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MATHEMATICS

Classification of One-Dimensional Attractors of Diffeomorphisms of Surfaces by Means of Pseudo-Anosov Homeomorphisms

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Abstract—Axiom A diffeomorphisms of closed 2-manifold of genus $p \geq 2$ whose nonwandering set contains a perfect spaciously situated one-dimensional attractor are considered. It is shown that such diffeomorphisms are topologically semiconjugate to a pseudo-Anosov homeomorphism with the same induced automorphism of fundamental group. The main result of this paper is as follows. Two diffeomorphisms from the given class are topologically conjugate on perfect spaciously situated attractors if and only if the corresponding homotopic pseudo-Anosov homeomorphisms are topologically conjugate by means of a homeomorphism that maps a certain subset of one pseudo-Anosov homeomorphism onto a subset of the other.

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We consider diffeomorphisms defined on a closed orientable 2-manifold M^2 of genus $p \geq 2$ that satisfy Smale's axiom A [10] (A-diffeomorphisms). According to Smale's spectral theorem, the nonwandering set $NW(f)$ of an A-diffeomorphism f can be represented in the form of a finite union of pairwise disjoint closed invariant basic sets, each containing an dense trajectory.

Examples of nontrivial (different from periodic orbits) basic sets of diffeomorphisms of 2-manifolds are a two-dimensional basic set of f on M^2 (in this case, the basic set coincides with M^2, which is a two-dimensional torus, and f is an Anosov diffeomorphism) and a one-dimensional basic set of a DA-diffeomorphism of the two-dimensional torus obtained from an Anosov diffeomorphism by applying surgery [10]. In [5] Smale's surgery is generalized to pseudo-Anosov homeomorphisms of surfaces in such a way that the surgery applied to any pseudo-Anosov homeomorphism defined on an orientable surface of genus $p \geq 2$ yields a structurally stable diffeomorphism of the same surface whose nonwandering set consists of precisely one one-dimensional attractor and a finite number of source periodic points.

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In constructing a topological classification of A-diffeomorphisms with nontrivial basic sets, the primary task is to find effective topological invariants describing the complex structure of the restriction of diffeomorphisms to a basic set and an embedding of the basic sets in an ambient manifold.

Following [8], a one-dimensional basic set Λ is called spaciously situated on M^2 if, for different points $x, y \in \Lambda$, any closed curve made up of the arcs $[x, y]^s \subset W^s_x$ and $[x, y]^u \subset W^u_x$ is not homotopic to zero on M^2. The topological conjugacy of A-diffeomorphisms of a 2-manifold on one-dimensional spaciously situated basic sets was completely analyzed by Grines and R.V. Plykin, while Grines and Kh.Kh. Kalai obtained a complete topological invariant for restrictions of diffeomorphisms to arbitrary one-dimensional basic sets by reducing this problem to constructing an algebraic classification of automorphisms of fundamental groups for the supports of the basic sets. Moreover, A.Yu. Zhirov obtained a complete combinatorial description of the dynamics of restrictions of diffeomorphisms to one-dimensional basic sets (see [1, 2] for more details and references).

In this work, we consider A-diffeomorphisms of a closed orientable 2-manifold M^2 of genus $p \geq 2$ whose nonwandering set contains a one-dimensional spaciously situated basic set Λ such that the set $M^2 \setminus \Lambda$ consists of a finite number of domains homeomorphic to the two-dimensional disk (such a basic set is perfect (see Definition 2)). It is shown that the problem of a topological classification of such diffeomorphisms is closely related to a topological classification of pseudo-Anosov homeomorphisms of surfaces.

On M^2 we introduce an analytic structure turning M^2 into a Riemannian surface. Consider a conformal map π of the universal covering of $\bar{M}^2$ to M^2, where $\bar{M}^2$ is the Lobachevsky plane in the Poincaré model on the interior of the disk $|z| < 1$ in the complex z-plane. It is well known that M^2 is associated with a uniquely determined discrete group Γ of non-Euclidean translations such that M^2 is conformally equivalent to $\bar{M}^2/\Gamma$ and Γ is isomorphic to the fundamental group $\pi_1(M^2)$ of M^2.

