CHAPTER II

# STOCHASTIC CALCULUS

## § 1. Stochastic integration with respect to Brownian motion

In this section we present the basic facts of the theory of stochastic integration in the case when the integrator is a Brownian motion $W$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions (see § 4 Ch. I) and $W(t)$, $t \in [0,T]$, be an $\mathcal{F}_t$-measurable Brownian motion in this space. We also assume that for all $v > t$ the increments $W(v) - W(t)$ are independent of the $\sigma$-algebra $\mathcal{F}_t$. For $\{\mathcal{F}_t\}$ one can take the completed natural filtration, i.e., the family of the $\sigma$-algebras $\mathcal{G}_0^t = \sigma\{W(s), s \in [0,t]\}$, generated by the Brownian motion $W$ up to the time $t$.

The goal is to give some meaning to the *stochastic integrals* of the type

$$\int_0^t f(s)\, dW(s). \tag{1.1}$$

Since the Brownian motion $W$ has an infinite variation on any interval, it is not possible to define such integrals by means of classical approaches of the theory of integration. The approach proposed here is that the stochastic integral (1.1) can be defined via an isometry. The notion to which this approach leads us is called the *Itô integral* and the theory is called *stochastic calculus*. For a nonrandom function $f$, the integral (1.1) can be considered (see § 9 Ch. I) as the integral with respect to the orthogonal stochastic measure defined by $Z(\Delta) := W(b) - W(a)$, $\Delta = [a,b)$, and having the structure function $G(\Delta) = b - a$.

Consider the *class* $\mathcal{H}_2[0,T]$ of progressively measurable with respect to $\{\mathcal{F}_t\}$ stochastic processes $f(t)$, $t \in [0,T]$, satisfying the condition

$$\int_0^T \mathbf{E} f^2(s)\, ds < \infty. \tag{1.2}$$

In the present description we does not exclude the case $T = \infty$. In this case the interval $[0,T]$ is replaced by $[0,\infty)$.

Consider the class of *simple processes* of the form

$$\bar{f}(s) = \sum_{k=0}^{m-1} f_k \mathbb{I}_{[s_k, s_{k+1})}(s), \qquad s \in [0,T], \tag{1.3}$$

where $0 = s_0 < s_1 < \cdots < s_m = T$, the random variables $f_k$ are $\mathcal{F}_{s_k}$-measurable, and $\mathbf{E} f_k^2 < \infty$, $k = 0, \dots m-1$. In the case $T = \infty$, we set $f_{m-1} \equiv 0$. Obviously, the function $\bar{f}$ belongs to $\mathcal{H}_2[0,T]$.



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The stochastic integral of $\bar{f}$ with respect to $W$ is defined to be

$$\int_0^T \bar{f}(s)\,dW(s) := \sum_{k=0}^{m-1} f_k(W(s_{k+1}) - W(s_k)). \tag{1.4}$$

For arbitrary constants $\alpha$ and $\beta$,

$$\int_0^T (\alpha\bar{f}_1(s) + \beta\bar{f}_2(s))\,dW(s) = \alpha\int_0^T \bar{f}_1(s)\,dW(s) + \beta\int_0^T \bar{f}_2(s)\,dW(s). \tag{1.5}$$

The mean of the stochastic integral defined by (1.4) equals zero, i.e.,

$$\mathbf{E}\int_0^T \bar{f}(s)\,dW(s) = 0. \tag{1.6}$$

Indeed, since $f_k$ is $\mathcal{F}_{s_k}$-measurable, the variables $f_k$ and $W(s_{k+1}) - W(s_k)$ are independent. Therefore, in view of (10.3) Ch. I, we have

$$\mathbf{E}\{f_k(W(s_{k+1}) - W(s_k))\} = \mathbf{E}f_k\mathbf{E}(W(s_{k+1}) - W(s_k)) = 0.$$

Hence the expectation of the sum (1.4) is zero and (1.6) holds.

For the variance of the stochastic integral we have

$$\mathbf{E}\Bigl(\int_0^T \bar{f}(s)\,dW(s)\Bigr)^2 = \int_0^T \mathbf{E}\bar{f}^2(s)\,ds. \tag{1.7}$$

Indeed, since $f_k$ and $W(s_{k+1}) - W(s_k)$ are independent, by (10.3) Ch. I, we have

$$\mathbf{E}\{f_k^2(W(s_{k+1}) - W(s_k))^2\} = \mathbf{E}f_k^2\mathbf{E}(W(s_{k+1}) - W(s_k))^2 = \mathbf{E}f_k^2(s_{k+1} - s_k).$$

For $k < l$ the random variables $f_k(W(s_{k+1}) - W(s_k))f_l$ are $\mathcal{F}_{s_l}$-measurable and the increments $W(s_{l+1}) - W(s_l)$ are independent of $\mathcal{F}_{s_l}$. Therefore,

$$\begin{aligned} I_{k,l} &:= \mathbf{E}\{f_k(W(s_{k+1}) - W(s_k))f_l(W(s_{l+1}) - W(s_l))\} \\ &= \mathbf{E}\{f_k(W(s_{k+1}) - W(s_k))f_l\}\mathbf{E}(W(s_{l+1}) - W(s_l)) = 0. \end{aligned}$$

Here to prove that the expectation is finite we used the estimate

$$\begin{aligned} \mathbf{E}|f_k(W(s_{k+1}) - W(s_k))f_l| &\le \mathbf{E}^{1/2}\{f_k^2(W(s_{k+1}) - W(s_k))^2\}\mathbf{E}^{1/2}\{f_l^2\} \\ &= \mathbf{E}^{1/2}\{f_k^2\}(s_{k+1} - s_k)^{1/2}\mathbf{E}^{1/2}\{f_l^2\} < \infty. \end{aligned}$$

Now it is easy to check (1.7):

$$\mathbf{E}\Big(\int_0^T \bar{f}(s)\,dW(s)\Big)^2 = \mathbf{E}\Big(\sum_{k=0}^{m-1} f_k(W(s_{k+1}) - W(s_k))\Big)^2$$
$$= \sum_{k=0}^{m-1} \mathbf{E}\{f_k^2(W(s_{k+1}) - W(s_k))^2\} + 2 \sum_{0\le k<l\le m-1} I_{k,l}$$
$$= \sum_{k=0}^{m-1} \mathbf{E}f_k^2(s_{k+1} - s_k) = \int_0^T \mathbf{E}\bar{f}^2(s)\,ds.$$

Formula (1.7) is of key importance for the definition of the stochastic integral for the class of random processes $\mathcal{H}_2[0,T]$.

Let $L^2(\mathbf{P})$ be the space of square integrable random variables. Then $L^2(\mathbf{P})$ is a Hilbert space when equipped with the norm $(\mathbf{E}X^2)^{1/2}$, $X \in L^2(\mathbf{P})$.

For a function $f \in \mathcal{H}_2[0,T]$, the norm is $\Big(\int_0^T \mathbf{E}f^2(s)\,ds\Big)^{1/2}$.

In view of (1.7), for a class of simple processes $\bar{f} \in \mathcal{H}_2[0,T]$ the mapping

$$\bar{f} \to \int_0^T \bar{f}(s)\,dW(s) \tag{1.8}$$

is an *isometry* from a subset of $\mathcal{H}_2[0,T]$ into $L^2(\mathbf{P})$.

**Proposition 1.1.** *The set of simple processes is dense in the space $\mathcal{H}_2[0,T]$, i.e., for any process $f \in \mathcal{H}_2[0,T]$ there exists a sequence of simple processes $\bar{f}_n \in \mathcal{H}_2[0,T]$ such that*

$$\lim_{n\to\infty} \int_0^T \mathbf{E}(f(s) - \bar{f}_n(s))^2\,ds = 0. \tag{1.9}$$

*Proof.* Without loss of generality, we can assume that $f$ is bounded. Otherwise we set $f_N(t) := f(t)\mathbb{1}_{[-N,N]}(f(t))$ and use the fact that

$$\lim_{N\to\infty} \int_0^T \mathbf{E}(f(s) - f_N(s))^2\,ds = 0.$$

For a continuous bounded $f$, set $\bar{f}_n(s) := f([ns]/n)$, where $[a]$ denotes the largest integer not exceeding $a$. Then (1.9) follows from the Lebesgue dominated convergence theorem for integrals of uniformly bounded functions.

Now to prove Proposition 1.1 it is enough to approximate a bounded progressively measurable process $f$ by continuous processes. Such processes are

$$\widehat{f}_n(s) := n \int_{(s-1/n)^+}^{s} f(v)\,dv, \qquad n = 1, 2, \ldots.$$

where $a^+ = \max\{0, a\}$. It is clear that $\widehat{f}_n$, $n = 1, 2, \dots$, are uniformly bounded progressively measurable processes, because they are continuous. Set $F(s) := \int_0^s f(v)\,dv$. Then $F$ is a.s. a function of bounded variation. By the Lebesgue differentiation theorem, for almost all $s \in [0, T]$ there exists $F'(s)$ and the equality $f(s) = F'(s) = \lim_{n\to\infty} \widehat{f}_n(s)$ holds. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^T \mathbf{E}(f(s) - \widehat{f}_n(s))^2\,ds = 0.$$

This completes the proof. □

In view of Proposition 1.1, the linear isometry (1.8) can be extended uniquely to a linear isometry from the whole $\mathcal{H}_2[0, T]$ into $L^2(\mathbf{P})$, thus defining the stochastic integral of $f \in \mathcal{H}_2[0, T]$ with respect to the Brownian motion.

This means the following. Consider the sequence $\{\bar{f}_n\}$ of functions, satisfying (1.9). Using the inequality

$$\int_0^T \mathbf{E}(\bar{f}_m(s) - \bar{f}_n(s))^2\,ds \le 2\int_0^T \mathbf{E}(f(s) - \bar{f}_m(s))^2\,ds + 2\int_0^T \mathbf{E}(f(s) - \bar{f}_n(s))^2\,ds$$

and formulas (1.5), (1.7), we have

$$\mathbf{E}\biggl(\int_0^T \bar{f}_m(s)\,dW(s) - \int_0^T \bar{f}_n(s)\,dW(s)\biggr)^2 = \int_0^T \mathbf{E}(\bar{f}_m(s) - \bar{f}_n(s))^2\,ds \underset{\substack{m\to\infty\\ n\to\infty}}{\longrightarrow} 0.$$

Thus the sequence $\int_0^T \bar{f}_n(s)\,dW(s)$ is Cauchy for the mean square convergence. Therefore, there exists a limit, which is assigned to be the stochastic integral of $f$ with respect to the Brownian motion $W$.

Thus, for a function $f \in \mathcal{H}_2[0, T]$ such that (1.9) holds we set

$$\int_0^T f(s)\,dW(s) := \mathrm{l.\,i.\,m.} \int_0^T \bar{f}_n(s)\,dW(s), \tag{1.10}$$

where l. i. m. denotes the limit in mean square.

By (1.10), the properties (1.5)–(1.7) are valid for all processes from the space $\mathcal{H}_2[0, T]$:

1) for any constants $\alpha$ and $\beta$,

$$\int_0^T (\alpha f_1(s) + \beta f_2(s))\,dW(s) = \alpha\int_0^T f_1(s)\,dW(s) + \beta\int_0^T f_2(s)\,dW(s) \qquad \text{a.s.;}$$

2) the mean of the stochastic integral equals zero, i.e.,

$$\mathbf{E}\int_0^T f(s)\,dW(s) = 0; \tag{1.11}$$

3) the variance of the stochastic integral satisfies the relation

$$\mathbf{E}\Bigl(\int_0^T f(s)\,dW(s)\Bigr)^2 = \int_0^T \mathbf{E}f^2(s)\,ds; \tag{1.12}$$

4) if

$$\lim_{n\to\infty}\int_0^T \mathbf{E}(f(s)-f_n(s))^2\,ds = 0,$$

then

$$\int_0^T f(s)\,dW(s) = \mathrm{l.\,i.\,m.}\int_0^T f_n(s)\,dW(s). \tag{1.13}$$

In addition to the first property, from the construction of the stochastic integral one can deduce that for any bounded $\mathcal{F}_v$-measurable random variable $\xi$ and any $t > v$

$$\int_0^T \xi \mathbb{1}_{[v,t)}(s) f(s)\,dW(s) = \xi \int_0^T \mathbb{1}_{[v,t)}(s) f(s)\,dW(s) \qquad \text{a.s.} \tag{1.14}$$

## § 2. Stochastic integrals with variable upper limit

Define a family of stochastic integrals with variable upper limit by setting

$$\int_0^t f(s)\,dW(s) := \int_0^T \mathbb{1}_{[0,t)}(s) f(s)\,dW(s), \qquad \text{for every} \quad t\in[0,T]. \tag{2.1}$$

Then the following problem arises. Formula (1.10) defines the stochastic integral uniquely up to a set $\Lambda_f$ of probability zero. This set depends on the integrand. Definition (2.1) involves a whole family of integrands depending on the time parameter $t$. Therefore, it is possible that the probability of the union of sets $\Lambda_{\mathbb{1}_{[0,t)}f}$ is not zero. In this case the integrals are not determined as a function of $t$ on a set of nonzero probability. We overcome this difficulty by proving that the stochastic integral, as a function of $t$, is a.s. continuous $\mathcal{F}_t$-measurable martingale.

For $v < t$ it is natural to set

$$\int_v^t f(s)\,dW(s) := \int_0^T \mathbb{1}_{[v,t)}(s) f(s)\,dW(s). \tag{2.2}$$

Then

$$\int_v^t f(s)\,dW(s) = \int_0^t f(s)\,dW(s) - \int_0^v f(s)\,dW(s),$$

since $\mathbb{1}_{[v,t)}(s) = \mathbb{1}_{[0,t)}(s) - \mathbb{1}_{[0,v)}(s)$ and the linearity property holds.

The following generalizations of the properties 2), 3) of § 1 hold: for every $v < t$

$$\mathbf{E}\Big\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_v\Big\} = 0 \qquad \text{a.s.,} \tag{2.3}$$

$$\mathbf{E}\Big\{\Big(\int_v^t f(s)\,dW(s)\Big)^2\Big|\mathcal{F}_v\Big\} = \int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_v\}\,ds \qquad \text{a.s.} \tag{2.4}$$

Indeed, for any $\mathcal{F}_v$-measurable bounded random variable $\xi$ we have

$$\mathbf{E}\Big\{\xi\mathbf{E}\Big\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_v\Big\}\Big\} = \mathbf{E}\Big\{\mathbf{E}\Big\{\xi\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_v\Big\}\Big\}$$

$$= \mathbf{E}\Big\{\xi\int_0^T \mathbb{1}_{[v,t)}(s)f(s)\,dW(s)\Big\} = \mathbf{E}\int_0^T \xi\mathbb{1}_{[v,t)}(s)f(s)\,dW(s) = 0,$$

where (1.14) and (1.11) were used. Since the random variable $\xi$ is arbitrary, this implies (2.3).

Similarly, using (1.14) and (1.12), we have

$$\mathbf{E}\Big\{\xi^2\mathbf{E}\Big\{\Big(\int_v^t f(s)\,dW(s)\Big)^2\Big|\mathcal{F}_v\Big\}\Big\} = \mathbf{E}\Big(\int_0^T \xi\mathbb{1}_{[v,t)}(s)f(s)\,dW(s)\Big)^2$$

$$= \int_0^T \mathbf{E}\{\xi^2\mathbb{1}_{[v,t)}(s)f^2(s)\}\,ds = \mathbf{E}\Big\{\xi^2\int_v^t f^2(s)\,ds\Big\} = \mathbf{E}\Big\{\xi^2\int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_v\}\,ds\Big\}.$$

This implies (2.4).

**Theorem 2.1.** *Let $f \in \mathcal{H}_2[0,T]$. Then the process $I(t) := \int_0^t f(s)\,dW(s)$, $t \in [0,T]$, is an* a.s. *continuous martingale such that for any $\varepsilon > 0$*

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t f(s)\,dW(s)\Big| \ge \varepsilon\Big) \le \frac{1}{\varepsilon^2}\int_0^T \mathbf{E}f^2(s)\,ds, \tag{2.5}$$

$$\mathbf{E}\sup_{0\le t\le T}\left|\int_0^t f(s)\,dW(s)\right|^2 \le 4\int_0^T \mathbf{E}f^2(s)\,ds. \tag{2.6}$$

*Proof.* The case $T=\infty$ can be considered as the limiting case for $T_n=n$. So we can assume that $T<\infty$. We first prove the theorem for the simple processes defined by (1.3). For such processes, for $t\in[s_l,s_{l+1})$, $l=0,\dots,m-1$, we have

$$I(t)=\int_0^T \mathbb{1}_{[0,t)}(s)\bar f(s)\,ds=\sum_{k=0}^{l-1} f_k(W(s_{k+1})-W(s_k))+f_l(W(t)-W(s_l)). \tag{2.7}$$

Since the Brownian motion is a.s. continuous, the process $I(t)$ is also continuous.

From (2.3) it follows that for $v<t$

$$\mathbf{E}\Big\{\int_0^t \bar f(s)\,dW(s)\Big|\mathcal{F}_v\Big\}=\int_0^v \bar f(s)\,dW(s),$$

i.e., for simple processes $I(t)$ is a martingale. By Doob's inequality for martingales (5.11), $p=2$, Ch. I,

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f(s)\,dW(s)\Big|\ge\varepsilon\Big)\le\frac{1}{\varepsilon^2}\mathbf{E}\Big(\int_0^T \bar f(s)\,dW(s)\Big)^2=\frac{1}{\varepsilon^2}\int_0^T \mathbf{E}\bar f^2(s)\,ds.$$

The equality on the right-hand side is due to (1.7). This proves (2.5). Similarly, from the second Doob inequality for martingales (see (5.12), $p=2$, Ch. I) it follows that (2.6) is also valid. Thus for simple processes the theorem is proved.

For an arbitrary $f\in\mathcal{H}_2[0,T]$, using (1.9) we can choose a subsequence of the integer numbers $n_k$ such that

$$\int_0^T \mathbf{E}(f(s)-\bar f_{n_k}(s))^2\,ds\le\frac{1}{2^k}.$$

Then

$$\int_0^T \mathbf{E}(\bar f_{n_{k+1}}(s)-\bar f_{n_k}(s))^2\,ds\le 2\int_0^T \mathbf{E}(f(s)-\bar f_{n_{k+1}}(s))^2\,ds$$

$$+2\int_0^T \mathbf{E}(f(s)-\bar f_{n_k}(s))^2\,ds\le\frac{3}{2^k}.$$

The process $\bar f_{n_{k+1}}(s)-\bar f_{n_k}(s)$ is simple, therefore, (2.5) applies. We have

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f_{n_{k+1}}(s)\,dW(s)-\int_0^t \bar f_{n_k}(s)\,dW(s)\Big|\ge\frac{1}{k^2}\Big)$$

$$= \mathbf{P}\Big( \sup_{0\le t\le T} \Big| \int_0^t \big(\bar f_{n_{k+1}}(s) - \bar f_{n_k}(s)\big)\, dW(s)\Big| \ge \frac{1}{k^2}\Big)$$

$$\le k^4 \int_0^T \mathbf{E}(\bar f_{n_{k+1}}(s) - \bar f_{n_k}(s))^2\, ds \le \frac{3k^4}{2^k}.$$

Since the series of these probabilities converges, the first part of the Borel–Cantelli lemma, shows that there exists a.s. a number $k_0 = k_0(\omega)$ such that for all $k > k_0$

$$\sup_{0\le t\le T} \Big| \int_0^t \bar f_{n_{k+1}}(s)\, dW(s) - \int_0^t \bar f_{n_k}(s)\, dW(s)\Big| < \frac{1}{k^2}.$$

Then the sequence of integrals

$$\int_0^t \bar f_{n_m}(s)\, dW(s) = \int_0^t \bar f_{n_0}(s)\, dW(s) + \sum_{k=0}^{m-1} \Big( \int_0^t \bar f_{n_{k+1}}(s)\, dW(s) - \int_0^t \bar f_{n_k}(s)\, dW(s)\Big)$$

converges a.s. uniformly in $[0,T]$ to some limit, which, by definition, is a stochastic integral $I(t)$, i.e.,

$$\sup_{0\le t\le T} \Big| I(t) - \int_0^t \bar f_{n_m}(s)\, dW(s)\Big| \to 0, \qquad \text{as} \quad m\to\infty.$$

Since a uniform limit of continuous functions is continuous, the process $I(t)$, $t \in [0,T]$ is a.s. continuous. From (2.3) it follows that $I(t)$ is a martingale and the estimates (2.5), (2.6) hold. □

A very important property of stochastic integrals follows from (2.5) and (2.6). Let

$$\lim_{n\to\infty} \int_0^T \mathbf{E}(f(s) - f_n(s))^2\, ds = 0, \quad f_n, f \in \mathcal{H}_2[0,T].$$

Then

$$\sup_{t\in[0,T]} \Big| \int_0^t f(s)\, dW(s) - \int_0^t f_n(s)\, dW(s)\Big| \to 0 \quad \text{as } n\to\infty \tag{2.8}$$

in probability and in mean square.

This property enables us to justify the passage to a limit in different schemes involving stochastic integrals.

Here is a simple example of an interesting class of Gaussian processes expressed via the stochastic integral. For nonrandom functions $h(s)$ and $g(s)$, $s \in [0,T]$, set

$$\overline{W}(t) := x + h(t) + \int_0^t g(s)\, dW(s).$$

It is clear that $\overline{W}(t)$, $t \in [0, T]$, is a Gaussian process with the mean $x + h(t)$ and the covariance

$$\operatorname{Cov}(\overline{W}(s), \overline{W}(t)) = \int_0^s g^2(v)\,dv, \qquad \text{for } s \le t.$$

This is a process with independent increments, it is identical in law to the process $h(t) + W\Big(\int_0^t g^2(s)\,ds\Big)$, $W(0) = x$.

It is easy to understand (see (11.21) Ch. I) that for $h(0) = 0$ the process

$$\overline{W}^{\circ}_{x,t,z}(s) := \overline{W}(s) - \frac{\int_0^s g^2(v)\,dv}{\int_0^t g^2(v)\,dv}(\overline{W}(t) - z) \tag{2.9}$$

is the bridge from $x$ to $z$ of the process $\overline{W}$.

For every $\mu \in \mathbf{R}$, the process $\overline{W}$ with $h(t) = \mu \int_0^t g^2(s)\,ds$ has the same bridge as for $\mu = 0$.

**Exercises.**

**2.1.** Compute the conditional distribution of $\int_0^t s\,dW(s)$ given $W(t) = z$.

**2.2.** Check whether the following equalities hold true for some $\varepsilon > 0$:

1) $\quad \mathbf{E}\Big\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_{v+\varepsilon}\Big\} = 0 \qquad$ a.s.

2) $\quad \mathbf{E}\Big\{\Big(\int_v^t f(s)\,dW(s)\Big)^2\Big|\mathcal{F}_{v+\varepsilon}\Big\} = \int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_{v+\varepsilon}\}\,ds \qquad$ a.s.

3) $\quad \mathbf{E}\Big\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_{v-\varepsilon}\Big\} = 0 \qquad$ a.s.

4) $\quad \mathbf{E}\Big\{\Big(\int_v^t f(s)\,dW(s)\Big)^2\Big|\mathcal{F}_{v-\varepsilon}\Big\} = \int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_{v-\varepsilon}\}\,ds \qquad$ a.s.

**2.3.** Prove directly from the definition of the Itô integral that

$$\int_0^t s\,dW(s) = tW(t) - \int_0^t W(s)\,ds$$

(the integration by parts formula).

**2.4.** Deduce directly from the definition of the Itô integral that

$$2\int_s^t W(v)\,dW(v) = W^2(t) - W^2(s) - (t-s).$$

Hint: Use the result about the quadratic variation of the Brownian motion.

## § 3. Extension of the class of integrands

The condition that processes from $\mathcal{H}_2[0,T]$ must have a finite second moment is rather restrictive. Using an approach based on the truncation of integrands, the definition of the stochastic integral can be generalized to a class of stochastic processes broader than $\mathcal{H}_2[0,T]$.

Let $\mathcal{L}_2[0,T]$ be a *class* of progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$ stochastic processes $f(t)$, $t\in[0,T]$, satisfying the condition

$$\mathbf{P}\Big(\int_0^T f^2(s)\,ds < \infty\Big) = 1. \tag{3.1}$$

Clearly, $\mathcal{H}_2[0,T]\subset\mathcal{L}_2[0,T]$.

For simple processes from $\mathcal{L}_2[0,T]$ of the form (1.3), where it is not supposed that the second moments of $f_k$, $k=0,\dots m-1$, are finite, the stochastic integral with variable upper limit is defined by (2.7).

For further arguments we need the following estimate. For any simple process $\bar f\in\mathcal{L}_2[0,T]$ and any $C>0$, $N>0$,

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f(s)\,dW(s)\Big|\ge C\Big) \le \mathbf{P}\Big(\int_0^T \bar f^2(s)\,ds > N\Big) + \frac{N}{C^2}. \tag{3.2}$$

To prove this inequality define $f_N(t) := \bar f(t)\mathbb{1}_{[0,N]}\Big(\int_0^t \bar f^2(v)\,dv\Big)$. It is clear that the process $f_N(t)$ is progressively measurable with respect to the $\sigma$-algebras $\{\mathcal{F}_t\}$ and $\int_0^T \bar f_N^2(s)\,ds \le N$. Therefore, $f_N(t)\in\mathcal{H}_2[0,T]$ and the estimate (2.5) can be applied. Then

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f(s)\,dW(s)\Big|\ge C\Big) \le \mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t f_N(s)\,dW(s)\Big|\ge C\Big)$$

$$+\mathbf{P}\big(\bar f(t)\ne f_N(t) \text{ for some } t\in[0,T]\big) \le \frac{N}{C^2} + \mathbf{P}\Big(\int_0^T \bar f^2(s)\,ds > N\Big).$$

Here the obvious inclusion

$$\Big\{\int_0^t \bar{f}^2(s)\,ds > N \text{ for some } t\in[0,T]\Big\} \subseteq \Big\{\int_0^T \bar{f}^2(s)\,ds > N\Big\}$$

was taken into account. The inequality (3.2) is proved.

**Proposition 3.1.** *The set of simple processes is dense in the space $\mathcal{L}_2[0,T]$, i.e., for any process $f\in\mathcal{L}_2[0,T]$ there exists a sequence of simple processes $\bar{f}_n\in\mathcal{L}_2[0,T]$ such that*

$$\lim_{n\to\infty}\int_0^T (f(s)-\bar{f}_n(s))^2\,ds = 0 \qquad \text{a.s.} \tag{3.3}$$

The proof of this statement is analogous to the proof of Proposition 1.1. It is only necessary to replace the mean square convergence by the a.s. convergence.

From (3.3) it follows that

$$\int_0^T (\bar{f}_m(s)-\bar{f}_n(s))^2\,ds \to 0, \qquad \text{as} \ \ m\to\infty, \quad n\to\infty,$$

in probability. For every $m$, $n$, $\varepsilon>0$ and $\delta>0$, letting in (3.2) $C=\varepsilon$, $N=\delta\varepsilon^2$, we have

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar{f}_m(s)\,dW(s) - \int_0^t \bar{f}_n(s)\,dW(s)\Big| \ge \varepsilon\Big)$$
$$\le \mathbf{P}\Big(\int_0^T (\bar{f}_m(s)-\bar{f}_n(s))^2\,ds > \delta\varepsilon^2\Big) + \delta. \tag{3.4}$$

Letting first $m\to\infty$, $n\to\infty$, and then $\delta\to 0$, we obtain that the sequence of processes $\int_0^t \bar{f}_n(s)\,dW(s)$, $t\in[0,T]$, is Cauchy in the uniform norm $\sup_{t\in[0,T]}|\cdot|$ for the convergence in probability.

Therefore, there exists a stochastic process $I(t)$, $t\in[0,T]$, such that

$$\sup_{t\in[0,T]}\Big|I(t) - \int_0^t \bar{f}_n(s)\,dW(s)\Big| \to 0$$

in probability. We set $I(t) := \int_0^t f(s)\,dW(s)$.

Since according to Proposition 1.1 in Ch. I the convergence in probability is equivalent to a.s. convergence for some subsequences, we see that the process $I(t)$ is a.s. continuous.

Now we can prove by passage to the limit as $n\to\infty$ in (3.2), applied for the processes $\bar{f}_n$, that (3.2) is also valid for all processes $f\in\mathcal{L}_2[0,T]$.

As a result, we have the following theorem.

**Theorem 3.1.** *Let* $f \in \mathcal{L}_2[0,T]$. *Then the process* $I(t) = \int_0^t f(s)dW(s)$, $t \in [0,T]$, *is* a.s. *continuous, and for any* $C > 0$, $N > 0$,

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t f(s)\,dW(s)\Big| \ge C\Big) \le \mathbf{P}\Big(\int_0^T f^2(s)\,ds > N\Big) + \frac{N}{C^2}. \tag{3.5}$$

We conclude by pointing out an important property following from (3.5). Let

$$\lim_{n\to\infty}\int_0^T (f(s) - f_n(s))^2\,ds = 0, \quad f_n, f \in \mathcal{L}_2[0,T],$$

in probability. Then

$$\sup_{t\in[0,T]}\Big|\int_0^t f(s)\,dW(s) - \int_0^t f_n(s)\,dW(s)\Big| \to 0 \qquad \text{as} \quad n\to\infty, \tag{3.6}$$

in probability.

In addition to the stochastic integral with variable upper limit, we define an integral with a random upper limit.

Let $\rho$ be a stopping time with respect to the filtration $\{\mathcal{F}_t, t \in [0,\infty)\}$. Let $f(s)$, $s \in [0,\infty)$, be a progressively measurable stochastic process satisfying the condition

$$\mathbf{P}\Big(\int_0^\infty f^2(s)\,ds < \infty\Big) = 1. \tag{3.7}$$

Then

$$\int_0^\rho f(s)\,dW(s) := \int_0^\infty \mathbb{1}_{[0,\rho)}(s) f(s)\,dW(s). \tag{3.8}$$

Note that, by the definition of a stopping time, $\{\rho \le s\} \in \mathcal{F}_s$ for every $s$. Then $\mathbb{1}_{[0,\rho)}(s) = 1 - \mathbb{1}_{[0,s]}(\rho)$ is an $\mathcal{F}_s$-measurable right continuous process. Therefore, it is progressively measurable and the stochastic integral on the right-hand side of (3.8) is well defined. The variable $\int_0^\rho f(s)\,dW(s)$ has mean zero, if $\int_0^\infty \mathbf{E} f^2(s)\,ds < \infty$, and it is $\mathcal{F}_\rho$-measurable, because the integral as the process of the upper limit is continuous.

For finite stopping times ($\mathbf{P}(\rho < \infty) = 1$) instead of (3.7) it is enough to assume that for any $T > 0$

$$\mathbf{P}\Big(\int_0^T f^2(s)\,ds < \infty\Big) = 1, \tag{3.9}$$

since in this case

$$\mathbf{P}\Big(\int_0^\infty \mathbb{1}_{[0,\rho)}(s) f^2(s)\,ds < \infty\Big) = 1.$$

## § 4. Itô's formula

It is often of interest to study the properties of the process $f(W(t))$, $t \ge 0$, where $f$ is a given smooth function. For the investigation of such processes the technique of stochastic differentiation is very effective. Here we present some results due to K. Itô.

Let $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$ and let for all $v > t$ the increments $W(v) - W(t)$ be independent of the $\sigma$-algebra $\mathcal{F}_t$.

Let the stochastic processes $a(s)$, $b(s)$, $s \in [0,T]$, be progressively measurable with respect to the $\sigma$-algebras $\{\mathcal{F}_s\}$.

Assume that

$$\int_0^T |a(s)|\,ds < \infty, \qquad \int_0^T b^2(s)\,ds < \infty, \qquad \text{a.s.},$$

i.e., $\sqrt{|a(\cdot)|} \in \mathcal{L}_2[0,T]$, $b(\cdot) \in \mathcal{L}_2[0,T]$.

Let $X(t)$, $t \in [0,T]$, be a stochastic process such that $X(0)$ is $\mathcal{F}_0$-measurable. If

$$X(t) = X(0) + \int_0^t a(v)\,dv + \int_0^t b(v)\,dW(v) \tag{4.1}$$

holds a.s. for all $t \in [0,T]$, then we say that $X(t)$ has a *stochastic differential* of the form

$$dX(t) = a(t)\,dt + b(t)\,dW(t). \tag{4.2}$$

Formula (4.2) is the brief symbolic notation for (4.1).

**Theorem 4.1 (Itô's formula).** *Let $f(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function. Then*

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2} f''(W(t))\,dt. \tag{4.3}$$

*Proof.* According to the definition of the stochastic differential, it is sufficient to prove that for all $0 \le t \le T$

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(v))\,dW(v) + \frac{1}{2}\int_0^t f''(W(v))\,dv \qquad \text{a.s.} \tag{4.4}$$

The stochastic integral is well defined because $f'(W(\cdot)) \in \mathcal{L}_2[0,T]$. If equality (4.4) holds a.s. for a fixed $t$, then it holds a.s. for all $t \in [0,T]$, because all terms figuring in it are continuous processes.

We first assume that $f(x)$, $x \in \mathbf{R}$, is a three times continuously differentiable function with bounded derivatives $f'$, $f''$, $f'''$.

Consider an arbitrary sequence of subdivisions $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = t$ of the interval $[0,t]$, satisfying the condition

$$\lim_{n\to\infty} \max_{0\le k\le n-1} |t_{n,k+1} - t_{n,k}| = 0. \tag{4.5}$$

We use the equality

$$f(W(t)) - f(W(0)) = \sum_{k=0}^{n-1} \big(f(W(t_{n,k+1})) - f(W(t_{n,k}))\big).$$

Applying Taylor's formula to the function $f(x)$, $x \in \mathbf{R}$, we have that for every $k = 0,\dots,n-1$

$$f(W(t_{n,k+1})) - f(W(t_{n,k})) = f'(W(t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))$$

$$+\frac{1}{2} f''(W(t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))^2 + \frac{1}{6} f'''(W(\tilde{t}_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))^3,$$

where $\tilde{t}_{n,k}$ is some random point in the interval $[t_{n,k}, t_{n,k+1}]$.

By summing these expressions, we obtain

$$f(W(t)) - f(W(0)) = \sum_{k=0}^{n-1} f'(W(t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))$$

$$+\frac{1}{2}\sum_{k=0}^{n-1} f''(W(t_{n,k}))(t_{n,k+1} - t_{n,k}) + \frac{1}{6}\sum_{k=0}^{n-1} f'''(W(\tilde{t}_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))^3$$

$$+\frac{1}{2}\sum_{k=0}^{n-1} f''(W(t_{n,k}))\big[(W(t_{n,k+1}) - W(t_{n,k}))^2 - (t_{n,k+1} - t_{n,k})\big]$$

$$=: I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}. \tag{4.6}$$

Since

$$t_n(v) := \sum_{k=0}^{n-1} t_{n,k} \mathbb{1}_{[t_{n,k}, t_{n,k+1})}(v) \to v \qquad \text{as } n \to \infty$$

uniformly in $v \in [0,t]$, using the continuity of Brownian motion paths and of $f'$, we get

$$\int_0^t \big(f'(W(v)) - f'(W(t_n(v)))\big)^2 dv \to 0 \qquad \text{as } n \to \infty, \qquad \text{a.s.}$$

By the definition of the stochastic integral,

$$I_{n,1} = \int_0^t f'(W(t_n(v)))\,dW(v) \to \int_0^t f'(W(v))\,dW(v) \qquad \text{as } n \to \infty, \tag{4.7}$$

in probability.

Since $f''$ is continuous,

$$I_{n,2} = \frac{1}{2}\int_0^t f''(W(t_n(v)))\,dv \to \frac{1}{2}\int_0^t f''(W(v))\,dv \qquad \text{as } n \to \infty, \qquad \text{a.s.} \tag{4.8}$$

Using the assumption that $|f'''(x)| \le C$ for all $x \in \mathbf{R}$, we obtain

$$|I_{n,3}| \le \frac{C}{6}\sum_{k=0}^{n-1} |W(t_{n,k+1}) - W(t_{n,k})|^3$$

$$\le \frac{C}{6}\max_{0\le k\le n-1} |W(t_{n,k+1}) - W(t_{n,k})| \sum_{k=0}^{n-1} |W(t_{n,k+1}) - W(t_{n,k})|^2.$$

By the continuity of Brownian motion paths and condition (4.5) on the sequence of subdivisions of $\{t_{n,k}\}$, we have

$$\max_{0\le k\le n-1} |W(t_{n,k+1}) - W(t_{n,k})| \to 0 \qquad \text{a.s.}$$

Since the Brownian motion $W$ has the finite quadratic variation (see (10.23) Ch. I),

$$\sum_{k=0}^{n-1} |W(t_{n,k+1}) - W(t_{n,k})|^2 \to t \qquad \text{as } n \to \infty,$$

in mean square. Therefore, $I_{n,3} \to 0$ in probability.

To prove the convergence $I_{n,4} \to 0$ in probability we estimate $\mathbf{E}I_{n,4}^2$:

$$\mathbf{E}I_{n,4}^2 \le \frac{1}{4}\sum_{k=0}^{n-1} \mathbf{E}\big\{(f''(W(t_{n,k})))^2\big[(W(t_{n,k+1}) - W(t_{n,k}))^2 - (t_{n,k+1} - t_{n,k})\big]^2\big\}$$

$$+\frac{1}{2}\sum_{0\le k<l\le n-1} \mathbf{E}\big\{f''(W(t_{n,k}))\big[(W(t_{n,k+1}) - W(t_{n,k}))^2 - (t_{n,k+1} - t_{n,k})\big] f''(W(t_{n,l}))$$

$$\times\big[(W(t_{n,l+1}) - W(t_{n,l}))^2 - (t_{n,l+1} - t_{n,l})\big]\big\}. \tag{4.9}$$

For $k < l$ the random variables

$$f''(W(t_{n,k}))\big[(W(t_{n,k+1}) - W(t_{n,k})^2 - (t_{n,k+1} - t_{n,k})\big] f''(W(t_{n,l})) \tag{4.10}$$

are $\mathcal{F}_{t_{n,l}}$-measurable and the increments $W(t_{n,l+1}) - W(t_{n,l})$ are independent of $\mathcal{F}_{t_{n,l}}$. Therefore, the expectation after the sign of the double sum is equal to the product of the expectations of the random variables (4.10), and the expectation

$$\mathbf{E}\{(W(t_{n,l+1}) - W(t_{n,l}))^2 - (t_{n,l+1} - t_{n,l})\} = 0.$$

Thus the second sum on the right-hand side of (4.9) equals zero. Since $|f''(x)| \le C$, $x \in \mathbf{R}$, we obtain

$$\mathbf{E}I_{n,4}^2 \le \frac{C^2}{4} \sum_{k=0}^{n-1} \mathbf{E}\big[(W(t_{n,k+1}) - W(t_{n,k}))^2 - (t_{n,k+1} - t_{n,k})\big]^2$$

$$= \frac{C^2}{4} \sum_{k=0}^{n-1} \mathrm{Var}\{(W(t_{n,k+1}) - W(t_{n,k}))^2\} \le \frac{C^2}{2} \max_{0\le k\le n-1} |t_{n,k+1} - t_{n,k}|\, t.$$

Here we used the estimate (10.24) Ch. I. Using condition (4.5), we finally have

$$I_{n,4} \to 0 \tag{4.11}$$

in mean square and, consequently, in probability.

From (4.6), using the limits (4.7), (4.8) and the convergence of the random variables $I_{n,3}$, $I_{n,4}$ to zero in probability, we get (4.4).

The convergence (4.11) plays a very important role in the whole theory of stochastic differentiation, because it enables us to replace the second-order increments $(W(t_{n,k+1})-W(t_{n,k}))^2$ by the first-order ones $t_{n,k+1}-t_{n,k}$, when applying Taylor's formula. In the limiting case this can be expressed as follows: the square of the differential of the Brownian motion $\big((dW(t))^2\big)$ coincides with $dt$, i.e., $(dW(t))^2 = dt$.

To prove (4.4) without the assumption that the derivatives $f'$, $f''$, and $f'''$ are bounded, we can use the approximation of $f$ by a sequence of functions with bounded derivatives up to the third order.

