

Bachet's game with lottery moves

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Abstract

Bachet's game is a variant of the game of Nim. There are n objects in one pile. Two players make moves one after another. On every move, a player is allowed to take any positive number of objects not exceeding some fixed number m . The player who takes the last object loses. We consider a variant of Bachet's game in which each move is a lottery over set $\{1, 2, \dots, m\}$. Outcome of a lottery is the number of objects that player takes from the pile. We show that under some nondegeneracy assumptions on the set of available lotteries the probability that the first player wins in subgame perfect Nash equilibrium converges to $1/2$ as n tends to infinity.

Keywords: game theory; Bachet's game; backward induction; lotteries.

1 Introduction and main result

Bachet's game was formulated in [1] as follows. Starting from 1, two players add one after another some integer number not exceeding 10 to the sum. The player who is the first to reach 100, wins. This game can be considered as a variant of the game of Nim [3] (other variants of the game of Nim can be found, for example, in [6, 5, 2, 4]). One can easily find subgame perfect Nash equilibrium (SPNE) in Bachet's game with backward induction [1].

Now assume that at every move instead of choosing the exact number not exceeding some m , the player chooses some lottery (i.e. probability distribution) over numbers $\{1, 2, \dots, m\}$ from some set of available lotteries, observes realization of the lottery and then makes the corresponding move. Below we provide formal rules of the game that will be considered in this paper.

Bachet's game with lottery moves (BGLM). The game is defined by the natural number n of objects in the pile, the natural number m and a set of available lotteries $K \subset S_m$, where S_m is a simplex of all lotteries over numbers $\{1, 2, \dots, m\}$. Two players make moves one after another. On each move, the

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player chooses a lottery from the set K . After making the choice, the player observes realization of the lottery and then takes the corresponding number of objects from the pile. The player who takes the last object loses, including the case when they have to take more objects than remains in the pile. Both players want to maximize the probability of their own victory.

Our main result is the following theorem.

Theorem 1. *Fix arbitrary integer $m > 1$ and some compact set $K \subset S_m$ with the following properties:*

$$\eta := \max_{(\pi_1, \dots, \pi_m) \in K} \max_{i \in \{1, \dots, m\}} \pi_i < 1; \quad (1)$$

$$\nu := \min_{i \in \{1, \dots, m\}} \max_{(\pi_1, \dots, \pi_m) \in K} \pi_i > 0. \quad (2)$$

For any initial number of objects n , consider BGLM with parameters n, m, K . This game has a non-empty set of SPNE. Denote by p_n the probability that the first player wins in arbitrary SPNE.

Then p_n does not depend on the choice of SPNE and

$$\lim_{n \rightarrow \infty} p_n = \frac{1}{2}. \quad (3)$$

Remark. *It can be easily proved that if limit (3) exists, it has to be equal to $1/2$. The interesting part is the existence of this limit.*

Remark. *Theorem 1 allows the following interpretation. Assume that the players play classical Bachel's game, but from time to time they make random mistakes. Condition (1) says that mistakes are unavoidable. Condition (2) says that any move is allowed (with some probability). It follows that presence of unavoidable mistakes drastically changes the analysis of the game for large n .*

Conjecture. *We believe Theorem 1 holds true even without assumption (2).*

2 Proof of the main result

2.1 Existence of SPNE

We find SPNE by backward induction. Fix m and K . Obviously, for $n = 1$, any move leads to losing, as the player has to take at least one object in any case. Therefore, any move of the first player is in the set of all SPNE and $p_1 = 0$.

For convenience reasons, put $p_s = 1$ for any $s \leq 0$.