For a diffeomorphism $f: M^2 \to M^2$, let $\bar{f}: \bar{M}^2 \to \bar{M}^2$ denote a diffeomorphism covering f, i.e., a diffeomorphism for which $\pi\bar{f} = f\pi$. The transformation $\bar{f}_*: \Gamma \to \Gamma$ defined as $\bar{f}_*(\gamma) = \bar{f}\gamma\bar{f}^{-1}$ is an automorphism of Γ.

Definition 1. The automorphism $\bar{f}_*$ of the group Γ is called *hyperbolic* if, for any $\gamma_1, \gamma_2 \in \Gamma$ ($\gamma_1 \neq id$) and any $n \in \mathcal{N}$, we have $\bar{f}_*^n(\gamma_1) \neq \gamma_2\gamma_1\gamma_2^{-1}$.

A basic set of an A-diffeomorphism that is different from a periodic orbit is called nontrivial. According to Smale's spectral theorem [10], the nonwandering set $NW(f)$ of a diffeomorphism f is represented as a finite union of pairwise disjoint closed invariant sets, called basic sets, each containing a dense trajectory. Moreover, according to Anosov and R. Bowen, each basic set Λ can be represented as a finite union $\Lambda_1 \cup \ldots \cup \Lambda_k$ of closed subsets $(k \geq 1)$ such that $f^k(\Lambda_i) = \Lambda_i, f(\Lambda_i) = \Lambda_{i+1}$ $(\Lambda_{k+1} = \Lambda_1)$. The sets $\Lambda_1, \ldots, \Lambda_k$ are called periodic components of Λ, and the number k is their period.

Definition 2. A nontrivial basic set Λ of an A-diffeomorphism $f: M^2 \to M^2$ is called *perfect* if its complement $M^2 \backslash \Lambda$ consists of a finite number of domains Δ homeomorphic to the disk.

Lemma 1. *Given an A-diffeomorphism $f: M^2 \to M^2$ of a closed surface M^2, a perfect basic set Λ of f is a connected one-dimensional set and, hence, consists of precisely one periodic component.*

It follows from [8, Theorem 3] that a one-dimensional basic set of an A-diffeomorphism of a two-dimensional surface is either an attractor or a repeller and, by Theorem 1 from [8], contains an unstable and a stable manifold of its points, respectively.[1] Following [8], we give the following definition.

Definition 3. A basic set Λ of an A-diffeomorphism $f: M^2 \to M^2$ is said to be *spaciously situated* if, for different points $x, y \in \Lambda$, any closed curve made up of the arcs $[x, y]^s \subset W_x^s$ and $[x, y]^u \subset W_x^u$ is not homotopic to zero.[2]

Theorem 1. *Let $f: M^2 \to M^2$ be a diffeomorphism with a perfect spaciously situated attractor or repeller, and let $\bar{f}: \bar{M}^2 \to \bar{M}^2$ be a diffeomorphism covering f. Then the automorphism $\bar{f}_*$ is hyperbolic.*

Definition 4. A homeomorphism $P: M^2 \to M^2$ is called *pseudo-Anosov* if on M^2 there exists a pair of P-invariant transversal foliations $\mathcal{F}^s, \mathcal{F}^u$ with a set $\mathfrak{S}$ of saddle singularities and transversal measures μ^s, μ^u such that

(i) each saddle singularity from $\mathfrak{S}$ has at least three separatrices;

(ii) there exists a number $\lambda > 1$ such that $\mu^s(P(\alpha)) = \lambda\mu^s(\alpha)$ $(\mu^u(P(\alpha)) = \lambda^{-1}\mu^u(\alpha))$ for any arc α transversal to $\mathcal{F}^s$ $(\mathcal{F}^u)$.

