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2017 J. Phys. A: Math. Theor. 50 205202

(http://iopscience.iop.org/1751-8121/50/20/205202)

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **50** (2017) 205202 (12pp) https://doi.org/10.1088/1751-8121/aa6a6d

KZ-Calogero correspondence revisited

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Received 4 February 2017, revised 25 March 2017
Accepted for publication 31 March 2017
Published 24 April 2017

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Abstract

We discuss the correspondence between the Knizhnik–Zamolodchikov equations associated with $GL(N)$ and the n-particle quantum Calogero model in the case when n is not necessarily equal to N. This can be viewed as a natural 'quantization' of the quantum-classical correspondence between quantum Gaudin and classical Calogero models.

Keywords: integrable systems, Knizhnik–Zamolodchikov equations, quantum-classical correspondence

1. Introduction

The rational Knizhnik–Zamolodchikov (KZ) equations [10] have the form

$$\hbar\partial_{x_i}\Big|\Phi\Big\rangle = \left(\mathbf{g}^{(i)} + \kappa\sum_{j\neq i}^{n}\frac{\mathbf{P}_{ij}}{x_i - x_j}\right)\Big|\Phi\Big\rangle \tag{1}$$

where $\Big|\Phi\Big\rangle = \Big|\Phi\Big\rangle(x_1,\ldots,x_n)$ belongs to the tensor product $\mathcal{V} = V\otimes V\otimes\ldots\otimes V = V^{\otimes n}$ of the vector spaces $V = \mathbb{C}^N$, $\mathbf{P}_{ij}$ is the permutation of the ith and jth factors, $\mathbf{g} = \mathrm{diag}\,(g_1,\ldots,g_N)$ is a diagonal $N\times N$ matrix and $\mathbf{g}^{(i)}$ is the operator in $\mathcal{V}$ acting as $\mathbf{g}$ on the ith factor (and identically on all other factors).

The remarkable correspondence of the KZ equations with the quantum Calogero model [5] defined by the Hamiltonian

$$\hat{\mathcal{H}} = \hbar^2 \sum_{i=1}^{n} \partial_{x_i}^2 - \sum_{i\neq j}^{n} \frac{\kappa(\kappa-\hbar)}{(x_i-x_j)^2} \tag{2}$$

was established by Matsuo and Cherednik in [6, 11] (see also [7, 17]) in the case $N = n$. In this case one can find solutions to (1) in the form

$$\Big|\Phi\Big\rangle = \sum_{\sigma\in S_n} \Phi_\sigma \Big|e_\sigma\Big\rangle, \qquad \Big|e_\sigma\Big\rangle = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \ldots \otimes e_{\sigma(n)},$$

where e_a are standard basis vectors in $V = \mathbb{C}^n = \mathbb{C}^N$ and S_n is the symmetric group. If such $\Big|\Phi\Big\rangle$ solves the KZ equations, then the function

$$\Psi = \sum_{\sigma\in S_n} \Phi_\sigma \tag{3}$$

is an eigenfunction of the Calogero Hamiltonian:

$$\hat{\mathcal{H}}\Psi = E\Psi, \qquad E = g_1^2 + g_2^2 + \ldots + g_N^2. \tag{4}$$

This correspondence can be extended to the trigonometric versions of the both models.

We will show that a similar correspondence exists also in the case when the number of marked points n is not necessarily equal to $N = \dim V$. In this form it looks like a quantum deformation of the quantum-classical correspondence [1, 2, 9, 13, 18] between the quantum Gaudin and classical Calogero models (see [4] for a discussion of the Matsuo–Cherednik map in this context).

The system of KZ equations is a non-stationary version of the quantum Gaudin model, with $\hbar$ being the parameter of non-stationarity. We denote it as $\hbar$ because it becomes the true Planck constant in the corresponding quantum Calogero model. The spectral problem for the Gaudin model is a 'quasiclassical' limit of KZ as $\hbar \to 0$. Indeed, as $\hbar \to 0$ the KZ solutions have the asymptotic form [14]

$$\Big|\Phi\Big\rangle = \Big(\Big|\phi_0\Big\rangle + \hbar\Big|\phi_1\Big\rangle + \ldots\Big)\, \mathrm{e}^{S/\hbar}$$

which, upon substitution to the KZ equations (1), leads, in the leading order, to the joint eigenvalue problems

$$\mathbf{H}_i\Big|\phi_0\Big\rangle = p_i\Big|\phi_0\Big\rangle, \qquad p_i = \frac{\partial S}{\partial x_i}, \qquad i = 1, \ldots, n,$$

for the commuting Gaudin Hamiltonians $\mathbf{H}_i = \mathbf{g}^{(i)} + \kappa \sum_{j\neq i}^{n} \frac{\mathbf{P}_{ij}}{x_i - x_j}$ with the Planck constant κ. In the quantum-classical correspondence, the eigenvalues p_i are identified with momenta of the Calogero–Moser particles with coordinates x_i.

The plan of the paper is as follows. In section 2, we describe the rational Gaudin model with a formal Planck constant κ and the associated KZ equations. In section 3 the KZ-Calogero correspondence is established. Section 4 is devoted to the trigonometric version of the correspondence. Finally, in section 5 we discuss the interpretation of the results as a 'quantum' deformation of the quantum-classical correspondence. Section 6 is the conclusion. In the appendix we show that the wave function from section 3 is also an eigenfunction of the higher Calogero Hamiltonian $\hat{\mathcal{H}}_3$.