Now assume we proved existence of SPNE for all BGLM with no more than $n = k - 1$ objects. Consider BGLM with $n = k$ objects. Assume that after the move of the first player, i objects is taken from the pile. The second player now plays BGLM with $n = k - i$ objects (becoming 'first player' in this subgame) and wins it with probability p_{k-i} by induction hypothesis. If the second player wins, the first player loses. Therefore, the probability that the first player wins in this

case is $1 - p_{k-i}$. By the law of total probability, for move $\pi = (\pi_1, \dots, \pi_m) \in K$, the probability that the first player wins is given by:

$$\tilde{p}_k(\pi) = 1 - \sum_{i=1}^m \pi_i p_{k-i}. \quad (4)$$

The player wants to maximize this probability by choosing optimal π . Function \tilde{p}_k is continuous with respect to π and therefore attains its maximum value on compact set K . Then

$$p_k = \max_{\pi \in K} \tilde{p}_k(\pi) \quad (5)$$

and $\operatorname{argmax}_{\pi} \tilde{p}_k(\pi)$ is non-empty. Obviously, p_k does not depend on the choice of the move. After the move, the number of objects in the pile will be reduced, hence, existence of SPNE now follows from the induction hypothesis.

2.2 Limit behaviour

In this section we prove (3).

2.2.1 Notation and idea of the proof

First, introduce some notation. Let

$$\mathcal{D}_n := p_n - \frac{1}{2}, \quad \Delta_n := |\mathcal{D}_n|, \\ W_k = \{k, k-1, \dots, k-m+1\}, \quad \bar{\Delta}_k = \max_{j \in W_k} \Delta_j.$$

It is easy to show that sequence $\{\bar{\Delta}_k\}$ is non-increasing (see [Lemma 1](#) and [Corollary 1](#)). Our goal is to show that it is strictly decreasing and has zero limit.

Consider the state of game with $k+1$ objects in the pile. Due to (4)-(5), \mathcal{D}_{k+1} is a convex combination of values \mathcal{D}_j , $j \in W_k$, taken with a negative sign. If some of these values taken with nontrivial weights are less by absolute value than their maximum possible value $\bar{\Delta}_k$, their convex combination is also less than $\bar{\Delta}_k$ by absolute value and $\Delta_{k+1} < \bar{\Delta}_k$. Moreover, the gap can be estimated from below. This suggests a way to prove that sequence $\{\bar{\Delta}_k\}$ is strictly decreasing and tends to zero.

However, it is also possible that the convex combination for \mathcal{D}_{k+1} includes (with nontrivial weights) only those \mathcal{D}_j 's which absolute values are (almost) equal to $\bar{\Delta}_k$. In this case, $\Delta_{k+1} \approx \bar{\Delta}_k$ and no significant drop occurs. Such cases should be considered separately.

Due to condition (2), the player is allowed to put nontrivial weight on any move j . Due to rationality, the player tends to put larger weights on moves with smaller \mathcal{D}_j 's. The 'worst case' scenario is when all \mathcal{D}_j 's, $j \in W_k$, are positive and (almost) equal to $\bar{\Delta}_k$. We show that in this case \mathcal{D}_{k-m} should be negative and significantly larger by absolute value than $\bar{\Delta}_k$, see details in [Lemma 3](#). This gives us a drop between $\bar{\Delta}_{k-m}$ and Δ_{k+1} .

Another case that needs special attention is when there are several negative values of $\mathcal{D}_j \approx -\bar{\Delta}_j$, $j \in W_k$. This case is covered by [Lemma 6](#). There we prove that significant drops in Δ_k occur at least for every additional $3m$ objects in the pile, and the sequence $\{\Delta_k\}$ can be estimated from above by decreasing geometric progression. This finishes the poof.

2.2.2 Preliminary considerations

Lemma 1 (Monotonicity lemma). *For every integer $k > 1$, $\Delta_k \leq \bar{\Delta}_{k-1}$.*

Proof. It follows from (4)-(5) that

$$p_k = 1 - \sum_{i=1}^m \pi_i p_{k-i}.$$

for some $\pi \in S$. We have:

$$\begin{aligned} \Delta_k = |\mathcal{D}_k| &= \left| p_k - \frac{1}{2} \right| = \left| \frac{1}{2} - \sum_{i=1}^m \pi_i p_{k-i} \right| = \left| \sum_{i=1}^m \pi_i \left(\frac{1}{2} - p_{k-i} \right) \right| \leq \\ &= \sum_{i=1}^m \pi_i \left| \frac{1}{2} - p_{k-i} \right| = \sum_{i=1}^m \pi_i \Delta_{k-i} \leq \sum_{i=1}^m \pi_i \bar{\Delta}_{k-1} = \bar{\Delta}_{k-1}. \end{aligned} \quad (6)$$

