DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **39**, Number **8**, August **2019**

doi:10.3934/dcds.2019181

pp. **4443–4453**

WIND GENERATED EQUATORIAL GERSTNER-TYPE WAVES

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Abstract. A class of non-stationary surface gravity waves propagating in the zonal direction in the equatorial region is described in the f-plane approximation. These waves are described by exact solutions of the equations of hydrodynamics in Lagrangian formulation and are generalizations of Gerstner waves. The wave shape and non-uniform pressure distribution on a free surface depend on two arbitrary functions. The trajectories of fluid particles are circumferences. The solutions admit a variable meridional current. The dynamics of a single breather on the background of a Gerstner wave is studied as an example.

1. **Introduction.** Gerstner waves are trochoidal vortex gravity waves on deep water described by an exact solution to equations of a perfect incompressible fluid [16, 24]. That solution was later rediscovered by Froude [15], Rankine [30] and Reech [31] and found various applications in geophysics. Dubreil-Jacotin showed that Gerstner's solution also describes free surface waves on a fluid of arbitrary internal stratification [13]. Yih applied it to the edge waves along a sloping boundary [39] (see also discussions in [6, 26, 35, 36]). Pollard [29] and Mollo-Christensen [25, 27] modified Gerstner's solution to describe surface waves in deep water in a rotating fluid and gravitational billows on an interface between two fluids (or air masses) accordingly. Constantin obtained the three-dimensional solution for equatorial trapped waves generalizing the Gerstner wave in the β-plane approximation [9, 10, 11]. Henry extended this exact solution by allowing for a uniform current in the direction of propagation [20, 21]. Godin considered an extension of the Gerstner wave to waves in compressible three-dimensionally inhomogeneous moving fluids [17, 18].

All the mentioned theoretical studies were carried out assuming constant pressure on a free fluid surface. We generalize the Gerstner solution to the case of variable and non-uniform pressure, which can model the effect of wind blowing over a free surface.

In the present paper we consider waves in an equatorial region. We neglect the variations of Coriolis parameter and use the f-plane approximation. We study plane waves propagating in zonal direction. Their motion is investigated in Lagrangian formulation. Gerstner's solution with constant pressure on the profile gives one of exact solutions [9]. The incorporation of a constant underlying zonal current for

2010 *Mathematics Subject Classification.* Primary: 76B15; Secondary: 86A05, 74G05, 70H03.
Key words and phrases. Lagrangian variables, f-plane approximation, exact solution, Gerstner wave, vorticity.

The paper is for the special theme: Mathematical Aspects of Physical Oceanography, organized by Adrian Constantin.

Gerstner waves was achieved by Henry [20]. Kluczek studied the analogous three-dimensional flow taking into account a variable meridional current [23]. We present a class of exact solutions in the f-plane approximation generalizing Gerstner waves. The trajectories of the fluid particles are epicycloids (hypocycloids). These solutions were obtained earlier for a non-rotating fluid and were named Ptolemaic [1].

Gerstner wave is a particular case of Ptolemaic flows. Fluid particles in a Gerstner wave move around circumferences with radii decreasing exponentially with depth. Following Constantin and Monismith [12], waves in which the particle trajectories are closed will be called Gerstner-type waves. They include, in particular, equatorial [9, 10, 11] and interfacial [38] trapped waves. There also exists a subclass of Gerstner-type waves that possess the following properties: a) the fluid particles rotate around a circumference but the dependence of the radius of rotation on Lagrangian coordinates is different from the Gerstner one; b) the asymptotic behavior of wave shape is described by Gerstner's solution on both infinities. Consequently, the dynamics of our Gerstner-type waves differs from Gerstner's solution only on a limited section of a free surface. Pressure on this section is variable and we attribute this to the action of wind. The profile and pressure in Gerstner-type waves depend on two arbitrary analytical functions, so the obtained solution may describe a broad variety of initial wave shapes and pressure distributions. We propose a classification of Gerstner-type waves based on the form of these functions.