We first prove (4.4) for a twice continuously differentiable function $f$ with bounded support. Set

$$\hat{f}_n(x) = n \int_{x-1/n}^{x} f(v)\, dv, \qquad n = 1, 2, \dots.$$

These functions are three times continuously differentiable. They have bounded support and bounded third derivative. The first and the second derivatives are uniformly bounded and

$$\hat{f}_n(x) \to f(x), \quad \hat{f}'_n(x) \to f'(x), \quad \hat{f}''_n(x) \to f''(x), \qquad \text{as } n \to \infty$$

uniformly in $x \in \mathbf{R}$.

Indeed, by the mean value theorem for integrals, we have $\hat{f}_n(x) = f(x_n)$,

$$\hat{f}'_n(x) = \frac{f(x) - f(x-1/n)}{1/n} = f'(\tilde{x}_n), \quad \hat{f}''_n(x) = \frac{f'(x) - f'(x-1/n)}{1/n} = f''(\hat{x}_n),$$

where $x_n$, $\tilde{x}_n$, $\hat{x}_n$, are some points from the interval $[x, x-1/n]$. Using the fact that $f$ and its derivatives $f'$, $f''$ are uniformly continuous because they have bounded support, we obtain the desired approximation.

For the functions $\hat{f}_n(x)$ equality (4.4) holds. Now, taking into account (3.6) and the continuity of the Brownian motion, we can pass to the limit in (4.4) for the functions $\hat{f}_n(x)$. This proves (4.4) for twice continuously differentiable functions $f$ with bounded support.

As the second step we approximate a twice continuously differentiable function $f$ by the functions

$$f_n(x) = f(x)\mathbb{1}_{[-n,n]}(x) + g_n(x)\mathbb{1}_{(n,n+1]}(x) + g_n(x)\mathbb{1}_{[-n-1,-n)}(x)$$

with bounded support. Here the functions $g_n(x)$ are such that $f_n(x)$, $x \in \mathbf{R}$, is twice continuously differentiable function for every $n$.

From (2.5) for $f \equiv 1$ it follows that

$$\mathbf{P}\Big(\sup_{0\le t\le T} |W(t)| \ge n\Big) \le \frac{T}{n^2}. \tag{4.12}$$

Then for any $\varepsilon > 0$

$$\mathbf{P}\Big(\int_0^T (f'(W(v)) - f'_n(W(v)))^2\, dv > \varepsilon\Big) \le \mathbf{P}\Big(\sup_{0\le t\le T} |W(t)| \ge n\Big) \le \frac{T}{n^2} \to 0 \tag{4.13}$$

as $n \to \infty$. Similarly,

$$\mathbf{P}\Big(\int_0^T |f''(W(v)) - f''_n(W(v))|\, dv > \varepsilon\Big) \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.14}$$

Taking into account these estimates and (3.6), we can pass to the limit in (4.4) for functions $f_n$. Thus (4.4) holds for twice continuously differentiable functions and this completes the proof. □

**Remark 4.1.** The main feature of Itô's formula is that the second derivative appears in the expression for the first differential. This is impossible in the standard analysis. In stochastic analysis it is the consequence of the properties of Brownian motion.

The analog of (4.4) holds even if the function $f$ has no second derivative.

**Theorem 4.2.** *Let $f(x)$, $x \in \mathbf{R}$, be a differentiable function, whose first derivative has the form*

$$f'(x) = f'(0) + \int_0^x g(y)dy, \tag{4.15}$$

*where $g(x)$, $x \in \mathbf{R}$, is a measurable function bounded on any finite interval. Then* a.s. *for all* $0 \le t \le T$

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(v))\,dW(v) + \frac{1}{2}\int_0^t g(W(v))\,dv. \tag{4.16}$$

*Proof.* It is sufficient to prove (4.16) for a function $g$ with bounded support. Otherwise $g(x)$, $x \in \mathbf{R}$, can be approximated by the functions $g_n(x) = g(x)\mathbb{I}_{[-n,n]}(x)$ and we can apply the arguments used before in (4.12)–(4.14) for the proof of Theorem 4.1.

Assume that $\{x : g(x) \ne 0\} \subseteq [a, b]$ for some $a < b$. Set

$$\hat{f}_n(x) := n \int_{x-1/n}^{x} f(y)\,dy, \qquad n = 1, 2, \dots.$$

These are the twice continuously differentiable functions and

$$\hat{f}_n(x) \to f(x), \quad \hat{f}'_n(x) \to f'(x)$$

uniformly in $x \in \mathbf{R}$. Moreover,

$$\hat{f}''_n(x) = n \int_{x-1/n}^{x} g(y)\,dy \to g(x)$$

for almost all $x$.

Then

$$\int_0^T \mathbf{E}|g(W(s)) - \hat{f}''_n(W(s))|\,ds \le \int_0^T \int_a^b |g(x) - \hat{f}''_n(x)| \frac{e^{-(x-x_0)^2/2s}}{\sqrt{2\pi s}}\,dx ds \to 0$$

as $n \to \infty$, where $W(0) = x_0$.

For the functions $\hat{f}_n(x)$ equality (4.4) holds and we can pass to the limit. This proves the theorem. □

Further, we derive Itô's formula for the case when $f$ depends also on the time parameter $t$.

**Theorem 4.3.** *Let $f(t, x)$, $(t, x) \in [0, T] \times \mathbf{R}$, be a continuous function with continuous partial derivatives $\frac{\partial}{\partial t} f(t, x)$, $\frac{\partial}{\partial x} f(t, x)$ and with continuous partial derivatives $\frac{\partial^2}{\partial x^2} f(t, x)$ for $x \ne x_k$, where $\min_{k\in\mathbb{Z}}(x_{k+1} - x_k) \ge \delta > 0$ for some $\delta > 0$. Assume that at the points $x_k$ the second order partial derivatives have left and right limits uniformly bounded in $[0, T]$. Then*

$$df(t, W(t)) = \frac{\partial}{\partial t} f(t, W(t))\,dt + \frac{\partial}{\partial x} f(t, W(t))\,dW(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, W(t))\,dt, \tag{4.17}$$

*where at the points $x_k$ the second partial derivatives are treated as the left limits of the corresponding derivatives.*

*Proof.* According to the definition of a stochastic differential it is sufficient to prove that for all $0 \le t \le T$,

$$f(t, W(t)) - f(0, W(0)) = \int_0^t \frac{\partial}{\partial v} f(v, W(v))\, dv$$

$$+ \int_0^t \frac{\partial}{\partial x} f(v, W(v))\, dW(v) + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(v, W(v))\, dv. \tag{4.18}$$

We first prove (4.18) for the case when $f(t, x) = \sigma(t)g(x)$ and there exists the continuous derivatives $\sigma'$ and $g'$. Moreover, we assume that exists the continuous derivative $g''(x)$ for $x \in \mathbf{R} \setminus \{x_k\}_{k \in \mathbb{Z}}$ with bounded left and right limits at the points $x_k$. Using subdivisions of the interval $[0, t]$, as in the proof of Theorem 4.2, we can write

$$\sigma(t)g(W(t)) - \sigma(0)g(W(0)) = \sum_{k=0}^{n-1} \big(\sigma(t_{n,k+1})g(W(t_{n,k+1})) - \sigma(t_{n,k})g(W(t_{n,k}))\big)$$

$$= \sum_{k=0}^{n-1} g(W(t_{n,k+1}))(\sigma(t_{n,k+1}) - \sigma(t_{n,k})) + \sum_{k=0}^{n-1} \sigma(t_{n,k})(g(W(t_{n,k+1})) - g(W(t_{n,k}))). \tag{4.19}$$

By Theorem 4.2,

$$g(W(t_{n,k+1})) - g(W(t_{n,k})) = \int_{t_{n,k}}^{t_{n,k+1}} g'(W(v))\, dW(v) + \frac{1}{2} \int_{t_{n,k}}^{t_{n,k+1}} g''(W(v))\, dv.$$

Set

$$t_n^+(v) := \sum_{k=0}^{n-1} t_{n,k+1} \mathbb{1}_{[t_{n,k}, t_{n,k+1})}(v).$$

Then using the representation

$$\sigma(t_{n,k+1}) - \sigma(t_{n,k}) = \int_{t_{n,k}}^{t_{n,k+1}} \sigma'(v)\, dv$$

and the notation $t_n(v)$ introduced in the proof of Theorem 4.1, one can write (4.19) in the form

$$\sigma(t)g(W(t)) - \sigma(0)g(W(0)) = \int_0^t \sigma'(v) g(W(t_n^+(v))\, dv$$

$$+\int_0^t \sigma(t_n(v))g'(W(v))\,dW(v) + \frac{1}{2}\int_0^t \sigma(t_n(v))g''(W(v))\,dv. \tag{4.20}$$

Since $t_n(v) \to v$ and $t_n^+(v) \to v$ uniformly in $v \in [0,t]$, the passage to the limit in (4.20) proves (4.18) for the special case $f(t,x) = \sigma(t)g(x)$. Here to justify the passage to the limit for the stochastic integral we can apply (3.6).

It is clear that (4.18) is valid for the functions

$$f_n(t,x) := \sum_{k=0}^{n} \sigma_{n,k}(t) g_{n,k}(x), \tag{4.21}$$

where the functions $g_{n,k}$ have the same properties as the function $g$ above.

For an arbitrary smooth function $f(t,x)$ there exists a sequence of functions $f_n(t,x)$, of the form (4.21), such that for any $N > 0$

$$\lim_{n\to 0} \sup_{0\le t\le T} \sup_{|x|\le N} \Bigl(|f(t,x) - f_n(t,x)| + \Bigl|\frac{\partial}{\partial t} f(t,x) - \frac{\partial}{\partial t} f_n(t,x)\Bigr|\Bigr) = 0,$$

$$\lim_{n\to 0} \sup_{0\le t\le T} \sup_{|x|\le N} \Bigl|\frac{\partial}{\partial x} f(t,x) - \frac{\partial}{\partial x} f_n(t,x)\Bigr| = 0,$$

$$\lim_{n\to 0} \sup_{0\le t\le T} \sup_{|x|\le N, x\notin D} \Bigl|\frac{\partial^2}{\partial x^2} f(t,x) - \frac{\partial^2}{\partial x^2} f_n(t,x)\Bigr| = 0,$$

where $D := \{x_k\}_{k\in\mathbb{Z}}$. Using arguments similar to those stated in (4.12)–(4.14), it is not difficult to complete the proof of the theorem for the general case. □

We now consider the general form of the Itô formula for twice continuously differentiable functions of several arguments.

**Theorem 4.4.** *Let $f(t,\vec{x})$, $(t,\vec{x}) \in [0,T]\times \mathbf{R}^d$, be a continuous function with continuous partial derivatives $\frac{\partial}{\partial t} f(t,\vec{x})$, $\frac{\partial}{\partial x_i} f(t,\vec{x})$, $\frac{\partial^2}{\partial x_i \partial x_j} f(t,\vec{x})$, $i,j = 1,\dots,d$.*

*Suppose that the coordinates of the vector process $\vec{X}(t)$, $x \in [0,T]$, have the stochastic differentials*

$$dX_i(t) = a_i(t)\,dt + b_i(t)\,dW(t), \qquad i = 1,\dots,d,$$

*where the functions $a_i(t)$ and $b_i(t)$, $t\in[0,T]$, are right continuous and have left limits.*

*Then the process $f(t,\vec{X}(t))$, $x\in[0,T]$, has the stochastic differential given by*

$$df(t,\vec{X}(t)) = \frac{\partial}{\partial t} f(t,\vec{X}(t))\,dt + \sum_{i=1}^{d} a_i(t)\frac{\partial}{\partial x_i} f(t,\vec{X}(t))\,dt$$

$$+\sum_{i=1}^{d} b_i(t)\frac{\partial}{\partial x_i} f(t,\vec{X}(t))\,dW(t) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} b_i(t)b_j(t)\frac{\partial^2}{\partial x_i\partial x_j} f(t,\vec{X}(t))\,dt. \tag{4.22}$$

**Remark 4.2.** One can prove (4.22) under the assumption that the second-order partial derivatives $\frac{\partial^2}{\partial x_i \partial x_j} f(t, \vec{x})$, $i, j = 1, \dots, d$, do not exist at vector points $\vec{x}_k$, $k \in \mathbb{Z}$, with coordinates satisfying for some $\delta > 0$ the inequality

$$\min_{1 \le i \le d, k \in \mathbb{Z}} (x_{i,k+1} - x_{i,k}) \ge \delta > 0.$$

*Proof of Theorem* 4.4. According to the definition of the stochastic differential, it is sufficient to prove that a.s. for all $0 \le t \le T$

$$f(t, \vec{X}(t)) - f(0, \vec{X}(0)) = \int_0^t \frac{\partial}{\partial v} f(v, \vec{X}(v))\, dv + \sum_{i=1}^d \int_0^t a_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v))\, dv$$

$$+ \sum_{i=1}^d \int_0^t b_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v))\, dW(v) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t b_i(v) b_j(v) \frac{\partial^2}{\partial x_i \partial x_j} f(v, \vec{X}(v))\, dv. \tag{4.23}$$

We prove first (4.23) in the case when the processes $a_i$ and $b_i$, $i = 1, \dots, d$, are simple. Without loss of generality we can assume that the intervals of constancy are the same for all processes $a_i$, $b_i$, i.e.,

$$a_i(s) = \sum_{k=0}^{m-1} a_{i,k} \mathbb{1}_{[s_k, s_{k+1})}(s), \qquad b_i(s) = \sum_{k=0}^{m-1} b_{i,k} \mathbb{1}_{[s_k, s_{k+1})}(s), \qquad i = 1, \dots, d,$$

where $0 = s_0 < s_1 < \cdots < s_l < \cdots < s_m = T$, and the random variables $a_{i,k}$, $b_{i,k}$ are $\mathcal{F}_{s_k}$-measurable, $k = 0, \dots m-1$, $i = 1, \dots, d$.

In this case the coordinate process $X_i$ for $v \in [s_k, s_{k+1})$ has the form

$$X_i(v) = X_i(s_k) + a_{i,k}(v - s_k) + b_{i,k}(W(v) - W(s_k)), \qquad i = 1, \dots, d.$$

Set for $v \in [s_k, s_{k+1})$

$$g(v, x) := f(v, \vec{X}(s_k) + \vec{a}_k(v - s_k) + \vec{b}_k(x - W(s_k))),$$

where $\vec{a}_k = (a_{1,k}, \dots, a_{d,k})$, $\vec{b}_k = (b_{1,k}, \dots, b_{d,k})$.

We can apply Theorem 4.3, although in the definition of the function $g$ we have the random variables $X_i(s_k)$, $W(s_k)$, $a_{i,k}$, and $b_{i,k}$ (however, it is important that these random variables are $\mathcal{F}_{s_k}$-measurable). Since

$$\frac{\partial}{\partial v} g = \frac{\partial}{\partial v} f + \sum_{i=1}^d a_{i,k} \frac{\partial}{\partial x_i} f, \quad \frac{\partial}{\partial x} g = \sum_{i=1}^d b_{i,k} \frac{\partial}{\partial x_i} f, \quad \frac{\partial^2}{\partial x^2} g = \sum_{i=1}^d \sum_{j=1}^d b_{i,k} b_{j,k} \frac{\partial^2}{\partial x_i \partial x_j} f$$

for $v \in [s_k, s_{k+1})$, using (4.18) we have

$$f(s_{k+1}, \vec{X}(s_{k+1})) - f(s_k, \vec{X}(s_k)) = g(s_{k+1}, W(s_{k+1})) - g(s_k, W(s_k))$$

$$= \int_{s_k}^{s_{k+1}} \frac{\partial}{\partial v} g(v, W(v))\, dv + \int_{s_k}^{s_{k+1}} \frac{\partial}{\partial x} g(v, W(v))\, dW(v) + \frac{1}{2} \int_{s_k}^{s_{k+1}} \frac{\partial^2}{\partial x^2} g(v, W(v))\, dv$$

$$= \int_{s_k}^{s_{k+1}} \frac{\partial}{\partial v} f(v, \vec{X}(v)) dv + \int_{s_k}^{s_{k+1}} \sum_{i=1}^{d} a_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v))\, dv$$

$$+ \int_{s_k}^{s_{k+1}} \sum_{i=1}^{d} b_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v)) dW(v) + \frac{1}{2} \int_{s_k}^{s_{k+1}} \sum_{i=1}^{d} \sum_{i=j}^{d} b_i(v) b_j(v) \frac{\partial^2}{\partial x_i \partial x_j} f(v, \vec{X}(v)) dv.$$

If $t \in [s_l, s_{l+1})$ for some $l$, then summing these equalities for $k = 0, \dots, l-1$, and adding the analogous equality for the interval $[s_l, t)$, we obtain (4.23) in the case when $a_i$ and $b_i$, $i = 1, \dots, d$, are simple processes.

In the general case we can approximate $X_i$, $i = 1, \dots, d$, by the processes

$$X_{i,n}(t) = X_i(0) + \int_0^t a_{i,n}(v)\, dv + \int_0^t b_{i,n}(v)\, dW(v),$$

where the simple processes $a_{i,n}$ and $b_{i,n}$ are such that

$$\int_0^t |a_i(v) - a_{i,n}(v)|\, dv \to 0, \qquad \int_0^t (b_i(v) - b_{i,n}(v))^2\, dv \to 0, \qquad \text{as } n \to \infty \qquad \text{a.s.}$$

Passage to the limit as $n \to \infty$ in (4.23), done for $\vec{X}_n(t) = (X_{1,n}(t), \dots, X_{d,n}(t))$, completes the proof. $\square$

Notice that for $b_i(t) \equiv 0$, $t \in [0, T]$, $i = 1, \dots, d$, formula (4.22) turns into the classical formula of differentiation of composition of functions. However, in the case when the stochastic differential is included, the second derivatives of functions with respect to the spatial variables play an important role. This is due to the fact that when computing the principal values of the increments of functions of stochastic processes one can use Taylor's formula. Thus, when considering the squares of stochastic differentials, the term $(dW(t))^2$ has, in fact, the first order equal to $dt$.

We now give an informal description of the generalized Itô's formula, using the following rule:
the differential of function of several stochastic processes is computed by applying Taylor's formula, where one sets $(dt)^2 = 0$, $dt\, dW(t) = 0$, $(dW(t))^2 = dt$, and the differentials of higher orders must be equal to zero.

To illustrate this rule, consider a function with two spatial variables. Let $f(t, x, y)$, $t \in [0, \infty)$, $x, y \in \mathbf{R}$, be a continuous function with continuous partial derivatives $f'_t$, $f'_x$, $f'_y$, $f''_{x,x}$, $f''_{x,y}$, and $f''_{y,y}$.

Suppose that the processes $X$ and $Y$ have the stochastic differentials

$$dX(t) = a(t)\,dt + b(t)\,dW(t), \qquad\qquad dY(t) = c(t)\,dt + q(t)\,dW(t).$$

Then according to the rule stated above,

$$(dX(t))^2 = (a(t))^2(dt)^2 + 2a(t)b(t)\,dt\,dW(t) + (b(t))^2(dW(t))^2 = b^2(t)\,dt.$$

Similarly, $(dY(t))^2 = q^2(t)\,dt$ , $dX(t)dY(t) = b(t)q(t)\,dt$. It is clear that the differentials of higher orders of the processes $X$, $Y$ are equal to zero.

Applying Taylor's formula, we obtain

$$\begin{aligned} df(t, X(t), Y(t)) &= f'_t(t, X(t), Y(t))\,dt + f'_x(t, X(t), Y(t))\,dX(t) \\ &\quad + f'_y(t, X(t), Y(t))\,dY(t) + \frac{1}{2} f''_{x,x}(t, X(t), Y(t))(dX(t))^2 \\ &\quad + f''_{x,y}(t, X(t), Y(t))dX(t)dY(t) + \frac{1}{2} f''_{y,y}(t, X(t), Y(t))(dY(t))^2. \end{aligned}$$

Therefore,

$$\begin{aligned} df(t, X(t), Y(t)) &= f'_t(t, X(t), Y(t))\,dt + f'_x(t, X(t), Y(t))\{a(t)\,dt + b(t)\,dW(t)\} \\ &\quad + f'_y(t, X(t), Y(t))\{c(t)\,dt + q(t)\,dW(t)\} + \frac{1}{2} f''_{x,x}(t, X(t), Y(t))b^2(t)\,dt \\ &\quad + f''_{x,y}(t, X(t), Y(t))b(t)q(t)dt + \frac{1}{2} f''_{y,y}(t, X(t), Y(t))q^2(t)\,dt. \end{aligned} \tag{4.24}$$

**Remark 4.3.** One can consider independent Brownian motions $W_1(t)$ and $W_2(t)$, $t \geq 0$. Suppose that the processes $X$ the $Y$ have the stochastic differentials

$$dX(t) = a(t)\,dt + b(t)\,dW_1(t), \qquad\qquad dY(t) = c(t)\,dt + q(t)\,dW_2(t).$$

In this case $dX(t)dY(t) = 0$, since one must set $dW_1(t)dW_2(t) = 0$. This is a consequence of the fact that for any $s < t$

$$\mathbf{E}\{(W_1(t) - W_1(s))(W_2(t) - W_2(s))\} = \mathbf{E}(W_1(t) - W_1(s))\mathbf{E}(W_2(t) - W_2(s)) = 0.$$

This feature must be taken into account when applying Taylor's formula for computing the differential $df(t, X(t), Y(t))$.

As an application of Theorem 4.4, we derive the *Burkholder–Davis–Gundy inequality* for stochastic integrals.

**Lemma 4.1.** *Let $h(v)$, $v \in [s, t]$, be a progressively measurable process. Then for $k = 1, 2, \ldots$ the inequality*

$$\mathbf{E} \sup_{s \leq u \leq t} \biggl( \int_s^u h(v)\,dW(v) \biggr)^{2k} \leq 2^k k^{2k} \Bigl(\frac{2k}{2k-1}\Bigr)^{(2k-1)k} \mathbf{E}\biggl( \int_s^t h^2(v)\,dv \biggr)^k \tag{4.25}$$

*holds.*

*Proof.* Set

$$Z(u) := \int_s^u h(v)\,dW(v), \qquad s \le u \le t,$$

and $\tau_N := \inf\{u \ge s : |Z(u)| = N\}$, assuming $\tau_N = t$ for $\sup\limits_{s \le u \le t} |Z(u)| < N$. Then $\{\tau_N \ge v\} = \Big\{ \sup\limits_{s \le u \le v} |Z(u)| \le N \Big\} \in \mathcal{F}_v$ for every $v \in [0, t]$.

For a fixed $s$ the process

$$Z(u \wedge \tau_N) = \int_s^u 1\!\!\mathrm{I}_{\{v \le \tau_N\}} h(v)\,dW(v), \qquad s \le u \le t,$$

is a martingale with respect to the family of $\sigma$-algebras $\{\mathcal{F}_u\}$. By Doob's inequality for martingales (see (5.12) Ch. I),

$$\mathbf{E} \sup_{s \le u \le t \wedge \tau_N} Z^{2k}(u) = \mathbf{E} \sup_{s \le u \le t} Z^{2k}(u \wedge \tau_N) \le \Big(\frac{2k}{2k-1}\Big)^{2k} \mathbf{E} Z^{2k}(t \wedge \tau_N). \tag{4.26}$$

Applying to the process $Z^{2k}(t)$ Itô's formula and substituting $t \wedge \tau_N$ instead of $t$, we have

$$Z^{2k}(t \wedge \tau_N) = 2k \int_s^t 1\!\!\mathrm{I}_{\{v \le \tau_N\}} Z^{2k-1}(v) h(v)\,dW(v) + k(2k-1) \int_s^{t \wedge \tau_N} Z^{2k-2}(v) h^2(v)\,dv.$$

Since the expectation of the stochastic integral is zero,

$$\mathbf{E} Z^{2k}(t \wedge \tau_N) = k(2k-1)\mathbf{E}\Big( \int_s^{t \wedge \tau_N} Z^{2k-2}(v) h^2(v)\,dv \Big).$$

Next applying Hölder's inequality, we obtain

$$\mathbf{E} Z^{2k}(t \wedge \tau_N) \le k(2k-1)\mathbf{E}\Big( \sup_{s \le u \le t \wedge \tau_N} Z^{2k-2}(u) \int_s^t h^2(v))\,dv \Big)$$

$$\le k(2k-1)\mathbf{E}^{(k-1)/k}\Big( \sup_{s \le u \le t \wedge \tau_N} Z^{2k-2}(u) \Big)^{k/(k-1)} \mathbf{E}^{1/k}\Big( \int_s^t h^2(v))\,dv \Big)^k.$$

In view of (4.26), this yields

$$\mathbf{E} \sup_{s \le u \le t \wedge \tau_N} Z^{2k}(u) \le 2k^2 \Big(\frac{2k}{2k-1}\Big)^{2k-1} \mathbf{E}^{(k-1)/k} \sup_{s \le u \le t \wedge \tau_N} Z^{2k}(u)\, \mathbf{E}^{1/k}\Big( \int_s^t h^2(v))\,dv \Big)^k,$$

or

$$\mathbf{E}^{1/k}\sup_{s\le u\le t\wedge\tau_N} Z^{2k}(u)\le 2k^2\Bigl(\frac{2k}{2k-1}\Bigr)^{2k-1}\mathbf{E}^{1/k}\Bigl(\int_s^t h^2(v))\,dv\Bigr)^k.$$

By raising both sides of this inequality to the power $k$, letting $N\to\infty$ and applying Fatou's lemma (see (5.18) Ch. I), we get (4.25). □

As it was noticed by R. L. Stratonovich (1966), for special integrands it is possible to define a stochastic integral different from Itô's integral.

**Example 4.1.** Let $f(x)$, $x\in\mathbf{R}$, be a continuously differentiable function. Let $0=t_{n,0}<t_{n,1}<\cdots<t_{n,n}=T$ be an arbitrary sequence of subdivisions of the interval $[0,T]$, satisfying (4.5). Then the limits in probability

$$\int_0^T f(W(t))\diamond dW(t):=\lim_{n\to\infty}\sum_{k=0}^{n-1} f(W(t_{n,k+1}))(W(t_{n,k+1})-W(t_{n,k})), \tag{4.27}$$

$$\int_0^T f(W(t))\circ dW(t):=\lim_{n\to\infty}\sum_{k=0}^{n-1} f\Bigl(W\Bigl(\frac{t_{n,k}+t_{n,k+1}}{2}\Bigr)\Bigr)(W(t_{n,k+1})-W(t_{n,k})) \tag{4.28}$$

exist.

The existence is due to (4.11). Indeed, assuming that $f$ is a twice continuously differentiable function with bounded second derivative $f''$ and applying Taylor's formula, we have

$$\sum_{k=0}^{n-1} f(W(t_{n,k+1}))(W(t_{n,k+1})-W(t_{n,k}))=\sum_{k=0}^{n-1} f(W(t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))$$
$$+\sum_{k=0}^{n-1} f'(W(t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))^2+\frac12\sum_{k=0}^{n-1} f''(W(\tilde t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))^3.$$

The last sum tends to zero in probability analogously to $I_{n,3}$. In view of (4.11), the second sum on the right-hand side of this equality tends to $\int_0^T f'(W(t))\,dt$. We conclude that the limit (4.27) exists and

$$\int_0^T f(W(t))\diamond dW(t)=\int_0^T f(W(t))\,dW(t)+\int_0^T f'(W(t))\,dt. \tag{4.29}$$

Analogously,

$$\sum_{k=0}^{n-1} f\Bigl(W\Bigl(\frac{t_{n,k}+t_{n,k+1}}{2}\Bigr)\Bigr)(W(t_{n,k+1})-W(t_{n,k}))=\sum_{k=0}^{n-1} f(W(t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))$$
$$+\sum_{k=0}^{n-1} f'(W(t_{n,k}))\Bigl(W\Bigl(\frac{t_{n,k}+t_{n,k+1}}{2}\Bigr)-W(t_{n,k})\Bigr)(W(t_{n,k+1})-W(t_{n,k}))$$
$$+\frac12\sum_{k=0}^{n-1} f''(W(\tilde t_{n,k}))\Bigl(W\Bigl(\frac{t_{n,k}+t_{n,k+1}}{2}\Bigr)-W(t_{n,k})\Bigr)^2(W(t_{n,k+1})-W(t_{n,k})).$$

The main quantities on the right-hand side of this equality are the first sum, which tends to Itô's integral, and the term

$$\sum_{k=0}^{n-1} f'(W(t_{n,k}))\Big(W\Big(\frac{t_{n,k}+t_{n,k+1}}{2}\Big) - W(t_{n,k})\Big)^2,$$

which tends to $\frac{1}{2}\int_0^T f'(W(t))\,dt$. Therefore,

$$\int_0^T f(W(t)) \circ dW(t) = \int_0^T f(W(t))\,dW(t) + \frac{1}{2}\int_0^T f'(W(t))\,dt. \tag{4.30}$$

**Exercises.**

**4.1.** Use Itô's formula to prove that for a Brownian motion $W$ with $W(0) = 0$,

$$\int_0^t W^4(s)\,dW(s) = \frac{1}{5}W^5(t) - 2\int_0^t W^4(s)\,ds.$$

**4.2.** Use Itô's formula to compute the differentials:

1) $d\Big(W^3(t) - \frac{t^2}{2} + \int_0^t W^2(s)\,dW(s)\Big)$;

2) $d\big(W(t)\,\mathrm{sh}\,W(t)\big)$, where $\mathrm{sh}\,x := \dfrac{e^x - e^{-x}}{2}$;

3) $d\exp\big(W^2(t) + W^3(t)\big)$.

**4.3.** Prove that the process $e^{t/2}\cos W(t)$, $t \ge 0$, is a martingale.

**4.4.** Use Itô's formula to compute the differentials:

1) $d\exp\Big(W^5(t) + \int_0^t W^4(s)\,dW(s)\Big)$;

2) $d\big(W^3(t)\exp\big(W^2(t)\big)\big)$.

**4.5.** Suppose that the process $V$ has the differential

$$dV(t) = \mu V(t)\,dt + \sigma V(t)\,dW(t), \qquad V(0) = x > 0.$$

Write out $\ln V(t)$.

**4.6.** Suppose that the process $Z$ has the differential

$$dZ(t) = \big(n\sigma^2 - 2\gamma Z(t)\big)dt + 2\sigma\sqrt{Z(t)}\,dW(t).$$

Compute $d\sqrt{Z(t)}$.

**4.7.** Prove that the following stochastic processes are martingales:

1) $\big(c + \frac{1}{3}W(t)\big)^3 - \frac{1}{3}\int\limits_0^t \big(c + \frac{1}{3}W(s)\big)\,ds$ for any $c \in \mathbf{R}$;

2) $(W(t)+t)\exp\big(-W(t) - \frac{1}{2}t\big)$.

## §5. Brownian local time. Tanaka's formula

Let $X(t)$, $t \in [0,T]$, be a progressively measurable with respect to a filtration $\{\mathcal{F}_t\}$ stochastic process. The *occupation measure* of the process $X$ up to the time $t$ is the measure $\mu_t$ defined by

$$\mu_t(\Delta) := \int\limits_0^t \mathbb{1}_\Delta(X(s))ds, \qquad \Delta \in \mathcal{B}(\mathbf{R}), \qquad 0 \le t \le T, \tag{5.1}$$

where $\mathbb{1}_\Delta(\cdot)$ is the indicator function.

In other words, $\mu_t(\Delta)$ is equal to the Lebesgue measure (mes) of the time spent by a sample path of the process $X$ in the set $\Delta$ up to the time $t$ ($\mu_t(\Delta) = \text{mes}\{s : X(s) \in \Delta,\ s \in [0,t]\}$). This is a random measure that depends on the path of the process.

If a.s. for every $t$ the measure $\mu_t$ has a density, i.e., there exists a nonnegative random function $\ell(t,x)$ such that

$$\mu_t(\Delta) = \int\limits_\Delta \ell(t,x)\,dx \tag{5.2}$$

for any Borel set $\Delta$, then the density $\ell(t,x)$ is called the *local time* of the process $X$ at the level $x$ up to the time $t$.

In the special case when in (5.2) the process $\ell(t,x)$ is continuous in $x$, one has the following equivalent definition: if a.s. for all $(t,x) \in [0,T] \times \mathbf{R}$ there exists the limit

$$\ell(t,x) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\delta+\varepsilon}\int\limits_0^t \mathbb{1}_{[x-\delta,x+\varepsilon)}(X(s))ds \qquad \text{a.s.}, \tag{5.3}$$

then $\ell(t,x)$ is called the *local time* of the process $X$.

From (5.3) it follows that for any fixed $x$ the local time $\ell(t,x)$ is a nondecreasing random function with respect to $t$, which increases only on the set $\{t : X(t) = x\}$. As a rule, the Lebesgue measure of this set is zero and the most natural measure for such a set turned out to be the local time at the level $x$.

From the definition of $\mu_t$ it obviously follows that the support of $\mu_t$ is included in the set

$$\Big\{x : \inf_{0\le s\le t} X(s) \le x \le \sup_{0\le s\le t} X(s)\Big\}.$$

If the process $X$ has continuous paths, the support of $\mu_t$ is a.s. finite. Then

$$\int_0^t f(X(s))ds = \int_{-\infty}^{\infty} f(x)\mu_t(dx) \qquad \text{a.s.} \tag{5.4}$$

for any locally integrable function $f$. Indeed, by (5.1),

$$\int_0^t 1\!\!\mathrm{I}_\Delta(X(s))ds = \int_{-\infty}^{\infty} 1\!\!\mathrm{I}_\Delta(x)\mu_t(dx)$$

and $f$ can be approximated by the functions $\sum\limits_{k=1}^{n} c_{n,k} 1\!\!\mathrm{I}_{\Delta_{n,k}}(x)$, $\Delta_{n,k} \in \mathcal{B}(\mathbf{R})$.

In particular, if the local time $\ell(t,x)$ exists, then

$$\int_0^t f(X(s))ds = \int_{-\infty}^{\infty} f(x)\ell(t,x)dx \qquad \text{a.s.} \tag{5.5}$$

Let $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to a filtration $\{\mathcal{F}_t\}$ and let for all $v > t$ the increment $W(v) - W(t)$ be independent of the $\sigma$-algebra $\mathcal{F}_t$. Assume that $W(0) = x_0$.

The concept of a local time was introduced by P. Lévy (1939). G. Trotter (1958) proved that for a Brownian motion there exists a continuous local time (the *Brownian local time*). The following result is due to H. Tanaka.

**Theorem 5.1 (Tanaka's formula).** *The Brownian local time $\ell(t,x)$ exists. The local time $\ell(t,x)$ is an* a.s. *jointly continuous process in $(t,x) \in [0,T] \times \mathbf{R}$, and*

$$(W(t) - x)^+ - (W(0) - x)^+ = \int_0^t 1\!\!\mathrm{I}_{[x,\infty)}(W(s))\,dW(s) + \frac{1}{2}\ell(t,x), \tag{5.6}$$

*where $a^+ = \max\{a, 0\}$.*

*Proof.* We prove first that for the process

$$J_x(t) := \int_0^t 1\!\!\mathrm{I}_{[x,\infty)}(W(s))\,dW(s)$$

there exist a modification that is continuous in $(t,x) \in [0,T] \times \mathbf{R}$.

Note first that for a fixed $x$ the process $J_x(t)$ is continuous in $t$ by the property of the stochastic integral as a function of the upper limit. Let us consider $J_x(\cdot)$ as a random variable taking values in the space of continuous functions on $[0,T]$. This space is a Banach space when equipped with the norm $\|f\| := \sup\limits_{t\in[0,T]} |f(t)|$.

Analogously to the proof of Theorem 3.2 Ch. I for real-valued processes one can derive Kolmogorov's continuity criterion for processes with values in a Banach space.

This criterion implies that for any $N > 0$ the process $J_x$, $x \in [-N, N]$ is a.s. continuous with respect to the norm $\|\cdot\|$ if there exist positive constants $\alpha$, $\beta$, and $M_N$ such that

$$\mathbf{E}\|J_x - J_y\|^\alpha \le M_N|x-y|^{1+\beta}, \qquad |x|, |y| \le N. \tag{5.7}$$

For any $0 < \gamma < \beta/\alpha$, the sample paths of the process $J_x$, $x \in [-N, N]$ a.s. satisfy the Hölder condition

$$\|J_x - J_y\| \le L_{N,\gamma}(\omega)|x-y|^\gamma. \tag{5.8}$$

Indeed, from the proof of the analog of Theorem 3.2 Ch. I it follows that (5.8) is true for the set $D$ of dyadic rational points. By Cauchy's criterion, the process $J_y$, $y \in D\bigcap[-N, N]$, can be extended by continuity to the whole interval $[-N, N]$. Since

$$\lim_{y\to x}\int_0^T \mathbf{E}\big(\mathbb{1}_{[y,\infty)}(W(s)) - \mathbb{1}_{[x,\infty)}(W(s))\big)^2\, ds = 0,$$

we have by (2.8) that for all $x \in [-N, N]$ the process $J_x$ has the desired form as a stochastic integral. Moreover, in view of the a.s. continuity of stochastic integrals for a countable number of particular integrands and the uniform convergence in $t \in [0, T]$, the process $J_x(t)$, $t \in [0, T]$, is a.s. continuous with respect to $t$ for all $x$ simultaneously.

We now prove (5.7). We have

$$\mathbf{E}\|J_x - J_y\|^4 = \mathbf{E}\sup_{t\in[0,T]}\left|\int_0^t \mathbb{1}_{[x,y)}(W(s))\, dW(s)\right|^4 \qquad \text{for} \quad x < y.$$

By (4.25), $k = 2$,

$$\begin{aligned}
\mathbf{E}\|J_x - J_y\|^4 &\le 360\,\mathbf{E}\left|\int_0^T \mathbb{1}_{[x,y)}(W(s))\, ds\right|^2 \\
&= 720\int_0^T ds\int_s^T du\mathbf{E}\big[\mathbb{1}_{[x,y)}(W(s))\mathbb{1}_{[x,y)}(W(u))\big] \\
&= 720\int_0^T ds\int_s^T du\int_x^y\int_x^y \frac{e^{-(x_1-x_0)^2/2s}}{\sqrt{2\pi s}}\,\frac{e^{-(x_2-x_1)^2/2(u-s)}}{\sqrt{2\pi(u-s)}}\,dx_1 dx_2 \\
&\le \frac{360}{\pi}|x-y|^2\int_0^T ds\int_s^T du\frac{1}{\sqrt{s(u-s)}} = M_T|x-y|^2.
\end{aligned}$$

Thus for the process $J_x$ the Hölder condition (5.8) holds for $0 < \gamma < 1/4$.

Applying (4.25) for an arbitrary even power, we can prove the estimate

$$\mathbf{E}\|J_x - J_y\|^{2k} \le M_{k,T}|x-y|^k, \qquad k = 1, 2, \dots .$$

Therefore (5.8) holds for any $0 < \gamma < 1/2$.

The continuity of $J_x(t)$ in $(t,x)$ follows from (5.8), because

$$|J_x(t) - J_y(s)| \le |J_x(t) - J_x(s)| + \|J_x(\cdot) - J_y(\cdot)\|.$$

We now prove that for arbitrary $r \in \mathbf{R}$ there the limit

$$\ell(t,r) := \lim_{\alpha\uparrow r}\lim_{\beta\downarrow r}\frac{1}{\beta-\alpha}\int_0^t \mathrm{1\!I}_{[\alpha,\beta)}(W(s))ds \qquad \text{a.s.} \tag{5.9}$$

exists uniformly in $t \in [0,T]$ and (5.6) holds for $x = r$.

