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ON AFFINELY CLOSED, HOMOGENEOUS SPACES

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Abstract. Affinely closed, homogeneous spaces G/H, i.e., affine homogeneous spaces that admit only the trivial affine embedding, are characterized for an arbitrary affine algebraic group G. A description of affine G-algebras with finitely generated invariant subalgebras is obtained.

CONTENTS

1. Introduction 6133
2. Description of Affinely Closed Spaces 6134
3. G-Algebras with Finitely Generated Invariant Subalgebras 6135
4. Some Remarks on Affine Embeddings of Homogeneous Spaces for Nonreductive Groups . . . 6138
References 6138

1. Introduction

Let G be an affine algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero and H be an algebraic subgroup of G. By the Chevalley theorem, the homogeneous space G/H admits the canonical structure of a quasi-projective variety. An *embedding* of the homogeneous space G/H is an algebraic G-variety X with a base point $x \in X$ such that the orbit Gx is dense (and open) in X and the stabilizer G_x equals H. We denote an embedding by $G/H \hookrightarrow X$. We say that an embedding is *trivial* if $Gx = X$. An embedding $G/H \hookrightarrow X$ is said to be *affine* if the variety X is affine. It is easy to show (see, e.g., [9, Theorem 1.6]) that the space G/H admits an affine embedding if and only if G/H is a quasi-affine variety, or, equivalently, H can be realized as the stabilizer of a vector in a finite-dimensional G-module. In this case, the subgroup H is said to be *observable* in G. An effective description of observable subgroups in an algebraic group G was obtained by Sukhanov [10].

Definition 1 (see [2])**.** A homogeneous space G/H is said to be *affinely closed* if it admits only the trivial affine embedding.

An affinely closed, homogeneous space is automatically affine. The consideration of this class of homogeneous spaces is motivated by the following question: when does "the stabilizer of a point x on an affine G-variety X equals H" imply "the orbit Gx is closed"?

For a reductive group G, the homogeneous space G/H is an affine variety if and only if the subgroup H is reductive (the Matsushima criterion). Note that for an arbitrary affine algebraic group G, a description of affine homogeneous spaces G/H in group-theoretic terms for the pair (G, H) is an open problem (for details, see [5, Chap. 2]).

For a reductive group G, a description of affinely closed homogeneous spaces follows directly from the following result of Luna [7].

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Theorem 1. *Let G be a reductive group. A homogeneous space G/H is affinely closed if and only if the subgroup H is reductive and has finite index in its normalizer $N_G(H)$. Moreover, if G acts on an affine variety X and the stabilizer of a point $x \in X$ contains a reductive subgroup H such that the group $N_G(H)/H$ is finite, then the orbit Gx is closed.*

For example, for a maximal torus T of a reductive group G, the Weyl group $W = N_G(T)/T$ is finite, hence G/T is affinely closed. If $\rho : H \to SL(V)$ is an irreducible representation of a semisimple group, then $SL(V)/\rho(H)$ is affinely closed (by the Schur lemma, the group $N_{SL(V)}(\rho(H))/\rho(H)$ is finite). Thus, the class of affinely closed homogeneous spaces is sufficiently wide.

In [1], affinely closed, homogeneous spaces of a reductive group G play a key role in a classification of affine G-algebras such that any invariant subalgebra is finitely generated. A characterization of complex, affinely closed, homogeneous spaces of reductive groups in terms of compact transformation groups and invariant algebras on compact homogeneous spaces is given in [4, 6].

The aim of this paper is to generalize the result of Luna to the case of arbitrary affine, algebraic group G (Theorem 2) and to obtain a classification of affine G-algebras with finitely generated invariant subalgebras (Theorem 3). Note that a characteristic-free version of these results for a solvable group G is given in [11].

2. Description of Affinely Closed Spaces

Let us fix a Levi decomposition $G = LG^u$ of the group G in the semidirect product of a reductive subgroup L and the unipotent radical G^u. We denote by ϕ the homomorphism $\phi : G \to G/G^u$. We identify the image of ϕ with L. Set $K = \phi(H)$.

