

# ON CONJUGACY OF STABILIZERS OF REDUCTIVE GROUP ACTIONS

VLADIMIR L. POPOV

ABSTRACT. It is shown that the main result of N. R. Wallach, *Principal orbit type theorems for reductive algebraic group actions and the Kempf–Ness Theorem*, [arXiv:1811.07195v1](https://arxiv.org/abs/1811.07195v1) (17 Nov 2018) is a special case of a more general statement, which can be deduced, using a short argument, from the classical Richardson and Luna theorems.

1. In the recent preprint [W], the following main result is obtained using the Kempf–Ness theorem to reduce it to the principal orbit type theorem for compact Lie groups:

“Let  $G$  be a reductive, affine algebraic group and let  $(\rho, V)$  be a regular representation of  $G$ . Let  $X$  be an irreducible  $\mathbb{C}^\times G$  invariant Zariski closed subset such that  $G$  has a closed orbit that has maximal dimension among all orbits (this is equivalent to: generic orbits are closed). Then there exists an open subset,  $W$ , of  $X$  in the metric topology which is dense with complement of measure 0 such that if  $x, y \in W$  then  $(\mathbb{C}^\times G)_x$  is conjugate to  $(\mathbb{C}^\times G)_y$ . Furthermore, if  $Gx$  is a closed orbit of maximal dimension and if  $x$  is a smooth point of  $X$  then there exists  $y \in W$  such that  $(\mathbb{C}^\times G)_x$  contains a conjugate of  $(\mathbb{C}^\times G)_y$ .”

Below is shown that the more general statements can be deduced, using a short argument, from the classical Richardson and Luna theorems.

2. We fix an algebraically closed ground field  $k$  of characteristic 0 and use freely the standard notation of [B], [PV].

Let  $G$  be a reductive algebraic group such that  $G = CR$ , where  $C$  is a diagonalizable algebraic subgroup of the center of  $G$  and  $R$  is a reductive algebraic subgroup of  $G$ . We denote by  $\mathcal{X}(C)$  the character group of  $C$  and, given an algebraic  $C$ -module  $M$  and a character  $\alpha \in \mathcal{X}(C)$ , by  $M_\alpha$  the weight space of  $M$  of the weight  $\alpha$ . Since  $C$  is diagonalizable,  $M$  is the direct sum of the  $M_\alpha$ 's; see [B, III.8.17].

Let  $X$  be irreducible affine algebraic variety endowed with a regular (morphic) action of  $G$ .

**Theorem.** *In the above notation, assume that there is a closed  $R$ -orbit of maximal dimension among all  $R$ -orbits in  $X$ . Then the following hold:*

- (a) *There exists a dense open (in the Zariski topology) subset  $U$  of  $X$  such that if  $x, y \in U$ , then  $G_x$  is conjugate to  $G_y$ .*
- (b) *If the  $R$ -orbit  $R(z)$  of a point  $z \in X$  is closed, then there exists a point  $y \in U$  such that  $G_z$  contains a conjugate of  $G_y$ .*

*Proof.* (a) Let  $S$  be the singular locus of  $X$ . We may (and shall) assume that  $S \neq \emptyset$ , because otherwise the claim to be proved immediately follows from

the Richardson theorem [R, Prop. 5.3] (see also [L, Cor. 8]). As  $S$  is a closed  $G$ -stable subset of  $X$ , we have  $k[S] = \bigoplus_{\alpha \in \mathcal{X}(C)} (k[S])_{\alpha}$ . The assumption on  $R$ -orbit implies the existence of a dense open subset of  $X$  whose points have closed  $R$ -orbits of maximal dimension [P, Thm. 4]. Hence there is a closed  $R$ -orbit  $\mathcal{O}$  such that  $S \cap \mathcal{O} = \emptyset$ . This implies the existence of a function  $f \in k[X]^R$  such that  $f|_S = 0$ ,  $f|_{\mathcal{O}} = 1$  (see, e.g., [B, Lem. 8.19(ii)] or [PV, Thm. 4.7]). Since  $C$  centralizes  $R$ , the algebra  $k[X]^R$  is  $C$ -stable, so we have the weight decomposition  $k[X]^R = \bigoplus_{\alpha \in \mathcal{X}(C)} (k[X]^R)_{\alpha}$ . Let  $\pi_{\alpha}: k[X]^R \rightarrow (k[X]^R)_{\alpha}$  be the natural projection. Since  $f|_{\mathcal{O}} \neq 0$ , there is  $\alpha \in \mathcal{X}(C)$  such that for  $f_{\alpha} := \pi_{\alpha}(f)$ , we have  $f_{\alpha}|_{\mathcal{O}} \neq 0$ . Since  $G = CR$ , the function  $f_{\alpha}$  is a semi-invariant of  $G$ . As  $S$  is  $G$ -stable, the homomorphism  $\varrho: k[X] \rightarrow k[S]$ ,  $h \mapsto h|_S$ , is  $G$ -equivariant, hence  $\varrho(k[X]_{\alpha}) \subseteq k[S]_{\alpha}$ . In view of  $\varrho(f) = 0$ , this implies  $\varrho(f_{\alpha}) = 0$ . Thus  $f_{\alpha}$  is a nonzero semi-invariant of  $G$  vanishing on  $S$ . Hence  $X_{f_{\alpha}} := \{x \in X \mid f_{\alpha}(x) \neq 0\}$  is a  $G$ -stable dense open subset of  $X$ , which is a smooth affine variety. Now, by the Richardson theorem [R, Prop. 5.3] (see also [L, Cor. 8]), there is a dense open subset  $U$  of  $X_{f_{\alpha}}$  such that if  $x, y \in U$ , then  $G_x$  is conjugate to  $G_y$ . This proves (a).

(b) Let  $\overline{G(z)}$  be the closure of the  $G$ -orbit of  $z$  in  $X$ . Then  $B := \overline{G(z)} \setminus G(z)$  is a closed  $G$ -stable subset of  $X$ . If  $B = \emptyset$ , then the existence of  $y$  immediately follows from the Luna slice theorem, see [L, Rem. 4° on p. 98] (cf. [PV, Thm. 6.3]). Now consider the case  $B \neq \emptyset$ . Since  $B \cap R(z) = \emptyset$ , the same argument as in the above proof of (a) shows the existence of a  $G$ -semi-invariant  $f \in k[X]^R$  such that  $f|_B = 0$ ,  $f|_{R(z)} = 1$ . The latter equality implies that  $f$  vanishes nowhere on  $G(z)$ . Therefore,  $X_f$  is a  $G$ -stable dense open subset of  $X$  containing  $G(z)$ , and  $G(z)$  is closed in  $X_f$ . Now, since  $X_f$  is affine, the existence of  $y$  follows from the Luna slice theorem, as above. This proves (b).  $\square$

**3. Remark.** In [W, Sect. 6] is given an example of a linear action of a semi-simple group, which shows that the existence of a point with trivial stabilizer does not imply triviality of stabilizers of points in general position. It should be noted that this phenomenon is not new, similar examples has long been known (perhaps the earliest one belongs to Richardson, see [L, Rem. 4° on p. 98]).

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STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, GUBKINA 8, MOSCOW, 119991, RUSSIA

*E-mail address:* popovvl@mi-ras.ru