2. The Gaudin Hamiltonians and KZ equations

Let e_{ab}^{κ} be generators of the 'κ-dependent version' of the universal enveloping algebra $U(gl(N))$ with the commutation relations $[\mathrm{e}_{ab}^{\kappa}, \mathrm{e}_{a'b'}^{\kappa}] = \kappa(\delta_{a'b}\mathrm{e}_{ab'}^{\kappa} - \delta_{ab'}\mathrm{e}_{a'b}^{\kappa})$. Since at $\kappa = 0$ the operators e_{ab}^{κ} commute, the parameter κ plays the role of the formal Planck's constant. Let π be the

N-dimensional vector representation of $U^{(\kappa)}(gl(N))$. We have $\pi(\mathrm{e}^{\kappa}_{ab}) = \kappa e_{ab}$, where e_{ab} is the standard basis in the space of $N \times N$ matrices: the matrix e_{ab} has only one non-zero element (equal to 1) at the place ab: $(e_{ab})_{a'b'} = \delta_{aa'}\delta_{bb'}$. Note that $\mathbf{I} = \sum_a e_{aa}$ is the unity operator and $\mathbf{P} = \sum_{ab} e_{ab} \otimes e_{ba}$ is the permutation operator acting in the space $\mathbb{C}^N \otimes \mathbb{C}^N$.

In the tensor product $U^{(\kappa)}(gl(N))^{\otimes n}$ the generators e^{κ}_{ab} can be realized as $\mathrm{e}^{\kappa\,(i)}_{ab} := \mathbf{I}^{\otimes(i-1)} \otimes \mathrm{e}^{\kappa}_{ab} \otimes \mathbf{I}^{\otimes(n-i)}$. It is clear that they commute for any $i \neq j$ and any a,b because act non-trivially in different spaces. Similarly, for any matrix $\mathbf{g} \in$ End $(\mathbb{C}^N)$ we define $\mathbf{g}^{(i)}$ acting in the tensor product $\mathcal{V} = (\mathbb{C}^N)^{\otimes n}$: $\mathbf{g}^{(i)} = \mathbf{I}^{\otimes(i-1)} \otimes \mathbf{g} \otimes \mathbf{I}^{\otimes(n-i)} \in$ End $(\mathcal{V})$. In this notation, $\mathbf{P}_{ij} := \sum_{a,b} \mathrm{e}^{(i)}_{ab}\mathrm{e}^{(j)}_{ba}$ is the permutation operator of the ith and jth tensor factors in $\mathcal{V} = \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N$. Clearly, $\mathbf{P}_{ij} = \mathbf{P}_{ji}$ and $\mathbf{P}^2_{ij} = \mathbf{I}$.

Fix n distinct numbers $x_i \in \mathbb{C}$ and a diagonal $N \times N$ matrix $\mathbf{g} = \mathrm{diag}\,(g_1, \ldots, g_N)$. (We assume that $n \geqslant N$ and that the g_i's are all distinct and non-zero.) We will call $\mathbf{g}$ the twist matrix. The commuting Gaudin Hamiltonians are

$$\mathbf{H}_i = \frac{1}{\kappa}\left(\sum_{a=1}^{N} g_a \mathrm{e}^{\kappa\,(i)}_{aa} + \sum_{j\neq i}\sum_{a,b=1}^{N} \frac{\mathrm{e}^{\kappa\,(i)}_{ab}\mathrm{e}^{\kappa\,(j)}_{ba}}{x_i - x_j}\right), \quad i = 1, \ldots, n. \tag{5}$$

The Hamiltonians of the quantum Gaudin model [8] with the Hilbert space $\mathcal{V} = (\mathbb{C}^N)^{\otimes n}$ are restrictions of the operators (5) to the N-dimensional vector representation π:

$$\mathbf{H}_i = \sum_{a=1}^{N} g_a \mathrm{e}^{(i)}_{aa} + \kappa \sum_{j\neq i}\sum_{a,b=1}^{N} \frac{\mathrm{e}^{(i)}_{ab}\mathrm{e}^{(j)}_{ba}}{x_i - x_j} = \mathbf{g}^{(i)} + \kappa \sum_{j\neq i} \frac{\mathbf{P}_{ij}}{x_i - x_j}, \quad i = 1, \ldots, n \tag{6}$$

(for brevity we denote $\pi^{\otimes n}(\mathbf{H}_i)$ by the same letter $\mathbf{H}_i$). It is known that the Gaudin Hamiltonians form a commutative family: $[\mathbf{H}_i, \mathbf{H}_j] = 0$ for all $i, j = 1, \ldots, n$.

The operators

$$\mathbf{M}_a = \sum_{l=1}^{n} e^{(l)}_{aa} \tag{7}$$

commute among themselves and with the Gaudin Hamiltonians: $[\mathbf{H}_i, \mathbf{M}_a] = 0$. Clearly, $\sum_a \mathbf{M}_a = n\mathbf{I}$, and $\sum_{i=1}^{n} \mathbf{H}_i = \sum_{a=1}^{N} g_a \mathbf{M}_a$. The joint spectral problem is

$$\begin{cases} \mathbf{H}_i \big|\phi\big\rangle = H_i \big|\phi\big\rangle \\ \mathbf{M}_a \big|\phi\big\rangle = M_a \big|\phi\big\rangle \end{cases}$$

The common eigenstates of the Hamiltonians can be classified according to eigenvalues of the operators $\mathbf{M}_a$.