Theorem 2. *The following conditions are equivalent*:

(1) *G/H is affinely closed*;
(2) *L/K is affinely closed.*

Proof. The subgroup H is observable in G if and only if the subgroup K is observable in L (see [10] and [5, Theorem 7.3].

Assume that L/K admits a nontrivial affine embedding. Then there are an L-module V and a vector $v \in V$ such that the stabilizer L_v equals K and the orbit boundary $Y = Z \setminus Lv$, where $Z = \overline{Lv}$, is nonempty. Let $I(Y)$ be the ideal in $\mathbb{K}[Z]$ defining the subvariety Y. Recall that for an action of an algebraic group G on an affine variety X, any element $f \in \mathbb{K}[X]$ belongs to a finite-dimensional invariant subspace, or, equivalently, $\mathbb{K}[X]$ is the sum of its finite-dimensional G-submodules. Thus, there exists an L-submodule $V_1 \subset I(Y)$ that generates $I(Y)$ as an ideal. The inclusion $V_1 \subset \mathbb{K}[Z]$ defines an L-equivariant morphism $\psi : Z \to V_1^*$, $\psi^{-1}(0) = Y$. Then an L-equivariant morphism

$$\xi : Z \to V_2 = V_1^* \oplus V \otimes V_1^*, \quad z \to (\psi(z), z \otimes \psi(z)),$$

maps Y to the origin and is injective on the open orbit in Z. Hence we obtain an embedding of L/K in an L-module such that the closure of the image of this embedding contains the origin. We set $v_2 = \xi(v)$. By the Hilbert–Mumford criterion, there is a one-parameter subgroup $\lambda : \mathbb{K}^* \to L$ such that $\lim\limits_{t\to 0} \lambda(t)v_2 = 0$. Consider the weight decomposition $v_2 = v_2^{(i_1)} + \cdots + v_2^{(i_s)}$ of the vector v_2, where $\lambda(t)v_2^{(i_k)} = t^{i_k} v_2^{(i_k)}$ and all i_k are positive.

By the identification $G/G^u = L$, one may consider V_2 as a G-module. Let W be a finite-dimensional G-module with a vector w whose stabilizer equals H. Replacing the pair (W, w) by the pair $(W \oplus W \otimes W, w + w \otimes w)$, one can assume that the orbit Gw intersects the line $\mathbb{K}w$ only at the point w. For sufficiently large N, in the G-module $W \otimes V_2^{\otimes N}$ one has $\lim\limits_{t\to 0} \lambda(t)(w \otimes v_2^{\otimes N}) = 0$ ($\lambda(\mathbb{K}^*)$ can be considered as a subgroup of G). On the other hand, the stabilizer of $w \otimes v_2^{\otimes N}$ coincides with H. This implies that the space G/H is not affinely closed.

Conversely, assume that G/H admits a nontrivial affine embedding. This embedding corresponds to a G-invariant subalgebra $A \subset \mathbb{K}[G/H]$ containing a nontrivial G-invariant ideal I. Note that the algebra $\mathbb{K}[L]$ can be identified with the subalgebra in $\mathbb{K}[G]$ of (left- or right-) G^u-invariant functions, $\mathbb{K}[G/H]$ is realized in $\mathbb{K}[G]$ as the subalgebra of right H-invariants, and $\mathbb{K}[L/K]$ is the subalgebra of left G^u-invariants in $\mathbb{K}[G/H]$. Consider the action of G^u on the ideal I. By the Lie–Kolchin theorem, there is a nonzero G^u-invariant element in I. Thus, the subalgebra $A \cap \mathbb{K}[L/K]$ contains the nontrivial L-invariant ideal $I \cap \mathbb{K}[L/K]$. If the space L/K is affinely closed, then we obtain a contradiction with the following lemma.