The rest of this paper is organized as follows. In Section 2 we consider the mathematical formulation of the problem in Lagrangian coordinates. The expressions for the integrals of motion in a uniformly rotating fluid extending the Cauchy invariants to a non-rotating fluid [24] are found in Section III. Section IV concerns Ptolemaic flows. It is shown that, if an arbitrary meridional current is superimposed on Ptolemaic motion, then the resulting flow will also be described by an exact solution of the equations of hydrodynamics. The expressions for vorticity and invariants of such a 3D flow are presented. The properties of a homogeneous (stationary) Gerstner wave are discussed in Section V. It is shown that it may be generated by a traveling harmonic pressure wave. Gerstner-type waves generated by the region of non-stationary and non-uniform pressure are studied in Section VI. Inhomogeneous Gerstner waves and modulated Gerstner waves are classed as separate families of possible types of motions. A vortex breather oscillating on the background of a homogeneous Gerstner wave is considered as an example of arbitrary Gerstner-type waves.

2. **Lagrangian formulation of governing equations in rotating reference frame.** Consider the motion of a homogeneous incompressible fluid in the reference frame moving with constant angular velocity $\vec{\Omega}$. The equations of hydrodynamics in Euler variables are written in the following form

$$\operatorname{div} \vec{R}_t = 0, \tag{1}$$

$$\vec{R}_{tt} + 2\,\vec{\Omega} \times \vec{R}_t = -\frac{1}{\rho}\nabla p - \nabla\Phi - \vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\right). \tag{2}$$

Here, $\vec{R}\,(X, Y, Z)$ is the radius vector of elementary liquid volume (X, Y, Z are Cartesian coordinates), $\vec{R}_t$ is its velocity and $\vec{R}_{tt}$ acceleration, t is time, ρ is density, p is pressure, and Φ is the potential of external forces. Expression (1) is a continuity equation. The vector momentum equation (2) is the record of Newton's second law with allowance for the action on the fluid of Coriolis and centrifugal forces. The latter has a gradient character, so Eq. (2) can be written as

$$\vec{R}_{tt} + 2\,\vec{\Omega}\;\times \vec{R}_t = -\nabla H\;;\qquad H = \frac{p}{\rho} + \Phi + \Phi_c, \quad \nabla\Phi_c = -\frac{1}{2}\nabla\left(\vec{\Omega}\times\vec{R}\right)^2, \tag{3}$$

where Φ_c is the potential of the centrifugal forces.

Let us find representation of Eqs. (1), (3) in terms of the Lagrangian variables $\{a_i\} = \{a,\, b,\, c\}$. We consider vector $\vec{R}$ as a function of these variables, so this vector determines the position of an individual liquid particle. The continuity equation in these variables has a form of time-independent Jacobian transition from Euler to Lagrangian variables that can be written in the form

$$\frac{D\,(X,Y,Z)}{D\,(a,b,c)} = \; S_0\,(a,b,c)\,. \tag{4}$$

If the initial positions of the fluid particles are equal to Lagrangian variables

$$X_0 = a, \quad Y_0 = b, \quad Z_0 = c, \tag{5}$$

$S_0 = 1$. In a general case, it is a function of Lagrangian variables. For one-to-one correspondence between the fluid particle coordinates X, Y, Z and their labels a, b, c this function should not turn to zero in the flow region. This is an important property of the Lagrangian flow description. We note that for both, the two-dimensional Gerstners wave [7, 19] and a number of three-dimensional generalizations [32, 33, 34], a mixture of analytical and topological methods can be applied to prove that the Lagrangian flow-map describing these exact solutions is a global diffeomorphism, with the result that the flow is globally dynamically possible.