Set

$$f_{\alpha,\beta}(x) := \int_{-\infty}^{x}\int_{-\infty}^{z}\frac{\mathrm{1\!I}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy\,dz.$$

By the formula of stochastic differentiation (4.16), a.s. for all $t \in [0,T]$,

$$\frac{1}{2}\int_0^t \frac{\mathrm{1\!I}_{[\alpha,\beta)}(W(s))}{\beta-\alpha}\,ds = f_{\alpha,\beta}(W(t)) - f_{\alpha,\beta}(W(0)) - \int_0^t f'_{\alpha,\beta}(W(s))\,dW(s). \tag{5.10}$$

It is clear that

$$f'_{\alpha,\beta}(x) = \int_{-\infty}^{x}\frac{\mathrm{1\!I}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy = \begin{cases} 1, & \beta \le x, \\ \dfrac{x-\alpha}{\beta-\alpha}, & \alpha < x < \beta, \\ 0, & x \le \alpha, \end{cases} \underset{\alpha\uparrow r,\ \beta\downarrow r}{\longrightarrow} \mathrm{1\!I}_{[r,\infty)}(x), \quad \text{for } x \ne r,$$

$$f_{\alpha,\beta}(x) = \int_{-\infty}^{x}\int_{-\infty}^{z}\frac{\mathrm{1\!I}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy\,dz = \begin{cases} x - \dfrac{\beta+\alpha}{2}, & \beta \le x, \\ \dfrac{(x-\alpha)^2}{2(\beta-\alpha)}, & \alpha < x < \beta, \\ 0, & x \le \alpha, \end{cases} \underset{\alpha\uparrow r,\ \beta\downarrow r}{\longrightarrow} (x-r)^+.$$

Since

$$|\mathrm{1\!I}_{[r,\infty)}(x) - f'_{\alpha,\beta}(x)| \le \mathrm{1\!I}_{[\alpha,\beta)}(x), \qquad \alpha < r < \beta,$$

and, consequently,

$$|(x-r)^+ - f_{\alpha,\beta}(x)| \le |\beta-\alpha|,$$

we have

$$\sup_{t\in[0,T]}\bigl|(W(t)-r)^+ - f_{\alpha,\beta}(W(t))\bigr| \le |\beta-\alpha| \underset{\alpha\uparrow r,\ \beta\downarrow r}{\longrightarrow} 0 \qquad \text{a.s.} \tag{5.11}$$

Let us prove that

$$\int_0^t f'_{\alpha,\beta}(W(s))\,dW(s) = \frac{1}{\beta-\alpha}\int_\alpha^\beta J_y(t)\,dy \qquad \text{a.s.} \tag{5.12}$$

It is clear that

$$f'_{\alpha,\beta}(x) = \int_{-\infty}^{x} \frac{\mathbb{1}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbb{1}_{[y,\infty)}(x)\,dy.$$

Then (5.12) can be written in the form

$$\frac{1}{\beta-\alpha}\int_0^t\int_\alpha^\beta \mathbb{1}_{[y,\infty)}(W(s))\,dy\,dW(s) = \frac{1}{\beta-\alpha}\int_\alpha^\beta\int_0^t \mathbb{1}_{[y,\infty)}(W(s))\,dW(s)\,dy$$

and this is the switching the order of integration formula (analog of Fubini's theorem) and for the stochastic integral such formula must be proved.

Set

$$q_n(x) = \frac{1}{\beta-\alpha}\sum_{\alpha\le k/n\le\beta} \mathbb{1}_{[k/n,\infty)}(x)\frac{1}{n}.$$

Since

$$\left| q_n(x) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbb{1}_{[[yn]/n,\infty)}(x)\,dy\right| \le \frac{2}{n(\beta-\alpha)},$$

we have

$$|f'_{\alpha,\beta}(x) - q_n(x)| \le \frac{3}{n(\beta-\alpha)}. \tag{5.13}$$

Using the continuity of $J_x$ in $x$, we obtain

$$\int_0^t q_n(W(s))\,dW(s) = \frac{1}{\beta-\alpha}\sum_{\alpha\le k/n\le\beta}\int_0^t \mathbb{1}_{[k/n,\infty)}(W(s))\,dW(s)\frac{1}{n}$$

$$= \frac{1}{\beta-\alpha}\sum_{\alpha\le k/n\le\beta} J_{k/n}(t)\frac{1}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\beta-\alpha}\int_\alpha^\beta J_y(t)\,dy.$$

This together with (5.13) imply (5.12).

Substituting (5.12) into (5.10), we have

$$\frac{1}{2}\int_0^t \frac{\mathbb{1}_{[\alpha,\beta)}(W(s))}{\beta-\alpha}\,ds = f_{\alpha,\beta}(W(t)) - f_{\alpha,\beta}(W(0)) - \frac{1}{\beta-\alpha}\int_\alpha^\beta J_y(t)\,dy.$$

Applying (5.11) and taking into account the continuity of $J_x$ (see (5.8)), we see that the limit (5.9) exists uniformly in $t \in [0,T]$, and (5.6) holds for $x = r$. The statement that equality (5.6) holds for all $t$ and $x$ simultaneously follows from the continuity of $J_x(t)$ and $(W(t)-x)^+$ in $(t,x)$. This also implies the continuity of $\ell(t,x)$ in $(t,x) \in [0,T] \times \mathbf{R}$.

Moreover, since

$$\sup_{z\in\mathbf{R}} |(z-x)^+ - (z-y)^+| \le |x-y|,$$

from (5.6) and (5.8), it follows that for any $0 < \gamma < 1/2$ and $N > 0$

$$\|\ell(\cdot,x) - \ell(\cdot,y)\| \le B_{N,\gamma}(\omega)|x-y|^\gamma, \qquad x,y \in [-N,N]. \tag{5.14}$$

We can prove that Brownian local time paths with respect to $x$ are a.s. nowhere locally Hölder continuous of order $\gamma \ge 1/2$ (see Ch. V § 11). In particular, they are nowhere differentiable in $x$. The theorem is proved. □

Since the local time has the finite support

$$\Big\{x : \inf_{0\le s\le t} W(s) \le x \le \sup_{0\le s\le t} W(s)\Big\},$$

from (5.5) it follows that for any locally integrable function $f$ and any $t > 0$,

$$\int_0^t f(W(s))\,ds = \int_{-\infty}^{\infty} f(x)\ell(t,x)\,dx \qquad \text{a.s.}, \tag{5.15}$$

and the integral on the right-hand side is finite.

From (5.9) we have

$$\mathbf{E}\ell(t,x) = \lim_{\alpha\uparrow x}\lim_{\beta\downarrow x} \int_0^t \mathbf{E}\Big(\frac{\mathbb{1}_{[\alpha,\beta)}(W(s))}{\beta-\alpha}\Big)ds = \int_0^t \frac{1}{\sqrt{2\pi s}} e^{-(x-x_0)^2/2s}\,ds. \tag{5.16}$$

Here $\frac{1}{\sqrt{2\pi s}} e^{-(x-x_0)^2/2s}$ is the density of the variable $W(s)$, $W(0) = x_0$.

Using Tanaka's formula (5.6) one can generalize Itô's formula (4.16) as follows.

**Theorem 5.2.** *Let $b$ be a function of bounded variation on any finite interval. Set*

$$f(x) := f_0 + \int_0^x b(y)dy, \tag{5.17}$$

*where $f_0$ is a constant.*

*Then* a.s. *for all $t \in [0,T]$,*

$$f(W(t)) - f(W(0)) = \int_0^t b(W(s))\,dW(s) + \frac{1}{2}\int_{-\infty}^{\infty} \ell(t,x)\,b(dx), \tag{5.18}$$

*where $b(dx)$ is the signed measure (charge) associated to $b$ via its representation as a difference of two nondecreasing functions.*

**Remark 5.1.** The differential form of (5.18) is the following formula:

$$df(W(t)) = b(W(t))\,dW(t) + \frac{1}{2}\int_{-\infty}^{\infty} \ell(dt,x)\,b(dx).$$

**Remark 5.2.** If $b(dx)$ has a bounded density, then $b(dx) = g(x)\,dx$, $b(x) = f'(x)$ and, in view of (5.15), formula (5.18) transforms into (4.16).

**Remark 5.3.** Let the function $f$ be twice continuously differentiable except at the finite number of points $x_1 < x_2 < \cdots < x_m$, in which $f$ is assumed to have the right and left derivatives. Then from (5.18) it follows that

$$\begin{aligned} f(W(t)) - f(W(0)) &= \int_0^t \sum_{k=0}^{m} f'(W(s))\mathbb{1}_{(x_k,x_{k+1})}(W(s))\,dW(s) \\ &+ \frac{1}{2}\int_0^t \sum_{k=0}^{m} f''(W(s))\mathbb{1}_{(x_k,x_{k+1})}(W(s))\,ds \\ &+ \frac{1}{2}\sum_{k=1}^{m}(f'(x_k+0) - f'(x_k-0))\ell(t,x_k) \qquad \text{a.s.}, \end{aligned} \tag{5.19}$$

where we set $x_0 = -\infty$, $x_{m+1} = \infty$.

*Proof of Theorem* 5.2. It suffices to prove (5.18) only for a nondecreasing function $b$, since any function of bounded variation is the difference of two nondecreasing functions.

For the functions

$$b_n(x) := \sum_{k=1}^{n} c_{n,k}\,\mathbb{1}_{[r_{n,k},\infty)}(x) \tag{5.20}$$

equality (5.18) follows from Tanaka's formula (5.6).

Now set

$$f_n(x) := f_0 + \int_0^x b_n(y)dy = f_0 + \sum_{k=1}^{n} c_{n,k}(x - r_{n,k})^+.$$

Then, by (5.6),

$$f_n(W(t)) - f_n(W(0)) = \int_0^t b_n(W(s))\,dW(s) + \frac{1}{2}\int_{-\infty}^{\infty} \ell(t,x)\,b_n(dx) \qquad \text{a.s.} \tag{5.21}$$

It is clear that any nondecreasing function can be uniformly approximated on any compact set by functions of the form (5.20), i.e., for any $N > 0$

$$\sup_{|x|\le N} |b(x) - b_n(x)| \to 0 \qquad \text{as } n \to \infty. \tag{5.22}$$

One can ensure that $b_n(N) = b(N)$ and $b_n(-N) = b(-N)$. Of course, the sequence of functions $b_n$ depends on $N$. It is clear that

$$\sup_{|x|\le N} |f(x) - f_n(x)| \le 2N \sup_{|x|\le N} |b(x) - b_n(x)|. \tag{5.23}$$

Since, by (4.12),

$$\mathbf{P}\Big(\sup_{0\le t\le T} |W(t)| \ge N\Big) \le \frac{T}{N^2}, \tag{5.24}$$

and by the choice of $N$, this probability can be made sufficiently small, we can restrict ourselves to the consideration of the set $\Omega_N = \Big\{\sup_{0\le t\le T} |W(t)| < N\Big\}$. From (5.23), (5.22), and (3.6) it follows that

$$\sup_{t\in[0,T]} |f(W(t)) - f_n(W(t))| \underset{n\to\infty}{\longrightarrow} 0, \qquad |f(W(0)) - f_n(W(0))| \underset{n\to\infty}{\longrightarrow} 0, \tag{5.25}$$

$$\sup_{t\in[0,T]} \bigg| \int_0^t b(W(s))\,dW(s) - \int_0^t b_n(W(s))\,dW(s) \bigg| \underset{n\to\infty}{\longrightarrow} 0 \tag{5.26}$$

in probability given the set $\Omega_N$.

Let us prove that

$$\sup_{t\in[0,T]} \bigg| \int_{-N}^N \ell(t,x)\,b(dx) - \int_{-N}^N \ell(t,x)\,b_n(dx) \bigg| \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{5.27}$$

By (5.14),

$$\sup_{t\in[0,T]} \bigg| \int_{-N}^N \ell(t,x)\,b(dx) - \int_{-N}^N \ell(t,\tfrac{[xm]}{m})\,b(dx) \bigg|$$

$$\le \sup_{|x|\le N} \|\ell(\cdot,x) - \ell(\cdot,\tfrac{[xm]}{m})\| (b(N) - b(-N)) \le \frac{B_{N,\gamma}(\omega)}{m^\gamma}(b(N) - b(-N)).$$

Analogously, in view of (5.22),

$$\sup_{t\in[0,T]} \bigg| \int_{-N}^N \ell(t,x)\,b_n(dx) - \int_{-N}^N \ell(t,\tfrac{[xm]}{m})\,b_n(dx) \bigg|$$

$$\leq \sup_{|x|\leq N} \|\ell(\cdot,x)-\ell(\cdot,\tfrac{[xm]}{m})\|(b_n(N)-b_n(-N)) \leq \frac{B_{N,\gamma}(\omega)}{m^\gamma}(b(N)-b(-N)).$$

In addition, we have the estimate

$$\sup_{t\in[0,T]}\Big|\int_{-N}^{N} \ell(t,\tfrac{[xm]}{m})\, b(dx) - \int_{-N}^{N} \ell(t,\tfrac{[xm]}{m})\, b_n(dx)\Big|$$

$$= \sup_{t\in[0,T]}\Big|\int_{-N}^{N} \ell(t,\tfrac{[xm]}{m})\,(b(dx)-b_n(dx))\Big|$$

$$\leq \sum_{k=-[Nm]}^{[Nm]} \ell(T,\tfrac{k}{m})\big|b(\tfrac{k+1}{m})-b(\tfrac{k}{m})-b_n(\tfrac{k+1}{m})+b_n(\tfrac{k}{m})\big|.$$

Now letting first $n\to\infty$ and then $m\to\infty$, we obtain (5.27).

Taking into account (5.24)–(5.27) we see that the passage to the limit in (5.21) leads to (5.18). □

Similarly, we can prove the following generalization of the special case of Theorem 4.3 where $f(t,x)=\sigma(t)f(x)$.

**Theorem 5.3.** *Let $f$ be the function defined by* (5.17) *and $\sigma(t)$, $t\geq 0$, be a function with locally integrable derivative.*

*Then* a.s. *for all $t\in[0,T]$,*

$$\sigma(t)f(W(t))-\sigma(0)f(W(0)) = \int_0^t \sigma'(s)f(W(s))\,ds$$

$$+\int_0^t \sigma(s)b(W(s))\,dW(s)+\frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \sigma(s)\ell(ds,x)\,b(dx). \tag{5.28}$$

*Proof.* We can apply the method used to establish formulas (4.19) and (4.20). Considering subdivisions of $[0,t]$, as in the proof of Theorem 4.1, we can write, according to (5.18), that

$$f(W(t_{n,k+1}))-f(W(t_{n,k}))=\int_{t_{n,k}}^{t_{n,k+1}} f'(W(v))\,dW(v)+\frac{1}{2}\int_{-\infty}^{\infty}(\ell(t_{n,k+1},x)-\ell(t_{n,k}))\,b(dx).$$

The analog of (4.20) is the relation

$$\sigma(t)f(W(t))-\sigma(0)f(W(0)) = \int_0^t \sigma'(v)f(W(t_n^+(v))\,dv$$

$$+\int_0^t \sigma(t_n(v))f'(W(v))\,dW(v)+\frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \sigma(t_n(v))\ell(dv,x)\,b(dx). \tag{5.29}$$

The function $\sigma(v)$, $v\in[0,T]$, is uniformly continuous with a modulus of continuity $\Delta(\delta)\to 0$ as $\delta\to 0$. Using this, we have

$$\Bigl|\int_0^t (\sigma(t_n(v))-\sigma(v))\ell(dv,x)\Bigr|\le \Delta\Bigl(\max_{1\le k\le n}|t_{n,k}-t_{n,k-1}|\Bigr)\ell(t,x).$$

The subdivisions of the interval $[0,t]$ satisfy (4.5), therefore in (5.29) we can pass to the limit and get (5.28). □

**Example 5.1.** Compute for $b>0$ the stochastic differential $d|\,|W(t)-a|-b\,|$. It is obvious that

$$|\,|x-a|-b\,|=(a-x-b)\mathbb{1}_{(-\infty,a-b)}(x)+(x-a+b)\mathbb{1}_{[a-b,a)}(x)$$
$$+(a-x+b)\mathbb{1}_{[a,a+b)}(x)+(x-a-b)\mathbb{1}_{[a+b,\infty)}(x).$$

Applying (5.19), we have

$$d|\,|W(t)-a|-b\,|=\bigl(\mathbb{1}_{(a+b,\infty)}(W(t))-\mathbb{1}_{(a,a+b)}(W(t))+\mathbb{1}_{(a-b,a)}(W(t))$$
$$-\mathbb{1}_{(-\infty,a-b)}(W(t))\bigr)dW(t)+\ell(dt,a+b)-\ell(dt,a)+\ell(dt,a-b).$$

Since the expectation of a stochastic integral equals zero, from (5.16) it follows that

$$\frac{d}{dt}\mathbf{E}_{x_0}||W(t)-a|-b|=\frac{1}{\sqrt{2\pi t}}e^{-(a-b-x_0)^2/2t}+\frac{1}{\sqrt{2\pi t}}e^{-(a+b-x_0)^2/2t}-\frac{1}{\sqrt{2\pi t}}e^{-(a-x_0)^2/2t},$$

where the subscript in the expectation means that it is computed with respect to the process $W$ with $W(0)=x_0$.

**Exercises.**

**5.1.** Compute the differentials

1) $d\exp\Bigl(|W(t)|^3+\int_0^t W^2(s)\,dW(s)\Bigr)$;

2) $d\bigl(|W(t)|e^{|W(t)-r|}\bigr)$;

3) $d|\,|W(t)-a|^3-b^3|,\qquad 0<b<a.$

## § 6. Stochastic exponent

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space and $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$. Let for all $v > t$ the increments $W(v) - W(t)$ be independent of the $\sigma$-algebra $\mathcal{F}_t$.

For an arbitrary $b \in \mathcal{L}_2[0,T]$, consider the *stochastic exponent*

$$\rho(t) := \exp\Big(\int_0^t b(s)\, dW(s) - \frac{1}{2}\int_0^t b^2(s)\, ds\Big), \qquad t \in [0,T]. \tag{6.1}$$

Let us compute the stochastic differential of the process $\rho$. Applying Itô's formula (4.22), $d = 1$, for $f(t,x) = e^x$ and the process

$$X(t) = \int_0^t b(s)\, dW(s) - \frac{1}{2}\int_0^t b^2(s)\, ds,$$

we have

$$d\rho(t) = \rho(t)\Big[b(t)\, dW(t) - \frac{1}{2}b^2(t)\, dt + \frac{1}{2}b^2(t)\, dt\Big] = \rho(t)b(t)\, dW(t).$$

Therefore,

$$d\rho(t) = \rho(t)b(t)\, dW(t), \qquad \rho(0) = 1. \tag{6.2}$$

The process $\rho$ is called the stochastic exponent by analogy with the classical exponent $\tilde\rho(t) = \exp\big(\int_0^t b(s)\, ds\big)$, which is the solution of the equation

$$d\tilde\rho(t) = \tilde\rho(t)b(t)\, dt, \qquad \tilde\rho(0) = 1.$$

Equation (6.2) is the simplest form of so-called *stochastic differential equation* (see § 7). According to the definition of stochastic differentials, (6.2) is equivalent to the equation

$$\rho(t) = 1 + \int_0^t \rho(s)b(s)\, dW(s), \qquad t \in [0,T]. \tag{6.3}$$

We will prove that, under some conditions, $\rho(t)$ is a nonnegative martingale with respect to the filtration $\{\mathcal{F}_t\}$, with mean value $\mathbf{E}\rho(t) = 1$ for every $t \in [0,T]$.

**Proposition 6.1.** *Let $b$ be a continuous stochastic process from $\mathcal{L}_2[0,T]$. Suppose that for some $\delta > 0$*

$$\mathbf{E}\exp\Big((1+\delta)\int_0^T b^2(s)\, ds\Big) < \infty, \tag{6.4}$$

*or*

$$\sup_{0 \le s \le T} \mathbf{E} e^{\delta b^2(s)} < \infty. \tag{6.5}$$

*Then for any* $0 \le t_1 < t_2 \le T$,

$$\mathbf{E} \exp\Big( \int_{t_1}^{t_2} b(s)\, dW(s) - \frac{1}{2} \int_{t_1}^{t_2} b^2(s)\, ds \Big) = 1, \tag{6.6}$$

*and, in addition,*

$$\mathbf{E}\Big\{ \exp\Big( \int_{t_1}^{t_2} b(s)\, dW(s) - \frac{1}{2} \int_{t_1}^{t_2} b^2(s)\, ds \Big) \Big| \mathcal{F}_{t_1} \Big\} = 1 \qquad \text{a.s.} \tag{6.7}$$

**Remark 6.1.** The relations (6.6) and (6.7) are valid (see Novikov (1972), Liptser and Shiryaev (1974)) for an arbitrary process from $\mathcal{L}_2[0,T]$ under weaker assumptions than (6.4), (6.5), which are taken from Gihman and Skorohod (1972). In (6.4) the factor $1+\delta$ can be replaced by the factor $1/2$, but to improve it to the factor $1/2 - \delta$ is not possible.

*Proof of Proposition* 6.1. We assume first that $b(s) = \bar{b}(s)$, $s \in [0,T]$, is a simple process defined by (1.3) and $\sup_{0 \le s \le T} |\bar{b}(s)| \le M$, where $M$ is nonrandom. Then

$$\mathbf{E} \exp\Big( \int_{t_1}^{t_2} \bar{b}(s)\, dW(s) \Big) \le e^{M^2(t_2 - t_1)/2}.$$

This means that for every $m > 0$

$$\mathbf{E} \exp\Big( m \int_{t_1}^{t_2} \bar{b}(s)\, dW(s) \Big) \le e^{m^2 M^2(t_2 - t_1)/2}. \tag{6.8}$$

Indeed, since on the interval $[s_k, s_{k+1})$, $s_0 = t_1$, $s_m = t_2$, $k = 1, \dots, m-1$, the process $\bar{b}$ is equal to the $\mathcal{F}_{s_k}$-measurable random variable $b_k$, using the properties of conditional expectations, and (10.9) Ch. I we have

$$\mathbf{E} \exp\Big( \int_{t_1}^{t_2} \bar{b}(s)\, dW(s) \Big) = \mathbf{E}\Big\{ \mathbf{E}\Big\{ \exp\Big( \sum_{k=0}^{m-1} b_k (W(s_{k+1}) - W(s_k)) \Big) \Big| \mathcal{F}_{s_{m-1}} \Big\} \Big\}$$

$$= \mathbf{E}\Big\{ \exp\Big( \sum_{k=0}^{m-2} b_k (W(s_{k+1}) - W(s_k)) \Big) \exp\big( b_{m-1}^2 (s_m - s_{m-1})/2 \big) \Big\}$$

$$\le e^{M^2(s_m - s_{m-1})/2} \mathbf{E} \exp\Big( \sum_{k=0}^{m-2} b_k (W(s_{k+1}) - W(s_k)) \Big) \le e^{M^2(t_2 - t_1)/2}.$$

By (6.3),

$$\exp\Bigl(\int_{t_1}^{t_2} \overline{b}(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} \overline{b}^2(s)\,ds\Bigr) - 1$$

$$= \int_{t_1}^{t_2} \exp\Bigl(\int_{t_1}^{t} \overline{b}(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t} \overline{b}^2(s)\,ds\Bigr)\overline{b}(t)\,dW(t).$$

Since

$$\int_{t_1}^{t_2} \mathbf{E}\Bigl\{\Bigl(\exp\Bigl(\int_{t_1}^{t} \overline{b}(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} \overline{b}^2(s)\,ds\Bigr)\overline{b}(t)\Bigr)^2\Bigr\}dt < \infty,$$

the expectation of the stochastic integral is zero, and

$$\mathbf{E}\exp\Bigl(\int_{t_1}^{t_2} \overline{b}(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} \overline{b}^2(s)\,ds\Bigr) = 1. \tag{6.9}$$

Thus (6.6) is proved for bounded simple processes. Now, by (2.3), equation (6.3) and inequality (6.8), we get $\mathbf{E}\{\rho(t_2)|\mathcal{F}_{t_1}\} = \rho(t_1)$ a.s. Hence, (6.7) holds for the simple processes.

We turn to the proof of the statement for a continuous process $b$. Let $b(s) = 0$ for $s < 0$. We construct for the process $b$ a sequence of bounded simple processes $\overline{b}_n(s)$, $s \in [0, T]$, such that

$$\overline{b}_n^2(s) \le b^2(s - 1/n) \tag{6.10}$$

and

$$\lim_{n\to\infty} \int_0^T (b(s) - \overline{b}_n(s))^2\,dt = 0 \qquad \text{a.s.} \tag{6.11}$$

For $s \in [0, 1/n)$, we set $\overline{b}_n(s) = 0$. For $s \in [k/n, (k+1)/n)$, $k = 1, 2, \dots, [nT]$, we set $\overline{b}_n(s) := \min\Bigl\{\inf_{(k-1)/n\le s\le k/n} b(s), n\Bigr\}$ if $\inf_{(k-1)/n\le s\le k/n} b(s) > 0$, we set $\overline{b}_n(s) := \max\Bigl\{\sup_{(k-1)/n\le s\le k/n} b(s), -n\Bigr\}$, if $\sup_{(k-1)/n\le s\le k/n} b(s) < 0$, and we set $\overline{b}_n(s) = 0$ if in at least one point of the interval $[(k-1)/n, k/n]$ the process $b$ becomes equal zero. The simple bounded processes $\overline{b}_n$ are adapted to the filtration $\{\mathcal{F}_t\}$ and (6.10) is satisfied. Then (6.11) holds, because the process $b$ is uniformly continuous on $[0, T]$.

In view of (6.11) and (3.6), the sequence of random variables

$$\exp\Bigl(\int_{t_1}^{t_2} \overline{b}_n(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} \overline{b}_n^2(s)\,ds\Bigr)$$

converges in probability to the variable

$$\exp\Bigl(\int_{t_1}^{t_2} b(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} b^2(s)\,ds\Bigr).$$

If we prove that this sequence of random variables is uniformly integrable (see § 1 Ch. I), then by Proposition 1.3 of Ch. I, we can pass to the limit in (6.9), applied to the simple processes $\bar{b}_n$ and get (6.6). Equality (6.7) is proved analogously with the help of property 7′) of the conditional expectations (see § 2 Ch. I).

Choose $\gamma > 0$ such that $(1+\gamma)^2(1+\gamma/2) = 1+\delta$. Using Hölder's inequality and (6.9), we get

$$\mathbf{E}\Bigl(\exp\Bigl(\int_{t_1}^{t_2} \bar{b}_n(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} \bar{b}_n^2(s)\,ds\Bigr)\Bigr)^{1+\gamma}$$

$$= \mathbf{E}\Bigl\{\exp\Bigl((1+\gamma)\int_{t_1}^{t_2} \bar{b}_n(s)\,dW(s) - \frac{(1+\gamma)^3}{2}\int_{t_1}^{t_2} \bar{b}_n^2(s)\,ds + \frac{\gamma(1+\gamma)(2+\gamma)}{2}\int_{t_1}^{t_2} \bar{b}_n^2(s)\,ds\Bigr)\Bigr\}$$

$$\le \Bigl[\mathbf{E}\Bigl(\exp\Bigl((1+\gamma)^2\int_{t_1}^{t_2} \bar{b}_n(s)\,dW(s) - \frac{(1+\gamma)^4}{2}\int_{t_1}^{t_2} \bar{b}_n^2(s)\,ds\Bigr)\Bigr]^{1/(1+\gamma)}$$

$$\times\Bigl[\mathbf{E}\exp\Bigl(\frac{(1+\gamma)^2(2+\gamma)}{2}\int_{t_1}^{t_2} \bar{b}_n^2(s)\,ds\Bigr)\Bigr]^{\gamma/(1+\gamma)}$$

$$\le \Bigl[\mathbf{E}\exp\Bigl((1+\delta)\int_{t_1-1/n}^{t_2-1/n} b^2(s)\,ds\Bigr)\Bigr]^{\gamma/(1+\gamma)} < \infty.$$

By Proposition 1.2 Ch. I with $G(x) = x^{1+\gamma}$, this implies that the corresponding sequence of random variables is uniformly integrable. Proposition 6.1 is proved under the condition (6.4).

We turn to the proof of this assertion under the condition (6.5).

Since the function $g(x) := e^x$ is convex, by Jensen's inequality for the integral of the normalized measure (see (1.4) Ch. I), we have that for $v < u$ and $\delta > 0$

$$\exp\Bigl((1+\delta)\int_v^u b^2(s)\,ds\Bigr) = \exp\Bigl(\int_v^u (1+\delta)(u-v)b^2(s)\frac{ds}{u-v}\Bigr) \le \int_v^u e^{(1+\delta)(u-v)b^2(s)}\,\frac{ds}{u-v}.$$

By (6.5), for any $0 < u-v \le \dfrac{\delta}{1+\delta}$ there holds the estimate

$$\mathbf{E}\exp\Bigl((1+\delta)\int_v^u b^2(s)\,ds\Bigr) \le \frac{1}{u-v}\int_v^u \mathbf{E}e^{\delta b^2(s)}\,ds < \infty.$$

This is exactly the condition (6.4), therefore by the assertion proved above, we have for any $0 < u-v \le \dfrac{\delta}{1+\delta}$ the equality

$$\mathbf{E}\Bigl\{\exp\Bigl(\int_v^u b(s)\,dW(s) - \frac{1}{2}\int_v^u b^2(s)\,ds\Bigr)\Bigm|\mathcal{F}_v\Bigr\} = 1. \tag{6.12}$$

Divide the interval $[t_1, t_2]$ by points $t_1 = v_0 < v_1 < \cdots < v_m = t_2$ such that $\max\limits_{1\le r\le m} (t_k - t_{k-1}) \le \frac{\delta}{1+\delta}$. Under the assumption that (6.12) is proved for $v = v_0$, $u = v_{m-1}$, we prove (6.12) for $v = v_0$, $u = v_m$. Since (6.12) holds for $v = v_{m-1}$, $u = v_m$, we have

$$\mathbf{E}\Big\{\exp\Big(\int_{v_0}^{v_m} b(s)\,dW(s) - \frac{1}{2}\int_{v_0}^{v_m} b^2(s)\,ds\Big)\Big|\mathcal{F}_{v_0}\Big\}$$

$$= \mathbf{E}\Big\{\exp\Big(\int_{v_0}^{v_{m-1}} b(s)\,dW(s) - \frac{1}{2}\int_{v_0}^{v_{m-1}} b^2(s)\,ds\Big)$$

$$\times\,\mathbf{E}\Big\{\exp\Big(\int_{v_{m-1}}^{v_m} b(s)\,dW(s) - \frac{1}{2}\int_{v_{m-1}}^{v_m} b^2(s)\,ds\Big)\Big|\mathcal{F}_{v_{m-1}}\Big\}\Big|\mathcal{F}_{v_0}\Big\}$$

$$= \mathbf{E}\Big\{\exp\Big(\int_{v_0}^{v_{m-1}} b(s)\,dW(s) - \frac{1}{2}\int_{v_0}^{v_{m-1}} b^2(s)\,ds\Big)\Big|\mathcal{F}_{v_0}\Big\} = 1 \quad .$$

The induction base for $v = v_0$, $u = v_1$ is also valid. Therefore (6.7) holds. Proposition 6.1 is proved. □

**Remark 6.2.** Suppose that the process $b(s)$, $s \in [0,T]$, is adapted to the filtration $\{\mathcal{F}_s\}$, and $\sup\limits_{0\le s\le T} |b(s)| \le M$ for some nonrandom constant $M$. Then for any $m > 0$

$$\mathbf{E}\exp\Big(m\int_{t_1}^{t_2} b(s)\,dW(s)\Big) \le e^{m^2M^2(t_2-t_1)/2}. \tag{6.13}$$

Indeed, according to Proposition 1.1, the process $b$ can be approximated by a sequence of bounded simple processes $\overline{b}_n$ such that (2.8) holds. For a simple processes $\overline{b}_n$ we have (6.8). By Proposition 1.2 Ch. I with $G(x) = x^{1+\gamma}$, $\gamma > 0$, the corresponding sequence of random variables is uniformly integrable. Therefore, we can pass to the limit under the expectation sign in (6.8) applied for $\overline{b}_n$. This implies (6.13).

Equation (6.3) gives us the iterative procedure

$$\rho(t) = 1 + \int_0^t \Big(1 + \int_0^{t_1} \rho(s)b(s)\,dW(s)\Big)b(t_1)\,dW(t_1) = 1 + \int_0^t b(t_1)\,dW(t_1)$$

$$+ \int_0^t\int_0^{t_1} \Big(1 + \int_0^{t_2} \rho(s)b(s)\,dW(s)\Big)b(t_2)\,dW(t_2)b(t_1)\,dW(t_1)$$

$$= 1 + \int_0^t dW(t_1)\, b(t_1) + \int_0^t dW(t_1)\, b(t_1) \int_0^{t_1} dW(t_2)\, b(t_2)$$

$$+ \int_0^t dW(t_1)\, b(t_1) \int_0^{t_1} dW(t_2)\, b(t_2) \int_0^{t_2} dW(t_3)\, b(t_3) + \cdots .$$

Formally, we have the series

$$\rho(t) = \sum_{n=0}^{\infty} \rho_n(t), \tag{6.14}$$

where $\rho_0(t) \equiv 1$ and

$$\rho_n(t) := \int_0^t dW(t_1)\, b(t_1) \int_0^{t_1} dW(t_2)\, b(t_2) \cdots \int_0^{t_{n-1}} dW(t_n)\, b(t_n).$$

This is equivalent to the equality

$$\rho_n(t) = \int_0^t \rho_{n-1}(t_1) b(t_1)\, dW(t_1). \tag{6.15}$$

Of course, we need to prove that the series (6.14) converges a.s. We assume this first. Therefore, the stochastic exponent is represented as the sum of multiple Itô integrals of the process $b(t)$, $t \in [0, T]$.

The usual multiple integral has a simple expression, i.e.,

$$\int_0^t dt_1\, b(t_1) \int_0^{t_1} dt_2\, b(t_2) \cdots \int_0^{t_{n-1}} dt_n\, b(t_n) = \frac{1}{n!} \Big( \int_0^t b(s)\, ds \Big)^n .$$

For a multiple stochastic integral $\rho_n(t)$ the formula is not so simple. To derive it, we proceed as follows.

For further purposes we consider the *Hermite polynomials*

$$\mathrm{He}_n(t, x) := (-t)^n e^{x^2/2t} \frac{d^n}{dx^n} e^{-x^2/2t} = n! \sum_{0 \le k \le n/2} \frac{(-1)^k x^{n-2k} t^k}{2^k k! (n-2k)!}, \qquad n = 0, 1, 2, \ldots .$$

As to the right-hand side of this equality, see the corresponding example of formula 5 in Appendix 6. We set $\mathrm{He}_0(t, x) := 1$. It is easy to compute that $\mathrm{He}_1(t, x) = x$, $\mathrm{He}_2(t, x) = x^2 - t$, $\mathrm{He}_3(t, x) = x^3 - 3xt$, $\mathrm{He}_4(t, x) = x^4 - 6x^2 t + 3t^2$.

The generating function of the Hermite polynomials is determined by the formula

$$\sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \mathrm{He}_n(t, x) = e^{\gamma x - \gamma^2 t/2}, \qquad \gamma \in \mathbf{R}. \tag{6.16}$$

To prove (6.16), we note that the Taylor expansion of the function $e^{-(x+\Delta)^2/2t}$ is

$$e^{-(x+\Delta)^2/2t} = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} \frac{d^n}{dx^n} e^{-x^2/2t}, \qquad x \in \mathbf{R}.$$

Multiplying this equality by $e^{x^2/2t}$ and setting $\Delta = -\gamma t$, we have (6.16):

$$e^{\gamma x - \gamma^2 t/2} = \sum_{n=0}^{\infty} \frac{(-\gamma t)^n}{n!} e^{x^2/2t} \frac{d^n}{dx^n} e^{-x^2/2t}.$$

Using the generating function (6.16), it is easy to derive the formulas

$$\frac{\partial}{\partial t} \mathrm{He}_n(t,x) = -\frac{n(n-1)}{2} \mathrm{He}_{n-2}(t,x),$$

$$\frac{\partial^k}{\partial x^k} \mathrm{He}_n(t,x) = \frac{n!}{(n-k)!} \mathrm{He}_{n-k}(t,x), \qquad k = 1,2,\dots. \tag{6.17}$$

Substituting $x = \int\limits_0^t b(s)\,dW(s)$, $t = \int\limits_0^t b^2(s)\,ds$ in (6.16), we have

$$\exp\Big(\gamma \int\limits_0^t b(s)\,dW(s) - \frac{\gamma^2}{2} \int\limits_0^t b^2(s)\,ds\Big) = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \mathrm{He}_n\Big(\int\limits_0^t b^2(s)\,ds, \int\limits_0^t b(s)\,dW(s)\Big). \tag{6.18}$$

The series on the right-hand side converges a.s., since (6.16) converge for all $x \in \mathbf{R}$ and $t > 0$. The left-hand side of (6.18) is the stochastic exponent $\rho^{(\gamma)}(t)$ defined in (6.1) with the function $\gamma b(t)$ instead of $b(t)$. For this stochastic exponent the equality (6.14) has the form

$$\rho^{(\gamma)}(t) = \sum_{n=0}^{\infty} \gamma^n \rho_n(t), \tag{6.19}$$

where $\rho_n(t)$ is defined by (6.15). Comparing (6.19) with (6.18) we come to the conclusion that the multiple Itô integral $\rho_n(t)$ must be equal to

$$\rho_n(t) = \frac{1}{n!} \mathrm{He}_n\Big(\int\limits_0^t b^2(s)\,ds, \int\limits_0^t b(s)\,dW(s)\Big), \qquad n = 1,2,\dots. \tag{6.20}$$

Below we prove (6.20) directly, using Itô's differentiation formula. Then this implies that the series (6.19) converges a.s for arbitrary $\gamma$, since the series (6.18) converges a.s., and our assumption on the convergence of the series (6.14) will be proved.

We prove (6.20) by induction. It is clear that (6.20) holds for $n = 0$ and $n = 1$. Suppose that it holds for index $n-1$ and let us prove it for index $n$. It is also evident that (6.20) holds for $t = 0$, since $\mathrm{He}_n(0,0) = 0$, $n = 1,2,\dots$. The last equality follows from (6.16). Now it is sufficient to prove that the stochastic differentials of both sides of (6.20) coincide.

According to (6.15) and the induction hypothesis, the stochastic differential on the left-hand side of (6.20) equals

$$d\rho_n(t) = \rho_{n-1}(t)b(t)\,dW(t) = \frac{1}{(n-1)!}\,\mathrm{He}_{n-1}\ \ b(t)\,dW(t). \tag{6.21}$$

Here and in what follows we omit the arguments $\int_0^t b^2(s)\,ds$, $\int_0^t b(s)\,dW(s)$ in the notation of the Hermite polynomials in (6.20).