Let

$$\mathcal{V} = V^{\otimes n} = \bigoplus_{M_1,\ldots,M_N} \mathcal{V}(\{M_a\}) \tag{8}$$

be the weight decomposition of the Hilbert space $\mathcal{V}$ of the Gaudin model into the direct sum of eigenspaces for the operators $\mathbf{M}_a$ with the eigenvalues $M_a \in \mathbb{Z}_{\geqslant 0}$, $a = 1, \ldots, N$ (recall that $M_1 + \ldots + M_N = n$). Then the eigenstates of $\mathbf{H}_i$'s are in the spaces $\mathcal{V}(\{M_a\})$,

$$\dim\, \mathcal{V}(\{M_a\}) = \frac{n!}{M_1! \ldots M_N!}.$$

The basis vectors in $\mathcal{V}(\{M_a\})$ are $\Big|J\Big\rangle = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n}$, where the number of indices j_k such that $j_k = a$ is equal to M_a for all $a = 1, \ldots, N$. We also introduce dual vectors $\Big\langle J\Big| = e^{\dagger}_{j_1} \otimes e^{\dagger}_{j_2} \otimes \ldots \otimes e^{\dagger}_{j_n}$ such that $\Big\langle J\Big|J'\Big\rangle = \delta_{J,J'}$.

The system of KZ equations is a non-stationary version of the Gaudin model:

$$\hbar\partial_{x_i}\Big|\Phi\Big\rangle = \mathbf{H}_i\Big|\Phi\Big\rangle, \qquad i = 1, \ldots, n. \tag{9}$$

It respects the weight decomposition (8), hence the solutions belong to the weight subspaces $\mathcal{V}(\{M_a\})$. Equations (9) are compatible due to the flatness conditions

$$[\hbar\partial_{x_i} - \mathbf{H}_i,\, \hbar\partial_{x_j} - \mathbf{H}_j] = 0 \qquad \text{for all} \quad i, j = 1, \ldots, n. \tag{10}$$

3. The KZ-Calogero correspondence

We claim that for any solution of the KZ equations belonging to the space $\mathcal{V}(\{M_a\})$, $\Big|\Phi\Big\rangle = \sum_J \Phi_J \Big|J\Big\rangle$, the function

$$\Psi = \sum_J \Phi_J \tag{11}$$

is an eigenfunction of the Calogero Hamiltonian with the eigenvalue $E = \sum_{a=1}^{N} M_a g_a^2$:

$$\left(\hbar^2 \sum_{i=1}^{n} \partial_{x_i}^2 - \sum_{i\neq j}^{n} \frac{\kappa(\kappa - \hbar)}{(x_i - x_j)^2}\right)\Psi = E\Psi. \tag{12}$$

In particular, at $n = N$ and $M_1 = M_2 = \ldots = M_N = 1$ we get the result of [6, 7, 11, 17].

For the proof consider the covector equal to the sum of all basis (dual) vectors from the space $(\mathcal{V}(\{M_a\}))^*$:

$$\Big\langle\Omega\Big| = \sum_J \Big\langle J\Big|,$$

then $\Psi = \Big\langle\Omega\Big|\Phi\Big\rangle$. Applying the operator $\hbar\partial_{x_i}$ to the KZ equation (1), we get:

$$\begin{aligned}
\hbar^2\partial_{x_i}^2\Big|\Phi\Big\rangle &= -\hbar\kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{(x_i - x_j)^2}\Big|\Phi\Big\rangle + \left(\mathbf{g}^{(i)} + \kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{x_i - x_j}\right)\hbar\partial_{x_i}\Big|\Phi\Big\rangle \\
&= -\hbar\kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{(x_i - x_j)^2}\Big|\Phi\Big\rangle + \left(\mathbf{g}^{(i)} + \kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{x_i - x_j}\right)\left(\mathbf{g}^{(i)} + \kappa\sum_{l\neq i}\frac{\mathbf{P}_{il}}{x_i - x_l}\right)\Big|\Phi\Big\rangle \\
&= -\hbar\kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{(x_i - x_j)^2}\Big|\Phi\Big\rangle + \kappa^2\sum_{j\neq i}\frac{1}{(x_i - x_j)^2}\Big|\Phi\Big\rangle + (\mathbf{g}^{(i)})^2\Big|\Phi\Big\rangle \\
&\quad + \kappa^2\sum_{j\neq l\neq i}\frac{\mathbf{P}_{ij}\mathbf{P}_{il}}{(x_i - x_j)(x_i - x_l)}\Big|\Phi\Big\rangle + \kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{x_i - x_j}\mathbf{g}^{(i)}\Big|\Phi\Big\rangle + \kappa\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{x_i - x_j}\mathbf{g}^{(j)}\Big|\Phi\Big\rangle.
\end{aligned}$$

In the last lines we took into account that $\mathbf{P}_{ij}^2 = \mathbf{I}$ and $\mathbf{g}^{(i)}\mathbf{P}_{ij} = \mathbf{P}_{ij}\mathbf{g}^{(j)}$. Since $\Big\langle\Omega\Big|\mathbf{P}_{ij}\Big|J\Big\rangle = 1$ for all basis vectors $\Big|J\Big\rangle$, we have $\Big\langle\Omega\Big|\mathbf{P}_{ij} = \Big\langle\Omega\Big|$. Therefore, the permutation operators disappear after applying $\Big\langle \Omega \Big|$ from the left. Summing $\hbar^2 \Big\langle \Omega \Big| \partial_{x_i}^2 \Big| \Phi \Big\rangle = \hbar^2 \partial_{x_i}^2 \Psi$ over i and using the identities[6]

$$\sum_{j \neq l \neq i} \frac{1}{(x_i - x_j)(x_i - x_l)} = 0, \tag{13}$$

$$\sum_{i \neq j} \frac{\mathbf{g}^{(i)} + \mathbf{g}^{(j)}}{x_i - x_j} = 0, \tag{14}$$

$$\sum_i \Big\langle \Omega \Big| (\mathbf{g}^{(i)})^2 \Big| \Phi \Big\rangle = \left(\sum_{a=1}^{N} M_a g_a^2 \right) \Psi \tag{15}$$

we get (12).