In Eq. (3) the function H is differentiated by Euler variables X, Y, Z. To go to the derivatives with respect to the Lagrangian variables, we perform scalar multiplication of this equation by the vector $\vec{R}_{a_i}$:

$$\vec{R}_{tt}\vec{R}_{a_i} + 2\left(\vec{\Omega}\,\vec{R}_t\,\vec{R}_{a_i}\right) = -H_{a_i}, \quad i = 1, 2, 3. \tag{6}$$

Equations (4), (6) make up a system of hydrodynamic equations of a perfect incompressible fluid in Lagrangian variables in rotating reference frame.

3. **Lagrangian invariants.**

We will omit from Eqs. (6) the gradient term by taking its cross derivatives and calculating their difference:

$$\vec{R}_{tta_j}\vec{R}_{a_i} - \vec{R}_{tta_i}\vec{R}_{a_j} + 2\left(\vec{\Omega}\;\vec{R}_{ta_j}\;\vec{R}_{a_i}\right) - 2\left(\vec{\Omega}\,\vec{R}_{ta_i}\;\vec{R}_{a_j}\right) = 0. \tag{7}$$

With allowance for

$$\vec{R}_{tta_j}\vec{R}_{a_i} - \vec{R}_{tta_i}\vec{R}_{a_j} = \left(\vec{R}_{ta_j}\vec{R}_{a_i} - \vec{R}_{ta_i}\vec{R}_{a_j}\right)_t,$$

$$\left(\vec{\Omega}\,\vec{R}_{ta_j}\,\vec{R}_{a_i}\right) - \left(\vec{\Omega}\,\vec{R}_{ta_i}\,\vec{R}_{a_j}\right) = \left(\vec{\Omega}\,\vec{R}_{a_j}\,\vec{R}_{a_i}\right)_t.$$

Eq. (7) is rewritten in the form

$$\frac{\partial}{\partial\, t}\left(\vec{R}_{ta_j}\vec{R}_{a_i} - \vec{R}_{ta_i}\vec{R}_{a_j} + 2\left(\vec{\Omega}\,\vec{R}_{a_j}\,\vec{R}_{a_i}\right)\right) = 0,$$

that is equivalent to the conditions of conservation of three invariants S_1, S_2, S_3:

$$\vec{R}_{tb}\vec{R}_c - \vec{R}_{tc}\vec{R}_b + 2\left(\vec{\Omega}\,\vec{R}_b\,\vec{R}_c\right) = S_1\,(a,b,c)\,, \tag{8}$$

$$\vec{R}_{tc}\vec{R}_a - \vec{R}_{ta}\vec{R}_c + 2\left(\vec{\Omega}\,\vec{R}_c\,\vec{R}_a\right) = S_2\left(a,b,c\right), \tag{9}$$

$$\vec{R}_{ta}\vec{R}_b - \vec{R}_{tb}\vec{R}_a + 2\left(\vec{\Omega}\,\vec{R}_a\,\vec{R}_b\right) = S_3\left(a,b,c\right), \tag{10}$$

which are functions of Lagrangian coordinates only. Equations (8)–(10) follow from the momentum equations. Together with the continuity equation (4) they make up a system of hydrodynamic equations of a perfect incompressible fluid in a rotating frame of reference.

For $\vec{\Omega} = 0$, expressions (8)–(10) take on the following form

$$\vec{R}_{tb}\vec{R}_c - \vec{R}_{tc}\vec{R}_b = S_{10}\left(a,b,c\right), \tag{11}$$

$$\vec{R}_{tc}\vec{R}_a - \vec{R}_{ta}\vec{R}_c = S_{20}\left(a,b,c\right), \tag{12}$$

$$\vec{R}_{ta}\vec{R}_b - \vec{R}_{tb}\vec{R}_a = S_{30}\left(a,b,c\right). \tag{13}$$

Here, the index "0" denotes the motion in a non-rotating reference frame. These expressions were derived by Cauchy back in 1815 and were referred to by Lamb in his book [24]. The functions S_{10}, S_{20}, S_{30} are named the Cauchy invariants [2, 5, 14, 40]. They are equal to the circulation around each of the three infinitely small closed curves whose planes were originally normal to the coordinate axes [24]. Formulas (8)–(10) generalize the Cauchy invariants (11)–(13) for motions in a rotating frame of reference.