Lemma 1. *Let L/K be an affinely closed space of a reductive group L. Then any L-invariant subalgebra in $\mathbb{K}[L/K]$ is finitely generated and does not contain nontrivial L-invariant ideals.*

Proof. Let $B \subset \mathbb{K}[L/K]$ be a nonfinitely generated, invariant subalgebra. For any chain $W_1 \subset W_2 \subset W_3 \subset \dots$ of finite-dimensional L-invariant submodules in $\mathbb{K}[L/K]$ with $\bigcup_{i=1}^{\infty} W_i = \mathbb{K}[L/K]$, the chain of subalgebras $B_1 \subset B_2 \subset B_3 \subset \dots$ generated by W_i does not stabilize. Hence one can assume that all inclusions here are strict. Let Z_i be the affine L-variety corresponding the algebra B_i. The inclusion $B_i \subset \mathbb{K}[L/K]$ induces the dominant morphism $L/K \to Z_i$ and Theorem 1 implies that $Z_i = L/K_i$, $K \subset K_i$. But $B_1 \subset B_2 \subset B_3 \subset \dots$, and any K_i is strictly contained in K_{i-1}, a contradiction. This shows that B is finitely generated and, as was proved above, L acts transitively on the affine variety Z corresponding to B. But any nontrivial L-invariant ideal in B corresponds to a proper L-invariant subvariety in Z. $\square$

Theorem 2 is proved. $\square$

Corollary 1. *Let G/H be an affinely closed, homogeneous space. Then for any affine G-variety X and a point $x \in X$ such that $Hx = x$, the orbit Gx is closed.*

Proof. If G_x is observable in G, then $\phi(G_x)$ is observable in L. The subgroup $\phi(G_x)$ contains $K = \phi(H)$ and Theorem 1 implies that the space $L/\phi(G_x)$ is affinely closed. By Theorem 2, the space G/G_x is affinely closed. $\square$

In particular, we obtain the following assertion.

Corollary 2. *If X is an affine G-variety and a point $x \in X$ is T-fixed, where T is a maximal torus in the group G, then the orbit Gx is closed.*

3. G-Algebras with Finitely Generated Invariant Subalgebras

In what follows, an *affine algebra* over a field $\mathbb{K}$ means a finitely generated, associative, commutative $\mathbb{K}$-algebra with unit. Let F be a subgroup of the automorphism group of an affine algebra $\mathcal{A}$ and $\operatorname{rad}\mathcal{A}$ be the set of nilpotent elements of the algebra $\mathcal{A}$. Clearly, $\operatorname{rad}\mathcal{A}$ is an F-invariant ideal in $\mathcal{A}$.

Lemma 2. *The following conditions are equivalent*:

(1) *any F-invariant subalgebra in $\mathcal{A}$ is finitely generated*;
(2) *any F-invariant subalgebra in $\mathcal{A}/\operatorname{rad}\mathcal{A}$ is finitely generated and* $\dim \operatorname{rad}\mathcal{A} < \infty$.

Proof. Any finite-dimensional subspace in $\operatorname{rad}\mathcal{A}$ generates a finite-dimensional subalgebra in $\mathcal{A}$. Hence if $\dim \operatorname{rad}\mathcal{A} = \infty$, then the subalgebra generated by this subspace is not finitely generated. On the other hand, the preimage in $\mathcal{A}$ of any nonfinitely generated subalgebra in $\mathcal{A}/\operatorname{rad}\mathcal{A}$ is not finitely generated.

Conversely, assume that (2) holds. Then any subalgebra in $\mathcal{A}$ is generated by elements whose images generate the image of this subalgebra in $\mathcal{A}/\operatorname{rad}\mathcal{A}$ and a basis of the space $\operatorname{rad}\mathcal{A}$. $\square$

By definition, a G*-algebra* is an affine algebra $\mathcal{A}$ with an action (by automorphisms) of an algebraic group G such that any element $a \in \mathcal{A}$ is contained in a finite-dimensional G-invariant subspace, where G acts rationally. Our aim is to describe all G-algebras such that any G-invariant subalgebra is finitely generated. By Lemma 2, we can assume that $\operatorname{rad} \mathcal{A} = 0$.