Note that $\Psi = \Big\langle \Omega \Big| \Phi \Big\rangle$ is an eigenfunction of the total momentum operator $\hat{\mathcal{P}} = \hbar \sum_j \partial_{x_j}$ with the eigenvalue $\sum_a M_a g_a$. In the appendix it is shown that Ψ is also an eigenfunction of the cubic Calogero Hamiltonian $\hat{\mathcal{H}}_3$. We conjecture that Ψ is the common eigenfunction for all higher Calogero Hamiltonians $\hat{\mathcal{H}}_k$ with the eigenvalues $E_k = \sum_a M_a g_a^k$. The first four Hamiltonians are explicitly written in [16].

4. Trigonometric case

The trigonometric (hyperbolic) version of the system of KZ equations reads [7]

$$\hbar \partial_{x_i} \Big| \Phi \Big\rangle = \left(\mathbf{g}^{(i)} + \kappa\gamma \sum_{j \neq i}^{n} \Big(\coth \gamma(x_i - x_j) \mathbf{P}_{ij} + \mathbf{T}_{ij} \Big) \right) \Big| \Phi \Big\rangle, \tag{16}$$

where we use the same notation as in (1) and

$$\mathbf{T} = \sum_{a > b} (e_{ab} \otimes e_{ba} - e_{ba} \otimes e_{ab}).$$

This operator acts on basis vectors as follows:

$$\mathbf{T} e_a \otimes e_b = \begin{cases} e_b \otimes e_a & \text{if} \quad a < b \\ -e_b \otimes e_a & \text{if} \quad a > b \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Note that $\mathbf{T}_{ji} = -\mathbf{T}_{ij}$. In the limit $\gamma \to 0$ we recover the rational KZ equations (1).

A calculation similar to the one given above in the rational setting leads to the following statement. For any solution of the KZ equations (16) belonging to the space $\mathcal{V}(\{M_a\})$, $\Big| \Phi \Big\rangle = \sum_J \Phi_J \Big| J \Big\rangle$, the function $\Psi = \sum_J \Phi_J$ solves the spectral problem for the Calogero–Sutherland Hamiltonian

$$\left(\hbar^2 \sum_{i=1}^{n} \partial_{x_i}^2 - \sum_{i \neq j}^{n} \frac{\kappa(\kappa - \hbar)\gamma^2}{\sinh^2 \gamma(x_i - x_j)} \right) \Psi = E\Psi \tag{18}$$

[6] Identity (13) follows from $\frac{1}{(x_i - x_j)} \frac{1}{(x_i - x_l)} + \frac{1}{(x_l - x_j)} \frac{1}{(x_l - x_j)} + \frac{1}{(x_i - x_l)} \frac{1}{(x_j - x_l)} = 0$ applied to the sum symmetrized with respect to i, j, l.

with the eigenvalue

$$E=\sum_{a=1}^{N}M_a g_a^2+\frac{\kappa^2\gamma^2}{3}\sum_{a=1}^{N}M_a(M_a^2-1). \tag{19}$$

Here are some details of the calculation which is actually more involved than in the rational case. Applying the operator $\hbar\partial_{x_i}$ to the KZ equation (16), we get:

$$\hbar^2\partial_{x_i}^2\Big|\Phi\Big\rangle=-\hbar\kappa\gamma^2\sum_{j\neq i}\frac{\mathbf{P}_{ij}}{\sinh^2\gamma(x_i-x_j)}\Big|\Phi\Big\rangle$$

$$+\left(\mathbf{g}^{(i)}+\kappa\gamma\sum_{j\neq i}^{n}\Big(\coth\gamma(x_i-x_j)\mathbf{P}_{ij}+\mathbf{T}_{ij}\Big)\right)\left(\mathbf{g}^{(i)}+\kappa\gamma\sum_{l\neq i}^{n}\Big(\coth\gamma(x_i-x_l)\mathbf{P}_{il}+\mathbf{T}_{il}\Big)\right)\Big|\Phi\Big\rangle.$$

Again, in order to obtain an equation for $\Psi=\Big\langle\Omega\Big|\Phi\Big\rangle$ we apply $\Big\langle\Omega\Big|=\sum_J\Big\langle J\Big|$ from the left and sum over i. After opening brackets in the right hand side several different terms appear, 'wanted' and 'unwanted' ones. The 'wanted' terms are

$$-\hbar\kappa\gamma^2\sum_{j\neq i}\frac{1}{\sinh^2\gamma(x_i-x_j)}\Psi+\kappa^2\gamma^2\sum_{i\neq j}\coth^2\gamma(x_i-x_j)\Psi$$

$$=\sum_{i\neq j}^{n}\frac{\kappa(\kappa-\hbar)\gamma^2}{\sinh^2\gamma(x_i-x_j)}\Psi+n(n-1)\kappa^2\gamma^2\Psi.$$

It appears that the 'unwanted' terms either cancel or contribute to the eigenvalue. To see this, we need some identities. First of all, the trigonometric analog of identity (13) is[7]

$$\sum_{i\neq j\neq l}\coth\gamma(x_i-x_j)\coth\gamma(x_i-x_l)=\frac{1}{3}n(n-1)(n-2). \tag{20}$$