If the condition (5) is valid, then from (8)–(10) follows

$$S_1 = \left(\vec{\omega}_0 + 2\vec{\Omega},\ \vec{X}_*\right), \quad S_2 = \left(\vec{\omega}_0 + 2\vec{\Omega},\ \vec{Y}_*\right), \quad S_3 = \left(\vec{\omega}_0 + 2\vec{\Omega},\ \vec{Z}_*\right),$$

where $\vec{\omega}_0\left(\omega_{X0},\, \omega_{Y0},\, \omega_{Z0}\right)$ is the vorticity vector at the initial moment of time, and $\vec{X}_*, \vec{Y}_*, \vec{Z}_*$ are the unit vectors in the direction of the corresponding axes. In a general case, the relationship between the vector of invariants and absolute vorticity $\vec{\omega} + 2\vec{\Omega}$ is written in a more complicated form.

4. **Inertial Ptolemaic flows.** Let us consider the wave motion of a fluid in an equatorial region. Choose a rotating framework with the origin at a point on the Earth's surface with the spatial variable X corresponding to longitude, the variable Y to latitude, and the variable Z to the local vertical, respectively. For waves located close to the equator, the Coriolis parameter can be considered constant [8] (the so-called f-plane approximation). The angular velocity vector $\vec{\Omega}$ is directed along the Y axis and Eqs. (8)–(10) take on the form

$$\frac{D\left(X_t, X\right)}{D\left(b,c\right)} + \frac{D\left(Y_t, Y\right)}{D\left(b,c\right)} + \frac{D\left(Z_t, Z\right)}{D\left(b,c\right)} + 2\Omega\,\frac{D\left(Z, X\right)}{D\left(b,c\right)} = S_1\left(a,b,c\right), \tag{14}$$

$$\frac{D\left(X_t, X\right)}{D\left(c,a\right)} + \frac{D\left(Y_t, Y\right)}{D\left(c,a\right)} + \frac{D\left(Z_t, Z\right)}{D\left(c,a\right)} + 2\Omega\,\frac{D\left(Z, X\right)}{D\left(c,a\right)} = S_2\left(a,b,c\right), \tag{15}$$

$$\frac{D\left(X_t, X\right)}{D\left(a,b\right)} + \frac{D\left(Y_t, Y\right)}{D\left(a,b\right)} + \frac{D\left(Z_t, Z\right)}{D\left(a,b\right)} + 2\Omega\,\frac{D\left(Z, X\right)}{D\left(a,b\right)} = S_3\left(a,b,c\right), \tag{16}$$

where $\Omega = \left|\vec{\Omega}\right|$. Assume that the wavelength is rather small compared to the fluid depth, then the deep water approximation may be used. The speed at the bottom ($c = -\infty$) turns to zero and the boundary condition for pressure on a free surface ($c = 0$):

$$p|_{\,c=0} = p^*\left(a,\, t\right) \tag{17}$$

is fulfilled. We do not detail the form of pressure distribution and assume that it may vary depending on wind. For the free waves $p^* = const$, and for the plane waves $p^* = p^*(a,t)$. The non-uniform and non-stationary law of pressure variation is interpreted as the action of wind. In this sense, we will seek for a rather wide class of exact solutions that would meet the boundary condition (17).