Applying Itô's formula (4.22) for $\vec{X} = \Big(\int_0^t b^2(s)\,ds, \int_0^t b(s)\,dW(s)\Big)$ and taking into account formulae (6.17), we obtain the following expression for the differential on the right-hand side (6.20):

$$\begin{aligned} d\,\mathrm{He}_n &= \frac{\partial}{\partial t}\,\mathrm{He}_n\ \ b^2(t)\,dt + \frac{\partial}{\partial x}\,\mathrm{He}_n\ \ b(t)\,dW(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\,\mathrm{He}_n\ \ b^2(t)\,dt \\ &= -\frac{n(n-1)}{2}\,\mathrm{He}_{n-2}\ \ b^2(t)\,dt + n\,\mathrm{He}_{n-1}\ \ b(t)\,dW(t) + \frac{n(n-1)}{2}\,\mathrm{He}_{n-2}\ \ b^2(t)\,dt \\ &= n\ \mathrm{He}_{n-1}\ \ b(t)\,dW(t). \end{aligned}$$

After normalization by $n!$ this stochastic differential coincides with (6.21) and, consequently, (6.20) is proved, because $\rho_n(0) = 0$ and $\mathrm{He}_n(0,0) = 0$ for $n \geq 1$. $\square$

**Proposition 6.2.** *Let $b(s)$, $s \in [0,t]$, be a stochastic process from $\mathcal{L}_2[0,T]$. Then*

$$0.3\,\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^2 \leq \mathbf{E}\Big(\int_0^t b(s)\,dW(s)\Big)^4 \leq 30\,\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^2. \tag{6.22}$$

*Proof.* Since $\mathrm{He}_4(t,x) = x^4 - 6x^2t + 3t^2$, we have

$$24\rho_4(t) = \Big(\int_0^t b(s)\,dW(s)\Big)^4 - 6\Big(\int_0^t b(s)\,dW(s)\Big)^2 \int_0^t b^2(s)\,ds + 3\Big(\int_0^t b^2(s)\,ds\Big)^2.$$

We can assume that the function $b$ is bounded, otherwise we can apply the truncation procedure. Let $\sup_{0\leq s\leq t} |b(s)| \leq M$. According to (6.15) and the definition of the stochastic integral, in order to take the expectation of $\rho_4(t)$ we need to be sure that $\int_0^t \mathbf{E}(\rho_3(t_1)b(t_1))^2\,dt_1 < \infty$. In view of (6.15) and (1.12), the required estimate follows from the inequalities

$$\int_0^t \mathbf{E}(\rho_3(t_1)b(t_1))^2\,dt_1 \leq M^2 \int_0^t \mathbf{E}(\rho_3(t_1))^2\,dt_1 = M^2 \int_0^t dt_1 \int_0^{t_1} \mathbf{E}(\rho_2(t_2)b(t_2))^2\,dt_2$$

$$\leq M^6 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \mathbf{E}(\rho_1(t_3))^2\, dt_3 \leq M^8 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 = \frac{M^8 t^4}{4!}.$$

Since the expectation of a stochastic integral is zero, we have $\mathbf{E}\rho_4(t) = 0$. Now, from the expression for $24\rho_4(t)$ it follows that

$$\mathbf{E}\Big(\int_0^t b(s)\, dW(s)\Big)^4 = 6\mathbf{E}\Big\{\Big(\int_0^t b(s)\, dW(s)\Big)^2 \int_0^t b^2(s)\, ds\Big\} - 3\mathbf{E}\Big(\int_0^t b^2(s)\, ds\Big)^2.$$

Applying Hölder's inequality, we get

$$\mathbf{E}\Big(\int_0^t b(s)\, dW(s)\Big)^4 \leq 6\mathbf{E}^{1/2}\Big(\int_0^t b(s)\, dW(s)\Big)^4 \mathbf{E}^{1/2}\Big(\int_0^t b^2(s)\, ds\Big)^2 - 3\mathbf{E}\Big(\int_0^t b^2(s)\, ds\Big)^2.$$

Set

$$z := \mathbf{E}^{1/2}\Big(\int_0^t b(s)\, dW(s)\Big)^4 \Big/ \mathbf{E}^{1/2}\Big(\int_0^t b^2(s)\, ds\Big)^2.$$

Then the previous inequality can be written in the form $z^2 - 6z + 3 \leq 0$. This is equivalent to $3 - \sqrt{6} \leq z \leq 3 + \sqrt{6}$. For nonnegative $z$ this is equivalent to $15 - 6\sqrt{6} \leq z^2 \leq 15 + 6\sqrt{6}$. Finally this implies $0.3 \leq z^2 \leq 30$, and hence (6.22) is proved. □

**Exercises.**

**6.1.** Let $b(s)$, $s \in [0, t]$, be a stochastic process from $\mathcal{L}_2[0, T]$. Prove the estimate

$$c_1\, \mathbf{E}\Big(\int_0^t b^2(s)\, ds\Big)^3 \leq \mathbf{E}\Big(\int_0^t b(s)\, dW(s)\Big)^6 \leq c_2\, \mathbf{E}\Big(\int_0^t b^2(s)\, ds\Big)^3$$

for some positive constants $c_1$ and $c_2$.

## § 7. Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $W(t)$, $t \in [0, T]$, be a Brownian motion with a starting point $x \in \mathbf{R}$, and $\xi \in \mathbf{R}$ be a random variable independent of $W$. Let $\mathcal{F}_t := \sigma\{\xi, W(s), 0 \leq s \leq t\}$ be the $\sigma$-algebra of events generated by the random variable $\xi$ and by the Brownian motion in the interval $[0, t]$.

Let $a(t, x)$ and $b(t, x)$, $t \in [0, T]$, $x \in \mathbf{R}$, be measurable functions.

A process $X(t)$, $t \in [0, T]$, $X(0) = \xi$, is said to be a *strong solution* of the *stochastic differential equation*

$$dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t), \qquad X(0) = \xi, \tag{7.1}$$

if $X$ is a continuous $\mathcal{F}_t$-adapted process such that a.s. for all $t \in [0,T]$

$$\int_0^t (|a(s,X(s))| + |b(s,X(s))|^2)\,ds < \infty \tag{7.2}$$

and

$$X(t) = \xi + \int_0^t a(s,X(s))\,ds + \int_0^t b(s,X(s))\,dW(s). \tag{7.3}$$

Note that due to (7.2) the integrals in (7.3) are well defined. In this section we follow the presentation in the book Gihman and Skorohod (1972).

**1. Existence and uniqueness of solution.**

**Theorem 7.1.** *Suppose that functions $a$ and $b$ satisfy the Lipschitz condition: there exists a constant $C_T$ such that for all $t \in [0,T]$ and $x, y \in \mathbf{R}$,*

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le C_T|x-y|, \tag{7.4}$$

*and the linear growth condition: for all $t \in [0,T]$ and $x \in \mathbf{R}$*

$$|a(t,x)| + |b(t,x)| \le C_T(1+|x|). \tag{7.5}$$

*Let $\mathbf{E}\xi^2 < \infty$.*

*Then there exists a unique strong solution of* (7.1) *satisfying the condition*

$$\sup_{0\le t\le T} \mathbf{E}X^2(t) < \infty. \tag{7.6}$$

**Remark 7.1.** Condition (7.5) follows from (7.4) if $|a(t,0)| + |b(t,0)| \le C_T$ for all $t \in [0,T]$.

**Remark 7.2.** Conditions (7.4) and (7.5) are rather essential even for deterministic equations.

Indeed, the equation

$$\frac{dX(t)}{dt} = X^2(t), \qquad X(0) = 1$$

has the unique solution $X(t) = \dfrac{1}{1-t}$, $t \in [0,1]$. Thus it is impossible to find a solution, for example, in the interval $[0,2]$.

Generally speaking, condition (7.5) that the functions increase no faster than linearly guarantees that the solution $X$ of (7.3) does not *explode*, i.e., $|X(t)|$ does not tend to $\infty$ in a finite time.

Another important example concerns the fact that for $t \in [0,T]$ the equation

$$\frac{dX(t)}{dt} = 3X^{2/3}(t), \qquad X(0) = 0,$$

has more than one solution. Indeed, for any $t_0 \in [0,T]$ the function

$$X(t) = \begin{cases} 0, & \text{for } 0 \le t \le t_0, \\ (t-t_0)^3, & \text{for } t_0 \le t \le T, \end{cases}$$

is a solution. In this case the Lipschitz condition (7.4) is not satisfied.

*Proof of Theorem* 7.1. We first prove the uniqueness. Suppose that there are two continuous solutions satisfying (7.3) and (7.6), i.e.,

$$X_l(t) = \xi + \int_0^t a(s, X_l(s))\, ds + \int_0^t b(s, X_l(s))\, dW(s), \qquad l = 1, 2.$$

Then using the inequality $(g+h)^2 \le 2g^2 + 2h^2$, we get

$$\begin{aligned}\mathbf{E}(X_1(t) - X_2(t))^2 \le 2\mathbf{E}\Big(\int_0^t (a(s, X_1(s)) - a(s, X_2(s)))\, ds\Big)^2 \\ + 2\mathbf{E}\Big(\int_0^t (b(s, X_1(s)) - b(s, X_2(s)))\, dW(s)\Big)^2.\end{aligned}$$

Applying the Hölder inequality for the first term and the isometry property (1.12) for the second one, we have

$$\begin{aligned}\mathbf{E}(X_1(t) - X_2(t))^2 \le 2t \int_0^t \mathbf{E}(a(s, X_1(s)) - a(s, X_2(s))^2\, ds \\ + 2\int_0^t \mathbf{E}(b(s, X_1(s)) - b(s, X_2(s)))^2\, ds.\end{aligned}$$

Now using the Lipschitz condition (7.4) we obtain

$$\mathbf{E}(X_1(t) - X_2(t))^2 \le L \int_0^t \mathbf{E}(X_1(s) - X_2(s))^2\, ds \qquad \text{for all } t \in [0,T], \tag{7.7}$$

where $L = 2(T+1)C_T^2$.

We will often use Gronwall's lemma.

**Lemma 7.1 (Gronwall).** *Let $g(t)$ and $h(t)$, $0 \le t \le T$, be bounded measurable functions and let for some $K > 0$ and all $t \in [0,T]$*

$$g(t) \le h(t) + K \int_0^t g(s)\, ds.$$

*Then*

$$g(t) \le h(t) + K \int_0^t e^{K(t-s)} h(s)\,ds, \qquad t \in [0,T]. \tag{7.8}$$

*If $h$ is nondecreasing, then*

$$g(t) \le h(t)\,e^{Kt}, \qquad t \in [0,T]. \tag{7.9}$$

*Proof.* Set

$$\psi(t) := h(t) + K \int_0^t e^{K(t-s)} h(s)\,ds, \qquad \Delta(t) := \psi(t) - g(t),$$

and note that the function $\Delta(t)$, $t \in [0,T]$, is bounded. Since

$$\Big( \int_0^t e^{K(t-s)} h(s)\,ds \Big)' = h(t) + K \int_0^t e^{K(t-s)} h(s)\,ds = \psi(t),$$

the function $\psi$ satisfies the equation

$$\psi(t) = h(t) + K \int_0^t \psi(s)\,ds,$$

and

$$\Delta(t) \ge K \int_0^t \Delta(s)\,ds, \qquad t \in [0,T].$$

Since $K > 0$, by iteration, we get

$$\Delta(t) \ge K^2 \int_0^t \int_0^s \Delta(u)\,du\,ds = K^2 \int_0^t (t-u)\Delta(u)\,du \ge K^3 \int_0^t (t-u) \int_0^u \Delta(s)\,ds\,du$$

$$= K^3 \int_0^t \frac{(t-u)^2}{2} \Delta(s)\,ds \ge \cdots \ge \frac{K^{n+1}}{n!} \int_0^t (t-s)^n \Delta(s)\,ds.$$

The last term tends to zero as $n \to \infty$, consequently, $\Delta(t) \ge 0$, $t \in [0,T]$, and (7.8) holds. For a nondecreasing function $h$ inequality (7.9) is a simple consequence of (7.8), because

$$g(t) \le h(t) + Kh(t) \int_0^t e^{K(t-s)}\,ds = h(t)\,e^{Kt}.$$

$\square$

Since $\sup\limits_{0\le s\le T} \mathbf{E}\{X_1^2(t)+X_2^2(t)\}<\infty$, using Gronwall's lemma for $h\equiv 0$, we deduce from (7.7) that $\mathbf{E}(X_1(t)-X_2(t))^2=0$ and, consequently, $\mathbf{P}(X_1(t)=X_2(t))=1$ for every $t\in[0,T]$. Therefore, the solutions $X_1$, $X_2$ coincide a.s. for all rational moments of time, and by the continuity of paths $\mathbf{P}\Big(\sup\limits_{0\le t\le T}|X_1(t)-X_2(t)|=0\Big)=1$. The uniqueness is proved.

To prove that there exists a solution of the stochastic equation (7.3) we apply the method of successive approximations.

Set $X_0(t):=\xi$,

$$X_n(t):=\xi+\int_0^t a(s,X_{n-1}(s))\,ds+\int_0^t b(s,X_{n-1}(s))\,dW(s). \tag{7.10}$$

Note that $X_n(t)$ is a continuous $\mathcal{F}_t$-adapted process for every $n$.

By the linear growth condition (7.5), analogously to (7.7), we have

$$\mathbf{E}(X_1(t)-X_0(t))^2\le 2t\int_0^t \mathbf{E}a^2(s,\xi)\,ds+2\int_0^t \mathbf{E}b^2(s,\xi)\,ds\le Lt(1+\mathbf{E}\xi^2)=KLt.$$

We now make the inductive assumption that for $k=n-1$

$$\mathbf{E}(X_{k+1}(t)-X_k(t))^2\le\frac{K(Lt)^{k+1}}{(k+1)!}\qquad\text{for all } t\in[0,T]. \tag{7.11}$$

Then analogously to (7.7) we have

$$\mathbf{E}(X_{n+1}(t)-X_n(t))^2\le L\int_0^t \mathbf{E}(X_n(s)-X_{n-1}(s))^2\,ds$$

$$\le\frac{KL^{n+1}}{n!}\int_0^t s^n\,ds=\frac{K(Lt)^{n+1}}{(n+1)!}.$$

Thus (7.11) holds for $k=n$ and the proof of (7.11) for all $k=0,1,2\dots$ is completed by induction.

The estimate (7.11) will enable us to prove that the processes $X_n(t)$ converge a.s. uniformly in $t\in[0,T]$ to a limit. We apply the estimate

$$\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|\le\int_0^T|a(s,X_n(s))-a(s,X_{n-1}(s))|\,ds$$

$$+\sup_{0\le t\le T}\bigg|\int_0^t(b(s,X_n(s))-b(s,X_{n-1}(s)))\,dW(s)\bigg|.$$

Then using Doob's inequality (2.6) and (1.12), we obtain

$$\mathbf{E}\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|^2 \le 2\mathbf{E}\Bigl(\int_0^T |a(s,X_n(s))-a(s,X_{n-1}(s))|ds\Bigr)^2$$

$$+8\int_0^T \mathbf{E}(b(s,X_n(s))-b(s,X_{n-1}(s)))^2\,ds \le 4L\int_0^t \mathbf{E}(X_n(s)-X_{n-1}(s))^2\,ds$$

$$\le \frac{KL^{n+1}T^{n+1}}{(n+1)!}.$$

By the Chebyshev inequality,

$$\mathbf{P}\Bigl(\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|>\frac{1}{n^2}\Bigr)\le 4n^4K\frac{L^{n+1}T^{n+1}}{(n+1)!}.$$

Since the series of these probabilities converge, by the first part of the Borel–Cantelli lemma, there exists a.s. a number $n_0=n_0(\omega)$ such that for all $n>n_0$

$$\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|\le\frac{1}{n^2}.$$

This implies that the random variables

$$X_n(t)=\xi+\sum_{k=0}^{n-1}\bigl(X_{k+1}(t)-X_k(t)\bigr) \tag{7.12}$$

converge uniformly in $t\in[0,T]$ to

$$X(t)=\xi+\sum_{k=0}^{\infty}\bigl(X_{k+1}(t)-X_k(t)\bigr)$$

i.e.,

$$\mathbf{P}\Bigl(\lim_{n\to\infty}\sup_{0\le t\le T}|X_n(t)-X(t)|=0\Bigr)=1.$$

Therefore $X(t)$, $t\in[0,T]$, is a continuous $\mathcal{F}_t$-adapted process and, in view of (7.5), we see that (7.2) holds.

Using (7.4) and the uniform convergence of $X_n$ to $X$, one can pass to the limit in (7.10). We have $a(t,X_n(t))\to a(t,X(t))$ and $b(t,X_n(t))\to b(t,X(t))$ a.s. uniformly in $t\in[0,T]$, and

$$\int_0^T\bigl(b(t,X_n(t))-b(t,X(t))\bigr)^2\,dt\to 0 \qquad \text{a.s.}$$

We now can apply (3.6) and, by passage to the limit in (7.10), prove that the process $X$ is the solution of equation (7.3).

To complete the proof of Theorem 7.1 it remains to prove that this solution satisfies (7.6).

From the inequality $\Bigl(\sum_{k=1}^{n} c_k\Bigr)^2 \le n\sum_{k=1}^{n} c_k^2$, and (7.11), (7.12) we get that $\mathbf{E}X_n^2(t) \le C(n+1)$ for some $C$. From (7.10) we have

$$\mathbf{E}X_{n+1}^2(t) \le 3\mathbf{E}\xi^2 + 3\mathbf{E}\biggl(\int_0^t a(s, X_n(s))\,ds\biggr)^2 + 3\mathbf{E}\biggl(\int_0^t b(s, X_n(s))\,dW(s)\biggr)^2.$$

Applying Hölder's inequality to the second term and the isometry property (1.12) to the third term, and using (7.5), we obtain

$$\mathbf{E}X_{n+1}^2(t) \le 3\mathbf{E}\xi^2 + 3T\int_0^t \mathbf{E}a^2(s, X_n(s))\,ds + 3\int_0^t \mathbf{E}b^2(s, X_n(s))\,ds$$

$$\le 3\mathbf{E}\xi^2 + 3L\int_0^t (1 + \mathbf{E}X_n^2(s))\,ds \le M + M\int_0^t \mathbf{E}X_n^2(s)\,ds.$$

for some constant $M$. By iteration, we get that for all $t \in [0, T]$

$$\mathbf{E}X_{n+1}^2(t) \le M + M^2 t + M^3\frac{t^2}{2!} + \cdots + M^{n+2}\frac{t^{n+1}}{(n+1)!}.$$

Therefore, $\mathbf{E}X_{n+1}^2(t) \le Me^{Mt}$. By Fatou's lemma,

$$\mathbf{E}X^2(t) \le Me^{Mt}. \qquad (7.13)$$

This proves (7.6). □

**2. Local dependence of solutions on coefficients.**

The meaning of the assertions presented below is the following. If for two stochastic differential equations with the same initial value the coefficients coincide for all time moments and for the spatial variable from some interval, then the solutions of these equations coincide up to the first exit time from this interval.

**Theorem 7.2.** *Suppose that the coefficients $a_1$, $b_1$ and $a_2$, $b_2$ of the stochastic differential equations*

$$dX_l(t) = a_l(t, X_l(t))\,dt + b_l(t, X_l(t))\,dW(t), \qquad X_l(0) = \xi, \qquad (7.14)$$

$l = 1, 2$, *satisfy conditions* (7.4), (7.5) *and* $a_1(t,x) = a_2(t,x)$, $b_1(t,x) = b_2(t,x)$, *for* $(t,x) \in [0,T] \times [-N,N]$ *with some* $N > 0$. *Let* $\mathbf{E}\xi^2 < \infty$.

*Let* $X_l$, $l = 1, 2$, *be the strong solutions of* (7.14) *and* $H_l := \max\{t \in [0,T] : \sup_{0\le s\le t} |X_l(s)| \le N\}$. *Then* $\mathbf{P}(H_1 = H_2) = 1$ *and*

$$\mathbf{P}\Bigl(\sup_{0\le s\le H_1} |X_1(s) - X_2(s)| = 0\Bigr) = 1. \qquad (7.15)$$

*Proof.* Set

$$\varphi_1(t)=\begin{cases}1, & \text{if } \sup\limits_{0\le s\le t}|X_1(s)|\le N,\\ 0, & \text{if } \sup\limits_{0\le s\le t}|X_1(s)|>N.\end{cases}$$

It is clear that $\varphi_1(t)=1$ iff $t\in[0,H_1]$. Since given the event $\{\varphi_1(t)=1\}$ we have $a_1(s,X_1(s))=a_2(s,X_1(s))$ and $b_1(s,X_1(s))=b_2(s,X_1(s))$ for all $s\in[0,t]$, one can write

$$\varphi_1(t)(X_1(t)-X_2(t))=\varphi_1(t)\int_0^t(a_2(s,X_1(s))-a_2(s,X_2(s)))\,ds$$

$$+\,\varphi_1(t)\int_0^t(b_2(s,X_1(s))-b_2(s,X_2(s)))\,dW(s).$$

Since the equality $\varphi_1(t)=1$ implies $\varphi_1(s)=1$ for all $s\le t$, using the Lipschitz condition (7.4), we obtain analogously to (7.7) that for all $t\in[0,T]$

$$\mathbf{E}\big\{\varphi_1(t)(X_1(t)-X_2(t))^2\big\}\le L\int_0^t\mathbf{E}\big\{\varphi_1(s)(X_1(s)-X_2(s))^2\big\}\,ds. \tag{7.16}$$

By Gronwall's lemma, $\mathbf{E}\big\{\varphi_1(t)(X_1(t)-X_2(t))^2\big\}=0$. Since the processes $X_1(t)$ and $X_2(t)$ are continuous, we get

$$\mathbf{P}\Big(\sup_{0\le t\le T}(\varphi_1(t)(X_1(t)-X_2(t))^2)=0\Big)=1.$$

This implies that in the interval $[0,H_1]$ the processes $X_1(t)$ and $X_2(t)$ coincide a.s. Therefore, $H_2\ge H_1$ a.s. Switching the indices 1 and 2, we have that $H_1\ge H_2$ a.s. Consequently, $H_1=H_2$ a.s. and (7.15) holds. □

**3. Local Lipschitz condition.**

In Theorem 7.1 condition (7.4) can be weakened to the local Lipschitz condition.

**Theorem 7.3.** *Suppose that the functions $a(t,x)$ and $b(t,x)$ satisfy the local Lipschitz condition: for every $N>0$ there exists a constant $C_{N,T}$ such that for all $t\in[0,T]$ and $x,y\in[-N,N]$*

$$|a(t,x)-a(t,y)|+|b(t,x)-b(t,y)|\le C_{N,T}|x-y|, \tag{7.17}$$

*and the linear growth condition: for all $t\in[0,T]$ and $x\in\mathbf{R}$*

$$|a(t,x)|+|b(t,x)|\le C_T(1+|x|). \tag{7.18}$$

*Then there exists a unique strong solution of* (7.1).

**Remark 7.3.** The condition (7.17) holds if there exists $\frac{\partial}{\partial x}a(t,x)$ and $\frac{\partial}{\partial x}b(t,x)$ continuous in $(t,x)\in[0,T]\times\mathbf{R}$.

*Proof of Theorem* 7.3. The proof involves a truncation procedure. We prove first the existence of a solution of (7.1).

Set

$$\xi_N := \xi \mathbb{1}_{\{|\xi|\le N\}} + N \operatorname{sign} \xi \mathbb{1}_{\{|\xi|>N\}},$$

$$a_N(t,x) := a(t,x)\mathbb{1}_{[0,N]}(|x|) + a(t, N \operatorname{sign} x)\mathbb{1}_{(N,\infty)}(|x|)$$

and

$$b_N(t,x) := b(t,x)\mathbb{1}_{[0,N]}(|x|) + b(t, N \operatorname{sign} x)\mathbb{1}_{(N,\infty)}(|x|).$$

Let $X_N(t)$ be a solution of the stochastic differential equation

$$dX_N(t) = a_N(t, X_N(t))\,dt + b_N(t, X_N(t))\,dW(t), \qquad X_N(0) = \xi_N. \tag{7.19}$$

For equation (7.19) all conditions of Theorem 7.1 holds. Therefore there exists a unique continuous solution of this equation satisfying the estimate

$$\sup_{0\le t\le T} \mathbf{E} X_N^2(t) < \infty.$$

Set $H_N := \max\{t \in [0,T] : \sup\limits_{0\le s\le t} |X_N(s)| \le N\}$. Since for $N' > N$ we have $a_N(t,x) = a_{N'}(t,x)$ and $b_N(t,x) = b_{N'}(t,x)$ for $x \in [-N,N]$, by Theorem 7.2, $X_N(t) = X_{N'}(t)$ for all $t \in [0, H_N]$ a.s. Therefore,

$$\{H_N = T\} \subseteq \Big\{ \sup_{N'>N} \sup_{0\le t\le T} |X_N(t) - X_{N'}(t)| = 0 \Big\}$$

and, consequently,

$$\mathbf{P}\Big( \sup_{N'>N} \sup_{0\le t\le T} |X_N(t) - X_{N'}(t)| > 0 \Big) \le \mathbf{P}(H_N < T) = \mathbf{P}\Big( \sup_{0\le t\le T} |X_N(t)| > N \Big).$$

Next we will prove that

$$\lim_{N\to\infty} \mathbf{P}\Big( \sup_{0\le t\le T} |X_N(t)| > N \Big) = 0. \tag{7.20}$$

Once this is done, then from the previous estimate and the first part of the Borel–Cantelli lemma, it follows that for a sufficiently scarce subsequence $N_n$ there exists a.s. a number $n_0 = n_0(\omega)$ such that for all $N_n \ge N_{n_0}$

$$\sup_{N'>N_n} \sup_{0\le t\le T} |X_{N_n}(t) - X_{N'}(t)| = 0.$$

Therefore, by Cauchy's criterion, the sequence of processes $X_{N_n}(t)$, $t \in [0,T]$, is Cauchy in the uniform norm for the a.s. convergence. Thus, $X_{N_n}(t)$ converges to a limit process $X(t)$ uniformly in $t \in [0,T]$. In the stochastic equation

$$X_N(t) = \xi_N + \int_0^t a_N(s, X_N(s))\,ds + \int_0^t b_N(s, X_N(s))\,dW(s). \tag{7.21}$$

we can pass to the limit as $N_n \to \infty$. The usual integral converges in view of the estimates (7.17), (7.18). To justify the convergence of the stochastic integral we can use the same estimates and (3.6). As a result, we see that the process $X(t)$, $t \in [0,T]$, is the strong solution of equation (7.1).

Thus it is enough to prove (7.20). Using (7.21), it is easy to prove that for any $t \in [0,T]$

$$\sup_{0\le s\le t} |X_N(s)|^2 \le 3\xi_N^2 + 3T\int_0^t a_N^2(s, X_N(s))\, ds + 3 \sup_{0\le s\le t}\biggl(\int_0^s b_N(u, X_N(u))\, dW(u)\biggr)^2.$$

We multiply this inequality by $\psi(\xi)$, where $\psi(x) = \dfrac{1}{1+x^2}$. Then using (2.6), (7.18) and the estimate $\xi_N^2\psi(\xi) \le 1$, we get

$$\mathbf{E}\bigl\{\psi(\xi)X_N^2(t)\bigr\} \le \mathbf{E}\Bigl\{\psi(\xi) \sup_{0\le s\le t} X_N^2(s)\Bigr\} \le 3 + 3TC_T^2 \int_0^t \mathbf{E}\bigl\{\psi(\xi)(1+X_N^2(s))\bigr\}\, ds$$

$$+ 12C_T^2 \int_0^t \mathbf{E}\bigl\{\psi(\xi)(1+X_N^2(s))\bigr\}\, ds.$$

By Gronwall's lemma, we have $\mathbf{E}\bigl\{\psi(\xi)X_N^2(t)\bigr\} \le C$ for $t \in [0,T]$ and for some constant $C$. Consequently,

$$\mathbf{E}\Bigl\{\psi(\xi) \sup_{0\le s\le T} X_N^2(s)\Bigr\} \le C_1$$

for some constant $C_1$ independent of $N$.

One has the estimates

$$\mathbf{P}\Bigl(\sup_{0\le t\le T} |X_N(t)| > N\Bigr) = \mathbf{P}\Bigl(\psi(\xi) \sup_{0\le t\le T} X_N^2(t) > N^2\psi(\xi)\Bigr)$$

$$\le \mathbf{P}\Bigl(\psi(\xi) \sup_{0\le t\le T} X_N^2(t) > \delta N^2\Bigr) + \mathbf{P}(\psi(\xi) \le \delta) \le \frac{C_1}{\delta N^2} + \mathbf{P}(\psi(\xi) \le \delta),$$

for any $\delta > 0$. This implies

$$\limsup_{N\to\infty} \mathbf{P}\Bigl(\sup_{0\le t\le T} |X_N(t)| > N\Bigr) \le \mathbf{P}(\psi(\xi) \le \delta) = \mathbf{P}(\xi^2 \ge (1-\delta)/\delta).$$

But $\lim\limits_{\delta\downarrow 0} \mathbf{P}(\xi^2 \ge (1-\delta)/\delta) = 0$. This proves (7.20) and, consequently, the existence of the solution of (7.1).

We now prove uniqueness of a solution of (7.1). Let $X_1(t)$ and $X_2(t)$ be two a.s. continuous solutions of (7.1), satisfying the initial condition $X_1(0) = X_2(0) = \xi$. Set

$$\varphi_N(t) := 1\!\!1_{[0,N]}\Bigl(\sup_{0\le v\le t} |X_1(v)|\Bigr) 1\!\!1_{[0,N]}\Bigl(\sup_{0\le v\le t} |X_2(v)|\Bigr).$$

Using the local Lipshitz condition, the isometry property (1.12) for stochastic integrals, and the fact that the equality $\varphi_N(t) = 1$ implies the equalities $\varphi_N(s) = 1$ for all $s \le t$, we can obtain (see the analogous estimate (7.7))

$$\mathbf{E}\big\{\varphi_N(t)(X_1(t) - X_2(t))^2\big\} \le 2t \int_0^t \mathbf{E}\big\{\varphi_N(s)(a(s, X_1(s)) - a(s, X_2(s)))^2\big\}\, ds$$

$$+ 2 \int_0^t \mathbf{E}\big\{\varphi_N(s)(b(s, X_1(s)) - b(s, X_2(s)))^2\big\}\, ds$$

$$\le 2(T + 1)C_N^2 \int_0^t \mathbf{E}\big\{\varphi_N(s)(X_1(s) - X_1(s))^2\big\}\, ds.$$

By Gronwall's lemma, $\mathbf{E}\big\{\varphi_N(t)(X_1(t) - X_2(t))^2\big\} = 0$. Therefore, for arbitrary $N$

$$\mathbf{P}\Big(\sup_{0\le t\le T} |X_1(t) - X_2(t)| > 0\Big) \le \mathbf{P}\Big(\sup_{0\le t\le T} |X_1(t)| > N\Big) + \mathbf{P}\Big(\sup_{0\le t\le T} |X_2(t)| > N\Big).$$

From the continuity of the solutions $X_1$ and $X_2$ it follows that their suprema are finite. This implies that the probabilities on the right-hand side of this estimate tend to zero as $N \to \infty$. Thus $\mathbf{P}\Big(\sup_{0\le t\le T} |X_1(t) - X_2(t)| = 0\Big) = 1$, and this means uniqueness of the solution of (7.1). □

It is convenient to have estimates for the moments of even order of the solution of the stochastic differential equation (7.1).

**Theorem 7.4.** *Suppose that the functions $a(t, x)$ and $b(t, x)$ satisfy the conditions of Theorem 7.3. Let $\mathbf{E}\xi^{2m} < \infty$, where $m$ is a positive integer. Then*

$$\mathbf{E}X^{2m}(t) \le \big(\mathbf{E}\xi^{2m} + Kt\big)e^{2Kt}, \tag{7.22}$$

*and for $s < t$*

$$\mathbf{E}(X(t) - X(s))^{2m} \le \widetilde{K}T\big(1 + Kt + \mathbf{E}\xi^{2m}\big)(1 + (t - s)^m)(t - s)^m e^{2Kt}, \tag{7.23}$$

*for some constants $K$ and $\widetilde{K}$, depending only on $m$ and $C_T$.*

*Proof.* We use the notations from the proof of Theorem 7.3. Since, by (7.18), the variables $\xi_N$ and the functions $a_N$ and $b_N$ are bounded by the constant $C_T(1+N)$, then from (7.21) and (4.25) it follows that $\mathbf{E}X_N^{2m}(t) \le C_T N^{2m}(1+t^{2m})$. Applying Itô's formula to $X_N^{2m}(t)$, we get

$$X_N^{2m}(t) = \xi_N^{2m}$$

$$+ \int_0^t \Big(2mX_N^{2m-1}(s)a_N(s, X_N(s)) + m(2m - 1)X_N^{2m-2}(s)b_N^2(s, X_N(s))\Big)ds$$

$$+\int_0^t 2mX_N^{2m-1}(s)b_N(s,X_N(s))\,dW(s).$$

Then

$$\mathbf{E}X_N^{2m}(t)=\mathbf{E}\xi_N^{2m}+\int_0^t\Big(2m\mathbf{E}\big\{X_N^{2m-1}(s)a_N(s,X_N(s))\big\}$$

$$+m(2m-1)\mathbf{E}\big\{X_N^{2m-2}(s)b_N^2(s,X_N(s))\big\}\Big)ds$$

$$\le \mathbf{E}\xi^{2m}+(2m+3)mC_T^2\int_0^t \mathbf{E}\big\{\big(1+X_N^2(s)\big)X_N^{2m-2}(s)\big\}\,ds.$$

Applying the obvious inequality $x^{2m-2}\le 1+x^{2m}$, we have

$$\mathbf{E}X_N^{2m}(t)\le \mathbf{E}\xi^{2m}+(2m+3)mC_T^2t+2(2m+3)mC_T^2\int_0^t \mathbf{E}X_N^{2m}(s)\,ds.$$

By Gronwall's lemma (see (7.9)),

$$\mathbf{E}X_N^{2m}(t)\le\big(\mathbf{E}\xi^{2m}+(2m+3)mC_T^2t\big)\exp\big(2(2m+3)mC_T^2t\big).$$

Therefore, by Fatou's lemma this implies (7.22), since $X_N(t)\to X(t)$.

We now prove (7.23). Obviously

$$\mathbf{E}(X(t)-X(s))^{2m}\le \mathbf{E}\bigg(\int_s^t a(v,X(v))\,dv+\int_s^t b(v,X(v))\,dW(v)\bigg)^{2m}$$

$$\le 2^{2m-1}\bigg(\mathbf{E}\bigg(\int_s^t a(v,X(v))\,dv\bigg)^{2m}+\mathbf{E}\bigg(\int_s^t b(v,X(v))\,dW(v)\bigg)^{2m}\bigg).$$

Using (4.25) with $L_k=2^kk^{2k}\Big(\frac{2k}{2k-1}\Big)^{(2k-1)k}$, Hölder's inequality, (7.18), and (7.22), we get

$$\mathbf{E}(X(t)-X(s))^{2m}\le(2(t-s))^{2m-1}\int_s^t\mathbf{E}a^{2m}(v,X(v))\,dv+L_m\mathbf{E}\bigg(\int_s^t b^2(v,X(v))\,dv\bigg)^m$$

$$\le(2(t-s))^{2m-1}\int_s^t\mathbf{E}a^{2m}(v,X(v))\,dv+L_m(t-s)^{m-1}\int_s^t\mathbf{E}b^{2m}(v,X(v))\,dv$$

$$\leq 2^{2m-1}C_T^{2m}(L_m+(t-s)^m)(t-s)^{m-1}\int_s^t (1+\mathbf{E}X^{2m}(v))\,dv$$

$$\leq 2^{4m-2}(t-s)^{m-1}C_T^{2m}(L_m+(t-s)^m)\int_s^t \big(1+(\mathbf{E}\xi^{2m}+Kv)e^{2Kv}\big)\,dv$$

$$\leq C_T^{2m}L_mK2^{4m-1}(1+(t-s)^m)(t-s)^m(1+Kt+\mathbf{E}\xi^{2m})e^{2Kt}.$$

□

**4. Multi-dimensional stochastic differential equations.**

Consider the vector-valued stochastic differential equations.

Let $\vec{W}(t)=(W_1(t),\dots,W_m(t))$, $t\in[0,T]$, be $m$-dimensional Brownian motion with independent coordinates, which are one-dimensional Brownian motions with the initial values $W_k(0)=x_k$, $k=1,2,\dots,m$. Let the random vector $\vec{\xi}\in\mathbf{R}^n$ be independent of the process $\vec{W}$ and let $\mathcal{F}_t:=\sigma\{\vec{\xi},\vec{W}(s),0\leq s\leq t\}$ be the $\sigma$-algebra of events generated by $\vec{\xi}$ and the Brownian motions $W_k$, $k=1,2,\dots,m$, in $[0,t]$.

Let $\vec{a}(t,\vec{x})$, $t\in[0,T]$, $\vec{x}\in\mathbf{R}^n$, be a measurable function with the state space $\mathbf{R}^n$ and $\mathbb{B}(t,\vec{x})$ be an $n\times m$ matrix with measurable real-valued functions as elements. Denote by $|\vec{a}|$ the Euclidean norm of the vector $\vec{a}$. Set $|\mathbb{B}|:=\Big(\sum_{k=1}^n\sum_{l=1}^m b_{k,l}^2\Big)^{1/2}$ for matrixes $\mathbb{B}$ with elements $\{b_{k,l}\}_{k=1,l=1}^{n,m}$.

Consider the $n$-dimensional stochastic differential equation

$$d\vec{X}(t)=\vec{a}(t,\vec{X}(t))\,dt+\mathbb{B}(t,\vec{X}(t))\,d\vec{W}(t),\qquad \vec{X}(0)=\vec{\xi}. \tag{7.24}$$

In coordinates this equation becomes the system of stochastic differential equations

$$dX_k(t)=a_k(t,X_1(t),\dots,X_n(t))dt$$

$$+\sum_{l=1}^m b_{k,l}(t,X_1(t),\dots,X_n(t))dW_l(t),\quad X_k(0)=\xi_k,\qquad k=1,2,\dots,n.$$

The process $\vec{X}$ is a *strong solution* of (7.24) if it is a continuous $\mathcal{F}_t$-adapted process such that a.s. for all $t\in[0,T]$,

$$\int_0^t \big(|\vec{a}(s,\vec{X}(s))|+|\mathbb{B}(s,\vec{X}(s))|^2\big)\,ds<\infty \tag{7.25}$$

and

$$\vec{X}(t)=\vec{\xi}+\int_0^t \vec{a}(s,\vec{X}(s))\,ds+\int_0^t \mathbb{B}(s,\vec{X}(s))\,d\vec{W}(s). \tag{7.26}$$

**Theorem 7.5.** *Suppose that $\vec{a}(t,\vec{x})$ and $\mathbb{B}(t,\vec{x})$ satisfy the Lipschitz condition: there exists a constant $C_T$ such that for all $t \in [0,T]$ and $\vec{x},\vec{y} \in \mathbf{R}^n$*

$$|\vec{a}(t,\vec{x}) - \vec{a}(t,\vec{y})| + |\mathbb{B}(t,\vec{x}) - \mathbb{B}(t,\vec{y})| \le C_T|\vec{x} - \vec{y}|, \tag{7.27}$$

*and the condition: for all $t \in [0,T]$ and $\vec{x} \in \mathbf{R}^n$*

$$|\vec{a}(t,\vec{x})| + |\mathbb{B}(t,\vec{x})| \le C_T(1 + |\vec{x}|). \tag{7.28}$$

*Let $\mathbf{E}|\vec{\xi}|^2 < \infty$. Then there exists a unique strong solution of* (7.24) *satisfying the condition*

$$\sup_{0\le t\le T} \mathbf{E}|\vec{X}(t)|^2 < \infty. \tag{7.29}$$

By standard techniques of linear algebra, the proof of this theorem follows essentially the proof of Theorem 7.1 for the one-dimensional case.

**Remark 7.4.** The $m$-dimensional Brownian motion $\vec{W}_\circ(t)$, $t \in [0,T]$, with dependent coordinates can be obtained from $\vec{W}(t)$ with independent coordinates by a linear transformation. This means that there exists an $m \times m$ matrix $\mathbb{C}$ such that $\vec{W}_\circ(t) = \mathbb{C}\vec{W}(t)$.

This is due to the fact that the matrix of variances of the Brownian motion $\vec{W}_\circ(t)$ is positive definite, and then

$$\mathrm{Var}(\vec{W}_\circ(1)) = \mathbb{C}^T\,\mathbb{C}$$

for some matrix $\mathbb{C}$. Here the symbol $T$ stands for the transposition of matrices. It is unnecessary to consider the analog of equation (7.24) for the Brownian motion $\vec{W}_\circ$, since

$$\mathbb{B}(t,\vec{X}(t))\,d\vec{W}_\circ(t) = \mathbb{B}(t,\vec{X}(t))\mathbb{C}\,d\vec{W}(t).$$

**Exercises.**

**7.1.** Let $\vec{W}(t)$, $t \in [0,T]$, be an $m$-dimensional Brownian motion with independent coordinates and $\mathcal{F}_t := \sigma\{\vec{W}(s), 0 \le s \le t\}$ be the $\sigma$-algebra of events generated by the Brownian motions $W_k$, $k = 1,2,\dots,m$, in $[0,t]$.

Let $\mathbb{B}(t)$, be an $n\times m$ matrix with progressively measurable processes as elements. Let

$$d\vec{X}(t) = \mathbb{B}(t)\,d\vec{W}(t), \qquad \vec{X}(0) = \vec{x}_0.$$

Prove that the process

$$M(t) := |\vec{X}(t)|^2 - \int_0^t |\mathbb{B}(s)|^2 ds$$

is a martingale with respect to the filtration $\mathcal{F}_t$.

**7.2.** Under the assumptions of Exercise 7.1, prove for $r \in \mathbb{N}$ the formula

$$\begin{aligned} d|\vec{X}(t)|^{2r} &= 2r|\vec{X}(t)|^{2r-2}(\vec{X}(t))^T\mathbb{B}(t)\,d\vec{W}(t) \\ &\quad + \Big(2r(r-1)|\vec{X}(t)|^{2r-4}|(\vec{X}(t))^T\mathbb{B}(t)|^2 + r|\vec{X}(t)|^{2r-2}|\mathbb{B}(t)|^2\Big)dt. \end{aligned}$$

## § 8. Methods of solving of stochastic differential equations

**1. Stochastic exponent.** We already have an example of a stochastic differential equation and its solution. This is the stochastic exponent (see § 6)

$$\rho(t) = \exp\biggl(\int_0^t b(s)\,dW(s) - \frac{1}{2}\int_0^t b^2(s)\,ds\biggr),$$

which is the solution of the equation

$$d\rho(t) = b(t)\rho(t)\,dW(t), \qquad \rho(0) = 1. \tag{8.1}$$

The state space of the solution of this stochastic equation is the positive real line.