An obvious trigonometric analog of (14) is

$$\sum_{i\neq j}\coth\gamma(x_i-x_j)(\mathbf{g}^{(i)}+\mathbf{g}^{(j)})=0. \tag{21}$$

Using (17), one can prove the identity

$$(\mathbf{g}^{(i)}-\mathbf{g}^{(j)})\mathbf{T}_{ij}+\mathbf{T}_{ij}(\mathbf{g}^{(i)}-\mathbf{g}^{(j)})=0. \tag{22}$$

The most non-trivial identities are

$$\sum_{i\neq j}\Big\langle\Omega\Big|\mathbf{T}_{ij}^2\Big|\Phi\Big\rangle=-\left(n(n-1)-\sum_a M_a(M_a-1)\right)\Psi, \tag{23}$$

$$\sum_{i\neq j\neq l}\Big\langle\Omega\Big|\mathbf{T}_{ij}\mathbf{T}_{il}\Big|\Phi\Big\rangle=-\frac{1}{3}\left(n(n-1)(n-2)-\sum_a M_a(M_a-1)(M_a-2)\right)\Psi. \tag{24}$$

[7] Similarly to (13) identity (20) follows from the summation formula for coth function: $\coth\gamma(x_i-x_j)\coth\gamma(x_i-x_l)+\coth\gamma(x_i-x_j)\coth\gamma(x_l-x_j)+\coth\gamma(x_i-x_l)\coth\gamma(x_j-x_l)=1$ and $\sum_{i\neq j\neq l}1=n(n-1)(n-2)$.

They are derived from the definition (17). Consider first (23). The operator $\mathbf{T}_{ij}^2$ acts on arbitrary $e_a^{(i)}e_b^{(j)}$ entering $|\Phi\rangle \in \mathcal{V}(\{M_a\})$ as follows:

$$\mathbf{T}_{ij}^2\, e_a^{(i)}e_b^{(j)} = \begin{cases} -e_a^{(i)}e_b^{(j)} & \text{if} \quad a \neq b \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Therefore, we compute $\sum_{i\neq j} 1 = n(n-1)$ for all $|J\rangle$ and subtract the terms corresponding to the second line of (25). To prove (24), it is convenient to symmetrize $\mathbf{T}_{ij}\mathbf{T}_{il}$ with respect to permutations of i, j, l (keeping in mind that $\mathbf{T}_{ij} = -\mathbf{T}_{ji}$):

$$\sum_{i\neq j\neq l} \left\langle \Omega \middle| \mathbf{T}_{ij}\mathbf{T}_{il} \middle| \Phi \right\rangle = \frac{1}{3} \sum_{i\neq j\neq l} \left\langle \Omega \middle| \mathbf{T}_{ij}\mathbf{T}_{il} + \mathbf{T}_{lj}\mathbf{T}_{ij} + \mathbf{T}_{il}\mathbf{T}_{jl} \middle| \Phi \right\rangle$$

It can be verified directly that the operator $\mathbf{T}_{ij}\mathbf{T}_{il} + \mathbf{T}_{lj}\mathbf{T}_{ij} + \mathbf{T}_{il}\mathbf{T}_{jl}$ acts on arbitrary $e_a^{(i)}e_b^{(j)}e_c^{(l)}$ entering $|\Phi\rangle \in \mathcal{V}(\{M_a\})$ as follows:

$$(\mathbf{T}_{ij}\mathbf{T}_{il} + \mathbf{T}_{lj}\mathbf{T}_{ij} + \mathbf{T}_{il}\mathbf{T}_{jl})\; e_a^{(i)}e_b^{(j)}e_c^{(l)} = \begin{cases} 0 & \text{if} \;\; a = b = c \\ -e_a^{(l)}e_b^{(i)}e_c^{(j)} & \text{otherwise.} \end{cases} \tag{26}$$

Therefore, we again compute $\sum_{i\neq j\neq l} 1 = n(n-1)(n-2)$, then subtract the cases corresponding to the first line of (26) and put the common minus sign.

5. Relation to the quantum-classical correspondence

We have established the correspondence between solutions to the KZ equations in different weight subspaces of $V^{\otimes n}$ and solutions to the spectral problem for the n-body Calogero model. It extends the previously known Matsuo–Cherednik map to the case when $\dim V$ is not necessarily equal to n. In this more general form, the correspondence can be understood as a natural 'quantization' of the quantum-classsical correspondence [2, 9, 13, 15] between the quantum Gaudin model and the classical Calogero–Moser system of particles.

The Hamiltonian of the latter has the form

$$\mathcal{H} = \sum_{i=1}^{n} p_i^2 - \sum_{i\neq j}^{n} \frac{\kappa^2}{(x_i - x_j)^2}$$

with the usual Poisson brackets $\{p_i, x_j\} = \delta_{ij}$ (for simplicity we consider the rational case). The model is known to be integrable [12], with the Lax matrix

$$L_{ij} = p_i\delta_{ij} + \frac{\kappa(1-\delta_{ij})}{x_i - x_j}.$$

The higher Hamiltonians in involution are given by traces of powers of the Lax matrix: $\mathcal{H}_k = \operatorname{tr} L^k$, $\mathcal{H}_2 = \mathcal{H}$. The correspondence with the quantum Gaudin model goes as follows. Consider the level set of all classical Hamiltonians,

$$\mathcal{H}_k = \sum_{a=1}^{N} M_a g_a^k, \qquad M_a \in \mathbb{Z}_{\geqslant 0},$$

with fixed coordinates x_i. (This means that eigenvalues of the $n\times n$ Lax matrix are g_a with multiplicities M_a.) Then the admissible values of momenta, p_i, coincide with eigenvalues of the Gaudin Hamiltonians $\mathbf{H}_i$ in the weight subspace $\mathcal{V}(\{M_a\})$ for the model with the marked points x_i and the twist matrix $\mathbf{g} = \mathrm{diag}\,(g_1, \ldots, g_N)$. In fact the admissible values of p_i's obey a system of algebraic equations. Different solutions of this system correspond to different eigenstates of the Gaudin Hamiltonians. The coupling constant κ plays the role of the formal Planck constant in the Gaudin model.