Consider a particular solution of the form

$$X = X(a,c,t), \quad Y = b + \sigma(a,c)\, t, \quad Z = Z(a,c,t). \tag{18}$$

It is a superposition of a meridional current having profile $\sigma(a,c)$ on a non-stationary flow in the X, Z plane. The incorporation of such a flow into exact solutions to f-plane equations was first presented in [22, 24]. From the equation of incompressibility (4) follows

$$\frac{D(X,Z)}{D(a,c)} = S_0(a,c), \tag{19}$$

but Eqs. (14)–(16) give

$$\frac{D(X_t,X)}{D(a,c)} + \frac{D(Z_t,Z)}{D(a,c)} = -S_2(a,c) - 2\Omega S_0(a,c), \tag{20}$$

$$S_1 = -\sigma_c^{'}(a,c), \quad S_3 = \sigma_a^{'}(a,c).$$

The invariants S_1, S_3 are determined only by the form of the meridional current. The invariant S_2, on the contrary, is determined by solving Eqs. (19), (20). It depends on the motion of liquid particles in the X, Z plane and on the magnitude of the angular velocity.

Equations (19), (20) define a plane non-stationary flow. We introduce a complex coordinate of a fluid particle trajectory:

$$W = X + iZ, \quad \overline{W} = X - iZ,$$

where the overline "–" means complex conjugation, i is the imaginary unit, and complex Lagrangian coordinates are

$$\chi = a + ic, \quad \overline{\chi} = a - ic.$$

In the new variables, Eqs. (19), (20) are written as [1]:

$$\frac{D\left(W,\overline{W}\right)}{D(\chi,\overline{\chi})} = S_0, \tag{21}$$

$$\frac{D\left(W_t,\overline{W}\right)}{D(\chi,\overline{\chi})} = -i\left(\frac{S_2}{2} + \Omega S_0\right). \tag{22}$$

Equation (22) is written taking into consideration that the time derivative of Eq. (21) is equal to zero. The system of equations (21), (22) possesses an interesting property noted by Aleman and Constantin [4]: If an unknown function W satisfies Eq. (22), it will always be a solution to Eq. (21).

Equations (21), (22) have an exact solution [1]:

$$W = G(\chi)\exp(i\delta_1 t) + F(\bar{\chi})\exp(i\delta_2 t), \tag{23}$$

where F, G are analytic functions, and δ_1, δ_2 are real constants. The trajectories of the fluid particles in the X, Z plane are epicycloids (hypocycloids) as planet orbits in the Ptolemaic system of the world, so the flows (23) were named Ptolemaic [1]. In the most general case, when an inertial meridional current $\sigma(a,c)$ is superimposed on the Ptolemaic motion (23), the liquid particles gyrate. The flows (18), (23) may be called inertially Ptolemaic.

We will study a particular case $\delta_1 = 0$:

$$W = G(\chi) + F(\overline{\chi}) \exp(-i\mu t), \tag{24}$$

when the fluid particles move clockwise in the X, Z plane ($\mu = -\delta_2 > 0$). The flow region corresponds to the $c = \operatorname{Im}\chi \le 0$ domain. The fluid is motionless at the bottom, so $|F| \to 0$ as $\operatorname{Im}\chi \to -\infty$. The function G should be bijective, so $G' \neq 0$ in the flow region. One more requirement to the functions F, G is to maintain the sign of $S_0 = |G'|^2 - |F'|^2$, i.e. $S_0 \neq 0$ in the fluid region. It is the condition of bijection between the Euler and Lagrangian variables. Let for simplicity

