Validation method of maturity adjustment formula for Basel II capital requirement

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In recent years considerable progress has been observed in the development of credit risk models. The Revised Framework on International Convergence of Capital Measurement and Capital Standards (2004) (Basel II) raised the standards of risk management to a new level. The validation methodologies of internal rating-based systems have emerged as an important issue for the implementation of Basel II. One of the less examined problems is the theoretical investigation of maturity effects and the probability of default time structure. The Basel Committee recommendations include maturity adjustment for capital requirements. However, the complete derivation of the proposed adjustment formula remains undisclosed. In this paper the authors describe a method of maturity adjustment calculation directly from open data published by rating agencies. In addition, analytical expressions revealing the probability of default time structure are proposed. In order to validate the Basel II recommendation a comparison of the results found with the Basel maturity adjustment formula is performed. The character of the presented dependences is close enough, but it was discovered that for low probabilities of default (for high ratings) and maturities of two to three years there may exist considerable underestimation of risk capital.

The unknown credit losses a bank will suffer can be represented by two components: expected loss and unexpected loss. While a bank can forecast the average level of expected loss and manage it, unexpected losses are peak losses that exceed expected levels. Economic capital is needed to cover the risks of such peak losses, and therefore it has a loss-absorbing function.

In June 2004, the Basel Committee issued the first version of the Revised Framework on International Convergence of Capital Measurement and Capital Standards (or Basel II) (see Basel Committee on Banking Supervision (2004)). In this document the Basel II internal ratings-based approach was introduced. This approach is built on the following main parameters of credit risk: probability of default (PD), loss given default (LGD), exposure at default (EAD) and effective maturity (M). Under the advanced internal ratings-based approach, institutions are allowed to use their own internal models for these base risk parameters as primary inputs to the capital requirement calculation.

Banks generally employ a one-year planning horizon. The majority of well known portfolio models (CreditPortfolioView, CreditRisk+, CreditPortfolioManager, CreditMetrics, etc) as well as Basel II, agree that the value of the credit portfolio is only observed with respect to a predefined time horizon (typically one year). In fact this time horizon generally does not correspond with the actual maturity of loans in the credit portfolio. It is obvious that long-term credits are riskier. With respect to a three-year term loan, for example, taking account of the one-year horizon could mean that more than two-thirds of the credit risk is potentially ignored. So maturity is one of the most important parameters of risk. As a consequence, it is necessary to account for a real risk horizon to estimate precisely multi-year probability of default, unexpected loss and consequently sufficient capital requirement.

The topic of maturity effects has been widely discussed in recent literature. A number of authors have worked on multi-horizon economic capital allocation on the basis of the mark-to-market paradigm (Kalkbrener and Overbeck (2002); Barco (2004); Grundke (2003)). Under these, the models changes in portfolio value are caused by changes in credit spreads, which in their turn strongly depend on credit rating migration. Although the Markov assumption for probability of default time dependence is not proved, there are many works on Markov chain application for maturity effects (see, for example, Jarrow *et al* (1997), Inamura (2006) and Frydman and Schuermann (2005)). Bluhm and Overbeck (2007) do not reject the Markov assumption but adopt it by dropping the homogeneity assumption with non-homogeneity continuous-time Markov chains. For models based on the default-mode paradigm there exists little literature analyzing the accounting of long risk horizons (Gurtler and Heithecker (2005)).

The Basel risk weight functions used for the calculation of supervisory capital charges are based on a specific one-factor model adopted by the Basel Committee on Banking Supervision (2005). This model relies on the results of Merton (1974) and Vasicek (2002).

To account for the maturity effect the Basel Committee proposes a special maturity adjustment formula (Basel Committee on Banking Supervision (2005)). However, there is no available detailed explication and no initial data used for its deviation.

Thus the following process of validation on the basis of open data is considered:

1) The Merton-type one-factor model of Vasicek is considered. The capital requirement formula is adjusted to account for maturities longer than one year.
2) Cumulative default rates published by rating agencies are analyzed.
3) Special functions are proposed to approximate continuously cumulative default rates.
4) An appropriate maturity adjustment is calculated.
5) The maturity adjustment found is compared with the Basel maturity adjustment.

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CAPITAL REQUIREMENT CALCULATION

We briefly consider the derivation of the capital requirement formula. It should be noted that in contrast to Basel II the one-year time horizon is not fixed. Voluntary maturity T is considered.

It is assumed that a loan defaults if the value of the borrower's assets at the loan maturity T falls below the contractual value B of its obligations payable. Let A be the value of borrower's assets, described by the Wiener process:

$$\mathrm{d}A = \mu A \mathrm{d}t + \sigma A \mathrm{d}x$$

Here the asset value at T can be represented as:

$$\log A(T) = \log(A) + \mu T - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}X$$

where X is a standard normal variable.