In the trigonometric case eigenvalues of the Lax matrix

$$L^{\mathrm{trig}}_{ij} = p_i \delta_{ij} + \frac{\kappa\gamma(1-\delta_{ij})}{\sinh \gamma(x_i - x_j)}$$

should form 'strings' of lengths M_a centered at g_a (see [3]):

$$g_a^{(\alpha)} = g_a - (M_a - 1 - 2\alpha)\kappa\gamma, \qquad \alpha = 0, 1, \ldots, M_a - 1.$$

Then p_i are eigenvalues of the trigonometric Gaudin Hamiltonians. The formula (19) for the eigenvalue E of the Calogero–Sutherland Hamiltonian agrees with this since it is actually equal to the sum of squares of all n eigenvalues of the trigonometric Lax matrix:

$$E = \sum_{a=1}^{N} \sum_{\alpha=0}^{M_a-1} (g_a^{(\alpha)})^2,$$

as one can easily check. Again, we conjecture that the function Ψ is a common eigenfunction for all higher Calogero–Sutherland Hamiltonians with eigenvalues $\sum_{a=1}^{N} \sum_{\alpha=0}^{M_a-1} (g_a^{(\alpha)})^k$.

We see that the quantization of the classical Calogero system of particles with the Planck constant $\hbar$ ($p_i \to \hbar\partial_{x_i}$) corresponds to the non-autonomous deformation of the Gaudin model which is the system of KZ equations with the twist matrix.

6. Conclusion

In this paper we have discussed the Matsuo–Cherednik type correspondence between solutions to the rational or trigonometric KZ equations in $(\mathbb{C}^N)^{\otimes n}$ and solutions to the spectral problem for the n-body Calogero model (respectively, rational or trigonometric). The previously known construction [6, 11] is extended to the case when n is not necessarily equal to N. The wave function of the Calogero model is simply a sum of all components of a solution to the KZ equation in a given weight subspace. We also conjecture that this wave function is a common eigenfunction for all higher commuting Calogero Hamiltonians. This is checked by a direct calculation for the third (cubic) Calogero Hamiltonian.

It is important to note that this result sheds some new light on the quantum-classical correspondence between the quantum Gaudin model and the classical Calogero system of particles [2, 9, 13, 15]. Namely, it suggests what happens with the other side of the correspondence when the Calogero system gets quantized: the Gaudin spectral problem should be substituted by its non-stationary version which is just the system of KZ equations.

Acknowledgments

The work of A Zabrodin has been funded by the Russian Academic Excellence Project '5-100'. It was also supported in part by RFBR grant 15-01-05990 and by joint RFBR grant 15-52-50041 YaF$_a$. The work of A Zotov was supported by RFBR grant 15-31-20484 mol_a_ved and by joint RFBR project 15-51-52031 HHC$_a$.

Appendix. The cubic Hamiltonian

In this appendix we show that the wave function Ψ (11) is an eigenfunction of the third (cubic) Calogero Hamiltonian

$$\hat{\mathcal{H}}_3 = \sum_i \hbar^3 \partial_{x_i}^3 - 3\hbar\kappa(\kappa-\hbar) \sum_{i\neq j} \frac{1}{(x_i-x_j)^2}\,\partial_{x_i}.$$

It is convenient to introduce the KZ connection

$$\nabla_i = \hbar\partial_{x_i} - \mathbf{g}^{(i)} - \kappa \sum_{j\neq i}^{n} \frac{\mathbf{P}_{ij}}{x_i-x_j}. \tag{A.1}$$

Then the KZ equations (1) are of the form:

$$\nabla_i \Big|\Phi\Big\rangle = 0, \qquad i=1,..,n. \tag{A.2}$$

Below we will omit the subscript $j \neq i$ implying that all summation indices do not equal to i. Direct calculations yield

$$\nabla_i^2 = \hbar^2\partial_{x_i}^2 - 2\hbar\left(\mathbf{g}^{(i)} + \kappa\sum_j \frac{\mathbf{P}_{ij}}{x_i-x_j}\right)\partial_{x_i} + \left(\mathbf{g}^{(i)}\right)^2 + \kappa\sum_j \frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}+\mathbf{P}_{ij}\mathbf{g}^{(i)}}{x_i-x_j}$$
$$+\hbar\kappa\sum_j \frac{\mathbf{P}_{ij}}{(x_i-x_j)^2} + \kappa^2\sum_{j,k}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}}{(x_i-x_j)(x_i-x_k)}, \tag{A.3}$$