$$|G'|^2 - |F'|^2 > 0. \tag{25}$$

The invariant S_2 is defined by

$$S_2 = 2\left[(\Omega - \mu)|F'|^2 - \Omega |G'|^2\right]. \tag{26}$$

The vorticity of the flow (18), (23) is written as

$$\begin{aligned}
\omega_X &= \frac{\partial Z_t}{\partial Y} - \frac{\partial Y_t}{\partial Z} = \frac{1}{S_0}\left[\frac{D(Z_t, Z, X)}{D(a,b,c)} + \frac{D(Y_t, Y, X)}{D(a,b,c)}\right] = \frac{1}{S_0}\frac{D(\sigma, \operatorname{Re}W)}{D(a,c)},\\
\omega_Y &= \frac{\partial X_t}{\partial Z} - \frac{\partial Z_t}{\partial X} = \frac{1}{S_0}\left[\frac{D(Z_t, Z, Y)}{D(a,b,c)} + \frac{D(X_t, X, Y)}{D(a,b,c)}\right] = -\frac{2\mu|F'|^2}{|G'|^2 - |F'|^2}, \qquad (27)\\
\omega_Z &= \frac{\partial Y_t}{\partial X} - \frac{\partial X_t}{\partial Y} = \frac{1}{S_0}\left[\frac{D(Y_t, Y, Z)}{D(a,b,c)} + \frac{D(X_t, X, Z)}{D(a,b,c)}\right] = \frac{1}{S_0}\frac{D(\sigma, \operatorname{Im}W)}{D(a,c)}.
\end{aligned}$$

Two vorticity components, ω_X and ω_Z, depend harmonically on time. The component ω_Y is an invariant of the flow that does not depend on meridional current $\sigma(a,c)$.

The expression for the pressure differential is found from Eqs. (6):

$$dp = -\rho \operatorname{Re}(W_{tt} - 2i\Omega W_t + ig)\, d\overline{W},$$

where the centrifugal force potential is neglected. The substitution into this equality of expression (24) and integration gives

$$\frac{p - p_0}{\rho} = g\operatorname{Im}\overline{G} + \frac{\mu(\mu + 2\Omega)}{2}|F|^2 + \operatorname{Re}\left[\int \mu(\mu + 2\Omega) F\overline{G}'\, d\overline{\chi} + igF\right]\exp(i\mu t), \tag{28}$$

where p_0 is a constant. The pressure is a sum of a stationary component (the first two terms in (29)) and a non-stationary component varying harmonically with time. Depending on the choice of the functions G and F, the form of the non-uniform pressure distribution on the free surface $p(a, c = 0, t)$ may be arbitrary to a large degree.

Liquid particles do not drift in zonal direction in the waves described by (24). The substitution $W \to W - U_0 t$ in expression (24), where U_0 is a real constant, allows considering the dynamics of wave perturbations against the background of a uniform underlying current. The region of non-uniform pressure on a free surface will, apparently, move with the same speed in zonal direction.

5. **Gerstner wave generated by a running harmonic pressure wave.** We choose a complex trajectory of the particles in the form

$$W = \chi + iA \exp i\left(k\overline{\chi} - \mu t\right), \quad \operatorname{Im} \chi \leq 0. \tag{29}$$

This expression describes a Gerstner wave having amplitude A, wave number k, and frequency μ [24]. A trochoidally shaped stationary wave is traveling to the right with speed $U = \mu k^{-1}$. The motion of a fluid is known to be stationary, if Lagrangian values are invariant under time translation [5]. The wave motion (29) is the only Ptolemaic flow (24) that has a stationary profile on a free surface.

The invariant S_2 for the Gerstner wave is

$$S_2 = 2\left[(\Omega - \mu)\, k^2 A^2 e^{2kc} - \Omega\right].$$

The pressure on the wave profile ($\operatorname{Im}\chi = 0$) is found by substituting expression (29) into (28):

$$\frac{p - p_0}{\rho} = \frac{\mu\,(\mu + 2\Omega)}{2} A^2 + \left[\mu\,(\mu + 2\Omega)\, k^{-1} - g\right] A \cos\,(ka - \mu t)\,. \tag{30}$$

For waves of the form (29), the traditionally imposed boundary condition is constant pressure on the free surface. Hence, zeroing the multiplier of cosine in expression (30) yields a dispersion wave equation [23].