The probability of default on risk horizon T (PD_T) then equals the probability that assets fall below the level of the borrower's obligations:

$$PD_T = \mathbf{P}[A(T) < B] = \mathbf{P}[X < c] = N(c)$$

where:

$$c = \frac{\log B - \log A - \mu T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and $N(\cdot)$ is a cumulative normal distribution function.

The variable X is standard normal and can therefore be represented as:

$$X = Y\sqrt{\rho} + Z\sqrt{1-\rho}$$

where Y, Z are mutually independent standard normal variables. The variable Y can be interpreted as a common factor, such as an economic index, over the interval $(0, T)$. Then ρ represents correlation of a borrower with the state of the economy. The term $Y\sqrt{\rho}$ is the company's exposure to the common factor and the term $Z_i\sqrt{1-\rho}$ represents the company's specific risk.

The probability of default is evaluated as the expectation over the common factor Y. When it is fixed, the conditional probability of default is:

$$\begin{aligned} pd(Y) &= \mathbf{P}[A(T) < B \mid Y] = \mathbf{P}[X \le c \mid Y] = \mathbf{P}[Y\sqrt{\rho} + Z\sqrt{1-\rho} \le c \mid Y] \\ &= \mathbf{P}\left[Z \le \frac{c - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right] = \mathbf{P}\left[Z \le \frac{N^{-1}(PD_T) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right] \\ &= N\left(\frac{N^{-1}(PD_T) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right) \end{aligned}$$

For the worst economical scenario the common factor takes the magnitude given by $-N^{-1}(\alpha)$ with some confidence level α ($\alpha = 0.999$ under Basel II). Then the

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worst conditional probability of default is:

$$pd(\alpha) = N\left(\frac{N^{-1}(PD_T) + N^{-1}(\alpha)\sqrt{\rho}}{\sqrt{1-\rho}}\right)$$

Under this worst case scenario the losses will also be the most serious. The capital requirement for a loan is then given by:

$$\begin{aligned}
&\text{Capital requirement}(PD_T, \alpha, \rho, EAD, LGD)\\
&\quad = \text{Worst loss} - \text{Expected loss}\\
&\quad = EAD \cdot LGD \cdot pd(\alpha) - EAD \cdot LGD \cdot PD_T\\
&\quad = EAD \cdot LGD \cdot \left(N\left(\frac{N^{-1}(PD_T) + N^{-1}(\alpha)\sqrt{\rho}}{\sqrt{1-\rho}}\right) - PD_T\right)\\
&\quad = EAD \cdot LGD \cdot (FDaR(T, \alpha, \rho) - PD_T) \qquad (1)
\end{aligned}$$

where $FDaR(T, \alpha, \rho) = N((N^{-1}(PD_T) + N^{-1}(\alpha)\sqrt{\rho})/\sqrt{\rho})$.

Thus, given the probability of default on time horizon T, the capital requirement could be calculated on the same time horizon.

Figure 1 (see page 5) illustrates the dependence of capital requirement on probability of default for a maturity of one year. For simplicity, EAD and LGD equal one.

BASEL MATURITY ADJUSTMENT

The Basel II capital requirement formula includes a component responsible for maturity (Basel maturity adjustment). It is noted that this adjustment follows from the regression of the output of the KMV Portfolio Manager™.

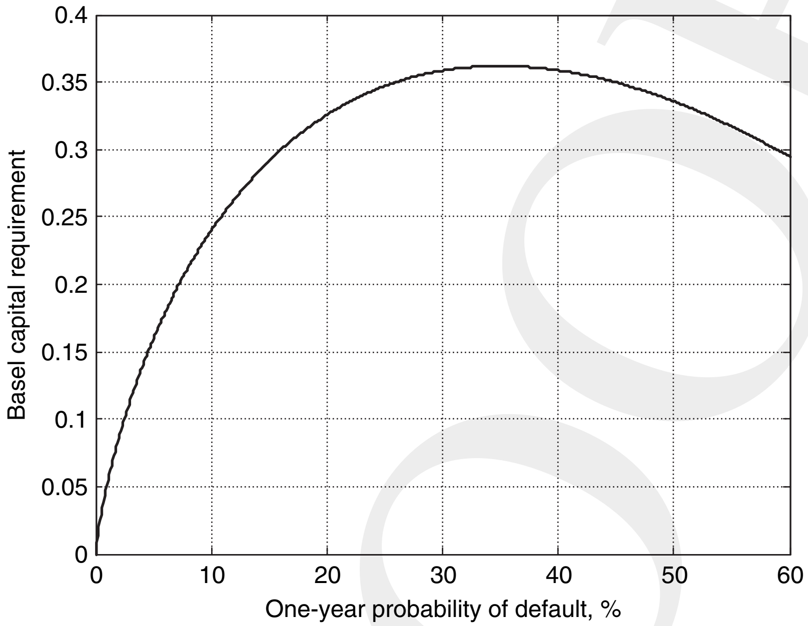
By its sense this adjustment is a penalty for the exceeding of one-year maturity. The dependence on maturity is linear for changes of risk horizon from one to five years and has the following form:

$$\begin{aligned}
\text{Basel maturity adjustment} &= \frac{1 + (T - 2.5) \cdot b(PD)}{1 - 1.5 \cdot b(PD)}\\
b(PD) &= (0.11852 - 0.05478 \log(PD))^2 \qquad (2)
\end{aligned}$$

where PD is one-year probability of default.