$$\nabla_i^3 = \hbar^3\partial_{x_i}^3 - 3\hbar^2\left(\mathbf{g}^{(i)} + \kappa\sum_j \frac{\mathbf{P}_{ij}}{x_i-x_j}\right)\partial_{x_i}^2 + 3\hbar^2\kappa\sum_j \frac{\mathbf{P}_{ij}}{(x_i-x_j)^2}\,\partial_{x_i} - \left(\mathbf{g}^{(i)}\right)^3$$
$$+3\hbar\kappa^2\sum_{j,k}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}}{(x_i-x_j)(x_i-x_k)}\,\partial_{x_i} + 3\hbar\left(\mathbf{g}^{(i)}\right)^2\,\partial_{x_i} + 3\hbar\kappa\sum_j \frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}+\mathbf{P}_{ij}\mathbf{g}^{(i)}}{x_i-x_j}\,\partial_{x_i}$$
$$-2\hbar^2\kappa\sum_j \frac{\mathbf{P}_{ij}}{(x_i-x_j)^3} - \hbar\kappa\sum_j \frac{2\mathbf{g}^{(i)}\mathbf{P}_{ij}+\mathbf{P}_{ij}\mathbf{g}^{(i)}}{(x_i-x_j)^2} - 3\hbar\kappa^2\sum_{j,k}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}}{(x_i-x_j)^2(x_i-x_k)}$$
$$-\kappa\sum_j \frac{\left(\mathbf{g}^{(i)}\right)^2\mathbf{P}_{ij}+\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{g}^{(i)}+\mathbf{P}_{ij}\left(\mathbf{g}^{(i)}\right)^2}{(x_i-x_j)} - \kappa^3\sum_{j,k,l}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{P}_{il}}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}$$
$$-\kappa^2\sum_{j,k}\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{g}^{(i)}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{g}^{(i)}}{(x_i-x_j)(x_i-x_k)}. \tag{A.4}$$

Now let us use (A.2). Substitute $\partial_{x_i}^2$ from equation $\nabla_i^2\Big|\Phi\Big\rangle = 0$ with ∇_i^2 written as in (A.3):

$$\nabla_i^3 = \hbar^3\partial_{x_i}^3 + 3\hbar^2\kappa\sum_j \frac{\mathbf{P}_{ij}}{(x_i-x_j)^2}\,\partial_{x_i} + 2\kappa^3\sum_{j,k,l}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{P}_{il}}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}$$
$$-3\hbar\kappa^2\sum_{j,k}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}}{(x_i-x_j)(x_i-x_k)}\,\partial_{x_i} - 3\hbar\left(\mathbf{g}^{(i)}\right)^2\partial_{x_i} - 3\hbar\kappa\sum_j\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}+\mathbf{P}_{ij}\mathbf{g}^{(i)}}{x_i-x_j}\,\partial_{x_i}$$
$$+\hbar\kappa\sum_j\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}-\mathbf{P}_{ij}\mathbf{g}^{(i)}}{(x_i-x_j)^2} + 2\kappa^2\sum_{j,k}\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{g}^{(i)}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{g}^{(i)}}{(x_i-x_j)(x_i-x_k)}$$
$$+2\kappa\sum_j\frac{\left(\mathbf{g}^{(i)}\right)^2\mathbf{P}_{ij}+\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{g}^{(i)}+\mathbf{P}_{ij}\left(\mathbf{g}^{(i)}\right)^2}{(x_i-x_j)} - 2\hbar^2\kappa\sum_j\frac{\mathbf{P}_{ij}}{(x_i-x_j)^3} + 2\left(\mathbf{g}^{(i)}\right)^3. \tag{A.5}$$

Here and below we imply that the operators act to a solution $\Big|\Phi\Big\rangle$ of the KZ equation (we do not write the vector $\Big|\Phi\Big\rangle$ for brevity). In the same way make the following substitutions into the r.h.s. of (A.5) (using $\nabla_i\Big|\Phi\Big\rangle = 0$):

$$-3\hbar\left(\mathbf{g}^{(i)}\right)^2\partial_{x_i} = -3\hbar\left(\mathbf{g}^{(i)}\right)^3 - 3\kappa\sum_j\frac{\left(\mathbf{g}^{(i)}\right)^2\mathbf{P}_{ij}}{(x_i-x_j)},$$
$$-3\hbar\kappa\sum_j\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}+\mathbf{P}_{ij}\mathbf{g}^{(i)}}{x_i-x_j}\,\partial_{x_i}$$
$$= -3\kappa^2\sum_{j,k}\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{g}^{(i)}\mathbf{P}_{ik}}{(x_i-x_j)(x_i-x_k)} - 3\kappa\sum_j\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{g}^{(i)}+\mathbf{P}_{ij}\left(\mathbf{g}^{(i)}\right)^2}{(x_i-x_j)}.$$

Then we get

$$\nabla_i^3 = \hbar^3\partial_{x_i}^3 + 3\hbar^2\kappa\sum_j \frac{\mathbf{P}_{ij}}{(x_i-x_j)^2}\,\partial_{x_i} + 2\kappa^3\sum_{j,k,l}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{P}_{il}}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}$$
$$+\hbar\kappa\sum_j\frac{\mathbf{g}^{(i)}\mathbf{P}_{ij}-\mathbf{P}_{ij}\mathbf{g}^{(i)}}{(x_i-x_j)^2} + \kappa^2\sum_{j,k}\frac{-\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{P}_{ik}-\mathbf{P}_{ij}\mathbf{g}^{(i)}\mathbf{P}_{ik}+2\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{g}^{(i)}}{(x_i-x_j)(x_i-x_k)}$$
$$+2\kappa\sum_j\frac{\left(\mathbf{g}^{(i)}\right)^2\mathbf{P}_{ij}+\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{g}^{(i)}+\mathbf{P}_{ij}\left(\mathbf{g}^{(i)}\right)^2}{(x_i-x_j)} - 2\hbar^2\kappa\sum_j\frac{\mathbf{P}_{ij}}{(x_i-x_j)^3} - \left(\mathbf{g}^{(i)}\right)^3$$
$$-3\hbar\kappa^2\sum_{j,k}\frac{\mathbf{P}_{ij}\mathbf{P}_{ik}}{(x_i-x_j)(x_i-x_k)}\,\partial_{x_i}. \tag{A.6}$$