□

Corollary 1. *For every integer $k > 1$, $\bar{\Delta}_k \leq \bar{\Delta}_{k-1}$.*

Proof. Indeed,

$$\begin{aligned} \bar{\Delta}_k &= \max\{\Delta_k, \Delta_{k-1}, \dots, \Delta_{k-m+1}\} \leq \max\{\bar{\Delta}_{k-1}, \Delta_{k-1}, \dots, \Delta_{k-m+1}\} = \\ &= \max\{\max\{\Delta_{k-1}, \dots, \Delta_{k-m}\}, \Delta_{k-1}, \dots, \Delta_{k-m+1}\} = \\ &= \max\{\Delta_{k-1}, \dots, \Delta_{k-m}\} = \bar{\Delta}_{k-1}. \end{aligned} \quad (7)$$

□

Lemma 2 (No long winning series). *Assume that for some integer $k > m$ and for all $j \in W_k$, $p_j > \frac{1}{2}$. Then*

$$p_{k+1} < \frac{1}{2} \quad (8)$$

and

$$p_{k-m} \leq \frac{1}{2}. \quad (9)$$

Proof. First, let us prove (8). Indeed, for some $\pi \in K$,

$$p_{k+1} = 1 - \sum_{i=1}^m \pi_i p_{k-i+1} < 1 - \sum_{i=1}^m \pi_i \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Now prove (9) by contradiction. Assume $p_{k-m} > \frac{1}{2}$. Then one can apply (8) with k decreased by 1 and prove that p_k have to be less than $\frac{1}{2}$. Contradiction. □

2.2.3 Worst case analysis

Lemma 3. Assume that for some $\varkappa \in (0, 1)$, for some integer $k > 1$ and for all $j \in W_k$ the following inequality holds:

$$p_j \geq \frac{1}{2} + (1 - \varkappa)\Delta_{k+1}. \quad (10)$$

Then the following inequality holds:

$$\Delta_{k+1} \leq \frac{\eta}{(2 - \eta)(1 - \varkappa)} \Delta_{k-m}. \quad (11)$$

Proof. First note that due to Lemma 2, $p_{k-m} \leq \frac{1}{2}$ and therefore $p_{k-m} = \frac{1}{2} - \Delta_{k-m}$. Now consider strategy $\pi = (\pi_1, \dots, \pi_m) \in K$ that allows the player facing k object to reach the winning probability of p_k . It follows from definition that

$$p_k = 1 - \left(\pi_m \left(\frac{1}{2} - \Delta_{k-m} \right) + \sum_{i=1}^{m-1} p_{k-i} \pi_i \right). \quad (12)$$

Then,

$$\begin{aligned} \pi_m \Delta_{k-m} &= p_k - 1 + \frac{\pi_m}{2} + \sum_{i=1}^{m-1} p_{k-i} \pi_i \geq \\ &\frac{1}{2} + (1 - \varkappa)\Delta_{k+1} - 1 + \frac{\pi_m}{2} + (1 - \pi_m) \left(\frac{1}{2} + (1 - \varkappa)\Delta_{k+1} \right), \end{aligned} \quad (13)$$

where the inequality follows from the lemma assumption (10). Simplifying the right-hand side of inequality, we get:

$$\pi_m \Delta_{k-m} \geq \Delta_{k+1}(1 - \varkappa)(2 - \pi_m),$$

or

$$\Delta_{k-m} \geq (1 - \varkappa) \frac{2 - \pi_m}{\pi_m} \Delta_{k+1} \geq (1 - \varkappa) \frac{2 - \eta}{\eta} \Delta_{k+1} \quad (14)$$

(from definition of η and Theorem assumption (see (1)), it follows that $\pi_m \leq \eta < 1$). Then (11) follows from (14). \square

2.2.4 Drop down for losing positions

In this part we show that for every *losing position* (i.e. position with winning probability less than $1/2$), there is a ‘drop down’ in the value of Δ_k .

Lemma 4. There exists $\delta < 1$ such that the following holds: if $p_{k+1} < 1/2$ for some k , then

$$\Delta_{k+1} \leq \delta \bar{\Delta}_{k-m}. \quad (15)$$

We need the following lemma for the proof.