Figure 2(a) (see page 6) illustrates the dependence of Basel maturity adjustment on one-year probability of default. Maturity is fixed to three years. The maturity effect is stronger for higher probabilities of default than for lower probabilities of default. Figure 2(b) (see page 6) illustrates how the Basel maturity adjustment

FIGURE 1 Dependence of Basel capital requirement on one-year probability of default.



formula changes with maturity. One-year probabilities of default equal 0.1%, 1% and 10%. The adjustment is linear and increasing with the maturity.

Probability of default time structure

From cumulative default rates published by major rating agencies, such as Fitch Ratings (2006), Moody's (2006) and Standard and Poor's (2007), it directly follows that probability of default increases with the increase of risk horizon (Table 1, see page 7; Figure 3, see page 8). So we need to perform an adjustment in one-year probability of default if we want to take into account maturities longer than one year. Consequently, adjustment in capital requirement is necessary when the one-year time horizon is exceeded.

It should be noted that similar results can be found on the basis of data provided by all of the rating agencies mentioned earlier. However, Moody's statistical data is mainly considered in this paper (see Table 1 on page 7).

There are some potential errors in this data (see Credit Metrics™ (1997)):

- Output cumulative default likelihoods violate proper rank order. For instance, Table 1 shows that Aaas have defaulted more often at the 10-year horizon than have Aas. This is true also for B1 and Ba3 ratings.

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FIGURE 2(a) Dependence of Basel maturity adjustment on one-year probability of default.

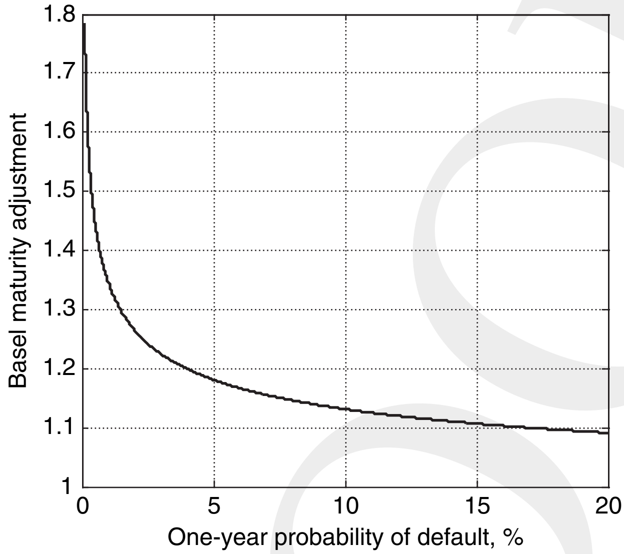

FIGURE 2(b) Dependence of Basel maturity adjustment on maturity.

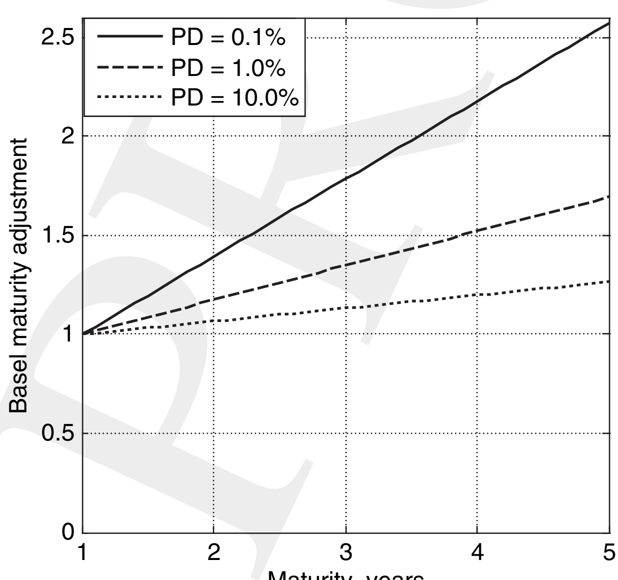

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TABLE 1 Example of average cumulative corporate default rates for several ratings/years.