The Nielsen–Thurston theory (see, e.g., [3, 4]) implies that, given a diffeomorphism $\bar{f}$ covering f, if the automorphism $\bar{f}_*$ is hyperbolic, then there exists a pseudo-Anosov homeomorphism P_f homotopic to f.

In what follows, to be definite, we assume (unless otherwise stated) that the one-dimensional basic set Λ under consideration is an attractor (in the case of a repeller, it suffices to consider the diffeomorphism f^{-1}).

Following [6, 9], a periodic point $p \in \Lambda$ is said to be a boundary periodic point of a basic set if one of the path connected components of the set $W^s(p)\backslash p$ is disjoint with Λ. According to [6], a one-dimensional attractor has a finite number of boundary periodic points and, for each domain Δ belonging to $M^2\backslash\Lambda$, its boundary accessible from within[3] consists of a finite number of one-dimensional unstable manifolds $W_{p_1}^u, \ldots, W_{p_{r_C}}^u$ $(r_C \geq 1)$ of boundary periodic points $p_1, \ldots, p_{r_C}$ of Λ and is called a bunch of degree r_C.

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[1] A basic set Λ of an A-diffeomorphism f is called an attractor if there exists a closed neighborhood U of Λ such that $f(U) \subset \text{int}\, U$, $\bigcap_{j \geq 0} f^j(U) = \Lambda$. An attractor of the diffeomorphism f^{-1} is called a repeller of the diffeomorphism f.

[2] $[x, y]^s, [x, y]^s, (x, y)^u, (x, y)^u$ denote closed and open intervals bounded by the points x, y that are contained in the one-dimensional stable W_x^s and unstable W_x^u manifolds, respectively.

[3] For a domain Δ, its boundary accessible from within is defined as a subset $C \subset \Lambda$ with the following property: for any point $y \in C$, there exists a path $\psi_y: I \to \Delta \cup C$ such that $\psi_y(1) = y$ and $\psi_y(t) \in \Delta$ for any $t \in [0, 1)$.

Theorem 2. *Let $f : M^2 \to M^2$ be a diffeomorphism with a perfect spaciously situated attractor Λ. Then there exists a continuous map $h : M^2 \to M^2$ homotopic to identity such that*

- $hf = P_f h$;
- *the restriction of h to Λ is one-to-one, except for the set $\Gamma^u = \{W^u(q_1), \ldots, W^u(q_k)\}$ $(k \geq 6)$ consisting of the unstable manifolds of all boundary periodic points $Q = \{q_1, \ldots, q_k\}$ from Λ;*
- *the set $h(Q)$ contains the set $\mathcal{S}$;*
- *for points $q_{i_1}, q_{i_2} \in Q$, the condition $h(q_{i_1}) = h(q_{i_2})$ holds if and only if q_{i_1}, q_{i_2} belong to the same bunch of Λ.*

Let $B = h(Q)$. The main result of this paper is as follows.

Theorem 3. *A homeomorphism $\varphi: M^2 \to M^2$ such that $f'|_{\Lambda'} = \varphi f \varphi^{-1}|_{\Lambda'}$ exists if and only if there exists a homeomorphism $\psi: M^2 \to M^2$ such that $\psi(B) = B'$ and $P_{f'} = \psi P_f \psi^{-1}$.*

Remark 1. If the carrying manifold M^2 for an A-diffeomorphism is a two-dimensional torus, then, in view of [7], Theorem 3 remains valid, but the role of pseudo-Anosov homeomorphisms P_f, $P_{f'}$ is played by corresponding algebraic Anosov automorphisms defined on the torus and induced by second-order hyperbolic unimodular matrices, i.e., by integer matrices with a determinant equal to +1 or −1 and with eigenvalues different from unity in absolute value.

Remark 2. Theorem 3 implies that the problem of constructing a topological classification of diffeomorphisms of surfaces of genus $p \geq 2$ with perfect spaciously situated basic sets is reduced to the problem of classifying pseudo-Anosov homeomorphisms with distinguished points. According to Zhirov (personal communication), an algorithm for solving this problem can be found in [11].

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