**2. Linear stochastic differential equation.** The general form of the linear stochastic differential equation is

$$dX(t) = (a(t)X(t) + r(t))\,dt + (b(t)X(t) + q(t))\,dW(t), \qquad X(0) = x_0. \tag{8.2}$$

This equation also has an explicit solution.

For the product of the stochastic exponent $\rho(t)$ and the ordinary exponent $\rho_0(t) = \exp\Bigl(\int_0^t a(s)\,ds\Bigr)$ we have

$$d\bigl(\rho_0(t)\rho(t)\bigr) = \rho(t)\,d\rho_0(t) + \rho_0(t)\,d\rho(t) = \rho_0(t)\rho(t)\bigl(a(t)\,dt + b(t)\,dW(t)\bigr).$$

Therefore, the solution of the homogeneous linear stochastic differential equation

$$dY(t) = a(t)Y(t)\,dt + b(t)Y(t)\,dW(t), \qquad Y(0) = 1. \tag{8.3}$$

is the product of the ordinary exponent and the stochastic one:

$$Y(t) = \exp\biggl(\int_0^t b(s)\,dW(s) + \int_0^t \bigl(a(s) - \tfrac{1}{2}b^2(s)\bigr)\,ds\biggr).$$

It is well known how the solutions of the ordinary nonhomogeneous linear equations are expressed via the solutions of homogeneous ones. We can expect that the solution of equation (8.2) has the same structure. It can be checked by direct computation that the solution of (8.2) is

$$X(t) = Y(t)\biggl\{x_0 + \int_0^t q(s)Y^{-1}(s)\,dW(s) + \int_0^t (r(s) - b(s)q(s))Y^{-1}(s)\,ds\biggr\}. \tag{8.4}$$

Indeed,

$$dX(t) = X(t)\{a(t)\,dt + b(t)\,dW(t)\} + q(t)\,dW(t) + (r(t) - b(t)q(t))\,dt$$

$$+b(t)q(t)\,dt = (a(t)X(t)+r(t))\,dt + (b(t)X(t)+q(t))\,dW(t).$$

Note that for the case $q(t)\equiv 0$ and $r(t)\geq 0$, $t\geq 0$, the state space of the linear stochastic differential equation with initial value $x_0>0$ is the positive real line.

If the ratio $q(t)/b(t)$ is well defined for all $t\geq 0$, then equation (8.2) can be transformed to the equation with $q(t)\equiv 0$ by means of a shift of the variable $X$. Indeed, (8.2) can be rewritten in the form

$$dX(t)=\Big\{a(t)\big(X(t)+\tfrac{q(t)}{b(t)}\big)+r(t)-a(t)\tfrac{q(t)}{b(t)}\Big\}dt+b(t)\Big(X(t)+\tfrac{q(t)}{b(t)}\Big)dW(t),\quad X(0)=x_0.$$

Setting $Z(t):=X(t)+\dfrac{q(t)}{b(t)}$ and assuming that $\dfrac{q(t)}{b(t)}$ is a differentiable function, we have

$$dZ(t)=\Big\{a(t)Z(t)+r(t)-a(t)\tfrac{q(t)}{b(t)}+\Big(\tfrac{q(t)}{b(t)}\Big)'\Big\}\,dt+b(t)Z(t)\,dW(t),\quad Z(0)=x_0+\tfrac{q(0)}{b(0)}. \tag{8.5}$$

Therefore, by (8.4) with $q(s)\equiv 0$, the solution of (8.2) can be written in the form

$$X(t)=Y(t)\Big\{x_0+\frac{q(0)}{b(0)}+\int_0^t\Big(r(s)-a(s)\frac{q(s)}{b(s)}+\Big(\frac{q(s)}{b(s)}\Big)'\Big)Y^{-1}(s)\,ds\Big\}-\frac{q(t)}{b(t)}. \tag{8.6}$$

For the case when $r(t)-a(t)\dfrac{q(t)}{b(t)}+\Big(\dfrac{q(t)}{b(t)}\Big)'\equiv 0$, $t\geq 0$, or, equivalently,

$$\frac{q(t)}{b(t)}=\frac{q(0)}{b(0)}\exp\Big(\int_0^t a(s)\,ds\Big)-\int_0^t r(s)\exp\Big(\int_s^t a(v)\,dv\Big)\,ds,$$

we have the *shifted stochastic exponent*

$$X(t)=\Big(x_0+\frac{q(0)}{b(0)}\Big)\exp\Big(\int_0^t b(s)\,dW(s)+\int_0^t\Big(a(s)-\frac{1}{2}b^2(s)\Big)\,ds\Big)-\frac{q(t)}{b(t)}$$

as the solution of equation (8.2).

**3. Nonrandom time change.** For a nonrandom function $h(t)$ that is different from zero for all $t\geq 0$ and satisfies $\int_0^t h^2(s)\,ds<\infty$, the processes $\int_0^t h(s)\,dW(s)$ and $W\Big(\int_0^t h^2(s)\,ds\Big)$, $W(0)=0$, are identical in law. Indeed, they are Gaussian processes with mean 0 and the variance $\int_0^t h^2(s)\,ds$. Also, these processes have independent increments.

Moreover, we can write

$$\int_0^t h(s)\,d\widetilde{W}(s) = W\Big(\int_0^t h^2(s)\,ds\Big)$$

for some new Brownian motion $\widetilde{W}(t)$, $t \ge 0$. Indeed,

$$\widetilde{W}(t) = \int_0^t h^{-1}(s)\,dW\Big(\int_0^s h^2(v)\,dv\Big).$$

The process $W\Big(\int_0^t h^2(s)\,ds\Big)$ has independent increments, and there are well-developed techniques of integration with respect to such processes (see §9 Ch. I).

Making the time substitution $t \to \int_0^t h^2(s)\,ds$ in the stochastic differential equation

$$dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dW(t), \qquad X(0) = x_0, \tag{8.7}$$

we have that the process $V(t) := X\Big(\int_0^t h^2(s)\,ds\Big)$ satisfies the following stochastic differential equation:

$$dV(t) = a\Big(\int_0^t h^2(s)\,ds, V(t)\Big)h^2(t)\,dt + b\Big(\int_0^t h^2(s)\,ds, V(t)\Big)h(t)\,d\widetilde{W}(t), \quad V(0) = x_0. \tag{8.8}$$

Such a time substitution enables us to change the coefficients $a(t,x)$ and $b(t,x)$ as functions of the time variable.

**4. Random time change.** Let $W(t)$, $t \in [0,\infty)$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t\ge0}$. Suppose that the increments $W(v) - W(t)$ are independent of the $\sigma$-algebra $\mathcal{F}_t$ for all $v > t$.

Let $b(t)$, $t \in [0,\infty)$, be a progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\ge0}$ process such that $\theta(t) := \int_0^t b^2(s)\,ds < \infty$ a.s. for every $t > 0$. Consider the stochastic integral $Y(t) := \int_0^t b(s)\,dW(s)$, $t \in [0,\infty)$. Suppose that $\int_0^\infty b^2(s)\,ds = \infty$. Let $\theta^{(-1)}(t) := \min\{s : \theta(s) = t\}$ be the left continuous inverse function to $\theta$, defined for all $t \ge 0$. Since for every $s$

$$\{\theta^{(-1)}(t) > s\} = \Big\{\int_0^s b^2(v)\,dv < t\Big\} \in \mathcal{F}_s,$$

the variable $\theta^{(-1)}(t)$ is a stopping time with respect to the filtration $\{\mathcal{F}_s\}_{s\ge0}$.

**Theorem 8.1 (Lévy).** *The process* $\widetilde{W}(t) := Y(\theta^{(-1)}(t))$, $t \ge 0$ *is a Brownian motion.*

*Proof.* For $\vec{\gamma} := (\gamma_1, \dots, \gamma_n) \in \mathbf{R}^n$ and $0 < t_1 < \cdots < t_n$ denote $B := \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_k \gamma_l (t_k \wedge t_l)$, where $s \wedge t := \min\{s, t\}$. To prove that $\widetilde{W}(t)$, $t \in [0, \infty)$, is a Brownian motion, it is enough to verify the following formula for the characteristic function:

$$\varphi(\vec{\gamma}) := \mathbf{E} \exp\Big( i \sum_{k=1}^{n} \gamma_k \widetilde{W}(t_k) \Big) = e^{-B/2},$$

since then $\widetilde{W}$ is a Gaussian process with mean zero and the covariance function $\mathrm{Cov}(\widetilde{W}(s), \widetilde{W}(t)) = s \wedge t$, but it is a Brownian motion.

We have

$$\varphi(\vec{\gamma}) e^{B/2} = \mathbf{E} \exp\Big( i \sum_{k=1}^{n} \gamma_k \int_0^{\theta^{(-1)}(t_k)} b(s)\, dW(s) + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_k \gamma_l (t_k \wedge t_l) \Big)$$

$$= \mathbf{E} \exp\Big( i \sum_{k=1}^{n} \gamma_k \int_0^{\theta^{(-1)}(t_k)} b(s)\, dW(s) + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_k \gamma_l \int_0^{\theta^{(-1)}(t_k) \wedge \theta^{(-1)}(t_l)} b^2(s)\, ds \Big)$$

$$= \mathbf{E} \exp\Big( i \int_0^{\infty} g(s)\, dW(s) + \frac{1}{2} \int_0^{\infty} g^2(s)\, ds \Big),$$

where $g(s) := b(s) \sum_{k=1}^{n} \gamma_k \mathbb{1}_{[0, \theta^{(-1)}(t_k)]}(s)$. Note that from the above transformations it follows that $\int_0^{\infty} g^2(s)\, ds = B$. The process

$$\rho(t) := \mathbf{E} \exp\Big( i \int_0^{t} g(s)\, dW(s) + \frac{1}{2} \int_0^{t} g^2(s)\, ds \Big)$$

is a uniformly integrable stochastic exponent (in formula (6.1) instead of $b(s)$ we have $ig(s)$). By (6.3) and (6.4), it is a complex-valued martingale (the real and the imaginary parts are martingales), and $\mathbf{E}\rho(t) = 1$ for every $t$. Letting here $t \to \infty$, we get $\mathbf{E}\rho(\infty) = 1$. As a result, we have that $\varphi(\vec{\gamma}) e^{B/2} = 1$. This proves the theorem. □

Consider a nonhomogeneous stochastic differential equation

$$dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t), \qquad X(0) = x_0, \tag{8.9}$$

with $b(t, x) > 0$ for all $(t, x) \in [0, \infty) \times \mathbf{R}$.

Let $\theta(t) := \int\limits_0^t b^2(s, X(s))\,ds < \infty$ a.s. for every $t > 0$ and $\theta(\infty) = \infty$. Let $\theta^{(-1)}(t) := \min\{s : \theta(s) = t\}$, $t \ge 0$, be the left continuous function inverse of $\theta$.

Set $Y(t) := X(\theta^{(-1)}(t))$, $\widetilde{W}(t) := \int\limits_0^{\theta^{(-1)}(t)} b(s, X(s))\,dW(s)$. By Theorem 8.1, the process $\widetilde{W}$ is a Brownian motion. In equation (8.9) we can make the random time substitution. This yields

$$Y(t) - Y(0) = \int\limits_0^{\theta^{(-1)}(t)} a(s, X(s))\,ds + \int\limits_0^{\theta^{(-1)}(t)} b(s, X(s))\,dW(s)$$

$$= \int\limits_0^t a(\theta^{(-1)}(s), X(\theta^{(-1)}(s)))\,d\theta^{(-1)}(s) + \widetilde{W}(t) = \int\limits_0^t \frac{a(\theta^{(-1)}(s), Y(s))}{b^2(\theta^{(-1)}(s), Y(s))}\,ds + \widetilde{W}(t),$$

because

$$d\theta^{(-1)}(s) = \frac{ds}{\theta'(\theta^{(-1)}(s))} = \frac{ds}{b^2(\theta^{(-1)}(s), Y(s))}.$$

Thus in equation (8.9), by the random time substitution, the coefficient before the stochastic differential is transformed to 1. We get the equation

$$dY(t) = \frac{a(\theta^{(-1)}(t), Y(t))}{b^2(\theta^{(-1)}(t), Y(t))}\,dt + d\widetilde{W}(t), \qquad Y(0) = x_0. \tag{8.10}$$

A feature of this equation is that the coefficient $\frac{a(\theta^{(-1)}(t), Y(t))}{b^2(\theta^{(-1)}(t), Y(t))}$ depends on the stopping time $\theta^{(-1)}(t)$. Since these stopping times are increasing in $t$, there exists an increasing family of $\sigma$-algebras $\mathcal{A}_t := \mathcal{F}_{\theta^{(-1)}(t)}$ connected with them (see § 4 Ch. I). The process $\theta^{(-1)}(t)$, $t \ge 0$, is adapted (see property 8 § 4 Ch. I) to the filtration $\{\mathcal{A}_t\}_{t\ge 0}$. Since the process $\int\limits_0^t b(s, X(s))\,dW(s)$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$, the Brownian motion $\widetilde{W}(t)$ is also adapted to the filtration $\{\mathcal{A}_t\}_{t\ge 0}$ (see § 4 Ch. I). In addition, for all $v > t$ the increments $\widetilde{W}(v) - \widetilde{W}(t)$ are independent of the $\sigma$-algebra $\mathcal{A}_t$. This can be proved in the following way. Analogously to the proofs of properties (2.3) and (2.4), we can verify that a.s.

$$\mathbf{E}\{\widetilde{W}(v) - \widetilde{W}(t) \big| \mathcal{A}_t\} = \mathbf{E}\left\{ \int\limits_{\theta^{(-1)}(t)}^{\theta^{(-1)}(v)} b(s, X(s))\,dW(s) \middle| \mathcal{A}_t \right\} = 0,$$

$$\mathbf{E}\{(\widetilde{W}(v) - \widetilde{W}(t))^2 \big| \mathcal{A}_t\} = \mathbf{E}\left\{ \left( \int\limits_{\theta^{(-1)}(t)}^{\theta^{(-1)}(v)} b(s, X(s))\,dW(s) \right)^2 \middle| \mathcal{A}_t \right\}$$

$$= \mathbf{E}\Biggl\{ \int\limits_{\theta^{(-1)}(t)}^{\theta^{(-1)}(v)} b^2(s, X(s))\, ds \Bigg| \mathcal{A}_t \Biggr\} ds = v - t.$$

Since the process of $\widetilde{W}$ is continuous, by Lévy's characterization (see Theorem 10.1 Ch. I) these two properties guarantee that $\widetilde{W}$ is a Brownian motion. This is a different proof than the one given above in Theorem 8.1. At the end of the proof of Theorem 10.1 Ch. I it was established that for any $0 \le t < v$ and $\alpha \in \mathbf{R}$,

$$\mathbf{E}\bigl\{e^{i\alpha(\widetilde{W}(v)-\widetilde{W}(t))}\big|\mathcal{A}_t\bigr\} = e^{-\alpha^2(v-t)/2} \qquad \text{a.s.}$$

This equality is equivalent to the fact that the random variables $\widetilde{W}(v) - \widetilde{W}(t)$ are independent of the $\sigma$-algebra $\mathcal{A}_t$.

The adaptivity of the random coefficient of equation (8.10) with the filtration $\{\mathcal{A}_t\}_{t\ge0}$ and the independence of $\widetilde{W}(v) - \widetilde{W}(t)$ for every $v > t$ of the $\sigma$-algebra $\mathcal{A}_t$ enables us to prove that there exists a unique solution of such an equation.

**5. Reduction to linear stochastic differential equations.** Consider the homogeneous stochastic differential equation

$$dX(t) = a(X(t))\, dt + b(X(t))\, dW(t), \qquad X(0) = x_0, \tag{8.11}$$

where the coefficients $a$ and $b$ are independent of the time variable.

For this equation we describe a method of reduction to a linear stochastic differential equation. Let $f(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function which has the inverse $f^{(-1)}(x)$.

Set $\widetilde{X}(t) := f(X(t))$, $t \ge 0$. Write out the stochastic differential equation for the process $\widetilde{X}(t)$. Applying Itô's formula, we have

$$df(X(t)) = f'(X(t))a(X(t))\, dt + f'(X(t))b(X(t))\, dW(t) + \frac{1}{2}f''(X(t))b^2(X(t))\, dt.$$

Setting

$$\tilde{a}(x) := f'(f^{(-1)}(x))a(f^{(-1)}(x)) + \frac{1}{2}f''(f^{(-1)}(x))b^2(f^{(-1)}(x)), \tag{8.12}$$

$$\tilde{b}(x) := f'(f^{(-1)}(x))b(f^{(-1)}(x)), \tag{8.13}$$

and using the equality $X(t) = f^{(-1)}(\widetilde{X}(t))$, we obtain

$$d\widetilde{X}(t) = \tilde{a}(\widetilde{X}(t))\, dt + \tilde{b}(\widetilde{X}(t))\, dW(t), \qquad \widetilde{X}(0) = f(x_0). \tag{8.14}$$

Therefore, the substitution $\widetilde{X}(t) = f(X(t))$ reduces equation (8.11) to equation (8.14).

We derive conditions on the coefficients $a(x)$ and $b(x)$, $x \in \mathbf{R}$, under which equation (8.11) can be further reduced to a linear one, i.e., $\tilde{a}(x) = \alpha x + r$ and

$\tilde{b}(x) = \beta x + q$ for some constants $\alpha$, $\beta$, $r$, and $q$. For these coefficients the linear stochastic differential equation has (see (8.6)) the solution of the form

$$\widetilde{X}(t) = \Big(f(x_0) + \frac{q}{\beta}\Big) e^{\beta(W(t)-W(0))+(\alpha-\beta^2/2)t}$$

$$+ \frac{\beta r - \alpha q}{\beta} \int_0^t e^{\beta(W(t)-W(s))+(\alpha-\beta^2/2)(t-s)}\, ds - \frac{q}{\beta}.$$

Using expressions (8.12) and (8.13), for the coefficients $\tilde{a}$ and $\tilde{b}$, we get

$$f'(x)a(x) + \frac{1}{2} f''(x) b^2(x) = \alpha f(x) + r, \tag{8.15}$$

$$f'(x)b(x) = \beta f(x) + q. \tag{8.16}$$

Suppose that $b(x)$ is a continuously differentiable function such that $b(x) \neq 0$ for all $x$ from the state space of the process $X$.

In the case $\beta \neq 0$ the first-order differential equation (8.16) has the solution

$$f(x) = \frac{c}{\beta} \exp\bigg(\beta \int_{x_0}^{x} \frac{dy}{b(y)}\bigg) - \frac{q}{\beta}, \tag{8.17}$$

where $c$ is some constant and $x_0$ is the starting point of $X$.

By (8.16),

$$f''(x)b(x) + b'(x)f'(x) = \beta f'(x),$$

or

$$f''(x)b^2(x) = (\beta - b'(x))(\beta f(x) + q). \tag{8.18}$$

Substituting this expression in equation (8.15), we have

$$\Big(\frac{a(x)}{b(x)} + \frac{1}{2}(\beta - b'(x))\Big)(\beta f(x) + q) = \alpha f(x) + r, \tag{8.19}$$

or

$$\Big(\frac{a(x)}{b(x)} - \frac{b'(x)}{2} + \frac{\beta}{2} - \frac{\alpha}{\beta}\Big)\Big(f(x) + \frac{q}{\beta}\Big) = \frac{\beta r - \alpha q}{\beta^2}.$$

In the case $\beta = 0$ equation (8.16) has the solution

$$f(x) = q \int_{x_0}^{x} \frac{dy}{b(y)} + h, \tag{8.20}$$

where $h$ is some constant, and (8.19) for $\beta = 0$ is transformed to the equality

$$\frac{a(x)}{b(x)} - \frac{b'(x)}{2} = \alpha \int_{x_0}^{x} \frac{dy}{b(y)} + \frac{\alpha h + r}{q}. \tag{8.21}$$

Thus, we can formulate the following statement.

**Proposition 8.1.** *For a continuously differentiable coefficient $b(x)$ that is different from zero for all $x$ from the state space of $X$, the homogeneous stochastic differential equation* (8.11) *is reducible to the homogeneous linear stochastic differential equation in the following cases.*

*If for some constants $\alpha$, $r$, $q$ and $\beta \neq 0$, $c \neq 0$*

$$a(x) = b(x)\biggl(\frac{b'(x)}{2} + \frac{\alpha}{\beta} - \frac{\beta}{2} + \frac{\beta r - \alpha q}{\beta c} \exp\biggl(-\beta \int\limits_{x_0}^{x} \frac{dy}{b(y)}\biggr)\biggr), \tag{8.22}$$

*then the process $\widetilde{X}(t) := \frac{c}{\beta} \exp\biggl(\beta \int\limits_{x_0}^{X(t)} \frac{dy}{b(y)}\biggr) - \frac{q}{\beta}$ satisfies the equation*

$$d\widetilde{X}(t) = (\alpha \widetilde{X}(t) + r)\,dt + (\beta \widetilde{X}(t) + q)\,dW(t), \qquad \widetilde{X}(0) = \frac{c-q}{\beta}. \tag{8.23}$$

*If for some constants $\alpha$, $r$, $h$, and $q \neq 0$,*

$$a(x) = b(x)\biggl(\frac{b'(x)}{2} + \alpha \int\limits_{x_0}^{x} \frac{dy}{b(y)} + \frac{\alpha h + r}{q}\biggr), \tag{8.24}$$

*then the process $\widetilde{X}(t) := q \int\limits_{x_0}^{X(t)} \frac{dy}{b(y)} + h$ satisfies the equation*

$$d\widetilde{X}(t) = (\alpha \widetilde{X}(t) + r)\,dt + q\,dW(t), \qquad \widetilde{X}(0) = h. \tag{8.25}$$

**6. Reduction to ordinary differential equations.** Consider a nonhomogeneous stochastic differential equation

$$dX(t) = a(t, X(t))\,dt + b(t)X(t)\,dW(t), \qquad X(0) = x_0. \tag{8.26}$$

Here it is important that the coefficient in front of the stochastic differential is linear. Let $\rho(t)$ be the stochastic exponent

$$\rho(t) = \exp\biggl(\int\limits_0^t b(s)\,dW(s) - \frac{1}{2}\int\limits_0^t b^2(s)\,ds\biggr).$$

Compute the stochastic differential of the process $X(t)\rho^{-1}(t)$. By Itô's formula,

$$d(X(t)\rho^{-1}(t)) = \rho^{-1}(t)\{a(t, X(t))\,dt + b(t)X(t)\,dW(t)\}$$
$$-X(t)\rho^{-2}(t)b(t)\rho(t)\,dW(t) + X(t)\rho^{-3}(t)b^2(t)\rho^2(t)\,dt$$
$$-\rho^{-2}(t)b(t)X(t)b(t)\rho(t)\,dt = \rho^{-1}(t)a(t, X(t))\,dt.$$

Setting $Z(t) := X(t)\rho^{-1}(t)$, we see that this process satisfies the ordinary differential equation

$$Z'(t) = \rho^{-1}(t)\, a(t, \rho(t)Z(t)), \qquad Z(0) = x_0. \tag{8.27}$$

Therefore, multiplying the solution of equation (8.26) by the factor $\rho^{-1}(t)$, we transform this stochastic differential equation into the "deterministic" differential equation, which is valid for each sample path of the process $\rho(t)$, $t \geq 0$.

Equation (8.27) is rather complicated for the investigations, since it has nowhere differentiable coefficient $\rho(t)$. An example of the class of functions $a(t,x)$, $t \geq 0$, $x \geq 0$, for which it has an explicit solution is

$$a(t,x) = a(t)x^{\gamma}.$$

Nevertheless, equation (8.27) can be useful for numerical computations.

Combining the approaches described above and in Subsection 5, we can reduce an arbitrary homogeneous stochastic differential equation

$$dX(t) = a(X(t))\, dt + b(X(t))\, dW(t), \qquad X(0) = x_0, \tag{8.28}$$

with continuously differentiable coefficient $b(x)$ that is different from zero for all $x$ from the state space of $X$, to a "deterministic" differential equation, which is valid for each sample path of the process.

Consider the function $f$, defined by (8.17) for $q = 0$, $\beta = 1$ and $c = 1$, i.e.,

$$f(x) = \exp\biggl(\int_{x_0}^{x} \frac{dy}{b(y)}\biggr). \tag{8.29}$$

The function $B(x) := \int_{x_0}^{x} \frac{dy}{b(y)}$ has the inverse $B^{(-1)}(x)$. This implies that the function $f(x)$ has the inverse $f^{(-1)}(x) = B^{(-1)}(\ln x)$.

According to (8.12), (8.16), and (8.18), the process $\widetilde{X}(t) = \exp(B(X(t)))$ satisfies the stochastic differential equation

$$d\widetilde{X}(t) = \tilde{a}(\widetilde{X}(t))\, dt + \widetilde{X}(t)\, dW(t), \qquad \widetilde{X}(0) = 1, \tag{8.30}$$

where

$$\tilde{a}(x) = x\Bigl(\frac{a(B^{(-1)}(\ln x))}{b(B^{(-1)}(\ln x))} + \frac{1}{2} - \frac{b'(B^{(-1)}(\ln x))}{2}\Bigr). \tag{8.31}$$

Then equation (8.30) takes the form (8.26) with $b(t) \equiv 1$. Therefore, the process $Z(t) = \widetilde{X}(t)\, e^{W(0)-W(t)+t/2}$ satisfies the ordinary differential equation

$$Z'(t) = e^{W(0)-W(t)+t/2}\, \tilde{a}(Z(t)\, e^{W(t)-W(0)-t/2}), \qquad Z(0) = 1. \tag{8.32}$$

Finally,

$$X(t) = B^{(-1)}(\ln \widetilde{X}(t)) = B^{(-1)}(\ln Z(t) + W(t) - W(0) - t/2).$$

**Exercises.**

**8.1.** Solve the linear stochastic differential equation

$$dV(t) = \mu V(t)\,dt + \sigma V(t)\,dW(t), \qquad V(0) = x_0 > 0.$$

**8.2.** Solve the linear stochastic differential equation

$$dU(t) = -\gamma U(t)\,dt + \sigma\,dW(t), \qquad U(0) = x_0 > 0.$$

**8.3.** Solve the linear stochastic differential equation

$$dZ(t) = (\beta Z(t) + \sigma)\,dW(t), \qquad Z(0) = x_0 > 0.$$

**8.4.** Solve the stochastic differential equation

$$dX(t) = \frac{1}{X(t)}\,dt + \beta X(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

**8.5.** Solve the stochastic differential equation

$$dX(t) = X^{-\gamma}(t)\,dt + \beta X(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

**8.6.** Solve the stochastic differential equation

$$dX(t) = aX(t)(1 - gX(t))\,dt + bX(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

**8.7.** Let $\sigma(x)$, $x \in \mathbf{R}$, be a function with bounded derivative such that the integral

$$S(x) := \int_{x_0}^{x} \frac{1}{\sigma(y)}\,dy$$

is finite for all $x \in \mathbf{R}$.

Let the process $X(t)$, $t \in [0,T]$, be a solution of the stochastic differential equation

$$dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))\,dt + \sigma(X(t))\,dW(t), \qquad X(0) = x_0.$$

Prove that $Z(t) := S(X(t))$, $t \in [0,T]$, is a Brownian motion.

**8.8.** Solve for $\gamma \neq 1$ the stochastic differential equation

$$dX(t) = \big(aX(t) - gX^{\gamma}(t)\big)\,dt + bX(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

**8.9.** Solve for integer $m \neq 1$ the stochastic differential equation

$$dX(t) = \Big(\frac{m}{2}X^{2m-1}(t) + \mu X^{m}(t)\Big)\,dt + X^{m}(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

**8.10.** Under what conditions on the parameters $a$, $b$, $n$, for $m \neq 1$ is the stochastic differential equation

$$dX(t) = aX^n(t)\,dt + bX^m(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

reducible to the linear one? What is the solution?

**8.11.** Let $X(t)$ be a solution of the stochastic differential equation

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t), \qquad X(0) = x_0,$$

with a continuously differentiable coefficient $b(x)$ different from zero for all $x$ from the state space of $X$.

Compute the function $f(x)$ such that the process $\widetilde{X}(t) = f(X(t))$, $t \geq 0$, satisfies the equation containing only the pure stochastic differential (the factor before $dt$ is zero).

## § 9. Dependence of solutions of stochastic differential equations on initial values

Consider a stochastic differential equation with nonrandom initial value:

$$dX_x(t) = a(t, X_x(t))\,dt + b(t, X_x(t))\,dW(t), \qquad X_x(0) = x. \tag{9.1}$$

The integral analog of this equation is the following:

$$X_x(t) = x + \int_0^t a(s, X_x(s))\,ds + \int_0^t b(s, X_x(s))\,dW(s). \tag{9.2}$$

**1. Continuous dependence of solutions on initial values.** We want to consider solutions of (9.2) for all $x \in \mathbf{R}$ simultaneously. Moreover, it is better to consider them as a process of $(t, x)$.

Then the following problem arises. A solution of the stochastic differential equation (9.2), as it was proved in § 7, exists a.s. It can depend on the initial value. Therefore, there is a set $\Lambda_x$ of probability zero, where the solution does not exists. The probability of the union of the sets $\Lambda_x$ can be nonzero. In this case the solutions of (9.2) are not determined as a function of $x$ on a set of nonzero probability. We had the analogous situation when considering the stochastic integral as a function of the variable upper limit.

The main approach to overcome this difficulty is to prove that the process $X_x(t)$, $(t, x) \in [0, T] \times \mathbf{R}$, can be chosen to be continuous.

**Theorem 9.1.** *Suppose that the functions $a(t, x)$ and $b(t, x)$ satisfy the conditions* (7.4) *and* (7.5). *Then there exist a modification $X_x(t)$ of a solution of* (9.1) a.s. *jointly continuous in $(t, x) \in [0, T] \times \mathbf{R}$.*

*If $X(t)$, $t \in [0,T]$, is the solution of the equation*

$$X(t) = \xi + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dW(s), \tag{9.3}$$

*with a square integrable random initial value $\xi$ independent of the process $W(t)$, $t \in [0,T]$, then*

$$\mathbf{P}\Big( \sup_{0\le t\le T} |X(t) - X_\xi(t)| = 0 \Big) = 1. \tag{9.4}$$

*Proof.* From (9.2) it is easy to derive that

$$\sup_{0\le v\le t} |X_x(v) - X_y(v)| \le |x-y| + \int_0^t |a(s, X_x(s)) - a(s, X_y(s))|\, ds$$

$$+ \sup_{0\le v\le t} \Big| \int_0^v (b(s, X_x(s)) - b(s, X_y(s)))\, dW(s) \Big|.$$

Applying Doob's inequality (2.6) and estimating as in (7.7), we get

$$\mathbf{E} \sup_{0\le v\le t} (X_x(v) - X_y(v))^2 \le 3|x-y|^2 + 3\mathbf{E}\Big( \int_0^t |a(s, X_x(s)) - a(s, X_y(s))|\, ds \Big)^2$$

$$+12 \int_0^t \mathbf{E}(b(s, X_x(s)) - b(s, X_y(s)))^2 ds \le 3|x-y|^2 + 3(T+4)C_T^2 \int_0^t \mathbf{E}(X_x(s) - X_y(s))^2 ds$$

$$\le 3|x-y|^2 + 3(T+4)C_T^2 \int_0^t \mathbf{E} \sup_{0\le v\le s} (X_x(v) - X_y(v))^2 ds.$$

By (7.6), from the second inequality it follows that $\mathbf{E} \sup\limits_{0\le v\le t} (X_x(v) - X_y(v))^2$, $t \in [0,T]$, is bounded. Then by Gronwall's lemma (see (7.9)),

$$\mathbf{E} \sup_{0\le v\le t} (X_x(v) - X_y(v))^2 \le 3|x-y|^2 e^{3(T+4)C_T^2 t}, \qquad t \in [0,T]. \tag{9.5}$$

Now we can apply arguments similar to those used to prove the continuity of $J_x(t)$, $(t,x) \in [0,T] \times \mathbf{R}$, in § 5. For every fixed $x$ the process $X_x(t)$ is continuous in $t$. Let us consider $X_x(\cdot)$ as a random variable taking values in the space $C([0,T])$ of continuous functions on $[0,T]$. This space, when equipped with the uniform norm $\|f\| := \sup\limits_{t\in[0,T]} |f(t)|$ is a Banach space. Then (9.5) can be written as

$$\mathbf{E}\|X_x - X_y\|^2 \le M_T |x-y|^2. \tag{9.6}$$

Applying Kolmogorov's continuity criterion in the form (5.7), (5.8), we obtain that for any $0 < \gamma < 1/2$ and $N > 0$ a.s.

$$\|X_x - X_y\| \le L_{N,\gamma}(\omega)|x-y|^\gamma, \qquad x, y \in D \bigcap [-N, N], \tag{9.7}$$

where $D$ is the set of dyadic (binary rational) points $k/2^n$ of $\mathbf{R}$. Since $D$ is countable, Theorem 7.1 shows that a.s. for every $y \in D$ and for all $t \in [0,T]$ there exists a unique solution $X_y(t)$ of the equation

$$X_y(t) = y + \int_0^t a(s, X_y(s))\, ds + \int_0^t b(s, X_y(s))\, dW(s). \tag{9.8}$$

Using (9.7) we can a.s. extend the processes $X_y$ from the dyadic set $D$ to the whole real line by setting $X_x(t) = \lim_{y \to x, y \in D} X_y(t)$. This limit is uniform in $t \in [0,T]$, therefore, $X_x(t)$ is a.s. continuous in $t \in [0,T]$ simultaneously for all $x$. Due to this, (7.4), and (3.6), the passage to the limit in (9.8) as $y \to x$, $y \in D$, proves that $X_x(t)$ is a.s. the solution of equation (9.2) for all $(t,x) \in [0,T] \times \mathbf{R}$.

From (9.7) we get that for any $0 < \gamma < 1/2$ the processes $X_x$ a.s. satisfy the Hölder condition

$$\sup_{0 \le t \le T} |X_x(t) - X_y(t)| \le L_{N,\gamma}(\omega)|x-y|^\gamma, \qquad x, y \in [-N, N], \tag{9.9}$$

for all integer $N$.

The joint continuity of the solution $X_x(t)$ in $(t,x)$ follows from (9.9) and from the continuity of $X_x(t)$ in $t$ for all $x$. Indeed, for arbitrary $x, y \in [-N, N]$ and $s, t \in [0,T]$

$$|X_x(t) - X_y(s)| \le |X_x(t) - X_x(s)| + \|X_x(\cdot) - X_y(\cdot)\|. \tag{9.10}$$

Substituting in (9.2) instead of $x$ the random variable $\xi$, satisfying the condition of Theorem 9.1, we have a.s for all $t \in [0,T]$

$$X_\xi(t) = \xi + \int_0^t a(s, X_\xi(s))\, ds + \int_0^t b(s, X_\xi(s))\, dW(s). \tag{9.11}$$

To explain this equality we do the following. Set $\xi_n(\omega) := \sum_{-\infty}^{\infty} \frac{k}{n} \mathbb{1}_{\Omega_{k,n}}(\omega)$, where $\Omega_{k,n} = \bigl\{\omega : \xi(\omega) \in \bigl[\frac{k}{n}, \frac{k+1}{n}\bigr)\bigr\}$. Then from (9.2), applied for $x = \frac{k}{n}$ and a.s. all $\omega \in \Omega_{k,n}$, we have for all $t \in [0,T]$ the equation

$$X_{\xi_n}(t) = \xi_n + \int_0^t a(s, X_{\xi_n}(s))\, ds + \int_0^t b(s, X_{\xi_n}(s))\, dW(s).$$

Since $|\xi - \xi_n| \le 1/n$, applying (9.9), we can pass to the limit in this equation. As a result, we have (9.11). This stochastic equation coincides with (9.3), therefore by the uniqueness of the solution, the processes $X_\xi(t)$ and $X(t)$, $t \in [0,T]$, are indistinguishable in the sense of (9.4). $\square$

Now we are going to prove a generalization of Theorem 9.1. Consider the family $X_{s,x}(t)$, $0 \le s \le t \le T$, of solutions of the stochastic differential equation

$$X_{s,x}(t) = x + \int_s^t a(q, X_{s,x}(q))\,dq + \int_s^t b(q, X_{s,x}(q))\,dW(q), \qquad t \in [s,T]. \quad (9.12)$$

**Theorem 9.2.** *Suppose that the functions $a(t,x)$ and $b(t,x)$ satisfy conditions* (7.4), (7.5). *Then there exists a modification $X_{s,x}(t)$ of a solution of* (9.12) a.s. *jointly continuous in $(s,t,x)$, $0 \le s \le t \le T$, $x \in \mathbf{R}$. Moreover, for the solution $X(t)$, $t \in [0,T]$, of equation* (9.3) *the following equality holds* a.s. *for all $s \in [0,T]$ :*

$$X(t) = X_{s,X(s)}(t), \qquad t \in [s,T], \quad (9.13)$$

*Proof.* The main approach to the proof of this result is the same as for Theorem 9.1, but there are some technical differences. Set $a_s(q,x) := a(q,x)\mathbb{1}_{[s,\infty)}(q)$, $b_s(q,x) := b(q,x)\mathbb{1}_{[s,\infty)}(q)$. Consider the stochastic differential equation

$$\widetilde{X}_{s,x}(t) = x + \int_0^t a_s(q, \widetilde{X}_{s,x}(q))\,dq + \int_0^t b_s(q, \widetilde{X}_{s,x}(q))\,dW(q), \qquad t \in [0,T]. \quad (9.14)$$

The conditions of Theorem 7.1 and 7.4 are satisfied, so there exists a unique strong solution of (9.14), obeying the estimate

$$\sup_{0\le t\le T} \mathbf{E}\widetilde{X}_{s,x}^{2m}(t) < K_{m,x,T}, \quad (9.15)$$

where $m$ is a positive integer. It is clear that $\widetilde{X}_{s,x}(t) = x\mathbb{1}_{[0,s]}(t) + X_{s,x}(t)\mathbb{1}_{[s,T]}(t)$.

From (9.14) we deduce the estimate

$$\sup_{0\le v\le t} |\widetilde{X}_{s,x}(v) - \widetilde{X}_{u,y}(v)| \le |x-y| + \int_0^t |a_s(q, \widetilde{X}_{s,x}(q)) - a_u(q, \widetilde{X}_{u,y}(q))|\,dq$$

$$+ \sup_{0\le v\le t} \left| \int_0^v (b_s(q, \widetilde{X}_{s,x}(q)) - b_u(q, \widetilde{X}_{u,y}(q)))\,dW(q) \right|.$$

Note that for $s < u$

$$a_s(q,x) - a_u(q,y) = a(q,x)\mathbb{1}_{[s,u]}(q) + (a(q,x) - a(q,y))\mathbb{1}_{[u,T]}(q).$$

Using conditions (7.4), (7.5), we have

$$|a_s(q,x)-a_u(q,y)|\le C_T((1+|x|)1\!\!1_{[s,u]}(q)+|x-y|).$$

Then, applying the analogous inequality for $b_s(q,x)$, (4.25), (9.15) and the Hölder inequality, we get for $x,y\in[-N,N]$ and for $t\in[0,T]$ the estimate

$$\mathbf{E}\sup_{0\le v\le t}(\widetilde X_{s,x}(v)-\widetilde X_{u,y}(v))^{2m}$$

$$\le K_{m,T,N}\biggl(|x-y|^{2m}+|u-s|^m+\int_0^t\mathbf{E}\bigl(\widetilde X_{s,x}(q)-\widetilde X_{u,y}(q)\bigr)^{2m}dq\biggr).$$

$$\le K_{m,T,N}\biggl(|x-y|^{2m}+|u-s|^m+\int_0^t\mathbf{E}\sup_{0\le v\le q}\bigl(\widetilde X_{s,x}(v)-\widetilde X_{u,y}(v)\bigr)^{2m}dq\biggr).$$

Finally, by Gronwall's lemma (see (7.9)),

$$\mathbf{E}\sup_{0\le v\le t}(\widetilde X_{s,x}(v)-\widetilde X_{u,y}(v))^{2m}\le\widetilde K_{m,T,N}\bigl(|x-y|^{2m}+|u-s|^m\bigr).$$

Applying Kolmogorov's continuity criterion (the analog of Theorem 3.3 for processes with values in Banach spaces), we deduce that for every $0<\gamma<1/2$ and $N>0$, a.s. for all $s,u\in D\bigcap[0,T]$ and $x,y\in D\bigcap[-N,N]$

$$\sup_{0\le v\le t}\bigl|\widetilde X_{s,x}(v)-\widetilde X_{u,y}(v)\bigr|\le L_{N,T,\gamma}(\omega)\bigl(|u-s|^\gamma+|x-y|^\gamma\bigr),\tag{9.16}$$

where $D$ is the set of dyadic (binary rational) points.