Rating	Year 1	Year 2	Year 3	Year 4	...	Year 16	Year 17	Year 18	Year 19	Year 20
Aaa	0.000	0.000	0.000	0.039	...	0.208	0.208	0.208	0.208	0.208
Aa1	0.000	0.000	0.000	0.110	...	0.941	0.941	0.941	0.941	0.941
Aa2	0.000	0.011	0.048	0.120	...	0.970	1.177	1.414	1.684	1.710
...	...	...	...	...	...	...	...	...	...	...
B1	3.223	8.503	13.573	17.635	...	32.161	32.161	32.161	32.161	32.161
B2	5.457	12.067	17.141	21.057	...	29.598	29.680	29.756	29.756	29.756
B3	10.460	18.653	25.249	29.887	...	38.964	38.964	38.985	38.985	38.985
Caa-C	20.982	30.274	36.115	39.500	...	43.326	43.326	43.326	43.326	43.326

Note: see Moody's (2006), Exhibit 36.

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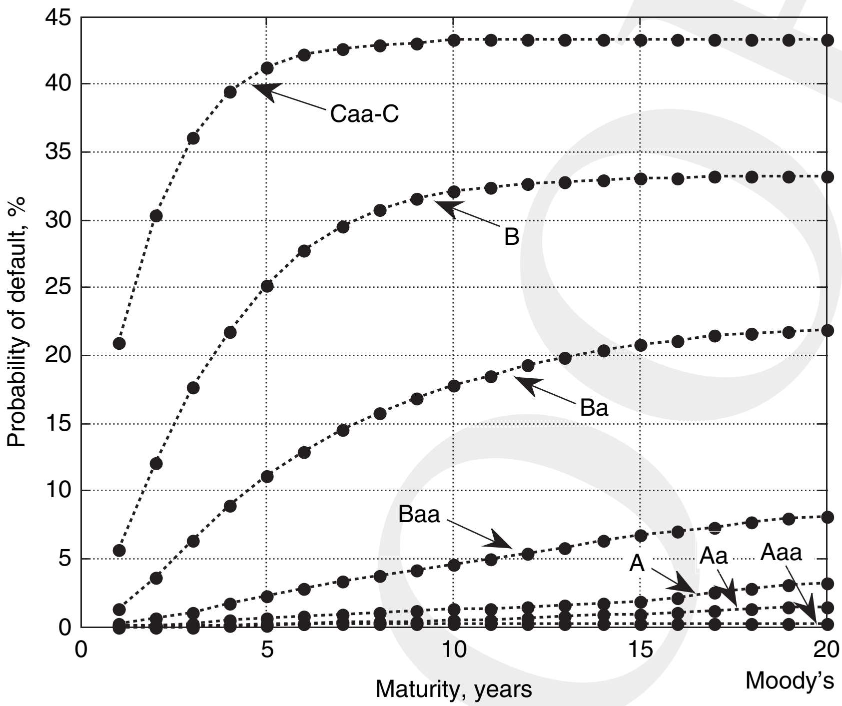
FIGURE 3 Average cumulative issue-weighted corporate default rates (Moody's data).



- Limited historical observation yields "granularity" in estimates. For instance, the Aaa row in Table 1 is supported by a limited number of firm-years worth of observation. In 1997 it was only 1,658 firm-years. This is enough to yield a "resolution" of 0.06% (ie, only probabilities in increments of 0.06% – or 1/1658 – are possible).
- This lack of resolution may erroneously suggest that some probabilities are identically zero. For instance, if there were truly a 0.01% chance of Aaa default, then we would have to wait another 80 years before there would be a 50% chance of tabulating a non-zero Aaa default probability.

In spite of these slight errors we suppose that the presented statistical data reflects well the time structure of the probability of default except, probably, for several first ratings for the reasons mentioned earlier.

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TABLE 2 Fitting results.

Alphanumeric rating	Numeric rating	PD one-year (%)	PDn	*a*	*b*	R^2
Aaa	1	0.000	0.000	0.364	0.404	0.915
Aa1	2	0.000	0.000	0.012	0.019	0.899
Aa2	3	0.000	0.027	0.000	0.006	0.975
Aa3	4	0.019	0.010	0.019	0.030	0.983
A1	5	0.003	0.033	0.040	0.060	0.970
A2	6	0.026	0.112	0.000	0.004	0.965
A3	7	0.037	0.063	0.000	0.016	0.951
Baa1	8	0.166	0.230	0.002	0.002	0.978
Baa2	9	0.161	0.157	0.041	0.252	0.998
Baa3	10	0.335	0.538	0.103	0.584	0.993
Ba1	11	0.753	1.072	0.084	1.106	0.996
Ba2	12	0.780	1.472	0.101	0.714	0.995
Ba3	13	2.069	4.117	0.162	0.762	0.986
B1	14	3.223	5.928	0.209	0.864	0.978
B2	15	5.457	7.325	0.297	1.252	0.992
B3	16	10.460	11.430	0.355	1.226	0.996
Caa-C	17	20.982	19.970	0.619	0.619	0.998