Let $X = \operatorname{Spec}(\mathcal{A})$ be the affine variety corresponding to affine algebra $\mathcal{A}$ without nilpotents. To fix a structure of G-algebra on $\mathcal{A} = \mathbb{K}[X]$ is nothing but to fix an (algebraic) G-action on X. Define the *dimension* of the algebra $\mathcal{A}$ as the dimension of the variety X.

Lemma 3. *If the dimension of $\mathcal{A}$ is less than or equal to one, then any subalgebra in $\mathcal{A}$ is finitely generated.*

Proof. The case where X is irreducible is considered in [1, Proposition 2]. If $X = X_1 \cup \cdots \cup X_m$ is the decomposition on irreducible components, then $\mathcal{A}$ is embedded into the direct sum of the algebras $\mathbb{K}[X_i]$ and any subalgebra in a summand is finitely generated. Now it is easy to complete the proof by induction on m considering the projection of $\mathcal{A}$ on $\mathbb{K}[X_1]$. □

Lemma 4. *Assume that $X = Z_1 \cup Z_2$, where Z_1 and Z_2 are closed invariant subvarieties. Then the following conditions are equivalent*:

(1) *any invariant subalgebra in $\mathcal{A}$ is finitely generated*;
(2) *any invariant subalgebra in $\mathbb{K}[Z_1]$ and in $\mathbb{K}[Z_2]$ is finitely generated.*

Proof. If there is a nonfinitely generated subalgebra in $\mathbb{K}[Z_1]$, then one can consider its preimage with respect to the restriction homomorphism $\mathbb{K}[X] \to \mathbb{K}[Z_1]$. To prove the converse, embed $\mathbb{K}[X]$ in $\mathbb{K}[Z_1] \oplus \mathbb{K}[Z_2]$ and use the arguments from the proof of the previous lemma. □

By Lemma 4, one can assume that G acts transitively on the set of irreducible components of the variety X.

In what follows, we generalize a construction from [1] to the case of nonconnnected groups and reducible varieties. Let Y be a closed subvariety of X. Consider the subalgebra

$$\mathcal{A}(X, Y) = \{f \in \mathbb{K}[X] \mid f(y_1) = f(y_2) \ \forall y_1, y_2 \in Y\}.$$

Lemma 5. *If Y contains an irreducible component of positive dimension that does not coincide with any irreducible component of X, then $\mathcal{A}(X, Y)$ is not finitely generated.*

Proof. Note that $\mathcal{A}(X, Y) = \mathbb{K} \oplus I(Y)$. If $\mathcal{A}(X, Y)$ is finitely generated, then one can assume that generators $f_1, \ldots, f_k$ are in $I(Y)$. Any monomial in $f_1, \ldots, f_k$ of degree s is in $I(Y)^s$. Hence it suffices to prove that for some l, the space $I(Y)/I(Y)^l$ is infinite-dimensional.

Let $Y = Y_1 \cup \cdots \cup Y_n$ and $X = X_1 \cup \cdots \cup X_m$ be the decompositions on irreducible components and $Y_1 \subset X_1$, $Y_1 \neq X_1$, and $\dim Y_1 > 0$. Assume that $f \in I(Y)$ and f is not identically zero on X_1. Let $\mathcal{O}_{X_1, Y_1}$ be the local ring of the subvariety Y_1 in X_1 and $\mathcal{I}$ be its maximal ideal. By the Nakayama lemma, $\bigcap_{i=1}^{\infty} \mathcal{I}^i = 0$, hence after restriction to X_1, the element f belongs to $\mathcal{I}^{l-1} \setminus \mathcal{I}^l$ for some $l \geq 2$. Let W be a subspace in $\mathbb{K}[X]$ complementary to $I(Y_1)$. Note that $\dim Y_1 > 0$ implies $\dim W = \infty$. The subspace fW can be considered as an infinite-dimensional subspace in $\mathcal{I}^{l-1}$ with zero intersection with $\mathcal{I}^l$. Hence fW determines an infinite-dimensional subspace in $I(Y)/I(Y)^l$. □

