time algorithm of finding a minimum $k$-path vertex cover and a maximum $k$-path packing in such graphs.

We denote by $(v_1, v_2, \ldots, v_k)$ a $k$-path that consists of vertices $v_1, v_2, \ldots, v_k$. We denote by $|G|$ the number of vertices in $G$. We denote by $G \cup H$ the graph, obtained from graphs $G$ and $H$ by their union.

For a given graph $G$ and its subgraph $H$, we denote by $G \backslash H$ the graph, obtained from $G$ by deleting each vertex of $H$ with all incident edges. For a given graph $G$ and $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of $G$, induced by the set $A$.

## 2 *k*-extended graphs

In this section, we describe $k$-extended graphs and prove the equality of $\beta_{P_k}$ and $\mu_{P_k}$ for such graphs.

**Definition 1** An induced subgraph $T$ of a graph $G$ is a *terminal subgraph* of $G$ if there is only one vertex $u$ of the graph $G \backslash T$, which is adjacent to one or more vertices of $T$. We call $u$ the *contact vertex* of $T$.

For any $k \geq 2$, we call a connected graph, which does not have a $k$-path, as a $F_k$-graph.

**Definition 2** Let $\mathcal{M}$ be a pseudograph (a graph with possible loops and multiple edges). Given an integer $k \geq 5$, the operation of a *$k$-extension* of $\mathcal{M}$ consists of the following. All edges of cycles, including loops and two or more edges between the same vertices, are subdivided, each with $k - 1$ vertices. For a vertex $v$, denote by $d(v)$ the distance between $v$ and a nearest vertex of $\mathcal{M}$. For each new vertex $x$ with $d(x) \geq 2$, several terminal $F_{d(x)}$-graphs with a contact vertex $x$ can be added. For each old vertex $y$, several terminal $F_k$-graphs with a contact vertex $y$ can be added.

We refer to the obtained graph as $k$-extended graph.

**Theorem 1** *For each integer $k \geq 5$, $\beta_{P_k}(G) = \mu_{P_k}(G)$ in every $k$-extended graph $G$.*

***Proof*** Let $G$ be obtained by the operation of a $k$-extention from a pseudograph $\mathcal{M}$.

We can assume that $G$ is a connected graph. Otherwise, we can consider any its connected component. The proof is by induction on the number of vertices in $G$. If $G$ does not have $k$-paths, then $\mu_{P_k}(G) = \beta_{P_k}(G) = 0$.

Let $\mu_{P_k}(G) \geq 1$ and $\beta_{P_k}(H) = \mu_{P_k}(H)$, for each graph $H$ with $|H| < |G|$.

Consider the following cases.

1. There exists a terminal $F_k$-subgraph $X$ with the contact vertex $a$ in $G$, such that $G[V(X) \cup \{a\}]$ contains a $k$-path. Or there exists terminal $F_k$ subgraphs $X_1$ and $X_2$ with a common contact vertex $a$ in $G$, such that none of $G[V(X_1) \cup \{a\}]$ and $G[V(X_2) \cup \{a\}]$ contains $k$-paths, but $G[V(X) \cup V(Y) \cup \{a\}]$ contains a $k$-path. Denote $X = X_1 \cup X_2$ in the second case. One can see that each $k$-path, which contains vertices of the set $V(X)$, contains also the vertex $a$.
   Consider an arbitrary $k$-path $P$ of $G[V(X) \cup \{a\}]$. Denote $G^{'} = G \backslash P$. By the induction hypothesis, there exist a $k$-path packing $M$ and a $k$-path vertex cover $C$ of $G^{'}$, such that $|M| = |C|$.
   Then $M \cup \{P\}$ is the $k$-path packing of $G$ of the cardinality $|M| + 1$ and $C \cup \{a\}$ is the $k$-path vertex cover of $G$ of the same cardinality. Therefore, $\mu_{P_k}(G) = \beta_{P_k}(G)$.
2. The graph $G$ has not terminal $F_k$-subgraphs with the properties above. Then $G$ contains a cycle of $nk$ vertices, where $n \in \mathbb{N}$. Consider a cycle $Y_0$ of $\mathcal{M}$. Denote by $Y_1$ a cycle of $G$, which is obtained from $Y_0$ by $k - 1$ vertex subdivisions of all its edges.
   Denote $G^{'} = G \backslash Y_1$. By the induction hypothesis, there exist a $k$-path packing $M$ and a $k$-path vertex cover $C$ of $G^{'}$, such that $|M| = |C|$.
   Denote $t = |Y_0|$. Then $|Y_1| = kt$, i.e. the cycle $Y_1$ can be split into $t$ pairwise vertex-disjoint $k$-paths. Denote such $k$-paths as $P_1, P_2, \dots, P_t$. Then $M \cup \{P_1, P_2, \dots, P_t\}$ is a $k$-path packing of $G$ of the cardinality $|M| + t$.
   We need to prove that $C \cup V(Y_0)$ is a $k$-path vertex cover of $G$. Note, there are not edges between the vertex sets $V(G \backslash Y_2)$ and $V(Y_2 \backslash Y_0)$. So, each path, containing vertices of the both sets, contains at least one vertex of $Y_0$.
   Consider a connected component $Z$ of the graph $Y_2 \backslash Y_0$. It consists of a $(k-1)$-path $P$ and some terminal subgraphs with the contact vertices from $P$. Let $x$ be a vertex of $P$, and $T$ be a terminal subgraph with the contact vertex $x$. One can see that $d(x)$ equals the difference between the radius of $P$ and its distance from the center of $P$. Since $T$ is a $F_{d(x)}$-graph, the graph $G[V(T) \cup \{x\}]$ has not $(d(x)+1)$-paths. Thus, none of the paths in $Z$ has length more than $k - 1$, i.e. $Z$ is the $F_k$-graph. Hence, each connected component of the graph $Y_2 \backslash Y_0$ is a $F_k$-graph.
   Hence, each $k$-path of $Y_2$ contains at least one vertex of $Y_0$. So, $C \cup V(Y_0)$ is a $k$-path vertex cover of $G$ of the cardinality $|C| + t = |M| + t$.
   Therefore, $\mu_{P_k}(G) = \beta_{P_k}(G)$.