First, Moody's cumulative probabilities are fitted with a special parametric function for every rating:

$$\begin{aligned} PD_T &= F(PDn, a, b, T) \\ &= \left[\frac{PDn}{100} \cdot \left(\frac{1-\exp(-T\cdot a)}{1-\exp(-a)}\right)\right. \\ &\quad \left. + \left\{\left(\frac{1-\exp(-T\cdot a)}{1-\exp(-a)}\right) - \left(\frac{1-\exp(-T\cdot b)}{1-\exp(-b)}\right)\right\} \cdot \frac{1-\exp(-b)}{100\cdot b}\right] \end{aligned} \tag{3}$$

The fitting function depends on three parameters, PDn, a and b, which are different for every rating.

The actual form of this function is chosen to satisfy several essential properties:

- For the maturity of one year, parameter PDn is equivalent to one-year probability of default taken as a percentage.
- For zero maturity PD_T equals zero.
- The function has an asymptotic value for large terms, which is not equivalent to 100%. This property follows from the notion that over time companies either default rather fast or attain higher ratings. So with time we have some kind of stabilization. The property is satisfied when parameter a is greater than b for every rating.
- The function has a change in shape (for low probabilities of default we have concavity, for high probabilities of default we have convexity). This property

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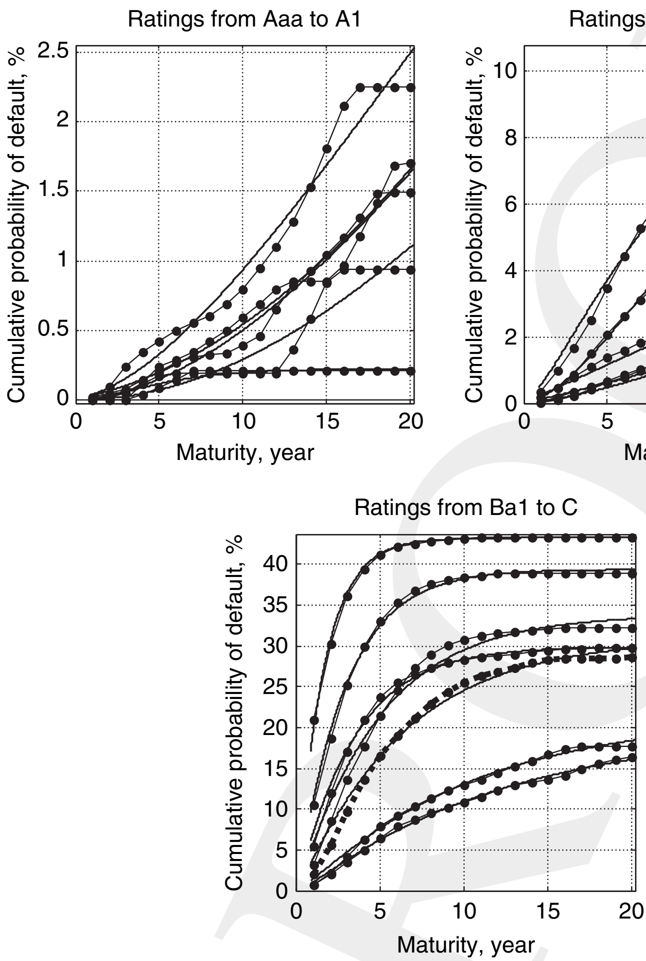
FIGURE 4 Results of cumulative default rates fitting (solid curves) and Moody's data (dots).



follows from the notion that companies with high ratings pass several lower ratings before default. So there exists some initial period where cumulative probability of default does not grow very fast (concavity). Companies with low ratings can come to default rather fast so we cannot observe such an effect and cumulative probability of default grows immediately (salience).

For proof, numerical ratings corresponding to alphanumeric ratings are introduced. The highest rating Aaa corresponds to the first numeric rating; Aa1 to the second, etc.