The two sums over j, k in (A.6) should be subdivided into two parts each—with $j=k$ and $j\neq k$. Then the last sum in (A.6) with $j\neq k$ should be transformed via $\nabla_i\Big|\Phi\Big\rangle = 0$. This yields

$$\nabla_i^3 = \hbar^3\partial_{x_i}^3 - 3\hbar\kappa\sum_j \tfrac{\kappa-\hbar\mathbf{P}_{ij}}{(x_i-x_j)^2}\,\partial_{x_i} + 2\kappa^3\sum_{j,k,l}\tfrac{\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{P}_{il}}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}$$

$$+\,\hbar\kappa\sum_j \tfrac{\mathbf{g}^{(i)}\mathbf{P}_{ij}-\mathbf{P}_{ij}\mathbf{g}^{(i)}}{(x_i-x_j)^2} - \kappa^2\sum_{j\neq k}\tfrac{\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{g}^{(i)}\mathbf{P}_{ik}+\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{g}^{(i)}}{(x_i-x_j)(x_i-x_k)}$$

$$+\,2\kappa\sum_j \tfrac{\left(\mathbf{g}^{(i)}\right)^2\mathbf{P}_{ij}+\mathbf{g}^{(i)}\mathbf{P}_{ij}\mathbf{g}^{(i)}+\mathbf{P}_{ij}\left(\mathbf{g}^{(i)}\right)^2}{(x_i-x_j)} - 2\hbar^2\kappa\sum_j\tfrac{\mathbf{P}_{ij}}{(x_i-x_j)^3} - \left(\mathbf{g}^{(i)}\right)^3$$

$$+\,\kappa^2\sum_j\tfrac{\mathbf{g}^{(i)}-\mathbf{g}^{(j)}}{(x_i-x_j)^2} - 3\kappa^3\sum_{j\neq k,l}\tfrac{\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{P}_{il}}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}. \qquad \text{(A.7)}$$

At last notice that in two sums over three indices j, k, l the terms corresponding to coinciding indices cancel out. Finally, we have

$$\nabla_i^3 = \hbar^3\partial_{x_i}^3 - 3\hbar\kappa\sum_j \tfrac{\kappa-\hbar\mathbf{P}_{ij}}{(x_i-x_j)^2}\,\partial_{x_i} - \kappa^3{\sum_{j,k,l}}'\tfrac{\mathbf{P}_{ij}\mathbf{P}_{ik}\mathbf{P}_{il}}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}$$

$$-\,\kappa\sum_j\tfrac{(\kappa-\hbar\mathbf{P}_{ij})(\mathbf{g}^{(j)}-\mathbf{g}^{(i)})}{(x_i-x_j)^2} - \kappa^2\sum_{j\neq k}\tfrac{\mathbf{P}_{ij}\mathbf{P}_{ik}\left[\mathbf{g}^{(i)}+\mathbf{g}^{(j)}+\mathbf{g}^{(k)}\right]}{(x_i-x_j)(x_i-x_k)}$$

$$+\,2\kappa\sum_j\tfrac{\mathbf{P}_{ij}\left[\left(\mathbf{g}^{(i)}\right)^2+\mathbf{g}^{(i)}\mathbf{g}^{(j)}+\left(\mathbf{g}^{(j)}\right)^2\right]}{(x_i-x_j)} - 2\hbar^2\kappa\sum_j\tfrac{\mathbf{P}_{ij}}{(x_i-x_j)^3} - \left(\mathbf{g}^{(i)}\right)^3, \qquad \text{(A.8)}$$

where ${\sum_{j,k,l}}'$ denotes summation over all distinct indices. Now we can write:

$$\sum_i\left\langle\Omega\middle|\nabla_i^3\middle|\Phi\right\rangle = 0$$

$$= \sum_i\hbar^3\partial_{x_i}^3\Psi - 3\hbar\kappa(\kappa-\hbar)\sum_{i\neq j}\tfrac{1}{(x_i-x_j)^2}\,\partial_{x_i}\Psi - \sum_i\left\langle\Omega\right|\left(\mathbf{g}^{(i)}\right)^3\left|\Phi\right\rangle \qquad \text{(A.9)}$$

or

$$\hat{\mathcal{H}}_3\Psi = E\Psi, \quad \hat{\mathcal{H}}_3 = \sum_i\hbar^3\partial_{x_i}^3 - 3\hbar\kappa(\kappa-\hbar)\sum_{i\neq j}\tfrac{1}{(x_i-x_j)^2}\,\partial_{x_i} \qquad \text{(A.10)}$$

where

$$E = \sum_{a=1}^N M_a g_a^3. \qquad \text{(A.11)}$$

In transition from (A.8) to (A.9) we have used $\left\langle\Omega\right|\mathbf{P}_{ij} = \left\langle\Omega\right|$ and the identities

$${\sum_{i,j,k}}'\tfrac{\mathbf{g}^{(i)}+\mathbf{g}^{(j)}+\mathbf{g}^{(k)}}{(x_i-x_j)(x_i-x_k)} = 0, \qquad \text{(A.12)}$$

$${\sum_{i,j,k,l}}'\tfrac{1}{(x_i-x_j)(x_i-x_k)(x_i-x_l)} = 0. \qquad \text{(A.13)}$$

The other terms cancel due to skew-symmetry with respect to i, j.

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