$\square$

## 3 Algorithms

Here we show that we can find a maximum $k$-path packing and a minimum $k$-path vertex cover in $k$-extended graphs in $O(n^2)$ time, where $n$ is the number of vertices in an input graph.

Let $\mathcal{M}(G)$ denote a pseudograph, such that $G$ is obtained by the operation of a $k$-extention from $\mathcal{M}(G)$. Let $A$ be the set of all cyclic vertices of $\mathcal{M}(G)$. The set $A$ can be found in $O(n^2)$ time, using the depth-first search (see Algorithm 1).

**Algorithm 1.**
**Input:** A $k$-extended connected graph $G = (V, E)$ with $|V| \geq k$.
**Output:** The set $A$ of all cyclic vertices of $\mathcal{M}(G)$.

1. $A = \emptyset$; $B = \emptyset$.
2. Choose an arbitrary vertex $z \in V$.
3. Build a DFS-tree $T$ of $G$ with the root $z$.
4. $E^{'} = E \setminus E(T)$.
5. **For** each $e \in E^{'}$ **do**
6.    Find a cycle $C$ in the graph $(V(T), E(T) \cup \{e\})$.
7.    **If** $|C| \geq k$, **then** add $V(C)$ into $B$ **End If**
8. **End For**
9. **For** each $v \in B$ **do**
10.    **If** $deg(v) \geq 2$ in $G[B]$, **then**
11.       add $v$ into $A$.
12.       **For** each $u \in B$ **do**
13.          **If** $dist(v, u)$ is divisible by $k$, **then** add $u$ into $A$ **End If**
14.       **End For**
15.    **End If**
16. **End For**

From the proof of Theorem 1, we can see that the following algorithm finds a maximum $k$-path packing and a minimum $k$-path vertex cover in a connected $k$-extended graph $G$.

**Algorithm 2.**
**Input:** A $k$-extended connected graph $G = (V, E)$ with $|V| \geq k$.
**Output:** A vertex set $C \subseteq V$, which is a minimum $k$-path vertex cover of $G$; a $k$-path set $M$, which is a maximum $k$-path packing of $G$.

1. $C = \emptyset$; $M = \emptyset$.
2. **For** each vertex $v \in V$ **do** $l(v) = 0$ **End For**
3. Find the set $A$.
4. Choose $z \in V$. **If** $A = B = \emptyset$, **then** $z$ is an arbitrary leaf of $G$, **else** $z$ is an arbitrary vertex of $A$.
5. Build a DFS-tree $T$ of $G$ with the root $z$.
   Denote by $p(v)$ the parent of the vertex $v$ in $T$.
   Denote by $Ch(v)$ the set of all children of the vertex $v$ in $T$.
6. **For** each leaf $v$ of $T$ **do** $l(v) = 1$ **End For**
7. **While** $|V(T)| \geq k$ **do**

Choose $v$, such that $l(v) = 0$ and $l(u) > 0$, for each $u \in Ch(v)$.
Choose $x \in Ch(v)$, where $l(x) \geq l(u)$, for each $u \in Ch(v)$.
**If** $l(x) = k - 1$, **then**
  Add $v$ into $C$.
  Find a $k$-path $P$ in the subtree with the root $v$.
  Add $P$ into $M$.
  Delete the subtree with the root $v$ from $T$.
  $l(p(v)) = 1$.
**Else**
  Choose $y \in Ch(v) \backslash \{x\}$, where $l(y) \geq l(u)$, for each $u \in Ch(v) \backslash \{x\}$.
  **If** $l(x) + l(y) \geq k - 1$, **then**
    Add $v$ into $C$.
    Find a $k$-path $P$ in the subtree with the root $v$.
    Add $P$ into $M$.
    Delete the subtree with the root $v$ from $T$.
    $l(p(v)) = 1$.
  **Else**
    $l(k) = l(x) + 1$.
  **End If**
**End If**
**End While**

Note that the complexity of building a DFS-tree for a graph is $O(n^2)$ and there is only one cycle in the other part of the algorithm. So, for any fixed $k$, the complexity of Algorithm 2 is $O(n^2)$. If the graph $G$ is not connected, then we can repeat this algorithm for each its connected component. Hence, a maximum $k$-path packing and a minimum $k$-path vertex cover can be found in $k$-extended graphs in time $O(n^2)$.

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