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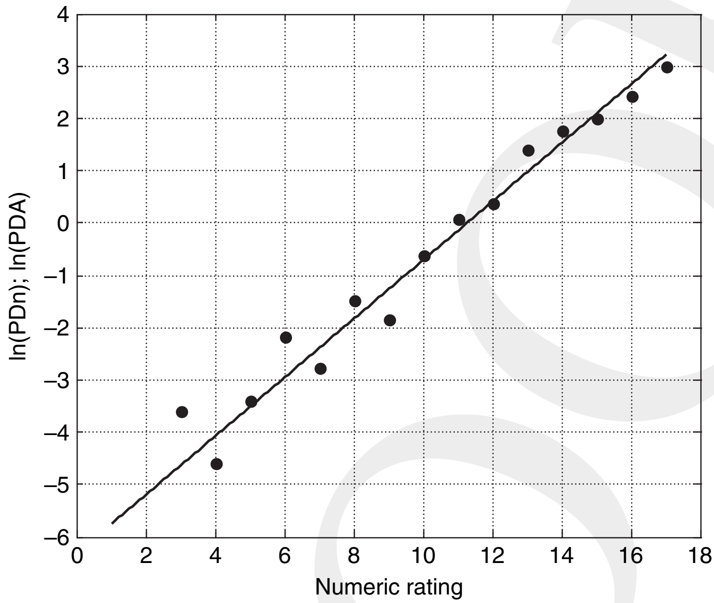
FIGURE 5 Transition from rating grades to continuous one-year probability of default.



Of course, the proposed function is not unique, but it shows very good fitting results (see Table 2 on page 9 and Figure 4 on page 10).

In Table 2 the set of received data is presented for every rating: for three parameters of the fitting function and one-year probabilities. R-squared shows that the proposed function precisely takes into account the particularities of the used data.

So far we have used probabilities of default that correspond to discrete ratings. However, probability of default is continuous by its nature. So it is necessary to pass from discrete ratings (and corresponding one-year probabilities of default) to continuous default probabilities.

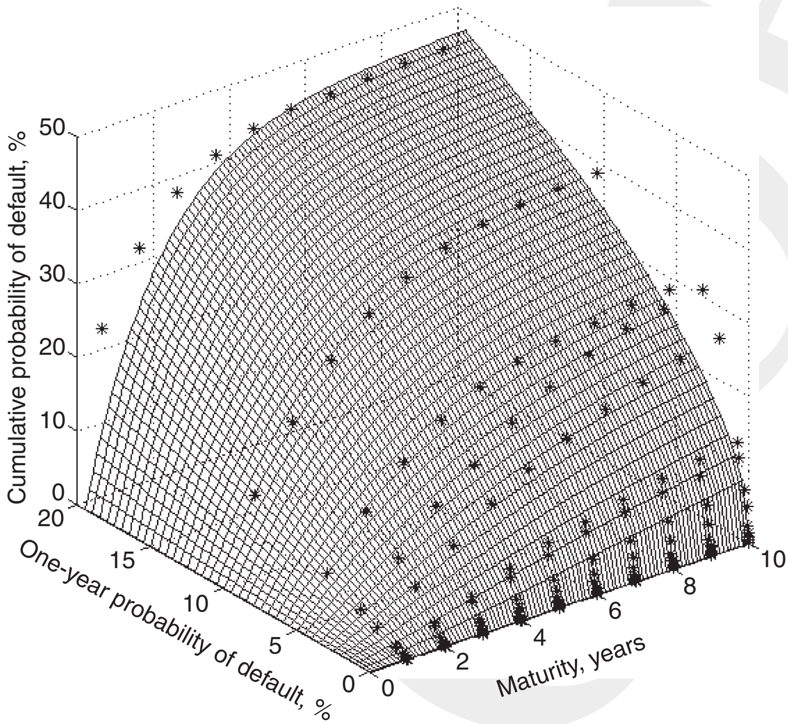
To do that we smooth the *PDn* parameter, which corresponds to one-year probability of default. The generally accepted logit function is used. Linear dependence is established between numeric ratings and the natural logarithm of *PDn* (see Figure 5). The quality of this approximation is rather high: R-squared equals 0.974:

$$PDA(\text{Numeric rating}) = \exp\,(0.561 \cdot (\text{Numeric rating}) - 6.307) \tag{4}$$

where *PDA* is the continuous approximation for the *PDn*.

To receive continuous dependency of cumulative default probabilities from one-year probability of default and maturity we also need to smooth two other parameters

FIGURE 6 Smoothed cumulative probabilities of default (surface) compared with Moody's data (dots).



(a and b). After the analysis of the dependence of parameters on PDA the following two fitting functions were proposed:

f_a depends on two parameters (α_a and β_a):

$$f_a(x) = \alpha_a \cdot \exp\,(\beta_a \cdot x)$$

f_b depends on three parameters (α_b, β_b, γ_b):

$$f_b(x) = \alpha_b \cdot \exp\,(-(\beta_b \cdot x + \gamma_b)^2)$$

Approximation of the parameters a and b gives the following results:

$$a(PD) = 0.080 \cdot \exp(0.639 \cdot \ln\,(100 \cdot PD)) \tag{5}$$

$$b(PD) = 1.278 \cdot \exp(-(0.293 \cdot \ln\,(100 \cdot PD) - 0.938)^2) \tag{6}$$

The constraint on a and b (a has to be greater than b) is fulfilled.