We conclude that any invariant subvariety Y satisfying the conditions of Lemma 5 determines the nonfinitely generated, invariant subalgebra $\mathcal{A}(X, Y)$ in $\mathbb{K}[X]$.

We denote by G^0 the connected component of unit of a group G.

Theorem 3. *Let $\mathcal{A}$ be a G-algebra without nilpotents with the nontrivial induced action of the subgroup G^u. The following conditions are equivalent*:

(1) *any G-invariant subalgebra in $\mathcal{A}$ is finitely generated*;
(2) *any G-invariant subalgebra in $\mathcal{A}$ does not contain nontrivial G-invariant ideals*;
(3) *any L-invariant subalgebra in $\mathcal{A}^{G^u}$ does not contain nontrivial L-invariant ideals*;
(4) $\mathcal{A} = \mathbb{K}[G/H]$, *where G/H is an affinely closed, homogeneous space*;
(5) $\mathcal{A}^{G^u} = \mathbb{K}[L/K]$, *where L/K is an affinely closed, homogeneous space.*

Proof. (1) $\Rightarrow$ (4) *Step* 1. Assume that the action $G : X$ is not transitive. The closure Y of a G-orbit on X is an invariant subvariety and we can apply Lemma 5 with the only exceptions $Y = X$ and $\dim Y = 0$. Hence (1) implies that G^0 acts on any component X_i with an open orbit and the boundary of this orbit is a finite set of points. In this case, the action $G^u : X$ is trivial [8, Theorem 3] (see also [1, Proposition 4]).

Step 2. Assume that the action $G : X$ is transitive. Then $X = G/H$. If G/H admits a nontrivial affine embedding $G/H \hookrightarrow X'$, then $\mathbb{K}[X']$ is an invariant subalgebra in $\mathcal{A}$. By Step 1, this subalgebra contains a nonfinitely generated, invariant subalgebra.

(2) $\Rightarrow$ (4) The absence of nontrivial invariant ideals in $\mathcal{A}$ implies that $X = G/H$ and the absence of nontrivial invariant ideals in invariant subalgebras implies that G/H does not admit nontrivial embeddings.

Proofs of (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) are similar to the proof of Lemma 1 (one should use Corollary 1). By the same arguments, we obtain (5) $\Rightarrow$ (3).

We know that $\mathbb{K}[G/H]^{G^u} = \mathbb{K}[L/K]$. Theorem 2 implies (4) $\Rightarrow$ (5).

(3) $\Rightarrow$ (2) Let $\mathcal{B}$ be an invariant subalgebra in $\mathcal{A}$ and I be a nontrivial invariant ideal in $\mathcal{B}$. Then $I \cap \mathcal{A}^{G^u}$ is a nontrivial invariant ideal in $\mathcal{B} \cap \mathcal{A}^{G^u}$. This completes the proof of Theorem 3. □

Remarks. (1) The conditions of Theorem 3 are not equivalent to the condition "any L-invariant subalgebra in $\mathcal{A}^{G^u}$ is finitely generated": one may consider $G = G^u = (\mathbb{K}, +)$ acting on $\mathbb{K}[x, y]$ by the formula $(a, f(x, y)) \to f(x + ay, y)$.

(2) The implication (1) $\Rightarrow$ (5) is invalid if $\mathcal{A} = \mathcal{A}^{G^u}$ (see [1]).