Using (9.16) we can a.s. extend the processes $\widetilde X_{u,y}$ from the dyadic set $D\times D$ to the whole real plane by setting

$$\widetilde X_{s,x}(t)=\lim_{y\to x,y\in D}\lim_{u\to s,u\in D}\widetilde X_{u,y}(t),\qquad s\in[0,T],\ x\in[-N,N].$$

This limit is uniform in $t\in[0,T]$, therefore $\widetilde X_{s,x}(t)$ is a.s. continuous in $t\in[0,T]$ for all $s,x$ simultaneously. Due to this, (7.4), and (3.6), the passage to the limit in (9.14) as $s_n\to s$, $x_n\to x$, $(s_n,x_n)\in D\times D$, proves that $\widetilde X_{s,x}(t)$ is a.s. the solution of equation (9.14) for all $(s,t,x)\in[0,T]^2\times\mathbf{R}$. Analogously to the proof of Theorem 9.1, one establishes that the process $\widetilde X_{s,x}(t)$, $(s,t,x)\in[0,T]^2\times\mathbf{R}$, is continuous and, consequently, the same is true for the process $X_{s,x}(t)$, $0\le s\le t\le T$, $x\in\mathbf{R}$. The equality (9.13) also holds (see the end of the proof of Theorem 9.1). □

**Remark 9.1.** Let $\tau$ be a stopping time with respect to the filtration $\mathcal{F}_t:=\sigma\{\xi,W(s),0\le s\le t\}$, where $\xi$ is a random variable independent of the Brownian motion $W$. Then for the set $\{\tau \le t \le T\}$ the following stochastic differential equation

$$X_{\tau,x}(t) = x + \int_\tau^t a(q, X_{\tau,x}(q))\, dq + \int_\tau^t b(q, X_{\tau,x}(q))\, dW(q), \qquad t \in [\tau, T], \quad (9.17)$$

makes sense.

This is due to the fact that for a fixed $x$ the processes $a_\tau(q,x) := a(q,x)1\!\!\mathrm{I}_{[\tau,\infty)}(q)$ and $b_\tau(q,x) := b(q,x)1\!\!\mathrm{I}_{[\tau,\infty)}(q)$ are adapted to the filtration $\mathcal{F}_q$, $q \ge 0$.

**2. Differentiability of solutions with respect to the initial value.** Consider the question of differentiability of the solution $X_x$ with respect to the initial value $x$. Since $X_x$ is a random function, we should treat the derivative with respect to $x$ in the mean square sense. If for a random function $Z(x)$, $x \in \mathbf{R}$, there exists the random function $V(x)$ such that

$$\lim_{\Delta \to 0} \mathbf{E}\Big(\frac{Z(x+\Delta) - Z(x)}{\Delta} - V(x)\Big)^2 = 0,$$

we call $V(x)$ the *mean square derivative* of $Z$ and set $\frac{d}{dx} Z(x) := V(x)$.

The mean square differentiability is important, for example, for the proof that the function $u(x) := \mathbf{E} f(X_x(t))$ is continuously differentiable, where $f(x)$, $x \in \mathbf{R}$, has a continuous bounded first derivative.

**Theorem 9.3.** *Suppose that the functions $a(t,z)$ and $b(t,z)$, $(t,z) \in [0,T] \times \mathbf{R}$, are continuous, and have continuous bounded partial derivatives $a'_z(t,z)$ and $b'_z(t,z)$ with respect to $z$.*

*Then the continuous solution $X_x(t)$ of* (9.1) *has a stochastically continuous in $(t,x) \in [0,T] \times \mathbf{R}$ mean square derivative $X^{(1)}_x(t) := \frac{d}{dx} X_x(t)$, which satisfies the equation*

$$X^{(1)}_x(t) = 1 + \int_0^t a'_z(s, X_x(s)) X^{(1)}_x(s)\, ds + \int_0^t b'_z(s, X_x(s)) X^{(1)}_x(s)\, dW(s). \quad (9.18)$$

**Remark 9.2.** Under the assumptions of Theorem 9.3, the functions $a$ and $b$ satisfy the Lipschitz condition (7.4) and the linear growth condition (7.5).

**Remark 9.3.** The function $X^{(1)}_x(t)$ satisfying equation (9.18) has the form (see (8.3))

$$X^{(1)}_x(t) = \exp\Big( \int_0^t b'_z(s, X_x(s))\, dW(s) + \int_0^t \big\{a'_z(s, X_x(s)) - \tfrac{1}{2}\big(b'_z(s, X_x(s))\big)^2\big\} ds \Big). \quad (9.19)$$

This derivative is positive and, therefore for every fixed $t$, the process $X_x(t)$ is a.s. an increasing function with respect to $x$.

*Proof of Theorem* 9.3. We start with an auxiliary result.

**Lemma 9.1.** *Let $a_\Delta(t)$ and $b_\Delta(t)$, $t \in [0,T]$, $\Delta \in [-1,1]$, be a family of uniformly bounded progressively measurable processes, i.e., $|a_\Delta(t)| \le K$, $|b_\Delta(t)| \le K$ for all $t \in [0,T]$ and some nonrandom constant $K$. Suppose that for every $\Delta$ a progressively measurable process $Y_\Delta(t)$, $t \in [0,T]$, satisfies the equation*

$$Y_\Delta(t) = 1 + \int_0^t a_\Delta(s) Y_\Delta(s)\, ds + \int_0^t b_\Delta(s) Y_\Delta(s)\, dW(s). \tag{9.20}$$

*Then for any $p \in \mathbf{R}$,*

$$\mathbf{E} Y_\Delta^p(t) \le e^{|p|(|p|K+1)Kt}. \tag{9.21}$$

*If $a_\Delta(t) \to a_0(t)$ and $b_\Delta(t) \to b_0(t)$ as $\Delta \to 0$ in probability for every $t \in [0,T]$, then $Y_\Delta(t) \to Y_0(t)$ in probability and in mean square, where $Y_0$ is the solution of* (9.20) *for $\Delta = 0$.*

*Proof.* We note first that $Y_\Delta(t)$ can be represented (see (8.3)) in the form

$$Y_\Delta(t) = \exp\biggl( \int_0^t b_\Delta(s)\, dW(s) + \int_0^t \bigl(a_\Delta(s) - \tfrac{1}{2} b_\Delta^2(s)\bigr)\, ds \biggr) \tag{9.22}$$

and, consequently, $Y_\Delta(t)$ is a nonnegative process.

Using the Hölder inequality and (6.13), we have

$$\mathbf{E} Y_\Delta^p(t) \le \mathbf{E}^{1/2} \exp\biggl( 2p \int_0^t b_\Delta(s)\, dW(s) \biggr) \mathbf{E}^{1/2} \exp\biggl( 2p \int_0^t a_\Delta(s)\, ds \biggr) \le e^{p^2 K^2 t} e^{|p|Kt}.$$

Note that the estimate (9.21) is valid for both positive and negative $p$.

Since the coefficients $a_\Delta$ and $b_\Delta$, $\Delta \in [-1,1]$, are uniformly bounded and they converge in probability, they converge also in mean square. Therefore,

$$\lim_{\Delta \to 0} \int_0^t \mathbf{E}(b_\Delta(s) - b_0(s))^2\, ds = 0, \qquad \lim_{\Delta \to 0} \int_0^t \mathbf{E}|a_\Delta(s) - a_0(s)|\, ds = 0.$$

Then, in view of (2.8), we can pass to the limit in (9.22) and get that $Y_\Delta(t) \to Y_0(t)$ in probability.

For arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} \mathbf{E}(Y_\Delta(t) - Y_0(t))^2 &= \mathbf{E}\{ \mathbb{1}_{[0,\varepsilon]}(|Y_\Delta(t) - Y_0(t)|)(Y_\Delta(t) - Y_0(t))^2 \} \\ &\quad + \mathbf{E}\{ \mathbb{1}_{(\varepsilon,\infty)}(|Y_\Delta(t) - Y_0(t)|)(Y_\Delta(t) - Y_0(t))^2 \} =: I_{1,\Delta} + I_{2,\Delta}. \end{aligned}$$

Using the convergence $Y_\Delta(t) \to Y_0(t)$ in probability, we see that the first term on the right-hand side of this equality tends to zero by the Lebesgue dominated convergence theorem. The second term is estimated by Hölder's inequality as follows:

$$I_{2,\Delta} \le \mathbf{P}^{1/2}(|Y_\Delta(t) - Y_0(t)| > \varepsilon) \mathbf{E}^{1/2}(Y_\Delta(t) - Y_0(t))^4.$$

This term also tends to zero in view of (9.21), $p = 4$, and the convergence $Y_\Delta(t) \to Y_0(t)$ in probability. Consequently, $Y_\Delta(t) \to Y_0(t)$ in mean square. Lemma 9.1 is proved. □

We continue the proof of the theorem. Since $a(t,x)$, $b(t,x)$ have bounded derivatives with respect to $x$, the functions

$$\tilde{a}(t,x,y) := \frac{a(t,y)-a(t,x)}{y-x}, \qquad \tilde{b}(t,x,y) := \frac{b(t,y)-b(t,x)}{y-x}, \qquad x \neq y,$$

can be extended continuously to the diagonal $x = y$ by setting $\tilde{a}(t,z,z) := a'_z(t,z)$, $\tilde{b}(t,z,z) := b'_z(t,z)$.

For a fixed $x$ and $\Delta \neq 0$, set $Y_\Delta(t) := \dfrac{X_{x+\Delta}(t) - X_x(t)}{\Delta}$, $t \in [0,T]$. Since

$$X_{x+\Delta}(t) - X_x(t) = \Delta + \int_0^t (a(s, X_{x+\Delta}(s)) - a(s, X_x(s)))\,ds$$
$$+ \int_0^t (b(s, X_{x+\Delta}(s)) - b(s, X_x(s)))\,dW(s),$$

the process $Y_\Delta(t)$ satisfies equation (9.20) with the coefficients

$$a_\Delta(t) := \tilde{a}(t, X_{x+\Delta}(t), X_x(t)), \qquad b_\Delta(t) := \tilde{b}(t, X_{x+\Delta}(t), X_x(t)).$$

These coefficients are uniformly bounded, because the functions $a(t,z)$ and $b(t,z)$ have continuous bounded derivatives with respect to $z$.

By (9.9), $X_{x+\Delta}(t) \longrightarrow X_x(t)$ as $\Delta \to 0$ a.s. Therefore,

$$a_\Delta(t) \to a'_z(t, X_x(t)), \qquad b_\Delta(t) \to b'_z(t, X_x(t)) \qquad \text{as } \Delta \to 0 \qquad \text{a.s.}$$

Finally, applying Lemma 9.1, we have that as $\Delta \to 0$ the variables $Y_\Delta(t)$ converge in probability and in mean square to the limit $Y_0(t)$ which is said to be the derivative $X_x^{(1)}(t) = \dfrac{d}{dx} X_x(t)$. The limit process $X_x^{(1)}(t)$ satisfies (9.19) and (9.18).

In view of (9.9), the conditions on the coefficients $a$, $b$, and (3.6),

$$\sup_{t\in[0,T]} \left| \int_0^t b'_z(t, X_y(t))\,dW(s) - \int_0^t b'_z(t, X_x(t))\,dW(s) \right| \longrightarrow 0 \qquad \text{as } y \to x$$

in probability. The ordinary integral in (9.19) is also continuous with respect to $x$, uniformly in $t \in [0,T]$. This and (9.19) imply that the process $X_x^{(1)}(t)$, $(t,x) \in [0,T] \times \mathbf{R}$, is stochastically continuous, and so continuous in mean square. □

**Remark 9.4.** Under the assumptions of Theorem 9.3,

$$\mathbf{E}\big(X_x^{(1)}(t)\big)^p \le e^{|p|(|p|K+1)Kt} \tag{9.23}$$

for any $p \in \mathbf{R}$.

Indeed, (9.23) is a consequence of (9.21) and Fatou's lemma.

**3. Second derivative of a solution with respect to the initial value.** Similarly to the proof of Theorem 9.3, we can prove the following result concerning the second-order derivative of $X_x(t)$ with respect to the initial value $x$.

**Theorem 9.4.** *Suppose that the functions $a(t,z)$ and $b(t,z)$, $(t,z)\in[0,T]\times\mathbf{R}$, are continuous, and have continuous partial derivatives $a'_z(t,z)$, $b'_z(t,z)$, $a''_{zz}(t,z)$, $b''_{zz}(t,z)$ with respect to $z$.*

*Then the continuous solution $X_x(t)$ of equation* (9.1) *has a stochastically continuous in $(t,x)\in[0,T]\times\mathbf{R}$ mean square second-order derivative $X_x^{(2)}(t):=\frac{d^2}{dx^2}X_x(t)=\frac{d}{dx}X_x^{(1)}(t)$, which satisfies the equation*

$$X_x^{(2)}(t)=\int_0^t a''_{zz}(s,X_x(s))(X_x^{(1)}(s))^2\,ds+\int_0^t a'_z(s,X_x(s))X_x^{(2)}(s)\,ds$$

$$+\int_0^t b''_{zz}(s,X_x(s))(X_x^{(1)}(s))^2\,dW(s)+\int_0^t b'_z(s,X_x(s))X_x^{(2)}(s)\,dW(s). \tag{9.24}$$

**Remark 9.5.** The function $X_x^{(2)}(t)$ satisfying this equation has (see (8.4)) the form

$$X_x^{(2)}(t)=X_x^{(1)}(t)\biggl\{\int_0^t b''_{zz}(s,X_x(s))X_x^{(1)}(s)\,dW(s)$$

$$+\int_0^t \bigl(a''_{zz}(s,X_x(s))-b'_z(s,X_x(s))b''_{zz}(s,X_x(s))\bigr)X_x^{(1)}(s)\,ds\biggr\}. \tag{9.25}$$

*Proof of Theorem* 9.4. From (9.18) for $\Delta\in[-1,1]$, it follows that

$$X_{x+\Delta}^{(1)}(t)-X_x^{(1)}(t)=\int_0^t \bigl(a'_z(s,X_{x+\Delta}(s))\,X_{x+\Delta}^{(1)}(s)-a'_z(s,X_x(s))\,X_x^{(1)}(s)\bigr)ds$$

$$+\int_0^t \bigl(b'_z(s,X_{x+\Delta}(s))\,X_{x+\Delta}^{(1)}(s)-b'_z(s,X_x(s))\,X_x^{(1)}(s)\bigr)dW(s). \tag{9.26}$$

Since $a(t,x)$ and $b(t,x)$ have bounded second derivatives with respect to $x$, the functions

$$\tilde a'(t,x,y):=\frac{a'_y(t,y)-a'_x(t,x)}{y-x},\qquad \tilde b'(t,x,y):=\frac{b'_y(t,y)-b'_x(t,x)}{y-x},\qquad x\ne y,$$

can be extended continuously to the diagonal $x=y$ by the equalities $\tilde a'(t,z,z):=a''_{zz}(t,z)$, $\tilde b'(t,z,z):=b''_{zz}(t,z)$. We also denote

$$a'_\Delta(t):=\tilde a'(t,X_{x+\Delta}(t),X_x(t)),\qquad\qquad b'_\Delta(t):=\tilde b'(t,X_{x+\Delta}(t),X_x(t)).$$

These coefficients are uniformly bounded, because the functions $a(t,x)$ and $b(t,x)$ have continuous bounded second derivatives with respect to $x$.

For a fixed $x$ and $\Delta \neq 0$ we set $Z_\Delta(t) := \dfrac{X^{(1)}_{x+\Delta}(t) - X^{(1)}_x(t)}{\Delta}$ and, as in the proof of Theorem 9.3, we set $Y_\Delta(t) := \dfrac{X_{x+\Delta}(t) - X_x(t)}{\Delta}$, $t \in [0,T]$.

Then (9.26) can be rewritten in the form

$$Z_\Delta(t) = \int_0^t a'_\Delta(s) Y_\Delta(s) X^{(1)}_{x+\Delta}(s)\, ds + \int_0^t b'_\Delta(s) Y_\Delta(s) X^{(1)}_{x+\Delta}(s)\, dW(s)$$

$$+ \int_0^t a'_z(s, X_x(s)) Z_\Delta(s)\, ds + \int_0^t b'_z(s, X_x(s)) Z_\Delta(s)\, dW(s). \tag{9.27}$$

This stochastic differential equation, as equation for the process $Z_\Delta$, has the form (8.2). The coefficients of its linear homogeneous part are of the same form as in (9.18). Therefore, according to (8.4) and the fact that in this case (8.3) is exactly (9.18), the solution (9.27) has the form

$$Z_\Delta(t) = X^{(1)}_x(t) \bigg\{ \int_0^t b'_\Delta(s) Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \big(X^{(1)}_x(s)\big)^{-1}\, dW(s)$$

$$+ \int_0^t \big(a'_\Delta(s) - b'_z(s, X_x(s))\, b'_\Delta(s)\big) Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \big(X^{(1)}_x(s)\big)^{-1}\, ds \bigg\}. \tag{9.28}$$

Using this representation, it is not hard to get the estimate

$$\sup_{0\le\Delta\le1}\ \sup_{0\le t\le T} \mathbf{E} Z^{2n}_\Delta(t) < \infty \tag{9.29}$$

for any positive integer $n$.

Indeed, taking into account the boundedness of the functions $a'_\Delta$, $b'_\Delta$, $b'_z$, the estimate (4.25), and the nonnegativity of the processes $Y_\Delta$, $X^{(1)}_x$, we get

$$\mathbf{E} Z^{2n}_\Delta(t) \le 2^{2n-1} \widetilde{C} \bigg\{ \mathbf{E}\bigg( \int_0^t Y^2_\Delta(s) \Big( X^{(1)}_{x+\Delta}(s) \frac{X^{(1)}_x(t)}{X^{(1)}_x(s)} \Big)^2 ds \bigg)^n$$

$$+ \mathbf{E}\bigg( \int_0^t Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \frac{X^{(1)}_x(t)}{X^{(1)}_x(s)} ds \bigg)^{2n} \bigg\} \le C_n \int_0^t \mathbf{E}\Big( Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \frac{X^{(1)}_x(t)}{X^{(1)}_x(s)} \Big)^{2n} ds.$$

Applying Hölder's inequality, we obtain

$$\mathbf{E} Z^{2n}_\Delta(t) \le C \int_0^t \mathbf{E}^{1/4} Y^{8n}_\Delta(s) \mathbf{E}^{1/4} \big(X^{(1)}_{x+\Delta}(s)\big)^{8n} \mathbf{E}^{1/4} \big(X^{(1)}_x(t)\big)^{8n} \mathbf{E}^{1/4} \big(X^{(1)}_x(s)\big)^{-8n} ds.$$

Now we can use estimates (9.21), (9.23), which leads to (9.29). The estimate (9.29) guarantees (see Proposition 1.1 Ch. I) the uniform integrability of the family of random variables $\{Z_\Delta^2(t)\}_{\Delta>0}$ for every $t \in [0,T]$.

By (9.9), $X_{x+\Delta}(t) \to X_x(t)$ as $\Delta \to 0$ a.s. Therefore,

$$a'_\Delta(t) \to a''_{zz}(t, X_x(t)), \qquad b'_\Delta(t) \to b''_{zz}(t, X_x(t)) \qquad \text{as } \Delta \to 0 \qquad \text{a.s.}$$

In turn, $Y_\Delta(t)$ converges as $\Delta \to 0$ in probability and in mean square to the derivative $X_x^{(1)}(t)$, and $X_{x+\Delta}^{(1)}(t) \to X_x^{(1)}(t)$. Consequently, in (9.28) we can pass to the limit as $\Delta \to 0$. Then we see that the processes $Z_\Delta(t)$ converge as $\Delta \to 0$ in probability and in mean square to the limit $Z_0(t)$, which is called the second-order derivative $X_x^{(2)}(t) = \dfrac{\partial^2}{\partial x^2} X_x(t)$. The limit process $X_x^{(2)}(t)$ satisfies (9.25), and hence it satisfies (9.24). By (9.25) and (9.29), $X_x^{(2)}(t)$ is stochastically continuous in $(t,x) \in [0,T]\times\mathbf{R}$, and it is continuous in the mean square, since the processes $X_x(t)$ and $X_x^{(1)}(t)$ are stochastically continuous with respect to $x$ uniformly in $t \in [0,T]$. $\square$

**Remark 9.6.** Under the assumptions of Theorem 9.4,

$$\sup_{0\le t\le T} \mathbf{E}\big(X_x^{(2)}(t)\big)^{2n} < \infty \tag{9.30}$$

for any positive integer $n$.

Indeed, (9.30) is a consequence of (9.29) and Fatou's lemma.

## § 10. Girsanov's transformation

To clarify the subject of this section we start with a simple example.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\zeta = \zeta(\omega)$ be a Gaussian random variable with mean zero and variance 1. The characteristic function of this variable is given by the formula

$$\mathbf{E}e^{iz\zeta} = \int_\Omega e^{iz\zeta(\omega)}\mathbf{P}(d\omega) = \int_{-\infty}^{\infty} e^{izx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = e^{-z^2/2}, \qquad z \in \mathbf{R}. \tag{10.1}$$

This equality holds also for a complex $z$.

Define a new probability measure by setting

$$\widetilde{\mathbf{P}}(A) := \int_A \exp\Big(-\mu\zeta(\omega) - \frac{\mu^2}{2}\Big)\mathbf{P}(d\omega)$$

for sets $A \in \mathcal{F}$. This relation has a brief expression in terms of the Radon–Nikodým derivative

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} := \frac{\widetilde{\mathbf{P}}(d\omega)}{\mathbf{P}(d\omega)} = \exp\Big(-\mu\zeta(\omega) - \frac{\mu^2}{2}\Big).$$

Note that $\widetilde{\mathbf{P}}(\Omega) = 1$, since by (10.1), for $z = i\mu$ we have $\widetilde{\mathbf{P}}(\Omega) = e^{-\mu^2/2}\mathbf{E}e^{-\mu\zeta} = 1$.

**Proposition 10.1.** *The random variable $\tilde\zeta = \zeta + \mu$ with respect to the measure $\widetilde{\mathbf{P}}$ is the Gaussian random variable with mean zero and variance* 1.

*Proof.* Denote by $\widetilde{\mathbf{E}}$ the expectation with respect to the measure $\widetilde{\mathbf{P}}$. Then for an arbitrary bounded random variable $\eta$,

$$\widetilde{\mathbf{E}}\eta := \int_\Omega \eta(\omega)\,\widetilde{\mathbf{P}}(d\omega) = \int_\Omega \eta(\omega)\exp\Big(-\mu\zeta(\omega) - \frac{\mu^2}{2}\Big)\mathbf{P}(d\omega) = \mathbf{E}\Big\{\eta\exp\Big(-\mu\zeta - \frac{\mu^2}{2}\Big)\Big\}.$$

Using this, we have

$$\begin{aligned}\widetilde{\mathbf{E}}e^{iz\tilde\zeta} &= \widetilde{\mathbf{E}}e^{iz(\zeta+\mu)} = \mathbf{E}\Big\{e^{iz(\zeta+\mu)}\exp\Big(-\mu\zeta - \frac{\mu^2}{2}\Big)\Big\}\\ &= e^{-\mu^2/2}\int_{-\infty}^{\infty} e^{iz(x+\mu)}e^{-\mu x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx\\ &= \int_{-\infty}^{\infty} e^{iz(x+\mu)}\frac{1}{\sqrt{2\pi}}e^{-(x+\mu)^2/2}\,dx = \int_{-\infty}^{\infty} e^{izy}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy = e^{-z^2/2}.\end{aligned}$$

This proves the statement. □

The main point of Proposition 10.1 can be formulated as follows: a special choice of the probability measure can compensate the shift of a Gaussian random variable.

The distribution of a random variable is uniquely determined by the characteristic function or by the family of expectations of a bounded measurable functions of this variable. The statement that the random variable $\tilde\zeta = \zeta + \mu$ with respect to the measure $\widetilde{\mathbf{P}}$ is again distributed as $\zeta$ can be expressed as follows: for an arbitrary bounded measurable function $f$ we have $\widetilde{\mathbf{E}}f(\tilde\zeta) = \mathbf{E}f(\zeta)$, or, in view of the definitions of $\widetilde{\mathbf{E}}$ and $\tilde\zeta$,

$$\mathbf{E}\Big\{f(\zeta+\mu)\exp\Big(-\mu\zeta - \frac{\mu^2}{2}\Big)\Big\} = \mathbf{E}f(\zeta). \tag{10.2}$$

As we saw, if instead of the abstract expectation with respect to the probability measure $\mathbf{P}$ we write (10.2) in terms of integrals with respect to the Gaussian distribution function $dG(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$, then formula (10.2) turns to the integration by substitution formula. We can rewrite (10.2) in another way. We apply (10.2) to the function $f(x)e^{\mu x - \mu^2/2}$ instead of $f(x)$ and get

$$\mathbf{E}f(\zeta+\mu) = \mathbf{E}\Big\{f(\zeta)\exp\Big(\mu\zeta - \frac{\mu^2}{2}\Big)\Big\}.$$

For $f(x) = \mathbb{I}_A(x)$, $A \in \mathcal{F}$, this formula has the brief equivalent

$$\frac{d\mathbf{P}_{\zeta+\mu}}{d\mathbf{P}_\zeta} = \exp\Big(\mu\zeta - \frac{\mu^2}{2}\Big),$$

where $\mathbf{P}_{\zeta+\mu}$ is the measure corresponding to the variable $\zeta+\mu$ and $\mathbf{P}_\zeta$ is the measure corresponding to the variable $\zeta$.

Results analogous to Proposition 10.1 hold also for some random variables taking values in functional spaces, i.e., for stochastic processes. The next result concerns the Brownian motion.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space. Let $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$. Suppose that for all $v > t$ the increments $W(v) - W(t)$ are independent of the $\sigma$-algebra $\mathcal{F}_t$.

For an arbitrary $b \in \mathcal{L}_2[0,T]$, consider the stochastic exponent

$$\rho(t) := \exp\Big( -\int_0^t b(s)\, dW(s) - \frac{1}{2}\int_0^t b^2(s)\, ds\Big), \qquad t \in [0,T].$$

Here compared with the exponent from § 6 we take the process $-b(s)$ instead of $b(s)$. The stochastic differential of $\rho$ is

$$d\rho(t) = -\rho(t)b(t)\, dW(t). \tag{10.3}$$

Therefore,

$$\rho(t) = 1 - \int_0^t \rho(v)b(v)\, dW(v).$$

Suppose that for some $\delta > 0$

$$\mathbf{E}\exp\Big((1+\delta)\int_0^T b^2(s)\, ds\Big) < \infty,$$

or

$$\sup_{0\le s\le T} \mathbf{E}e^{\delta b^2(s)} < \infty.$$

Then the stochastic exponent $\rho(t)$, $t \in [0,T]$, is (see Proposition 6.1) a nonnegative martingale with the mean $\mathbf{E}\rho(t) = 1$ for every $t \in [0,T]$.

Define the probability measure $\widetilde{\mathbf{P}}$ by setting

$$\widetilde{\mathbf{P}}(A) := \int_A \rho(T,\omega)\mathbf{P}(d\omega)$$

for $A \in \mathcal{F}$. Note that $\widetilde{\mathbf{P}}(\Omega) = \mathbf{E}\rho(T) = 1$.

Denote by $\widetilde{\mathbf{E}}$ the expectation with respect to the measure $\widetilde{\mathbf{P}}$. Then

$$\widetilde{\mathbf{E}}\eta := \int_\Omega \eta(\omega)\, \widetilde{\mathbf{P}}(d\omega) = \int_\Omega \eta(\omega)\rho(T,\omega)\, \mathbf{P}(d\omega) = \mathbf{E}\{\eta\rho(T)\}.$$

**Proposition 10.2.** *For any bounded $\mathcal{F}_t$-measurable random variable $\eta$ the following equalities hold*:

$$\widetilde{\mathbf{E}}\eta = \mathbf{E}\{\eta\rho(t)\}, \tag{10.4}$$

*and for $s < t$,*

$$\widetilde{\mathbf{E}}\{\eta|\mathcal{F}_s\} = \frac{1}{\rho(s)}\mathbf{E}\{\eta\rho(t)|\mathcal{F}_s\}. \tag{10.5}$$

*Proof.* Indeed, using the martingale property of $\rho(t)$, $t \in [0,T]$, we have

$$\widetilde{\mathbf{E}}\eta = \mathbf{E}\{\mathbf{E}\{\eta\rho(T)|\mathcal{F}_t\}\} = \mathbf{E}\{\eta\mathbf{E}\{\rho(T)|\mathcal{F}_t\}\} = \mathbf{E}\{\eta\rho(t)\}.$$

To prove (10.5) we consider an arbitrary bounded $\mathcal{F}_s$-measurable random variable $\xi$. We compute the expectation $\widetilde{\mathbf{E}}\{\xi\eta\}$ in two different ways. Using the properties of the conditional expectation and (10.4), we have

$$\widetilde{\mathbf{E}}\{\xi\eta\} = \widetilde{\mathbf{E}}\{\widetilde{\mathbf{E}}\{\xi\eta|\mathcal{F}_s\}\} = \widetilde{\mathbf{E}}\{\xi\widetilde{\mathbf{E}}\{\eta|\mathcal{F}_s\}\} = \mathbf{E}\{\xi\rho(s)\widetilde{\mathbf{E}}\{\eta|\mathcal{F}_s\}\}. \tag{10.6}$$

On the other hand, first applying (10.4) and then using the properties of the conditional expectation, we obtain

$$\widetilde{\mathbf{E}}\{\xi\eta\} = \mathbf{E}\{\xi\eta\rho(t)\} = \mathbf{E}\{\mathbf{E}\{\xi\eta\rho(t)|\mathcal{F}_s\}\} = \mathbf{E}\{\xi\mathbf{E}\{\eta\rho(t)|\mathcal{F}_s\}\}. \tag{10.7}$$

Since $\xi$ is an arbitrary bounded $\mathcal{F}_s$-measurable random variable, the coincidence of the right-hand sides of (10.6) and (10.7) implies (10.5). $\square$

The following result is due to I. V. Girsanov (1960) (for nonrandom $b$ see Cameron and Martin (1945)).

**Theorem 10.1.** *The process $\widetilde{W}(t) = W(t) + \int_0^t b(s)\,ds$ is a Brownian motion with respect to the measure $\widetilde{\mathbf{P}}$.*

*Proof.* By the characterization property (10.9) Ch. I, to prove that the process $\widetilde{W}$ is a Brownian motion it is sufficient to verify that for any $s < t$ and $z \in \mathbf{R}$,

$$\widetilde{\mathbf{E}}\{\exp(iz(\widetilde{W}(t) - \widetilde{W}(s)))|\mathcal{F}_s\} = e^{-z^2(t-s)/2} \qquad \text{a.s.} \tag{10.8}$$

We prove (10.8). We first assume that $\sup_{0\le s\le T}|b(s)| \le M$ for some nonrandom constant $M$. For any fixed $s$ and $t \ge s$, we set

$$\eta(t) := \exp\bigl(iz(\widetilde{W}(t) - \widetilde{W}(s))\bigr) = \exp\Bigl(iz(W(t) - W(s)) + iz\int_s^t b(u)\,du\Bigr).$$

Note that $\eta(s) = 1$. According to (10.5),

$$g(t) := \widetilde{\mathbf{E}}\{\eta(t)|\mathcal{F}_s\} = \frac{1}{\rho(s)}\mathbf{E}\{\eta(t)\rho(t)|\mathcal{F}_s\}.$$

We fix $s$ and for $t > s$ apply Itô's formula (4.24) for

$$f(t,x,y) = e^x y, \qquad X(t) = iz(W(t) - W(s)) + iz\int\limits_s^t b(u)\,du, \qquad Y(t) = \rho(t).$$

Then taking into account (10.3), we obtain

$$\begin{aligned} d(\eta(t)\rho(t)) &= \eta(t)\rho(t)\{iz\,dW(t) + izb(t)\,dt\} \\ &\quad - \eta(t)\rho(t)b(t)\,dW(t) - \frac{1}{2}z^2\eta(t)\rho(t)\,dt - iz\eta(t)\rho(t)b(t)\,dt. \end{aligned}$$

In the integral form this is written as follows: for every $t \ge s$,

$$\eta(t)\rho(t) = \rho(s) + iz\int\limits_s^t \eta(u)\rho(u)\,dW(u) - \int\limits_s^t \eta(u)\rho(u)b(u)\,dW(u) - \frac{z^2}{2}\int\limits_s^t \eta(u)\rho(u)\,du.$$

Since $|\eta(t)| \le 1$ and, by (6.13), the estimate $\mathbf{E}(\rho(u)b(u))^2 \le M^2e^{2M^2u}$ holds, we can use (2.3) and get

$$\mathbf{E}\{\eta(t)\rho(t)|\mathcal{F}_s\} = \rho(s) - \frac{z^2}{2}\int\limits_s^t \mathbf{E}\{\eta(u)\rho(u)|\mathcal{F}_s\}\,du \qquad \text{a.s.}$$

Using the definition of the function $g$, this can be written in the form

$$g(t) = 1 - \frac{z^2}{2}\int\limits_s^t g(u)\,du, \qquad t \ge s.$$

The solution of this differential equation is $g(t) = e^{-z^2(t-s)/2}$, which is the required result (10.8).

We will prove (10.8) for an arbitrary process $b \in \mathcal{L}_2[0,T]$, satisfying the assumptions stated above. There exists a sequence of bounded processes $b_n \in \mathcal{L}_2[0,T]$, such that

$$\lim_{n\to\infty}\int\limits_0^T (b(s) - b_n(s))^2\,ds = 0 \qquad \text{a.s.}$$

Then the processes $\widetilde{W}_n(t) = W(t) + \int\limits_0^t b_n(s)\,ds$ converge to the process $\widetilde{W}$. We denote $\rho_n(t)$ the stochastic exponent corresponding to the process $b_n$. Then, in view of (3.6), $\rho_n(t) \to \rho(t)$ in probability for every $t$. Since $\mathbf{E}\rho_n(t) = \mathbf{E}\rho(t) = 1$, we have

$$\mathbf{E}|\rho_n(t) - \rho(t)| = \mathbf{E}\big(|\rho(t) - \rho_n(t)| + \rho(t) - \rho_n(t)\big) = 2\mathbf{E}(\rho(t) - \rho_n(t))^+.$$

Since $(\rho(t) - \rho_n(t))^+ \le \rho(t)$, by the Lebesgue dominated convergence theorem, $\rho_n(t) \to \rho(t)$ in mean. For the processes $b_n$ the equality (10.8) has been already proved, i.e., in view of (10.5),

$$\mathbf{E}\{\exp(iz(\widetilde{W}_n(t) - \widetilde{W}_n(s)))\rho_n(t)|\mathcal{F}_s\} = \rho_n(s)e^{-z^2/2(t-s)}.$$

By property 7′) of conditional expectations (see § 2 Ch. I), we can pass to the limit in this equality and obtain (10.8) for the process $b$. □

**Remark 10.1.** For a nonrandom function $b$ Theorem 10.1 was first proved by Cameron and Martin (1945).

Girsanov's theorem can be presented in another form. Let $C([0,T])$ be the space of continuous functions on $[0,T]$. When equipped with the uniform norm $C([0,T])$ becomes a Banach space. Instead of an abstract probability measure $\mathbf{P}$ we can consider the Wiener measure $\mathbf{P}_W$, which for cylinder sets is determined by (10.1) Ch. I. Although the Wiener measure is concentrated on the sets of nowhere differentiable paths, it can be extended to the $\sigma$-algebra $\mathcal{B}(C([0,T]))$ of Borel sets of the space $C([0,T])$. This measure can be characterized by the expectations of a bounded measurable functionals of Brownian motion.

Girsanov's theorem can be recast as.

**Theorem 10.2.** *Let $\wp(X(s), 0 \le s \le t)$ be a bounded measurable functional on $C([0,t])$. Then*

$$\mathbf{E}\left\{\wp\left(W(s) + \int_0^s b(u)\,du, 0 \le s \le t\right)\rho(t)\right\} = \mathbf{E}\wp(W(s), 0 \le s \le t). \tag{10.9}$$

*Proof.* Indeed, the statement of Theorem 10.1 is equivalent to the following: for any bounded measurable functional $\wp(X(s), 0 \le s \le t)$,

$$\widetilde{\mathbf{E}}\wp(\widetilde{W}(s), 0 \le s \le t) = \mathbf{E}\wp(W(s), 0 \le s \le t).$$

In view of (10.4), the left-hand side of this equality coincides with the left-hand side of (10.9). □

Let us consider a very important application of Girsanov's transformation.

Let $X(t)$ and $Y(t)$, $t \in [0,T]$, be solutions of the stochastic differential equations

$$dX(t) = \sigma(t, X(t))\,dW(t) + \mu_1(t, X(t))\,dt, \tag{10.10}$$
$$dY(t) = \sigma(t, Y(t))\,dW(t) + \mu_2(t, Y(t))\,dt, \tag{10.11}$$

with the same nonrandom initial values. Suppose that the coefficients $\sigma$, $\mu_1$, $\mu_2$ satisfy the conditions of Theorem 7.1. Assume also that $\sigma(t,x) \ne 0$ for all $(t,x) \in [0,T] \times \mathbf{R}$.

Let $\mathcal{G}_0^t = \sigma(W(s), 0 \le s \le t)$ be the $\sigma$-algebra of events generated by the Brownian motion up to the time $t$. It was proved in § 7 that the processes $X$ and $Y$ are adapted to the natural filtration $\mathcal{G}_0^t$, i.e., for every $t$ the variables $X(t)$ and $Y(t)$ are measurable with respect to $\mathcal{G}_0^t$.

**Theorem 10.3.** *Let* $\alpha(t,x) := \dfrac{\mu_1(t,x)-\mu_2(t,x)}{\sigma(t,x)}$, *be a continuous function of the variables* $(t,x)\in[0,T]\times\mathbf{R}$. *Denote*

$$\rho(t) := \exp\biggl(-\int_0^t \alpha(s,X(s))\,dW(s) - \frac{1}{2}\int_0^t \alpha^2(s,X(s))\,ds\biggr)$$

*and suppose that for some* $\delta>0$

$$\mathbf{E}\exp\biggl((1+\delta)\int_0^T \alpha^2(t,X(t))\,dt\biggr) < \infty \qquad \text{or} \qquad \sup_{0\le t\le T}\mathbf{E}e^{\delta\alpha^2(t,X(t))} < \infty.$$

*Then for any bounded measurable functional* $\wp(Z(s), 0\le s\le t)$ *on* $C([0,t])$,

$$\mathbf{E}\wp(Y(s),0\le s\le t) = \mathbf{E}\bigl\{\wp\bigl(X(s),0\le s\le t\bigr)\rho(t)\bigr\} \tag{10.12}$$

*for every* $t\in[0,T]$.

**Remark 10.2.** Let $\mathbf{P}_X$ and $\mathbf{P}_Y$ be the measures associated with the processes $X(t)$ and $Y(t)$, $t\in[0,T]$, respectively. Then from (10.12) for the functional

$$\wp(Z(s),0\le s\le t) = \mathbb{1}_A(Z(s),0\le s\le t), \qquad A\in\mathcal{B}(C[0,t]),$$

it follows that the measure $\mathbf{P}_Y$ is absolutely continuous with respect to $\mathbf{P}_X$ when restricted to $\mathcal{G}_0^t$ and there exists the Radon–Nikodým derivative

$$\frac{d\mathbf{P}_Y}{d\mathbf{P}_X}\bigg|_{\mathcal{G}_0^t} = \rho(t) \qquad \text{a.s.} \tag{10.13}$$

*Proof of Theorem* 10.3. Since $\alpha$ is a continuous function, the process $\alpha(t,X(t))$, $t\in[0,T]$, is progressively measurable with respect to the filtration $\{\mathcal{G}_0^t\}$.