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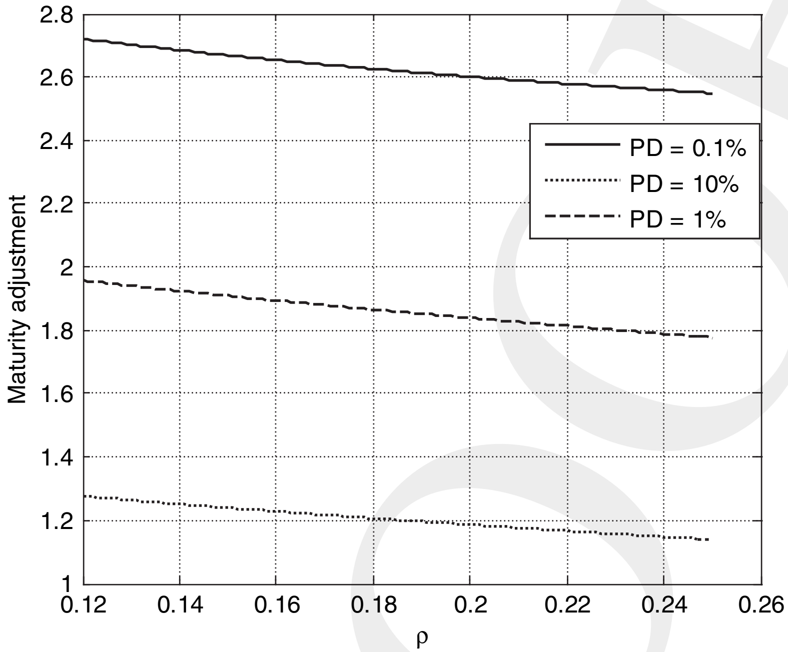
FIGURE 7 Dependence of maturity adjustment on correlation coefficient ρ.



The formula that gives the probability of default (PD_T) for every one-year default probability (PD) and maturity (T in years) follows from (3), (5) and (6):

$$PD_T = F(PD, a(PD), b(PD), T) \tag{7}$$

Figure 6 (see page 12) illustrates the correspondence of received continuous cumulative default probabilities (surface) with Moody's statistical data.

MATURITY ADJUSTMENT

Now, when the dependence of probability of default PD_T for every maturity is known we can construct maturity adjustment for the capital requirement in the following way:

$$\begin{aligned}\text{Maturity Adjustment }(PD, T) &= \frac{\text{Capital requirement }(PD_T, \alpha, \rho, EAD, LGD)}{\text{Capital requirement }(PD, \alpha, \rho, EAD, LGD)}\\ &= \frac{FDaR(T, \alpha, \rho) - PD_T}{FDaR(1, \alpha, \rho) - PD}\end{aligned}$$

where PD_T is calculated from (7).

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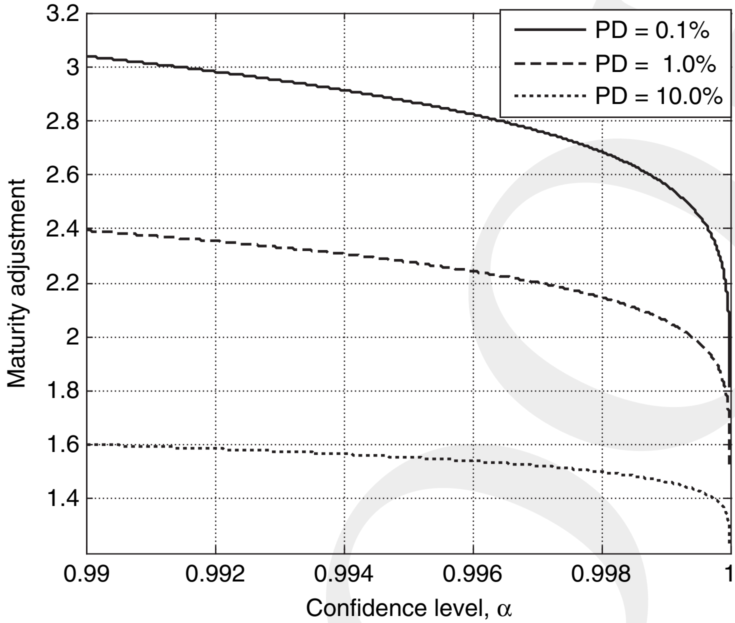
FIGURE 8 Dependence of maturity adjustment on confidence level α.