Lemma 5 (Corridor lemma). *Assume $p_{k+1} < 1/2$. Then*

$$\max_{i \in W_k} \left(p_i - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) \geq \frac{\nu}{1-\nu} \max_{i \in W_k} \left(\frac{1}{2} + \Delta_{k+1} - p_i \right). \quad (16)$$

The proof of Lemma 5 is rather technical. It can be found in Section 2.2.6.

Proof of Lemma 4. Fix arbitrary τ such that

$$0 < \tau < \frac{\nu}{1-\nu} \frac{2-2\eta}{2-\eta}. \quad (17)$$

Such τ exists since $\nu \in (0, 1)$ and $\eta \in (0, 1)$. We show that

$$\delta := \max \left\{ \frac{\eta}{2-\eta} \frac{\nu}{\nu - \tau + \nu\tau}, \frac{1}{1+\tau} \right\}$$

satisfy (15). Due to (17), $0 < \delta < 1$.

Consider separately two cases.

Case 1. For all $j \in W_k$

$$p_j - \frac{1}{2} \leq (1+\tau)\Delta_{k+1}. \quad (18)$$

This inequality can be rewritten as

$$p_j - \left(\frac{1}{2} + \Delta_{k+1} \right) \leq \tau\Delta_{k+1} \quad (19)$$

Since the latter inequality is true for any $j \in W_k$, we obtain:

$$\max_{j \in W_k} \left(p_j - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) \leq \tau\Delta_{k+1} \quad (20)$$

According to Corridor lemma 5,

$$\max_{j \in W_k} \left(p_j - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) \geq \frac{\nu}{1-\nu} \max_{j \in W_k} \left(\frac{1}{2} + \Delta_{k+1} - p_j \right). \quad (21)$$

From (20) and (21) it follows that

$$\max_{j \in W_k} \left(\frac{1}{2} + \Delta_{k+1} - p_j \right) \leq \frac{1-\nu}{\nu} \tau\Delta_{k+1}. \quad (22)$$

Hence, for any $j \in W_k$ it is true that

$$\frac{1}{2} + \Delta_{k+1} - p_j \leq \frac{1-\nu}{\nu} \tau\Delta_{k+1}, \quad (23)$$

or

$$p_j \geq \frac{1}{2} + \left(1 - \frac{1-\nu}{\nu} \tau \right) \Delta_{k+1}. \quad (24)$$

Applying [Lemma 3](#) with $\varkappa = \frac{1-\nu}{\nu}\tau$, we obtain that

$$\Delta_{k+1} \leq \frac{\eta}{(2-\eta)\left(1 - \frac{1-\nu}{\nu}\tau\right)} \Delta_{k-m}, \quad (25)$$

or

$$\Delta_{k+1} \leq \frac{\eta}{2-\eta} \frac{\nu}{\nu - \tau + \nu\tau} \Delta_{k-m} \leq \delta \Delta_{k-m} \leq \delta \bar{\Delta}_{k-m}. \quad (26)$$

Case 2. There exists $i \in W_k$ such that

$$p_i - \frac{1}{2} > (1 + \tau) \Delta_{k+1}. \quad (27)$$

Then,

$$\Delta_{k+1} < \frac{1}{1+\tau} \left(p_i - \frac{1}{2}\right) \leq \delta \Delta_i \leq \delta \bar{\Delta}_i \leq \delta \bar{\Delta}_{k-m}. \quad (28)$$

The last inequality is due to [Corollary 1](#) and the fact that $i > k - m$. \square

2.2.5 Drop down for any positions

Lemma 6. For δ from [Lemma 4](#) and for all integer $k > 2m$,

$$\Delta_{k+1} \leq \delta \bar{\Delta}_{k-2m}. \quad (29)$$

To prove [Lemma 6](#) we have to introduce new notation and prove auxiliary proposition. Let

$$\begin{aligned} \Delta_k^- &= \max \left\{ 0, \frac{1}{2} - p_k \right\}, & \Delta_k^+ &= \max \left\{ 0, p_k - \frac{1}{2} \right\}, \\ \bar{\Delta}_k^- &= \max_{i \in W_k} \Delta_i^-, & \bar{\Delta}_k^+ &= \max_{i \in W_k} \Delta_i^+. \end{aligned}$$

Obviously, $\bar{\Delta}_k = \max\{\bar{\Delta}_k^-, \bar{\Delta}_k^+\}$.