By Theorem 10.1, the process

$$\widetilde{W}(t) = W(t) + \int_0^t \alpha(s,X(s))\,ds$$

is a Brownian motion with respect to the measure $\widetilde{\mathbf{P}}$. Since

$$dX(t) = \sigma(t,X(t))\,d\widetilde{W}(t) + \mu_2(t,X(t))\,dt$$

and this stochastic differential equation coincides with (10.11), the finite-dimensional distributions of the process $X$ with respect to the measure $\widetilde{\mathbf{P}}$ coincide with those of the process $Y$ with respect to the measure $\mathbf{P}$. This implies that

$$\mathbf{E}\wp(Y(s),0\le s\le t) = \widetilde{\mathbf{E}}\wp(X(s),0\le s\le t).$$

In view of (10.4), the right-hand side of this equality coincides with the right-hand side of (10.12). □

The Radon–Nikodým derivative (10.13) can be rewritten as a functional of $X$. Indeed, from (10.10) it follows that

$$dW(t) = \frac{1}{\sigma(t, X(t))}(dX(t) - \mu_1(t, X(t))\, dt).$$

Then

$$\int_0^t \alpha(s, X(s))\, dW(s) = \int_0^t \frac{\alpha(s, X(s))}{\sigma(t, X(t))}\, dX(s) - \int_0^t \frac{\alpha(s, X(s))\mu_1(s, X(s))}{\sigma(t, X(t))}\, ds.$$

As a result, we have

$$\rho(t) = \exp\biggl(\int_0^t \frac{\mu_2(s, X(s)) - \mu_1(s, X(s))}{\sigma^2(s, X(s))}\, dX(s) - \frac{1}{2}\int_0^t \frac{\mu_2^2(s, X(s)) - \mu_1^2(s, X(s))}{\sigma^2(s, X(s))}\, ds\biggr).$$

Consider the particular case when $\sigma(t, x) \equiv 1$, $\mu_1(t, x) \equiv 0$, $\mu_2(t, x) \equiv \mu(x)$. Suppose that for some $\delta > 0$

$$\mathbf{E}\exp\biggl((1+\delta)\int_0^T \mu^2(W(t))\, dt\biggr) < \infty \qquad \text{or} \qquad \sup_{0\le t\le T} \mathbf{E}e^{\delta\mu^2(W(t))} < \infty.$$

Then for the process $Y(t) = W(t) + \int_0^t \mu(Y(s))\, ds$, $t \in [0, T]$, we have

$$\frac{d\mathbf{P}_Y}{d\mathbf{P}_W}\Big|_{\mathcal{G}_0^t} = \exp\biggl(\int_0^t \mu(W(s))\, dW(s) - \frac{1}{2}\int_0^t \mu^2(W(s))\, ds\biggr)$$

$$= \exp\biggl(\int_{W(0)}^{W(t)} \mu(y)dy - \frac{1}{2}\int_0^t \mu^2(W(s))\, ds - \frac{1}{2}\int_0^t \mu'(W(s))ds\biggr) \qquad \text{a.s.} \qquad (10.14)$$

Here the second equality follows from the Itô formula under the assumption that the function $\mu$ is differentiable.

In particular, for the Brownian motion with linear drift $\mu(x) \equiv \mu$, i.e., for the process $W^{(\mu)}(t) = \mu t + W(t)$ with $W(0) = x$, formula (10.14) has the form

$$\mathbf{E}\wp\bigl(W^{(\mu)}(s), 0 \le s \le t\bigr) = e^{-\mu x - \mu^2 t/2}\mathbf{E}\bigl\{e^{\mu W(t)}\wp(W(s), 0 \le s \le t)\bigr\}. \qquad (10.15)$$

**Exercises.**

**10.1.** Let $X(t)$ be a solution of the stochastic differential equation

$$dX(t) = a(X(t))\,dt + dW(t), \qquad X(0) = x_0.$$

Use the Girsanov theorem to prove that for all $K$, $x_0 \in \mathbf{R}$ and $t > 0$

$$\mathbf{P}(X(t) \geq K) > 0.$$

**10.2.** Let $Y(t) = W(t) + \mu t + \eta t^2$ be the Brownian motion with the quadratic drift, $W(0) = x$. Check that

$$\frac{d\mathbf{P}_Y}{d\mathbf{P}_W}\bigg|_{\mathcal{G}_0^t} = \exp\biggl(-\mu x - \frac{\mu^2 t}{2} - \mu\eta t^2 - \frac{2\eta^2 t^3}{3} + (\mu + 2\eta t)W(t) - 2\eta\int_0^t W(s)\,ds\biggr).$$

**10.3.** Let $Y(t)$ be a solution of the stochastic differential equation

$$dY(t) = -\theta Y(t)\,dt + dW(t), \qquad Y(0) = x, \quad \theta \in \mathbf{R}.$$

Compute $\dfrac{d\mathbf{P}_Y}{d\mathbf{P}_W}\bigg|_{\mathcal{G}_0^t}$.

## § 11. Probabilistic solution of the Cauchy problem

Let $X(t)$, $t \geq 0$, be a solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))\,dW(t) + \mu(X(t))\,dt, \qquad X(0) = x. \tag{11.1}$$

Suppose that for every $N > 0$ there exists a constant $K_N$ such that

$$|\sigma(x) - \sigma(y)| + |\mu(x) - \mu(y)| \leq K_N|x - y| \tag{11.2}$$

for all $x, y \in [-N, N]$. Introduce, in addition, the following restriction on the growth of the coefficients $\sigma$ and $\mu$: there exists a constant $K$ such that

$$|\sigma(x)| + |\mu(x)| \leq K(1 + |x|) \tag{11.3}$$

for all $x \in \mathbf{R}$. Then, by Theorem 7.3, equation (11.1) has a unique continuous solution defined for all $t \geq 0$.

Denote by $\mathbf{P}_x$ and $\mathbf{E}_x$ the probability and the expectation with respect to the process $X$ with the starting point $X(0) = x$.

Let $H_{a,b} := \min\{s : X(s) \notin (a, b)\}$ be the *first exit time* of the process $X$ from the interval $(a, b)$.

**Theorem 11.1.** *Let $\varPhi(x)$, $f(x)$, $x \in [a,b]$, be continuous functions, and let $f$ be nonnegative. Suppose that the coefficients $\sigma$, $\mu$ satisfy condition* (11.2) *for $N = \max\{|a|,|b|\}$, and $\sigma(x) > 0$ for $x \in [a,b]$.*

*Let $u(t,x)$, $(t,x) \in [0,\infty) \times [a,b]$, be a solution of the Cauchy problem*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}u(t,x) + \mu(x)\frac{\partial}{\partial x}u(t,x) - f(x)u(t,x), \tag{11.4}$$

$$u(0,x) = \varPhi(x), \tag{11.5}$$

$$u(t,a) = \varPhi(a), \qquad u(t,b) = \varPhi(b). \tag{11.6}$$

*Then*

$$u(t,x) = \mathbf{E}_x\biggl\{\varPhi(X(t \wedge H_{a,b}))\exp\biggl(-\int\limits_0^{t\wedge H_{a,b}} f(X(s))\,ds\biggr)\biggr\}. \tag{11.7}$$

*Proof.* We extend the functions $\sigma$ and $\mu$ outside the interval $[a,b]$ such that they satisfy (11.2), (11.3), and the condition $\sigma(x) > 0$ for $x \in \mathbf{R}$. In this case, by Theorem 7.2, the process $X$ is not changed in the interval $[0, H_{a,b}]$. We also extend $f$ to be a nonnegative continuous function outside $[a,b]$. One can extend the solution $u$ of the problem (11.4)–(11.6) outside $[a,b]$ such that it will be continuously differentiable in $(t,x) \in (0,\infty) \times \mathbf{R}$. Moreover, there exist the continuous second derivative in $x$ except in the points $a - k(b-a)$ and $b + k(b-a)$, $k \in \mathbb{N}$, and this derivative has the left and right limits at its points of discontinuity. It is not stated that $u(t,x)$ satisfies the equation (11.4) for $x \notin [a,b]$. For example, we can set $u(t,x) := -u(t, 2a-x)$ for $x \in [2a-b, a]$, $u(t,x) := -u(t, 2b-x)$ for $x \in [b, 2b-a]$, $u(t,x) := u(t, x+2b-2a)$ for $x \in [3a-2b, 2a-b]$ and so on.

For a fixed $t$ set

$$\eta(s) := u(t-s, X(s))\exp\biggl(-\int\limits_0^s f(X(v))\,dv\biggr), \qquad s \in [0,t].$$

Applying Itô's formula (4.22) for $d = 1$ in the integral form together with the Remark 4.2, we have for every $0 \le q \le t$ that

$$\eta(q) - \eta(0) = \int\limits_0^q \exp\biggl(-\int\limits_0^s f(X(v))\,dv\biggr)\biggl[\biggl(\frac{\partial}{\partial s}u(t-s, X(s))$$

$$+\frac{1}{2}\sigma^2(X(s))\frac{\partial^2}{\partial x^2}u(t-s,X(s)) + \mu(X(s))\frac{\partial}{\partial x}u(t-s,X(s))$$

$$-f(X(s))u(t-s,X(s))\biggr)ds + \sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)\biggr].$$

Replacing $q$ by the stopping time $t \wedge H_{a,b}$, we get

$$\eta(t\wedge H_{a,b}) - \eta(0) = \int\limits_0^{t\wedge H_{a,b}} \exp\biggl(-\int\limits_0^s f(X(v))\,dv\biggr)\biggl[\biggl(-\frac{\partial}{\partial t}u(t-s, X(s))$$

$$+\frac{1}{2}\sigma^2(X(s))\frac{\partial^2}{\partial x^2}u(t-s,X(s))+\mu(X(s))\frac{\partial}{\partial x}u(t-s,X(s))$$
$$-f(X(s))u(t-s,X(s))\Big)ds+\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)\Big].$$

Using the fact that $u(t,x)$ satisfies equation (11.4) for $x\in[a,b]$, we have

$$\eta(t\wedge H_{a,b})-\eta(0)=\int_0^{t\wedge H_{a,b}}\exp\Big(-\int_0^s f(X(v))\,dv\Big)\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)$$

$$=\int_0^t \mathbb{1}_{[0,H_{a,b})}(s)\exp\Big(-\int_0^s f(X(v))\,dv\Big)\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s). \quad (11.8)$$

It is important that $H_{a,b}$ is a stopping time with respect to the filtration $\mathcal{G}_0^t=\sigma(W(s),0\le s\le t)$. This ensures that the stochastic integral is well defined (see (3.8)). Note also that all integrands are bounded, because the process $X$ does not leave the interval $(a,b)$ up to the time $H_{a,b}$.

The expectation of the stochastic integral equals zero, and therefore

$$\mathbf{E}_x\eta(t\wedge H_{a,b})=\mathbf{E}_x\eta(0).$$

It is clear that

$$\mathbf{E}_x\eta(0)=\mathbf{E}\{u(t,X(0))|X(0)=x\}=u(t,x).$$

By the boundary conditions (11.5) and (11.6), we have

$$u(t-(t\wedge H_{a,b}),X(t\wedge H_{a,b}))=\Phi(X(t\wedge H_{a,b})),$$

and so

$$\mathbf{E}_x\eta(t\wedge H_{a,b})=\mathbf{E}_x\Big\{\Phi(X(t\wedge H_{a,b}))\exp\Big(-\int_0^{t\wedge H_{a,b}}f(X(v))\,dv\Big)\Big\}.$$

Thus (7.11) holds. □

**Remark 11.1.** It is very important that in Theorem 11.1 and in the following results of this section we assume that the solution of the corresponding differential problem exists.

The following generalization of Theorem 11.1 gives the probabilistic solution for the nonhomogeneous Cauchy problem.

**Theorem 11.2.** *Let $\Phi(x)$, $f(x)$ and $g(x)$, $x\in[a,b]$, be continuous functions, and $f$ be nonnegative. Suppose that the coefficients $\sigma$, $\mu$ satisfy condition* (11.2) *with $N=\max\{|a|,|b|\}$, and $\sigma(x)>0$ for $x\in[a,b]$.*

*Let $u(t,x)$, $(t,x)\in[0,\infty)\times[a,b]$, be a solution of the nonhomogeneous Cauchy problem*

$$\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}u(t,x)+\mu(x)\frac{\partial}{\partial x}u(t,x)-f(x)u(t,x)+g(x), \tag{11.9}$$

$$u(0,x)=\varPhi(x), \tag{11.10}$$

$$u(t,a)=\varPhi(a),\qquad u(t,b)=\varPhi(b). \tag{11.11}$$

*Then*

$$u(t,x)=\mathbf{E}_x\biggl\{\varPhi(X(t\wedge H_{a,b}))\exp\Bigl(-\int_0^{t\wedge H_{a,b}}f(X(s))\,ds\Bigr)$$
$$+\int_0^{t\wedge H_{a,b}}g(X(s))\exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)ds\biggr\}. \tag{11.12}$$

*Proof.* As in the proof of Theorem 11.1, we consider the extension of the solution of the problem (11.9)–(11.11) outside the interval $[a,b]$, assuming that the functions $f$ and $g$ are continuously extended outside $[a,b]$ so that $g$ is bounded and $f$ is nonnegative. For a fixed $t$, set

$$\eta(s):=u(t-s,X(s))\exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)+\int_0^s g(X(v))\exp\Bigl(-\int_0^v f(X(q))\,dq\Bigr)dv.$$

Applying Itô's formula and then substituting in it the stopping time $t\wedge H_{a,b}$, we get

$$\eta(t\wedge H_{a,b})-\eta(0)=\int_0^{t\wedge H_{a,b}}\exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\Bigl[\Bigl(-\frac{\partial}{\partial t}u(t-s,X(s))$$
$$+\frac{1}{2}\sigma^2(X(s))\frac{\partial^2}{\partial x^2}u(t-s,X(s))+\mu(X(s))\frac{\partial}{\partial x}u(t-s,X(s))-f(X(s))u(t-s,X(s))$$
$$+g(X(s))\Bigr)ds+\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)\Bigr].$$

We use the fact that for $x\in[a,b]$ the function $u(t,x)$ satisfies equation (11.9). As a result, we have (11.8). Now the proof is completed analogous to the proof of Theorem 11.1. □

Taking the Laplace transform with respect to $t$, we can reduce the problem (11.9)–(11.11) to a problem for an ordinary differential equation.

For any $\lambda>0$, set

$$U(x):=\lambda\int_0^\infty e^{-\lambda t}u(t,x)\,dt. \tag{11.13}$$

Then for every $x$ the function $u(t,x)$, $t \ge 0$, is uniquely determined by the function $U$, $\lambda > 0$, as the inverse Laplace transform.

Applying the integration by parts formula, taking into account the boundary condition (11.10) and the fact that the function $u(t,x)$, by virtue of the representation (11.12), obeys for some $K$ the estimate $|u(t,x)| \le K(1+t)$, we get

$$\lambda \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} u(t,x)\,dt = -\lambda \Phi(x) + \lambda^2 \int_0^\infty e^{-\lambda t} u(t,x)\,dt = -\lambda \Phi(x) + \lambda U(x).$$

In addition, we have

$$U'(x) = \lambda \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial x} u(t,x)\,dt, \qquad U''(x) = \lambda \int_0^\infty e^{-\lambda t} \frac{\partial^2}{\partial x^2} u(t,x)\,dt.$$

Now integrating both sides of (11.9) with the weight function $\lambda\, e^{-\lambda t}$, we get

$$\frac{1}{2}\sigma^2(x) U''(x) + \mu(x) U'(x) - (\lambda + f(x)) U(x) = -\lambda \Phi(x) - g(x), \quad x \in (a,b). \tag{11.14}$$

The boundary conditions (11.11) are transformed to the conditions

$$U(a) = \Phi(a), \qquad U(b) = \Phi(b). \tag{11.15}$$

One can give a natural probabilistic interpretation to the Laplace transform (formula (11.13)). Namely, let $\tau$ be an exponentially distributed random time independent of the Brownian motion $W$ and, consequently, of the process $X$. Let the density of $\tau$ have the form $\lambda e^{-\lambda t} 1\!\!1_{[0,\infty)}(t)$, $t \in \mathbf{R}$, $\lambda > 0$. Then applying Fubini's theorem, we get

$$\begin{aligned} U(x) = \mathbf{E} u(\tau, x) = \mathbf{E}_x \biggl\{ & \Phi(X(\tau \wedge H_{a,b})) \exp\biggl( - \int_0^{\tau \wedge H_{a,b}} f(X(s))\,ds \biggr) \\ & + \int_0^{\tau \wedge H_{a,b}} g(X(s)) \exp\biggl( - \int_0^s f(X(v))\,dv \biggr) ds \biggr\}. \end{aligned} \tag{11.16}$$

Therefore the function $U$ is equal to the expectation of the same random process as in formula (11.12), with the random time $\tau$ instead of a fixed time $t$.

As a result, we can formulate the following analog of Theorem 11.2.

**Theorem 11.3.** *Let $\Phi(x)$, $f(x)$ and $g(x)$, $x \in [a,b]$, be continuous functions and let $f$ be nonnegative. Suppose that the coefficients $\sigma$ and $\mu$ satisfy condition* (11.2) *with $N = \max\{|a|, |b|\}$ and $\sigma(x) > 0$ for $x \in [a,b]$.*

*Then the function $U(x)$, $x \in [a,b]$, defined by* (11.16) *is the unique continuous solution of the problem* (11.14), (11.15).

## § 12. Ordinary differential equations, probabilistic approach

The proof of Theorem 11.2 is based on the result that the nonhomogeneous Cauchy problem (11.9)–(11.11) has a solution. The proof of this result is very complicated and requires additional conditions on the functions $\sigma$, $\mu$, $f$ and $g$. As it was proved, by the Laplace transform with respect to $t$ the problem (11.9)–(11.11) is reduced to the ordinary differential equation (11.14) with the boundary conditions (11.15). The ordinary differential problem has a unique solution, which is not difficult to prove. The solution of (11.14), (11.15) has the probabilistic expression (11.16). Our aim, in particular, is to give a direct probabilistic proof for this expression, which is not based on the solution of the Cauchy problem.

We consider first the preliminary results concerning the solutions of ordinary second-order differential equations.

**Proposition 12.1.** *Let $g(x)$, $x \in (l,r)$, be a nonnegative continuous function that does not vanish identically, $l \geq -\infty$, $r \leq \infty$. Then the homogeneous equation*

$$\phi''(x) - g(x)\phi(x) = 0, \qquad x \in (l,r), \tag{12.1}$$

*has two nonnegative convex linearly independent solutions $\psi$ and $\varphi$ such that $\psi(x)$, $x \in (l,r)$, is increasing, and $\varphi(x)$, $x \in (l,r)$, is decreasing.*

*Proof.* Without loss of generality, we assume that $0 \in (l,r)$ and $g(0) > 0$. Consider for $x \in [0,r)$ the solution $\psi_+$ of equation (12.1) with the initial values $\psi_+(0) = 1$, $\psi'_+(0) = 1$. This solution is a convex function, therefore $\psi_+(x) \geq 1+x$. Another linearly independent solution, as is easily seen, has the form

$$\varphi(x) = \psi_+(x) \int_x^r \frac{dv}{\psi_+^2(v)} \leq \psi_+(x) \int_x^r \frac{dv}{(1+v)^2}, \qquad x \in [0,r).$$

Since $\psi'_+(x)$ is nondecreasing, the following estimates hold for $x \in [0,r)$:

$$\varphi'(x) = \psi'_+(x) \int_x^r \frac{dv}{\psi_+^2(v)} - \frac{1}{\psi_+(x)} < \int_x^r \frac{\psi'_+(v)}{\psi_+^2(v)}\, dv - \frac{1}{\psi_+(x)} = -\frac{1}{\psi_+(r)} \leq 0.$$

It follows that $\varphi(x)$, $x \in [0,r)$, is a nonnegative, convex, nonincreasing function, and $\varphi'(0) < 0$. We continue the solution $\varphi$ to the interval $(l,0]$ so that it satisfies (12.1). Since the solution is convex, $\varphi(x) \geq \varphi(0) + \varphi'(0)x$ for $x \in (l,0]$. Another linearly independent solution in this interval is given by

$$\psi(x) = \varphi(x) \int_l^x \frac{dv}{\varphi^2(v)} \leq \varphi(x) \int_l^x \frac{dv}{(\varphi(0) + \varphi'(0)v)^2}.$$

Arguing similarly, we find that this solution obeys the estimates

$$\psi'(x) = \frac{1}{\varphi(x)} + \varphi'(x) \int_l^x \frac{dv}{\varphi^2(v)} > \frac{1}{\varphi(x)} + \int_l^x \frac{\varphi'(v)}{\varphi^2(v)}\, dv = \frac{1}{\varphi(l)} \geq 0$$

and $\psi'(0) > 0$. We continue the solution $\psi$ to the interval $[0, r)$ so that it satisfies equation (12.1). Since the solution is convex, it is strictly increasing on the interval. Proposition 12.1 is proved. $\square$

Consider the homogeneous equation

$$\phi''(x) + q(x)\phi'(x) - h(x)\phi(x) = 0, \qquad x \in \mathbf{R}. \tag{12.2}$$

We set

$$p(x) := \exp\Bigl(\int_0^x q(v)\,dv\Bigr), \qquad y(x) := \int_0^x \frac{dv}{p(v)}, \qquad x \in \mathbf{R}.$$

The function $y(x)$, $x \in \mathbf{R}$, is strictly increasing and $y(0) = 0$; hence, it has the strictly increasing inverse function $y^{(-1)}(y)$, $y \in (l, r)$, where

$$l := -\int_{-\infty}^0 \frac{dv}{p(v)} \ge -\infty, \qquad r := \int_0^\infty \frac{dv}{p(v)} \le \infty.$$

By a change of variables, equation (12.2) can be reduced to the form (12.1) with

$$g(y) = p^2(y^{(-1)}(y))h(y^{(-1)}(y)), \qquad y \in (l, r). \tag{12.3}$$

Indeed, we change $x$ to $y(x)$. For this choice $\bar\phi(y) := \phi(y^{(-1)}(y))$, $y \in (l, r)$, i.e., we consider the function $\bar\phi$ such that $\phi(x) = \bar\phi(y(x))$, $x \in \mathbf{R}$. Since (12.2) can be written as

$$(p(x)\phi'(x))' - p(x)h(x)\phi(x) = 0, \qquad x \in \mathbf{R},$$

and $p(x)\phi'(x) = \bar\phi'(y(x))$, equation (12.2) is transformed to the following one

$$\frac{\bar\phi''(y(x))}{p(x)} - p(x)h(x)\phi(x) = 0, \qquad x \in \mathbf{R},$$

or, equivalently, to the equation

$$\bar\phi''(y(x)) - p^2(y^{(-1)}(y(x)))h(y^{(-1)}(y(x)))\bar\phi(y(x)) = 0, \qquad x \in \mathbf{R}.$$

This equation for the new variable $y$ is in the form (12.1).

An important question is when does equation (12.2) have nonzero bounded solutions on the whole real line? Since equation (12.1) considered on the whole real line does not have nonzero bounded solutions, the same is true for (12.2) if $l = -\infty$ and $r = \infty$.

Thus, we have proved the following statement.

**Proposition 12.2.** *Let $q(x)$ and $h(x)$, $x \in \mathbf{R}$, be continuous functions, and let $h$ be a nonnegative function that does not vanish identically. Then equation* (12.2) *has two nonnegative linearly independent solutions $\psi$ and $\varphi$ such that $\psi(x)$, $x \in \mathbf{R}$, is an increasing and $\varphi(x)$, $x \in \mathbf{R}$, is a decreasing solution.*

*If $l = -\infty$ and $r = \infty$, then equation* (12.2) *does not have nonzero bounded solutions.*

The functions $\psi(x)$ and $\varphi(x)$, $x \in \mathbf{R}$, are called *fundamental solutions* of (12.2). Their *Wronskian* $w(x) := \psi'(x)\varphi(x) - \psi(x)\varphi'(x)$ has the form

$$w(x) = w(0) \exp\Big( - \int_0^x q(y)\, dy \Big)$$

and it is a positive function.

Indeed, from (12.2) it follows that the Wronskian satisfies the equation

$$w'(x) = -q(x)\, w(x), \qquad w(0) > 0, \qquad x \in \mathbf{R},$$

which yields the desired formula.

If either $l > -\infty$ or $r < \infty$, then the answer to our question depends on the functions $q(x)$ and $h(x)$, $x \in \mathbf{R}$. Thus, if $l = -\infty$, $r < \infty$ and $\lim\limits_{y \uparrow r} g(y) < \infty$, where $g$ is defined by (12.3), equation (12.2) has a bounded solution on the whole real line, because in this case $g$ can be continued beyond the boundary $r$. Then the solution $\bar{\psi}(x)$ of equation (12.1) is bounded for $x \in (l, r)$ and $\psi(x) = \bar{\psi}(y(x))$, $x \in \mathbf{R}$, is a bounded solution of (12.2). If in this case $\lim\limits_{y \uparrow r} g(y) = \infty$, the equation may or may not have a nonzero bounded solution. The left boundary $l$ is treated analogously.

Let us give some examples. Consider for $\alpha \in \mathbf{R}$ the equation

$$\phi''(x) - \phi'(x) - e^{\alpha x}\phi(x) = 0, \qquad x \in \mathbf{R}.$$

By the change of variable $x = \ln(y+1)$, $y \in (-1, \infty)$, this equation is transformed to

$$\bar{\phi}''(y) - (y+1)^{\alpha-2}\bar{\phi}(y) = 0, \qquad y \in (-1, \infty).$$

If $\alpha = 2$, there exists the limit $\lim\limits_{y \downarrow -1} g(y) = 1$. The fundamental solutions of the transformed equation have the form $\bar{\varphi}(y) = e^{-y}$ and $\bar{\psi}(y) = e^{y}$. The solutions of the original equation then are $\varphi(x) = \exp\big(1 - e^x\big)$ and $\psi(x) = \exp\big(e^x - 1\big)$, $x \in \mathbf{R}$. The solution $\varphi$ is bounded.

If $\alpha = 0$, the fundamental solutions are $\bar{\varphi}(y) = (y+1)^{\sqrt{5/4}+1/2}$ and $\bar{\psi}(y) = (y+1)^{-\sqrt{5/4}+1/2}$, and the solutions of the original equation are $\varphi(x) = e^{x(\sqrt{5/4}+1/2)}$ and $\psi(x) = e^{-x(\sqrt{5/4}-1/2)}$, $x \in \mathbf{R}$. Hence, there are no nontrivial bounded solutions.

If $\alpha = 1$, the fundamental solutions of the transformed equation have the form $\bar{\varphi}(y) = \sqrt{y+1}\, K_1(2\sqrt{y+1})$ and $\bar{\psi}(y) = \sqrt{y+1}\, I_1(2\sqrt{y+1})$ (see Appendix 4, equation 6$a$ for $p = 1/2$). The solutions of the original equation are

$\varphi(x) = e^{x/2}K_1(2e^{x/2})$ and $\psi(x) = e^{x/2}I_1(2e^{x/2})$, $x \in \mathbf{R}$. The solution $\varphi$ is bounded, according to the asymptotic behavior of the modified Bessel function $K_1$ (see Appendix 2).

The conditions $l = -\infty$ and $r = \infty$, which guarantee unboundedness of nonzero solutions, are not always easy to check. In addition, they do not cover all cases. We prove the following useful result.

**Proposition 12.3.** *Let $q(x)$ and $h(x)$, $x \in \mathbf{R}$, be continuous functions and let the function $h$ be nonnegative. Suppose that for some $C > 0$*

$$|q(x)| \le C(1 + |x|) \qquad \text{for all} \quad x \in \mathbf{R}, \tag{12.4}$$

$$\liminf_{y\to\infty} \frac{1}{y}\int_0^y h(x)\,dx > 0, \qquad \liminf_{y\to\infty} \frac{1}{y}\int_{-y}^0 h(x)\,dx > 0. \tag{12.5}$$

*Then the homogeneous equation* (12.2) *has no nonzero bounded solutions.*

*Proof.* By Proposition 12.2, the homogeneous equation (12.2) has two linearly independent nonnegative solutions $\psi$ and $\varphi$ such that $\psi(x)$, $x \in \mathbf{R}$, is increasing and $\varphi(x)$, $x \in \mathbf{R}$, is decreasing. Assume that $\psi_+ := \lim\limits_{x\to\infty} \psi(x) < \infty$. The left condition in (12.5) implies the existence of $y_0 > 1$ and $h_0 > 0$ such that

$$\frac{1}{y}\int_0^y h(x)\,dx \ge h_0$$

for all $y > y_0$. Set $\varepsilon := \dfrac{h_0\psi_+}{2h_0 + 4C}$, where the constant $C$ is taken from condition (12.4). Let $y_0$ be so large that

$$\psi(x) \in (\psi_+ - \varepsilon, \psi_+)$$

for $x \ge y_0$. Set $y_1 := \dfrac{h_0 + 4C}{h_0(h_0 + 2C)}\displaystyle\int_0^{y_0} h(x)\,dx$. If $y > \max\{y_0, y_1\}$, then

$$\int_{y_0}^y \psi''(x)\,dx = \int_{y_0}^y (h(x)\psi(x) - q(x)\psi'(x))\,dx \ge \int_{y_0}^y (h(x)\psi(x) - 2Cx\psi'(x))\,dx$$

$$\ge \int_{y_0}^y h(x)\psi(x)\,dx - 2Cy(\psi(y) - \psi(y_0)) \ge (\psi_+ - \varepsilon)\Big(yh_0 - \int_0^{y_0} h(x)\,dx\Big) - 2Cy\varepsilon$$

$$= y\psi_+\frac{h_0}{2} - \psi_+\Big(\frac{h_0 + 4C}{2h_0 + 4C}\Big)\int_0^{y_0} h(x)\,dx = \psi_+\frac{h_0}{2}(y - y_1) > 0.$$

Consequently,

$$\psi'(y) - \psi'(y_0) = \int_{y_0}^{y} \psi''(x)\, dx \geq \psi_+ \frac{h_0}{2}(y - y_1)$$

for $y > \max\{y_0, y_1\}$, which contradicts the relation

$$\int_{y_0}^{\infty} \psi'(y)\, dy = \psi_+ - \psi(y_0) < \infty.$$

Therefore, the limit of the function $\psi_+$ cannot be finite and the solution $\psi$ cannot be bounded. A similar reasoning shows that the solution $\varphi$ cannot be bounded if the right condition in (12.5) holds. Hence, only the trivial solution can be bounded. □

We return to the problem (11.14), (11.15). Let $X$ be the solution of the stochastic differential equation (11.1) with coefficients satisfying (11.2) and (11.3). Suppose that $\sigma^2(x) > 0$ for all $x \in \mathbf{R}$. Let $H_{a,b} := \min\{s : X(s) \notin (a, b)\}$ be the first exit time of the process $X$ from the interval $(a, b)$.

We start with a simpler problem than (11.14), (11.15).

**Theorem 12.1.** *Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be continuous functions. Suppose that $\Phi$ is bounded and $f$ is nonnegative. Let $U(x)$ be a bounded solution of the equation*

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{12.6}$$

*Then*

$$U(x) = \mathbf{E}_x\Big\{\Phi(X(\tau)) \exp\Big( - \int_0^{\tau} f(X(s))\, ds\Big)\Big\}. \tag{12.7}$$

**Remark 12.1.** Under the conditions of Theorem 12.1 equation (12.6) has a unique bounded solution in $\mathbf{R}$, because it must have the probabilistic expression of the form (12.7). Consequently, the corresponding homogeneous equation has only the trivial bounded solution.

*Proof of Theorem* 12.1. Set

$$\eta(t) := U(X(t)) \exp\Big( - \lambda t - \int_0^{t} f(X(v))\, dv\Big).$$

Applying Itô's formula, we see that

$$\eta(r) - \eta(0) = \int_0^{r} \exp\Big( - \lambda t - \int_0^{t} f(X(v))\, dv\Big)\Big[U'(X(t))\sigma(X(t))\, dW(t)$$

$$+\Big(U'(X(t))\mu(X(t)+\tfrac{1}{2}U''(X(t))\sigma^2(X(t))-(\lambda+f(X(t)))U(X(t))\Big)dt\Big]$$

for any $0<r$. Taking into account (12.6), we can write

$$\eta(r\wedge H_{a,b})-U(x)=\int_0^r \mathbb{1}_{[0,H_{a,b})}(t)e^{-\lambda t}\exp\Big(-\int_0^t f(X(v))\,dv\Big)$$
$$\times\big[U'(X(t))\sigma(X(t))\,dW(t)-\lambda\Phi(X(t))\,dt\big]. \tag{12.8}$$

By reasons similar to those given for equation (11.8), one can take the expectation to the stochastic integral. Now, computing the expectation of both sides of (12.8) and taking into account that the expectation of the stochastic integral is equal to zero, we obtain

$$U(x)=\mathbf{E}_x\eta(r\wedge H_{a,b})+\mathbf{E}_x\int_0^{r\wedge H_{a,b}}\lambda e^{-\lambda t}\Phi(X(t))\exp\Big(-\int_0^t f(X(v))\,dv\Big)\,dt. \tag{12.9}$$

Since the diffusion process $X$ is continuous and defined for all time moments, $H_{a,b}\to\infty$ as $a\to-\infty$ and $b\to\infty$. By the Lebesgue dominated convergence theorem, one can pass to the limit in (12.9) as $a\to-\infty$ and $b\to\infty$. Next, we let $r\to\infty$. By the definition of the process $\eta$, the term $\mathbf{E}_x\eta(r)$ tends to zero. Hence, it follows from (12.9) that

$$U(x)=\mathbf{E}_x\int_0^\infty\lambda e^{-\lambda t}\Phi(X(t))\exp\Big(-\int_0^t f(X(v))\,dv\Big)\,dt.$$

Then, using the assumption that $\tau$ does not depend on the diffusion $X$ and has the density $\lambda e^{-\lambda t}\mathbb{1}_{[0,\infty)}(t)$, we conclude by Fubini's theorem that the above equality is identical to (12.7). □

We have the following version of Theorem 12.1.

**Theorem 12.2.** *Let $\Phi(x)$, $f(x)$ and $F(x)$, $x\in\mathbf{R}$, be continuous functions. Suppose that $\Phi$, $F$ are bounded and $f$ is nonnegative. Let $U(x)$, $x\in\mathbf{R}$, be a bounded solution of the equation*

$$\frac{1}{2}\sigma^2(x)U''(x)+\mu(x)U'(x)-(\lambda+f(x))U(x)=-\lambda\Phi(x)-F(x),\qquad x\in\mathbf{R}. \tag{12.10}$$

*Then*

$$U(x)=\mathbf{E}_x\Big\{\Phi(X(\tau))\exp\Big(-\int_0^\tau f(X(s))\,ds\Big)+\int_0^\tau F(X(s))\exp\Big(-\int_0^s f(X(v))\,dv\Big)ds\Big\}.$$

*Proof.* This result is a corollary of Theorem 12.1. Indeed, since $\tau$ is independent of $X$, Fubini's theorem shows that

$$\mathbf{E}_x\Big\{\int_0^\tau F(X(s))\exp\Big(-\int_0^s f(X(v))\,dv\Big)ds\Big\}$$

$$= \lambda \int_0^\infty e^{-\lambda t} \int_0^t \mathbf{E}_x \Big\{ F(X(s)) \exp\Big( -\int_0^s f(X(v))\,dv \Big) \Big\} ds\,dt$$

$$= \int_0^\infty e^{-\lambda s} \mathbf{E}_x \Big\{ F(X(s)) \exp\Big( -\int_0^s f(X(v))\,dv \Big) \Big\} ds$$

$$= \frac{1}{\lambda} \mathbf{E}_x \Big\{ F(X(\tau)) \exp\Big( -\int_0^\tau f(X(v))\,dv \Big) \Big\}.$$

Now we can apply Theorem 12.1 with the function $\Phi(x) + \frac{1}{\lambda} F(x)$ instead of $\Phi(x)$. $\square$

In the following result we can assume initially that the functions $\mu$ and $\sigma$ satisfy condition (11.2) only on the interval $(a, b)$, because one can continue $\mu$ and $\sigma$ outside $(a, b)$ in such a way that conditions (11.2) and (11.3) hold. In this case, by Theorem 7.2, the process $X(t)$ is not changed for $t \in [0, H_{a,b}]$.

**Theorem 12.3.** *Let $\Phi(x)$, $f(x)$ and $F(x)$, $x \in [a, b]$, be continuous functions, and let $f$ be nonnegative. Let $U(x)$, $x \in [a, b]$, be a solution of the problem*

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x) - F(x), \quad x \in (a, b), \tag{12.11}$$

$$U(a) = \Phi(a), \qquad U(b) = \Phi(b). \tag{12.12}$$

*Then*

$$U(x) = \mathbf{E}_x \Big\{ \Phi(X(\tau \wedge H_{a,b})) \exp\Big( -\int_0^{\tau\wedge H_{a,b}} f(X(s))\,ds \Big)$$

$$+ \int_0^{\tau\wedge H_{a,b}} F(X(s)) \exp\Big( -\int_0^s f(X(v))\,dv \Big)\,ds \Big\}. \tag{12.13}$$

*Proof.* We continue the solution of the problem (12.11), (12.12) to the whole real line. One can extend the functions $\Phi$, $f$ and $F$ to the real line so that the extensions are bounded and $f$ is nonnegative.

Let $\psi(x)$, $x \in \mathbf{R}$, and $\varphi(x)$, $x \in \mathbf{R}$, be linearly independent solutions of the homogeneous equation corresponding to (12.11), with $\psi$ increasing and $\varphi$ decreasing and nonnegative. Then $\psi(b)\varphi(a) - \psi(a)\varphi(b) > 0$.

The extension of the solution of (12.11), (12.12) to the real line can be written as

$$U(x) = U_p(x) + A_{a,b}\psi(x) + B_{a,b}\varphi(x), \tag{12.14}$$

where $U_p$ is a particular solution of equation (12.11) for $x \in \mathbf{R}$, and the constants $A_{a,b}$, $B_{a,b}$ satisfy the system of algebraic equations

$$\Phi(a) = U_p(a) + A_{a,b}\psi(a) + B_{a,b}\varphi(a),$$
$$\Phi(b) = U_p(b) + A_{a,b}\psi(b) + B_{a,b}\varphi(b).$$

This system has the following unique solution:

$$A_{a,b} = \frac{(\Phi(b) - U_p(b))\varphi(a) - (\Phi(a) - U_p(a))\varphi(b)}{\psi(b)\varphi(a) - \psi(a)\varphi(b)}, \tag{12.15}$$

$$B_{a,b} = \frac{\psi(b)(\Phi(a) - U_p(a)) - \psi(a)(\Phi(b) - U_p(b))}{\psi(b)\varphi(a) - \psi(a)\varphi(b)}. \tag{12.16}$$

We set

$$\eta(t) := U(X(t)) \exp\Big( - \lambda t - \int_0^t f(X(v))\,dv\Big)$$

$$+ e^{-\lambda t} \int_0^t F(X(s)) \exp\Big( - \int_0^s f(X(v))\,dv\Big)\,ds.$$

Applying Itô's formula, we see that

$$\eta(r) - \eta(0) = \int_0^r \exp\Big( - \lambda t - \int_0^t f(X(v))\,dv\Big)\Big[U'(X(t))\,\sigma(X(t))\,dW(t)$$

$$+\Big(U'(X(t))\mu(X(t)) + \frac{1}{2}U''(X(t))\,\sigma^2(X(t)) - (\lambda + f(X(t)))U(X(t)) + F(X(t))\Big)dt\Big]$$

$$- \lambda \int_0^r e^{-\lambda t} \int_0^t F(X(s)) \exp\Big( - \int_0^s f(X(v))\,dv\Big)\,ds\,dt$$

for every $r > 0$. Taking into account (12.11), we get the equality

$$\eta(H_{a,b}) - U(x) = \int_0^{H_{a,b}} \exp\Big( - \lambda t - \int_0^t f(X(v))\,dv\Big)\big[U'(X(t))\,\sigma(X(t))\,dW(t)$$

$$-\lambda\,\Phi(X(t))\big)\,dt\big] - \lambda \int_0^{H_{a,b}} e^{-\lambda t} \int_0^t F(X(s)) \exp\Big( - \int_0^s f(X(v))\,dv\Big)\,ds\,dt.$$

It is important that the process $\mathbb{1}_{[0,H_{a,b}]}(t)$, $t \geq 0$, is adapted to the filtration $\mathcal{G}_0^t = \sigma(W(s), 0 \leq s \leq t)$ and for $t \leq H_{a,b}$ the functions $U'(X(t))$ and $\sigma(X(t))$ are bounded by a constant depending on $a$, $b$. Therefore, we can take the expectation of the stochastic integral and this expectation is equal to zero.