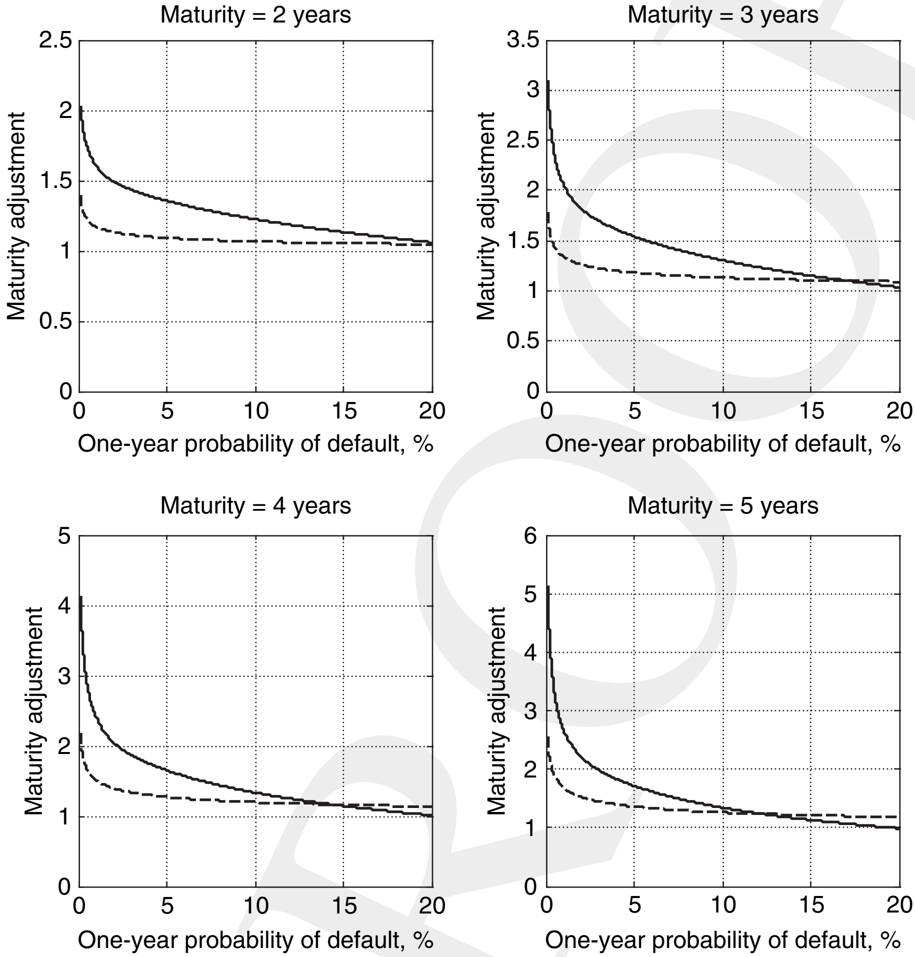
In addition, the dependence of maturity adjustment on the model parameters was analyzed. The maturity adjustment does not change strongly with the change of coefficient ρ (see Figure 7 on page 13). This fact confirms the absence of dependence on correlation in the received adjustment, as in the Basel maturity adjustment.

The dependence of maturity adjustment on the confidence level α is rather strong, particularly for low probabilities of default (see Figure 8). The level of adjustment declines with the convergence of α to one. The confidence level used for the derivation of the Basel maturity adjustment also remains undisclosed. However, under the Basel Committee recommendation we work with high confidence levels ($\alpha = 0.999$ or even $\alpha = 0.9999$).

Figure 9 (see page 15) illustrates the received maturity adjustment and the Basel maturity adjustment for several maturities, so it is possible to compare them.

Although the form of maturity adjustments is close enough, there is a difference in the Basel proposal and our results (see Figure 9). The received adjustment is higher for small probabilities of default (high ratings) and for maturities of about two or three years. It also reduces faster with the increase of one-year probability of default. The higher level of adjustment for high ratings is partly explainable and follows from the dependence of capital requirement on default probability (see Figure 1 on page 5). For small probabilities the slope of the curve is greater than for large

FIGURE 9 Received maturity adjustment (solid curves) and Basel maturity adjustment (dashed curves).



probabilities, so the same change in probability with time gives the greater change in capital requirement. However, at the moment there is no complete explanation for the difference between these two adjustments. If more exact information about methodology and data used for the Basel maturity adjustment were available it might be possible to explain these disagreements.

CONCLUSION

In this paper the dependence of default probability on time was continuously parameterized using data provided by Moody's. This approach gives results expressed analytically. The results correspond well with statistical data. The time structure of the probability of default allows the calculation of the maturity adjustment (or a penalty for excess of one-year maturity) for capital requirements.

The proposed approach of validation makes clearer the process of the maturity adjustment calculation.

It was shown that the character of the Basel approach maturity adjustment function can be explained rather well from open statistical data.

However, from the results found it follows that there exists the possibility to underestimate risk with the Basel maturity adjustment function. It is shown that the penalty is higher for assets with good rating (investment grade) and maturities of about two years. The precise estimation of unexpected loss is critical for bank stability. Although the Basel II recommendations are often regarded as rather conservative, the possible underestimation of risk may be as high as 50%.

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