Proposition 1. For any natural k the following holds:

$$\Delta_{k+1}^+ \leq \bar{\Delta}_k^-.$$

Proof. If $p_{k+1} \leq 1/2$, then $\Delta_{k+1}^+ = 0 \leq \bar{\Delta}_k^-$ by definition of $\bar{\Delta}_k^-$. Consider case $p_{k+1} \geq 1/2$. Then for some $\pi \in K$,

$$\begin{aligned} p_{k+1} - \frac{1}{2} &= \frac{1}{2} - \sum_{i=1}^m \pi_i p_{k-i+1} = \sum_{i=1}^m \pi_i \left(\frac{1}{2} - p_{k-i+1} \right) \\ &\leq \sum_{\substack{i=1, \\ p_{k-i+1} \leq 1/2}}^m \pi_i \left(\frac{1}{2} - p_{k-i+1} \right) \leq \sum_{\substack{i=1, \\ p_{k-i+1} \leq 1/2}}^m \pi_i \bar{\Delta}_k^- \\ &\leq \sum_{i=1}^m \pi_i \bar{\Delta}_k^- = \bar{\Delta}_k^-. \end{aligned} \quad (30)$$

\square

Now we can prove [Lemma 6](#).

Proof of Lemma 6. If $p_{k+1} < 1/2$, [Lemma 4](#) implies:

$$\Delta_{k+1} \leq \delta \bar{\Delta}_{k-m} \leq \delta \bar{\Delta}_{k-2m}$$

and lemma is proved. (The last inequality is due to [Corollary 1](#).)

Now assume $p_{k+1} \geq 1/2$. In this case $\Delta_{k+1} = \Delta_{k+1}^+ \leq \bar{\Delta}_k^-$ due to [Proposition 1](#). For all $j \in W_k$ such that $p_j < 1/2$, [Lemma 4](#) implies:

$$\Delta_j^- = \Delta_j \leq \delta \bar{\Delta}_{j-1-m} \leq \delta \bar{\Delta}_{k-2m}.$$

Again, the last inequality is due to [Corollary 1](#) since $j \geq k - m + 1$. Therefore, $\bar{\Delta}_k^- \leq \delta \bar{\Delta}_{k-2m}$. This finishes proof of [Lemma 6](#). \square

Corollary 2. For all integer $k > 3m$, $\bar{\Delta}_k \leq \delta \bar{\Delta}_{k-3m}$.

Proof. From definition of $\bar{\Delta}_k$, [Lemma 6](#) and [Corollary 1](#) it follows that

$$\bar{\Delta}_k = \max(\Delta_k, \dots, \Delta_{k-m+1}) \leq \delta \max(\bar{\Delta}_{k-2m-1}, \dots, \bar{\Delta}_{k-3m}) = \delta \bar{\Delta}_{k-3m}.$$

\square

Now we are ready to finish the proof of the main result. Let $k_N = 1 + 3mN$ for arbitrary integer N . Inductive application of [Corollary 2](#) implies:

$$\bar{\Delta}_{k_N} \leq \delta^N \bar{\Delta}_1 = \frac{1}{2} \delta^N \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Due to monotonicity of $\bar{\Delta}_k$, this implies:

$$\lim_{k \rightarrow \infty} \bar{\Delta}_k \rightarrow 0.$$

By definition of $\bar{\Delta}_k$, $\Delta_k \leq \bar{\Delta}_k$ and therefore:

$$\lim_{k \rightarrow \infty} \Delta_k \rightarrow 0$$

which is equivalent to (3). [Theorem 1](#) is proved modulo [Lemma 5](#).

2.2.6 Technical considerations

In this section we prove [Lemma 5](#).