It may be assumed, however, that under the action of wind, pressure distribution in the form of a harmonic traveling wave is maintained on the free surface:

$$p^* = p_1 + p_2 \cos\,(ka - \mu t)\,, \tag{31}$$

where p_1, p_2 are constant values satisfying the relations

$$p_1 = p_0 + \frac{\mu\,(\mu + 2\Omega)}{2} \rho A^2, \quad p_2 = \rho\left[\mu\,(\mu + 2\Omega)\, k^{-1} - g\right] A. \tag{32}$$

When these conditions are met, we can say that the exact solution (29) corresponds to the stationary trochoidal waves on a fluid surface maintained by the external pressure (31). If μ and k are known, we can find wave amplitude A from the second relation of system (32) and p_0 from the first one. The elevation of the free surface is defined by $Y = A \cos\,(ka - \mu t)$; hence, for positive values of p_2, the pressure changes in phase with the profile, and for negative p_2 in antiphase. The case $p_2 = 0$ corresponds to a Gerstner wave with constant pressure on the profile. Solving the quadratic equation (32) for μ leads us directly to the dispersion relation

$$\mu = \pm\sqrt{\Omega^2 + \left(g + \frac{p_2}{\rho A}\right) k} - \Omega. \tag{33}$$

We consider $(g + p_2/\rho A) > 0$. Taking the plus sign in (33) we obtain waves propagating eastwards. Taking the minus sign we get a wave propagating westwards. The plus or minus choice is not allowed in the β-plane [9] where the waves propagate eastwards only. The freedom in the sign of the phase speed is a consequence of the f-plane approximation. It is interesting to compare expression (33) with the dispersion relation of equatorial waves with an underlying current [23]. The quantity $\dfrac{p_2}{2\rho A\Omega}$ is analogous to the speed of the underlying current. Thus, depending on the p_2 sign, the wind either accelerates or slows down the waves.

The generation of Gerstner waves in laboratory conditions or in the real ocean has been actively discussed in the literature [12, 28, 37]. As follows from our analysis, wind may be a possible mechanism of their generation.

6. **Non-stationary waves.** Now consider the class of Ptolemaic waves (24) different from the Gerstner wave. As was mentioned above, such waves are non-stationary. We will assume for simplicity that there is no meridional current, and the flow is two-dimensional. Let the wind act only on a certain limited section of the free surface, outside which the pressure is constant and Gerstner's solution (29) is valid. Then, the asymptotic behavior

$$G(\chi) \to \chi,\ F(\overline{\chi}) \to iA \exp(ik\overline{\chi}),\ \text{if}\ \operatorname{Re}\chi \to \pm\infty \tag{34}$$

is valid for the functions G and F, and the wave frequency is defined by

$$\mu = \pm\sqrt{\Omega^2 + gk} - \Omega. \tag{35}$$

These waves may be classified as follows:

a) Inhomogeneous Gerstner waves: $G(\chi) \neq \chi,\ F(\overline{\chi}) = iA \exp(ik\overline{\chi})$. Fluid particles on Lagrangian horizon $\operatorname{Im}\chi = \text{const}$ rotate about circumferences of the same radius relative to the non-horizontal average level.

b) Modulated Gerstner waves: $G(\chi) = \chi,\ F(\overline{\chi}) \neq iA \exp(ik\overline{\chi})$. Fluid particles on Lagrangian horizon rotate about circumferences of different radii relative to the average level of oscillations $Z = \text{const}$. The non-stationary pressure component is determined only by the form of the function F.

c) Arbitrary Gerstner-type waves: $G(\chi) \neq \chi,\ F(\overline{\chi}) \neq iA \exp(ik\overline{\chi})$. Fluid particles on a free surface rotate about circumferences of radius $|F|$, the average level of surface oscillations $Z(X)$ is determined in parametric form by $X(a) = \operatorname{Re}\ G|_{c=0}$, $Z(a) = \operatorname{Im}\ G|_{c=0}$. The pressure distribution on the free surface depends on two arbitrary complex functions.