(3) The restriction $\mathcal{A} \neq \mathcal{A}^{G^u}$ is natural since the case of reductive group actions was studied in [1] under the assumptions that G is connected and X is irreducible. But Lemma 3 and the above arguments show that if a reductive group G acts transitively on the set of irreducible components of X, then any invariant subalgebra in $\mathbb{K}[X]$ is finitely generated if and only if either the G^0-algebras $\mathbb{K}[X_i]$ have (in the terminology of [1]) type C or HV or $X = G/H$ and G/H is affinely closed (in this case, the G^0-algebras $\mathbb{K}[X_i]$ may not have type N; see Example 5 below). Surprisingly, the restriction $\mathcal{A} \neq \mathcal{A}^{G^u}$ simplifies the main results.

Corollary 3. *Let G/H be a quasi-affine homogeneous space. Assume that H does not contain G^u. Then either G/H is affinely closed or there are infinitely many pairwise nonisomorphic, affine embeddings $G/H \hookrightarrow X_i$ and a sequence of dominant equivariant morphisms*

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots.$$

Proof. Let $G/H \hookrightarrow X$ be a nontrivial affine embedding and Y be the complement to the open orbit in X. The algebra $\mathcal{A}(X, Y)$ is not finitely generated. Let $g_0 \in I(Y)$ be a nonzero element and $f_1, \ldots, f_s$ be a set of generators of $\mathbb{K}[X]$. We set $g_i = g_0 f_i$, $i = 1, \ldots, s$, and extend the set $g_0, g_1, \ldots, g_s$ to an (infinite) generating set $g_0, g_1, \ldots, g_s, h_1, h_2, \ldots$ of the algebra $\mathcal{A}(X, Y)$. The affine G-varieties X_i corresponding to the algebras

$$\mathcal{B}_i = \mathbb{K}\left[\langle Gg_0, Gg_1, \ldots, Gg_s, Gh_1, \ldots, Gh_i\rangle\right] \subset \mathcal{A}(X, Y) \subset \mathbb{K}[X] \subset \mathbb{K}[G/H]$$

define embeddings $G/H \hookrightarrow X_i$ and the inclusions of algebras determine the desired chain of dominant morphisms. In the sequence X_i, there is a subsequence consisting of pairwise nonisomorphic embeddings. (By definition, an isomorphism of two embeddings sends the base point to the base point and is the unique equivariant morphism identical on the open orbit.) □

4. Some Remarks on Affine Embeddings of Homogeneous Spaces for Nonreductive Groups

In this section, we consider elementary examples that provide negative answers to some natural questions.

Let V be a finite-dimensional G-module and $v \in V$.

Example 1. The orbit Gv is closed but the orbit Lv is not closed. Consider

$$V = \mathbb{K}^2, \quad v = (1,1), \quad G = \left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} \right\}, \quad L = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Example 2. The orbit Lv is closed but the orbit Gv is not closed. Consider

$$V = \mathbb{K}^2, \quad v = (1,1), \quad G = \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \right\}, \quad L = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}.$$

Example 3. The space G/H is affinely closed does not imply that all L-orbits on G/H are closed. Consider

$$G = \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \right\}, \quad H = L = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}, \quad x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \overline{LxH} \neq LxH.$$

Example 4. If G^0/H^0 is affinely closed then G/H is affinely closed, but the converse is not true. One take $G = SL(2)$ with any finite non-Abelian subgroup H.

Example 5. If $G^0/(H \cap G^0)$ is affinely closed, then G/H is affinely closed, but the converse is not true. Consider

$$G = N_{SL(2)}T, \quad H = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

Example 6. The condition $0 \in \overline{Gv}$ does not imply that 0 can be obtained as the limit of a one-parameter subgroup in G. The corresponding example for a solvable group G is given in [3, Sec. 11].

An important open problem is to characterize affinely closed spaces (for both reductive and nonreductive groups) over an algebraically closed field of positive characteristic. In particular, it is not known if we have here Corollary 1. Some results in this direction can be found in [1, Sec. 8].

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