Proof. Take any $\pi = (\pi_1, \dots, \pi_m) \in K$. Since the players are rational (5),

$$p_{k+1} \geq 1 - \sum_{i=1}^m \pi_i p_{k-i+1},$$

or

$$\sum_{i=1}^m \pi_i p_{k-i+1} \geq 1 - p_{k+1} = 1 - \left(\frac{1}{2} - \Delta_{k+1} \right) = \frac{1}{2} + \Delta_{k+1}.$$

Then, the following inequality holds:

$$\sum_{i=1}^m \pi_i \left(p_{k-i+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) = \sum_{i=1}^m \pi_i p_{k-i+1} - \sum_{i=1}^m \pi_i \left(\frac{1}{2} + \Delta_{k+1} \right) \geq \left(\frac{1}{2} + \Delta_{k+1} \right) - \left(\frac{1}{2} + \Delta_{k+1} \right) = 0. \quad (31)$$

Now take arbitrary

$$j \in \operatorname{argmax}_{1 \leq i \leq m} \left(\frac{1}{2} + \Delta_{k+1} - p_{k-i+1} \right). \quad (32)$$

By definition of ν and theorem assumption $\nu > 0$ (see (2)), there exists a strategy $\hat{\pi} = (\hat{\pi}_1, \dots, \hat{\pi}_m) \in K$ such that

$$\hat{\pi}_j \geq \nu > 0 \quad (33)$$

From (31), it follows that

$$\sum_{\substack{1 \leq i \leq m \\ i \neq j}} \hat{\pi}_i \left(p_{k-i+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) + \hat{\pi}_j \left(p_{k-j+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) \geq 0.$$

By rearranging the terms, we get

$$\begin{aligned} -\hat{\pi}_j \left(p_{k-j+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) &\leq \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \hat{\pi}_i \left(p_{k-i+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) \leq \\ &\sum_{\substack{1 \leq i \leq m \\ i \neq j}} \hat{\pi}_i \max_{1 \leq t \leq m} \left(p_{k-t+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) = \\ &\left(\sum_{\substack{1 \leq i \leq m \\ i \neq j}} \hat{\pi}_i \right) \cdot \max_{1 \leq t \leq m} \left(p_{k-t+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) = \\ &(1 - \hat{\pi}_j) \max_{1 \leq t \leq m} \left(p_{k-t+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right), \quad (34) \end{aligned}$$

where the last equality follows from the fact that

$$\sum_{i=1}^m \hat{\pi}_i = 1.$$

From (34) we derive the lower estimate for the left-hand side of the Corridor lemma inequality (16):

$$\begin{aligned} \max_{1 \leq t \leq m} \left(p_{k-t+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) &\geq -\frac{\hat{\pi}_j}{1 - \hat{\pi}_j} \left(p_{k-j+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) = \\ &\frac{\hat{\pi}_j}{1 - \hat{\pi}_j} \left(\frac{1}{2} + \Delta_{k+1} - p_{k-j+1} \right). \quad (35) \end{aligned}$$

Note that

$$\frac{1}{2} + \Delta_{k+1} - p_{k-j+1} \geq 0.$$

Indeed, otherwise, from (32) it would follow that for all $i = 1, \dots, m$

$$\frac{1}{2} + \Delta_{k+1} - p_{k-i+1} < 0$$

or

$$p_{k-i+1} > \frac{1}{2} + \Delta_{k+1}.$$

However, this is impossible because any strategy would lead to $p_{k+1} < \frac{1}{2} - \Delta_{k+1}$ whereas $p_{k+1} = \frac{1}{2} - \Delta_{k+1}$ by definition.

Then, applying (33) to (35), we obtain

$$\begin{aligned} \max_{1 \leq t \leq m} \left(p_{k-t+1} - \left(\frac{1}{2} + \Delta_{k+1} \right) \right) &\geq \frac{\nu}{1-\nu} \left(\left(\frac{1}{2} + \Delta_{k+1} \right) - p_{k-j+1} \right) = \\ &= \frac{\nu}{1-\nu} \max_{1 \leq t \leq m} \left(\frac{1}{2} + \Delta_{k+1} - p_{k-t+1} \right). \end{aligned} \quad (36)$$

The last equality follows from (32). This finished the proof of the Lemma and the Main Result (Theorem 1). \square

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