Following the paper [3], we will study as an example arbitrary Gerstner-type waves. Consider the solution

$$W = \chi - \frac{i\gamma}{(\chi - i\alpha)^2} + \left[iAe^{i(k\overline{\chi}+\varphi_0)} + \frac{i\gamma}{(\overline{\chi} + i\alpha)^2}\right] e^{-i\mu t}. \tag{36}$$

Here $A, k, \mu, \alpha, \gamma$ are positive parameters, φ_0 is phase shift. When $\gamma = 0$, expression (36) describes a Gerstner wave. For Ptolemaic flows the superposition principle holds true. If the function F is a sum of functions, the resulting profile qualitatively corresponds to the superposition of the profiles defined by these functions. The terms in G, F (cf. (36) and (24)) have one pole of order 2, which corresponds to $c = \alpha > 0$, so it is outside the fluid region. The term with the pole in the function F describes a periodically appearing peak. The term with the pole in the function G compensates the peak of the wave profile at the initial moment of time. So, expression (36) corresponds to the peak standing out in the field of a Gerstner wave.

In solution (36) A is amplitude, μ is frequency, and k is the wave number of a Gerstner wave, $kA \leq 1$; $kA = 1$ corresponds to the wave with sharp crests on the profile; μ and k are related by the dispersion relation (35). The parameter φ_0 characterizes the phase shift between the crests of the Gerstner wave and the perturbation of its profile (vortex breather). If $\varphi_0 = \pi$, then their crests coincide and the amplitudes are summed. If $\varphi_0 = 0$, then the breather crest coincides with the trough of the Gerstner wave and their amplitudes are subtracted. This behavior of the solution can be interpreted as wave interference. According to (25) there is a constraint on the value of γ. A sufficient condition is formulated as $\gamma \leq (1 - kA)\,\alpha^3/4$. The parameter α characterizes the peak width. We will restrict our consideration to the case when the horizontal scale of the breather is less than the wavelength ($\alpha < 2\pi/k$).

Let $\varphi_0 = \pi$. At the time instant $t = 0$, there is no peak and the wave profile corresponds to the Gerstner wave exactly. Next, there appears a peak that rises up to a maximum value at the moment of time $t = \pi/\mu$, and then it decreases and eventually disappears (see the picture and more detailed analysis in [3]). The motion is periodic. Depending on external pressure, the peak height can be essentially greater than the amplitude of the Gerstner wave. The pressure distribution (28) on a free surface shall be calculated numerically. It is, actually, a trough in the center of the breather and two peaks on its edges relative to the level of constant pressure p_0 (see [3]).

The wave vorticity and the invariant S_2 are defined by (26), (27), where

$$|G'|^2 = 1 + \frac{4\gamma\left[\gamma - (\alpha - A)\left(3a^2 + (A-\alpha)^2\right)\right]}{\left[a^2 + (A-\alpha)^2\right]^3},$$

$$|F'|^2 = k^2A^2e^{2kc} + \frac{4\gamma\, I}{\left[a^2 + (A-\alpha)^2\right]^3},$$

$$I = \gamma - kAe^{kc}\left\{(c-\alpha)\left[3a^2 - (c-\alpha)^2\right]\cos(ka+\varphi_0) - \right.$$
$$\left. -a\left[a^2 - 3(\alpha - c)^2\right]\sin(ka+\varphi_0)\right\}.$$

When a meridional current $\sigma(a, c)$ is superimposed on the plane wave (36), the motion will be three-dimensional. The vorticity component ω_Y and the invariant S_2 will not change in this case.

7. **Conclusions.**

We have obtained an exact analytical description of the class of non-linear vortex waves propagating in the zonal direction in an equatorial region. They generalize the Gerstner wave and are called Gerstner-type waves. The pressure on the wave profile is variable. A distinguishing feature of the Gerstner-type waves is absence of fluid particle drift in the direction of wave propagation, which is a stringent restriction. At the same time, the considered waves describe a wide class of possible pressure distributions on a free surface. Consequently, it is reasonable to conjecture that the studied wave regimes may be implemented in real conditions, including the case when free surface oscillations are strongly non-linear.

Acknowledgments. The publication was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2018–2019 (grant 18-01-0006) and by the Russian Academic Excellence Project "5-100".

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Received July 2018; revised December 2018.

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