Since $\sup\limits_{x\in[a,b]} |\Phi(x)| < \infty$ and $\sup\limits_{x\in[a,b]} |F(x)| < \infty$, the expectations of the other terms of the difference $\eta(H_{a,b}) - U(x)$ are finite. Applying the expectation, we derive the equality

$$U(x) = \mathbf{E}_x \eta(H_{a,b}) + \lambda \mathbf{E}_x \int_0^{H_{a,b}} \exp\Big( - \lambda t - \int_0^t f(X(v))\,dv\Big)\Phi(X(t))\,dt$$

$$+\lambda \mathbf{E}_x \int_0^{H_{a,b}} e^{-\lambda t} \int_0^t F(X(s)) \exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\,ds\,dt.$$

Let us consider each of the terms on the right-hand side of this equality. We use the equality $U(X(H_{a,b})) = \Phi(X(H_{a,b}))$, and the fact that $\tau$ is independent of $X$ and has exponential distribution. By Fubini's theorem, these terms can be represented as follows:

$$\mathbf{E}_x \eta(H_{a,b}) = \mathbf{E}_x \Bigl\{\Phi(X(H_{a,b})) \exp\Bigl(-\int_0^{H_{a,b}} f(X(s))\,ds\Bigr) \mathbb{1}_{\{\tau > H_{a,b}\}}\Bigr\}$$

$$+\mathbf{E}_x \Bigl\{\int_0^{H_{a,b}} F(X(s)) \exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\,ds\, \mathbb{1}_{\{\tau > H_{a,b}\}}\Bigr\},$$

$$\lambda \mathbf{E}_x \int_0^{H_{a,b}} \exp\Bigl(-\lambda t - \int_0^t f(X(v))\,dv\Bigr) \Phi(X(t))\,dt$$

$$= \mathbf{E}_x \Bigl\{\Phi(X(\tau)) \exp\Bigl(-\int_0^{\tau} f(X(s))\,ds\Bigr) \mathbb{1}_{\{\tau \le H_{a,b}\}}\Bigr\},$$

and

$$\lambda \mathbf{E}_x \int_0^{H_{a,b}} e^{-\lambda t} \int_0^t F(X(s)) \exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\,ds\,dt$$

$$= \mathbf{E}_x \Bigl\{\int_0^{\tau} F(X(s)) \exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\,ds\, \mathbb{1}_{\{\tau \le H_{a,b}\}}\Bigr\}.$$

Summing these equalities, we see that $U$ takes the form (12.13). □

The analogs of Theorem 12.3 are of special interest in the cases when either $\tau \to \infty$, or $H_{a,b} \to \infty$, and when both limits take place. We begin with the analysis of the results for the second case.

**Theorem 12.4.** *Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be continuous functions, with $\Phi$ bounded and $f$ nonnegative.*

*Suppose that there exists the bounded on any finite interval derivative $\Bigl(\frac{\mu(x)}{\sigma^2(x)}\Bigr)'$, $x \in \mathbf{R}$. Then*

$$U(x) = \mathbf{E}_x \Bigl\{\Phi(X(\tau)) \exp\Bigl(-\int_0^{\tau} f(X(s))\,ds\Bigr)\Bigr\}, \qquad x \in \mathbf{R}, \tag{12.17}$$

*is the unique bounded solution of the equation*

$$\frac{1}{2}\,\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{12.18}$$

**Remark 12.2.** In contrast to Theorem 12.1, in this result it is not assumed that equation (12.18) has a bounded solution; instead, we state that the function (12.17) is such a solution.

*Proof of Theorem* 12.4. We set

$$U_{a,b}(x) := \mathbf{E}_x\Big\{\Phi(X(\tau \wedge H_{a,b}))\exp\Big(-\int_0^{\tau\wedge H_{a,b}} f(X(s))\,ds\Big)\Big\} \tag{12.19}$$

for $a < x < b$. By Theorem 12.3, the function $U_{a,b}(x)$, $x \in (a,b)$, is the solution of (12.11), (12.12) with $F \equiv 0$. We extend $U_{a,b}(x)$ to the whole real line by formula (12.14).

As it was mentioned above, $H_{a,b} \to \infty$ as $a \to -\infty$ and $b \to \infty$.

By the Lebesgue dominated convergence theorem,

$$\lim_{a\to-\infty,b\to\infty} U_{a,b}(x) = U(x), \qquad x \in \mathbf{R}. \tag{12.20}$$

We can assume that $a < 0 < b$. Integrating (12.11), we get the equation

$$\frac{1}{2}\,(U'_{a,b}(x) - U'_{a,b}(0)) + \frac{\mu(x)}{\sigma^2(x)}\,U_{a,b}(x) - \frac{\mu(0)}{\sigma^2(0)}\,U_{a,b}(0) - \int_0^x \Big(\frac{\mu(y)}{\sigma^2(y)}\Big)' U_{a,b}(y)\,dy$$

$$-\int_0^x \Big(\frac{\lambda + f(y)}{\sigma^2(y)}\Big)U_{a,b}(y)\,dy = -\lambda\int_0^x \frac{\Phi(y)}{\sigma^2(y)}\,dy. \tag{12.21}$$

Integrating this equation, we find that for $x \in \mathbf{R}$

$$\frac{1}{2}(U_{a,b}(x) - U_{a,b}(0)) - \frac{1}{2}\,U'_{a,b}(0)\,x + \int_0^x \frac{\mu(z)}{\sigma^2(z)}\,U_{a,b}(z)\,dz - \frac{\mu(0)}{\sigma^2(0)}\,U_{a,b}(0)x$$

$$-\int_0^x\int_0^z \Big(\Big(\frac{\mu(y)}{\sigma^2(y)}\Big)' + \frac{\lambda + f(y)}{\sigma^2(y)}\Big)U_{a,b}(y)\,dy\,dz = -\lambda\int_0^x\int_0^z \frac{\Phi(y)}{\sigma^2(y)}\,dy\,dz. \tag{12.22}$$

From (12.19) it follows that the functions $U_{a,b}(x)$, $x \in (a,b)$, are bounded by the same constant as the function $\Phi$. Using (12.20) and applying the Lebesgue dominated convergence theorem, we deduce from (12.22) that there exists the limit $\widetilde{U}_0 := \lim_{a\to-\infty,b\to\infty} U'_{a,b}(0)$, and

$$\frac{1}{2}(U(x) - U(0)) - \frac{1}{2}\widetilde{U}_0\,x + \int_0^x \frac{\mu(z)}{\sigma^2(z)}\,U(z)\,dz - \frac{\mu(0)}{\sigma^2(0)}U(0)\,x$$

$$-\int_0^x\int_0^z\Big(\Big(\frac{\mu(y)}{\sigma^2(y)}\Big)'+\frac{\lambda+f(y)}{\sigma^2(y)}\Big)U(y)\,dy\,dz=-\lambda\int_0^x\int_0^z\frac{\Phi(y)}{\sigma^2(y)}dy\,dz. \tag{12.23}$$

From this equality it follows that $U(x)$, $x\in\mathbf{R}$, is a continuous function. In addition, $U$ is differentiable for all $x$ including zero, and $\widetilde{U}_0=U'(0)$. Differentiating (12.23) with respect to $x$ and applying the integration by parts formula, we see that $U$ satisfies the equation

$$\frac{1}{2}(U'(x)-U'(0))+\int_0^x\frac{\mu(y)}{\sigma^2(y)}\,U'(y)\,dy-\int_0^x\frac{\lambda+f(y)}{\sigma^2(y)}U(y)\,dy=-\lambda\int_0^x\frac{\Phi(y)}{\sigma^2(y)}dy.$$

Differentiating this equation with respect to $x$, we get that $U$ is the solution of (12.18).

Now the fact that such bounded solution is unique follows from Remark 12.1. □

Consider the transformation of Theorem 12.3 as $\tau\to\infty$ and $H_{a,b}\to\infty$ simultaneously.

**Theorem 12.5.** *Let $f(x)$, $x\in\mathbf{R}$, be a nonnegative continuous function. Suppose that there exists the bounded on any finite interval derivative $\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'$, $x\in\mathbf{R}$. Then the function*

$$L(x):=\mathbf{E}_x\exp\Big(-\int_0^\infty f(X(s))\,ds\Big),\qquad x\in\mathbf{R}, \tag{12.24}$$

*is the solution of the homogeneous equation*

$$\frac{1}{2}\sigma^2(x)\phi''(x)+\mu(x)\phi'(x)-f(x)\phi(x)=0,\qquad x\in\mathbf{R}. \tag{12.25}$$

To prove this result one can repeat the proof of Theorem 12.4 for $\Phi\equiv1$, adding to it the passage to the limit as $\lambda\to0$. In this case, $\lim_{\lambda\to0}\tau=\infty$ in probability, since $\mathbf{P}(\tau>t)=e^{-\lambda t}$ for $t\ge0$.

This result has an important consequence.

**Corollary 12.1.** *Let $f(x)$, $x\in\mathbf{R}$, be a nonnegative continuous function. Suppose that there exists the bounded derivative $\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'$, $x\in\mathbf{R}$, and*

$$\liminf_{y\to\infty}\frac{1}{y}\int_0^y\frac{f(x)}{\sigma^2(x)}\,dx>0,\qquad \liminf_{y\to\infty}\frac{1}{y}\int_{-y}^0\frac{f(x)}{\sigma^2(x)}\,dx>0. \tag{12.26}$$

*Then*

$$\int_0^\infty f(X(s))\,ds=\infty\qquad\text{a.s.} \tag{12.27}$$

Indeed, according to Proposition 12.3, under these assumptions equation (12.25) has no nonzero bounded solutions. Therefore, $L\equiv0$ and we have (12.27).

Propositions 12.2 and 12.3 imply the following result.

**Corollary 12.2.** *Let $f(x)$, $x \in \mathbf{R}$, be a nonnegative continuous function. Suppose that conditions* (12.26) *hold and*

$$\left|\frac{\mu(x)}{\sigma^2(x)}\right| \le C(1+|x|) \qquad \text{for all} \quad x \in \mathbf{R} \tag{12.28}$$

*for some $C > 0$.*

*Then the homogeneous equation* (12.25) *has two nonnegative linearly independent solutions $\psi(x)$ and $\varphi(x)$ such that $\psi(x)$ is increasing and $\lim_{x\to\infty} \psi(x) = \infty$, while $\varphi(x)$ is decreasing and $\lim_{x\to-\infty} \varphi(x) = \infty$.*

Another extreme version of Theorem 12.3 is the case when only $\tau \to \infty$. As we have seen, this happens if $\lambda \to 0$. We precede the consideration of this case by the following important result.

**Lemma 12.1.** *For every $x \in [a, b]$,*

$$\mathbf{E}_x H_{a,b} < \infty. \tag{12.29}$$

**Remark 12.3.** From (12.29) it follows that $\mathbf{P}_x(H_{a,b} < \infty) = 1$ for $x \in [a, b]$.

*Proof of Lemma* 12.1. We consider the family $\{U_\lambda(x), x \in [a, b]\}_{\lambda \ge 0}$ of solutions of the problem

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - \lambda U(x) = -1, \qquad x \in (a, b), \tag{12.30}$$

$$U(a) = 0, \qquad U(b) = 0. \tag{12.31}$$

From Theorem 12.3 with $f \equiv 0$, $\Phi \equiv 0$ and $F \equiv 1$ it follows that $U_\lambda(x) = \mathbf{E}_x\{\tau \wedge H_{a,b}\}$ for $\lambda > 0$.

We will prove that for all $x \in [a, b]$

$$\sup_{\lambda>0} U_\lambda(x) \le U_0(x), \tag{12.32}$$

where $U_0(x)$ is the solution of (12.30), (12.31) for $\lambda = 0$. This estimate is useful for us due to the following reason. Since $\lim_{\lambda\to 0} \tau = \infty$ in probability, $\lim_{\lambda\to 0}\{\tau \wedge H_{a,b}\} = H_{a,b}$. Now from (12.32), by Fatou's lemma, it follows that

$$\mathbf{E}_x H_{a,b} \le \sup_{\lambda>0} \mathbf{E}_x\{\tau \wedge H_{a,b}\} \le U_0(x),$$

and this is what we want to prove.

To prove (12.32), we use the following result.

**Proposition 12.4.** *The solution of the problem*

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) = -F(x), \qquad x \in (a, b), \tag{12.33}$$

$$Q(a) = \Phi(a), \qquad Q(b) = \Phi(b), \tag{12.34}$$

*has the form*

$$Q(x) = \frac{S(b)-S(x)}{S(b)-S(a)}\Bigl(\Phi(a) + \int_a^x (S(y)-S(a))F(y)\,dM(y)\Bigr)$$

$$+\frac{S(x)-S(a)}{S(b)-S(a)}\Bigl(\Phi(b) + \int_x^b (S(b)-S(y))F(y)\,dM(y)\Bigr), \tag{12.35}$$

*where*

$$S(x) := \int^x \exp\Bigl(-\int^y \frac{2\mu(z)}{\sigma^2(z)}\,dz\Bigr)\,dy, \qquad dM(x) = \frac{2}{\sigma^2(x)}\exp\Bigl(\int^x \frac{2\mu(z)}{\sigma^2(z)}\,dz\Bigr)\,dx.$$

In the definition of the functions $S(x)$ and $M(x)$, the lower limit of integration can be arbitrary, but the same.

Formula (12.35) can be verified by direct differentiation since the function $S$ satisfies the equation

$$\frac{1}{2}\,\sigma^2(x)S''(x) + \mu(x)S'(x) = 0.$$

For the derivation of (12.35) see also the proof of formula (15.13) of Ch. IV.

The difference $U_0(x) - U_\lambda(x)$ is the solution of (12.33), (12.34) with $F(x) = \lambda U_\lambda(x)$, $\Phi(a) = 0$ and $\Phi(b) = 0$. Therefore, this difference is nonnegative. This proves (12.32) and thus completes the proof of Lemma 12.1. □

It is possible to pass to the limit as $\lambda \to 0$ in the problem (12.11), (12.12) and in (12.13) and get the following result.

**Theorem 12.6.** *Let $f(x)$ and $F(x)$, $x \in [a,b]$, be continuous functions and let $f$ be nonnegative. Let the function $\Phi$ be defined only at two points $a$ and $b$.*

*Then the function*

$$Q(x) := \mathbf{E}_x\Bigl\{\Phi(X(H_{a,b}))\exp\Bigl(-\int_0^{H_{a,b}} f(X(s))\,ds\Bigr)$$

$$+\int_0^{H_{a,b}} F(X(s))\exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\,ds\Bigr\}$$

*is the solution of the problem*

$$\frac{1}{2}\,\sigma^2(x)Q''(x) + \mu(x)Q'(x) - f(x)Q(x) = -F(x), \quad x \in (a,b), \tag{12.36}$$

$$Q(a) = \Phi(a), \qquad Q(b) = \Phi(b), \qquad x \in [a,b]. \tag{12.37}$$

The proof of this theorem repeats the proof of Theorem 12.3 for $\lambda = 0$ ($\tau = \infty$) with the function $U(x)$ replaced by $Q(x)$. Here an important point is the finiteness of the integral $\int_0^\infty \mathbf{E}_x \mathbb{1}_{[0,H_{a,b}]}(t)\,dt = \mathbf{E}_x H_{a,b}$. This enables us to take the expectation of the difference $\eta(H_{a,b}) - Q(x)$, which is expressed as a stochastic integral. As a result, this expectation is equal to zero, and we get the required equality $Q(x) = \mathbf{E}_x\eta(H_{a,b})$.

Theorem 12.6 and Proposition 12.4 imply the following assertions.

**Proposition 12.5.** *The probabilities of the first exit from the interval* $[a, b]$ *have the form*

$$\mathbf{P}_x(X(H_{a,b}) = a) = \frac{S(b) - S(x)}{S(b) - S(a)}, \qquad \mathbf{P}_x(X(H_{a,b}) = b) = \frac{S(x) - S(a)}{S(b) - S(a)}. \quad (12.38)$$

This corollary is obtained from Theorem 12.6 with $F \equiv 0$, $f \equiv 0$. For $\Phi(a) = 1$ and $\Phi(b) = 0$ we have the left equality in (12.38), while for $\Phi(a) = 0$ and $\Phi(b) = 1$ we have the right one.

**Proposition 12.6.** *The expectation* $\mathbf{E}_x H_{a,b}$ *is expressed by the formula*

$$\mathbf{E}_x H_{a,b} = \frac{S(b) - S(x)}{S(b) - S(a)} \int_a^x (S(y) - S(a))\, dM(y) + \frac{S(x) - S(a)}{S(b) - S(a)} \int_x^b (S(b) - S(y))\, dM(y). \quad (12.39)$$

To derive this result, we should use Theorem 12.6 with $F \equiv 1$, $f \equiv 0$, $\Phi(a) = 0$ and $\Phi(b) = 0$. Then $Q(x) = \mathbf{E}_x H_{a,b}$ is the solution of the problem (12.33), (12.34).

Now we consider another stopping time: the first hitting time of a level $z$ by the process $X$, i.e., $H_z = \min\{s : X(s) = z\}$. This stopping time can be either finite or infinite.

**Theorem 12.7.** *Let* $f(x)$, $x \in \mathbf{R}$, *be a nonnegative continuous function. Then*

$$L_z(x) := \mathbf{E}_x \Big\{ \exp\Big( -\int_0^{H_z} f(X(s))\, ds \Big) 1\!\!\mathrm{I}_{\{H_z < \infty\}} \Big\} = \begin{cases} \psi(x), & \text{for } x \le z, \\ \varphi(x), & \text{for } z \le x, \end{cases} \quad (12.40)$$

*where* $\varphi$ *is a positive decreasing solution and* $\psi$ *is a positive increasing solution of the homogeneous equation* (12.25) *that satisfy the equalities* $\varphi(z) = \psi(z) = 1$.

*Proof.* It is clear that a.s.

$$H_z = \begin{cases} \lim\limits_{a \to -\infty} H_{a,z}, & \text{for } x \le z, \\ \lim\limits_{b \to \infty} H_{z,b}, & \text{for } z \le x. \end{cases} \quad (12.41)$$

Denote

$$Q_{a,b}^{(y)}(x) := \mathbf{E}_x \Big\{ 1\!\!\mathrm{I}_y(W(H_{a,b})) \exp\Big( -\int_0^{H_{a,b}} f(X(s))\, ds \Big) \Big\}.$$

Here the presence of the indicator function of a one-point set reduces the expectation to the set of sample paths leaving the interval through the upper boundary $(y = b)$ or the lower boundary $(y = a)$.

Since a.s.

$$1\!\!\mathrm{I}_{\{H_z < \infty\}} = \begin{cases} \lim\limits_{a \to -\infty} 1\!\!\mathrm{I}_{\{z\}}(W(H_{a,z})), & \text{for } x \le z, \\ \lim\limits_{b \to \infty} 1\!\!\mathrm{I}_{\{z\}}(W(H_{z,b})), & \text{for } z \le x, \end{cases}$$

the Lebesgue dominated convergence theorem shows that

$$L_z(x) = \begin{cases} \lim\limits_{a\to-\infty} Q^{(z)}_{a,z}(x), & \text{for } x \le z, \\ \lim\limits_{b\to\infty} Q^{(z)}_{z,b}(x), & \text{for } z \le x. \end{cases} \tag{12.42}$$

We apply Theorem 12.6 with $F \equiv 0$. The solution of the problem (12.36), (12.37) has the form

$$Q^{(a)}_{a,b}(x) = \frac{\psi_0(b)\varphi_0(x) - \psi(x)\varphi_0(b)}{\psi_0(b)\varphi_0(a) - \psi_0(a)\varphi_0(b)} \tag{12.43}$$

for $\Phi(a) = 1$ and $\Phi(b) = 0$, and

$$Q^{(b)}_{a,b}(x) = \frac{\psi_0(x)\varphi_0(a) - \psi_0(a)\varphi_0(x)}{\psi_0(b)\varphi_0(a) - \psi_0(a)\varphi_0(b)} \tag{12.44}$$

for $\Phi(a) = 0$ and $\Phi(b) = 1$, where $\varphi_0(x)$ and $\psi_0(x)$, $x \in \mathbf{R}$, are fundamental solutions of the homogeneous equation (12.25) such that $\varphi_0(z) = \psi_0(z) = 1$.

From (12.44) for $b = z$ it follows that $\lim\limits_{a\to-\infty} Q^{(z)}_{a,z}(x) = \dfrac{\psi_0(x) - \rho_-\varphi_0(x)}{1-\rho_-}$, where $\rho_- = \lim\limits_{a\to-\infty} \dfrac{\psi_0(a)}{\varphi_0(a)}$. This limit exists and it is less than 1, because the ratio $\dfrac{\psi_0(a)}{\varphi_0(a)}$ is an increasing function. We set $\psi(x) := \dfrac{\psi_0(x) - \rho_-\varphi_0(x)}{1-\rho_-}$, $x \in \mathbf{R}$. It is clear that $\psi(x)$ is an increasing function. By (12.42) for $x \le z$, we have that $L_z(x) = \psi(x)$ and this function is positive.

We use similar arguments for the domain $x \ge z$. In this case from (12.43) for $a = z$ it follows that $\lim\limits_{b\to\infty} Q^{(z)}_{z,b}(x) = \dfrac{\varphi_0(x) - \rho_+\psi_0(x)}{1-\rho_+}$, where $\rho_+ = \lim\limits_{b\to\infty} \dfrac{\varphi_0(b)}{\psi_0(b)}$. This limit exists and it is less than 1. We set $\varphi(x) := \dfrac{\varphi_0(x) - \rho_+\psi_0(x)}{1-\rho_+}$, $x \in \mathbf{R}$. Then $L_z(x) = \varphi(x)$ for $x \ge z$ and the function $\varphi(x)$, $x \in \mathbf{R}$ is decreasing and positive.□

**Corollary 12.3.** *The following equality*

$$\mathbf{P}_x(H_z < \infty) = \begin{cases} \dfrac{S(x) - S(-\infty)}{S(z) - S(-\infty)}, & \text{for } x \le z, \\ \dfrac{S(\infty) - S(x)}{S(\infty) - S(z)}, & \text{for } z \le x, \end{cases}$$

*holds, where for $S(-\infty) = -\infty$ or $S(\infty) = \infty$ the corresponding ratio equals to 1.*

This follows from (12.40), (12.42) with $f \equiv 0$, and (12.35) with $F \equiv 0$.

## § 13. The Cauchy problem, existence of a solution

As mentioned above, the proof of existence of a solution of the Cauchy problem is very complicated. In this section we give a probabilistic proof of this existence.

Let the process $X_x(t)$, $t \in [0, T]$, be the solution of the homogeneous stochastic differential equation

$$X_x(t) = x + \int_0^t a(X_x(s))\,ds + \int_0^t b(X_x(s))\,dW(s). \tag{13.1}$$

We assume that the coefficients $a(x)$ and $b(x)$, $x \in \mathbf{R}$, are continuous, bounded, and have continuous bounded derivatives $a'(x)$, $b'(x)$, $a''(x)$, $b''(x)$.

**Theorem 13.1.** *Let $\varPhi(x)$, $x \in \mathbf{R}$, be a continuous bounded function with continuous bounded derivatives $\varPhi'(x)$, $\varPhi''(x)$. Then the function*

$$u(t,x) := \mathbf{E}\,\varPhi(X_x(t)) \tag{13.2}$$

*is differentiable with respect to $t$, twice continuously differentiable with respect to $x$, and it is the solution of the problem*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\,b^2(x)\frac{\partial^2}{\partial x^2}u(t,x) + a(x)\frac{\partial}{\partial x}u(t,x), \tag{13.3}$$

$$u(0,x) = \varPhi(x), \tag{13.4}$$

$(t,x) \in (0,T) \times \mathbf{R}$.

**Remark 1.13.** Equation (13.3) is called the *backward Kolmogorov equation.* For nonhomogeneous stochastic differential equations the analogue of Theorem 13.1 will be considered in § 2 Ch. IV. We also refer to Gihman and Skorohod (1969).

*Proof of Theorem* 13.1. Clearly, the function $u$ is bounded by the same constant as the function $\varPhi$.

The initial condition (13.4) is easily verified by passage to the limit under the expectation sign in (13.2) as $t \downarrow 0$.

Let us verify that for every fixed $t$ the function $u(t,x)$ is twice continuously differentiable with respect to $x$. By Theorems 9.2 and 9.3, the process $X_x(t)$, $t \in [0,T]$, has stochastically continuous mean square derivatives of the first and the second order with respect to $x$, which we denote by $X_x^{(1)}(t)$ and $X_x^{(2)}(t)$.

Denote $u_x^{(1)}(t,x) := \mathbf{E}\big\{\varPhi'(X_x(t))\,X_x^{(1)}(t)\big\}$ and prove that $\dfrac{\partial}{\partial x}u(t,x) = u_x^{(1)}(t,x)$. Set $Y_\Delta(t) := \dfrac{X_{x+\Delta}(t) - X_x(t)}{\Delta}$. We have

$$\Big|\frac{u(t,x+\Delta)-u(t,x)}{\Delta} - u_x^{(1)}(t,x)\Big| \le \mathbf{E}\Big|\frac{\varPhi(X_{x+\Delta}(t))-\varPhi(X_x(t))}{\Delta} - \varPhi'(X_x(t))\,X_x^{(1)}(t)\Big|$$

$$\le \mathbf{E}\Big|\frac{\varPhi(X_{x+\Delta}(t))-\varPhi(X_x(t))}{X_{x+\Delta}(t)-X_x(t)}\big(Y_\Delta(t) - X_x^{(1)}(t)\big)\Big|$$

$$+\,\mathbf{E}\Big\{\Big|\frac{\varPhi(X_{x+\Delta}(t))-\varPhi(X_x(t))}{X_{x+\Delta}(t)-X_x(t)} - \varPhi'(X_x(t))\Big|\,\big|X_x^{(1)}(t)\big|\Big\} \to 0 \quad \text{as } \Delta \to 0.$$

This relation is due to the fact that the ratio $\dfrac{\varPhi(y)-\varPhi(x)}{y-x}$ is bounded and converges to $\varPhi'(x)$ as $y \to x$, while the function $X_x(t)$ is continuous in $x$ (Theorem 9.1), and the fact that $\mathbf{E}\big(Y_\Delta(t) - X_x^{(1)}(t)\big)^2 \to 0$ (Theorem 9.3).

Thus we proved that the function $u(t,x)$ has a derivative

$$\frac{\partial}{\partial x}u(t,x) = \mathbf{E}\big\{\varPhi'(X_x(t))\,X_x^{(1)}(t)\big\}. \tag{13.5}$$

This derivative is continuous in $(t,x)$ thanks to the continuity of $X_x(t)$ and the mean square continuity of $X_x^{(1)}(t)$. Furthermore, according to Remark 9.4, $\mathbf{E}X_x^{(1)}(t) \le e^{K(K+1)t}$, therefore the derivative $\dfrac{\partial}{\partial x}u(t,x)$, $(t,x) \in (0,T) \times \mathbf{R}$, is a bounded function.

Similarly, we can prove that

$$\frac{\partial^2}{\partial x^2}u(t,x) = \mathbf{E}\big\{\varPhi''(X_x(t))\,\big(X_x^{(1)}(t)\big)^2\big\} + \mathbf{E}\big\{\,\varPhi'(X_x(t)\,X_x^{(2)}(t)\big\}, \tag{13.6}$$

$(t,x) \in (0,T)\times\mathbf{R}$. This derivative is a bounded continuous function.

For any $0 \le v \le t$, the solution $X_x(t)$ of equation (13.1) can be written in the form

$$X_x(t) = X_x(v) + \int_0^{t-v} a(X_x(v+s))\,ds + \int_0^{t-v} b(X_x(v+s))\,d\widetilde{W}_v(s), \tag{13.7}$$

where for a fixed $v$ the process $\widetilde{W}_v(s) = W(s+v) - W(v)$, $s \ge 0$, is a Brownian motion. Note that the process $\widetilde{W}_v$ does not depend on the $\sigma$-algebra $\mathcal{G}_0^v$ of events generated by the Brownian motion $W(s)$ for $0 \le s \le v$.

Consider the stochastic differential equation

$$\widetilde{X}_{v,x}(h) = x + \int_0^h a(\widetilde{X}_{v,x}(s))\,ds + \int_0^h b(\widetilde{X}_{v,x}(s))\,d\widetilde{W}_v(s), \tag{13.8}$$

which is similar to equation (13.1). It is clear that the process $\widetilde{X}_{v,x}$ is independent of the $\sigma$-algebra $\mathcal{G}_0^v$ and has the same finite-dimensional distributions as the process $X_x(h)$.

In (13.7) we set $t - v = h$. Then by the uniqueness of the solution of the stochastic differential equation, we have (see (9.4)) the equality

$$X_x(h+v) = \widetilde{X}_{v,X_x(v)}(h). \tag{13.9}$$

Let $0 < p < t < q$ and $\delta := q - p$. Further for a fixed $t$ we let $\delta \to 0$. Using the fourth property of the conditional expectations and Lemma 2.1 of Ch. I, we represent $u(q,x)$ as

$$\begin{aligned} u(q,x) &= \mathbf{E}\big\{\mathbf{E}\big\{\varPhi(X_x(q))\big|\mathcal{G}_0^\delta\big\}\big\} \\ &= \mathbf{E}\big\{\mathbf{E}\big\{\varPhi(\widetilde{X}_{\delta,X_x(\delta)}(p))\big|\mathcal{G}_0^\delta\big\}\big\} = \mathbf{E}\,u(p,X_x(\delta)). \end{aligned} \tag{13.10}$$

To obtain the last equality we used the fact that the random variable $X_x(\delta)$ is measurable with respect to the $\sigma$-algebra $\mathcal{G}_0^\delta$ and $\widetilde{X}_{\delta,z}(h)$, as a random function of the argument $z$, is independent of $\mathcal{G}_0^\delta$. Therefore, we can use Lemma 2.1 of Ch. I to compute the conditional expectation and to prove the last equality.

Since $X_x(0) = x$, Itô's formula yields

$$u(p,X_x(\delta)) - u(p,x) = \int_0^\delta b(X_x(s))\frac{\partial}{\partial x}u(p,X_x(s))\,dW(s)$$

$$+\int_0^\delta \Big(a(X_x(s))\frac{\partial}{\partial x}u(p,X_x(s))+\frac{1}{2}b^2(X_x(s))\frac{\partial^2}{\partial x^2}u(p,X_x(s))\Big)\,ds.$$

Taking the expectation of both sides of this equality, we get

$$\mathbf{E}u(p,X_x(\delta))-u(p,x)=\mathbf{E}\int_0^\delta \Big(a(X_x(s))\frac{\partial}{\partial x}u(p,X_x(s))+\frac{b^2(X_x(s))}{2}\frac{\partial^2}{\partial x^2}u(p,X_x(s))\Big)\,ds. \tag{13.11}$$

By the mean value theorem for integrals, we have

$$\mathbf{E}u(p,X_x(\delta))-u(p,x)=\mathbf{E}\Big(a(X_x(\tilde{s}))\frac{\partial}{\partial x}u(p,X_x(\tilde{s}))+\frac{1}{2}\,b^2(X_x(\tilde{s}))\frac{\partial^2}{\partial x^2}u(p,X_x(\tilde{s}))\Big)\,\delta,$$

where $\tilde{s}$ is some, possibly random, point of the interval $(0,\delta)$.

Since the derivatives $\frac{\partial}{\partial x}u(t,x)$, $\frac{\partial^2}{\partial x^2}u(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}$, are continuous and bounded, $X_x(\tilde{s})\to x$ as $p\to t$, $q\to t$, applying the Lebesgue dominated convergence theorem we obtain

$$\frac{\mathbf{E}u(p,X_x(\delta))-u(p,x)}{q-p}\to\frac{1}{2}\,b^2(x)\frac{\partial^2}{\partial x^2}u(t,x)+a(x)\frac{\partial}{\partial x}u(t,x). \tag{13.12}$$

Now, by (13.10),

$$\frac{u(q,x)-u(p,x)}{q-p}=\frac{\mathbf{E}u(p,X_x(\delta))-u(p,x)}{q-p},$$

and, thus it is proved that the function $u(t,x)$, $(t,x)\in(0,t)\times\mathbf{R}$, is the solution of (13.3). □

**Theorem 13.2.** *Let $\Phi(x)$, $f(x)$, $x\in\mathbf{R}$, be continuous bounded functions with continuous bounded derivatives $\Phi'(x)$, $\Phi''(x)$, $f'(x)$, and $f''(x)$. Assume, in addition, that the function $f$ is nonnegative.*

*Then the function*

$$u(t,x):=\mathbf{E}\Big\{\Phi(X_x(t))\exp\Big(-\int_0^t f(X_x(s))\,ds\Big)\Big\} \tag{13.13}$$

*is differentiable with respect to $t$, twice continuously differentiable with respect to $x$, and it is the solution of the problem*

$$\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\,b^2(x)\frac{\partial^2}{\partial x^2}u(t,x)+a(x)\frac{\partial}{\partial x}u(t,x)-f(x)u(t,x), \tag{13.14}$$

$$u(0,x)=\Phi(x), \tag{13.15}$$

$(t,x)\in(0,T)\times\mathbf{R}$.

*Proof.* Obviously, the function $u$ is bounded by the same constant as the function $\Phi$.

The initial condition (13.15) is easily verified by passing to the limit under the expectation sign in (13.13) as $t \downarrow 0$.

The existence of continuous bounded derivatives is proved similarly to (13.5). In particular, the function $u(t, x)$ has the continuous bounded first-order partial derivative

$$\frac{\partial}{\partial x} u(t,x) = \mathbf{E}\Big\{\Phi'(X_x(t))\, X_x^{(1)}(t) \exp\Big(-\int_0^t f(X_x(s))\, ds\Big)\Big\}$$

$$-\mathbf{E}\Big\{\Phi(X_x(t)) \exp\Big(-\int_0^t f(X_x(s))\, ds\Big) \int_0^t f'(X_x(s)) X_x^{(1)}(s)\, ds\Big\}. \tag{13.16}$$

In addition, the function $u(t, x)$ has a continuous bounded second-order partial derivative. We use for $0 < p < t < q$, $\delta := q - p$ the relation

$$\exp\Big(-\int_\delta^q f(X_x(s))\, ds\Big) - \exp\Big(-\int_0^q f(X_x(s))\, ds\Big)$$

$$= \int_0^\delta f(X_x(v)) \exp\Big(-\int_v^q f(X_x(s))\, ds\Big) dv. \tag{13.17}$$

Multiplying this equality by $\Phi(X_x(q))$ and taking into account (13.9), we have

$$\Phi\big(\widetilde{X}_{\delta,X_x(\delta)}(p)\big) \exp\Big(-\int_0^p f\big(\widetilde{X}_{\delta,X_x(\delta)}(s)\big)\, ds\Big) - \Phi(X_x(q)) \exp\Big(-\int_0^q f(X_x(s))\, ds\Big)$$

$$= \int_0^\delta f(X_x(v))\Phi\big(\widetilde{X}_{v,X_x(v)}(q-v)\big) \exp\Big(-\int_0^{q-v} f\big(\widetilde{X}_{v,X_x(v)}(s)\big)\, ds\Big) dv.$$

We take the expectation of both sides of this equality and use the fourth property of conditional expectations. Then we get

$$\mathbf{E}\Big\{\mathbf{E}\Big\{\Phi\big(\widetilde{X}_{\delta,X_x(\delta)}(p)\big) \exp\Big(-\int_0^p f\big(\widetilde{X}_{\delta,X_x(\delta)}(s)\big)\, ds\Big)\Big|\mathcal{G}_0^\delta\Big\}\Big\}$$

$$-\mathbf{E}\Big\{\Phi(X_x(q)) \exp\Big(-\int_0^q f(X_x(s))\, ds\Big)\Big\}$$

$$= \int_0^\delta \mathbf{E}\Big\{f(X_x(v))\mathbf{E}\Big\{\Phi\big(\widetilde{X}_{v,X_x(v)}(q-v)\big) \exp\Big(-\int_0^{q-v} f\big(\widetilde{X}_{v,X_x(v)}(s)\big)\, ds\Big)\Big|\mathcal{G}_0^v\Big\}\Big\} dv.$$

Using for the computation of these conditional expectations Lemma 2.1 Ch. I and the notation (13.13), the above equality can be written in the form

$$\mathbf{E}u(p, X_x(\delta)) - u(q,x) = \int_0^\delta \mathbf{E}\bigl\{f(X_x(v))\,u(q-v, X_x(v))\bigr\}dv. \tag{13.18}$$

Since $f(x)$ and $u(v,x)$, $(v,x) \in [0,T] \times \mathbf{R}$, are continuous bounded functions and the process $X_x(v)$ is continuous with respect to $v$, we have

$$\lim_{p\uparrow t, q\downarrow t} \frac{1}{q-p} \int_0^\delta \mathbf{E}\bigl\{f(X_x(v))\,u(q-v, X_x(v))\bigr\}dv = f(x)\,u(t,x).$$

Therefore,

$$\lim_{p\uparrow t, q\downarrow t} \frac{\mathbf{E}u(p, X_x(\delta)) - u(q,x)}{q-p} = f(x)\,u(t,x). \tag{13.19}$$

We now use the equality

$$\frac{u(q,x)-u(p,x)}{q-p} = \frac{\mathbf{E}u(p, X_x(\delta)) - u(p,x)}{q-p} - \frac{\mathbf{E}u(p, X_x(\delta)) - u(q,x)}{q-p}. \tag{13.20}$$

We can apply relation (13.12) to the first term in (13.20). Then from (13.20) and (13.19) we derive that $u(t,x)$, $(t,x) \in (0,T) \times \mathbf{R}$, is the solution of (13.14). □

**Exercises.**

In the following exercises all functions $u(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}$, have the corresponding probabilistic representation (13.13).

**13.1.** Verify that

$$u(t,x) = \exp\Bigl(-\gamma x t + \frac{\gamma^2 t^3}{6}\Bigr)$$

is the solution of (13.14), (13.15) with $b(x) \equiv 1$, $a(x) \equiv 0$, $f(x) = \gamma x$, and $\Phi(x) \equiv 1$.

**13.2.** Verify that

$$u(t,x) = \frac{1}{\sqrt{\operatorname{ch}(t\gamma)}} \exp\Bigl(-\frac{x^2\gamma \operatorname{sh}(t\gamma)}{2\operatorname{ch}(t\gamma)}\Bigr)$$

is the solution of (13.14), (13.15) with $b(x) \equiv 1$, $a(x) \equiv 0$, $f(x) = \frac{\gamma^2}{2}x^2$, and $\Phi(x) \equiv 1$.

**13.3.** Verify that

$$u(t,x) = \frac{1}{\sqrt{\operatorname{ch}(t\gamma) + 2\beta\gamma^{-1}\operatorname{sh}(t\gamma)}} \exp\Bigl(-\frac{x^2(\gamma \operatorname{sh}(t\gamma) + 2\beta\operatorname{ch}(t\gamma))}{2(\operatorname{ch}(t\gamma) + 2\beta\gamma^{-1}\operatorname{sh}(t\gamma))}\Bigr)$$

is the solution of (13.14), (13.15) with $b(x) \equiv 1$, $a(x) \equiv 0$, $f(x) = \frac{\gamma^2}{2}x^2$, and $\Phi(x) = e^{-\beta x^2}$.

**13.4.** Verify that

$$u(t,x)=\exp\Bigl(-\gamma xt-\gamma\mu\frac{t^2}{2}+\frac{\gamma^2t^3}{6}\Bigr)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv\mu$, $f(x)=\gamma x$, and $\Phi(x)\equiv 1$.

**13.5.** Verify that

$$u(t,x)=\frac{1}{\sqrt{\operatorname{ch}(t\gamma)}}\exp\Bigl(-\mu x-\frac{\mu^2t}{2}-\frac{(x^2\gamma^2-\mu^2)\operatorname{sh}(t\gamma)-2\mu x\gamma}{2\gamma\operatorname{ch}(t\gamma)}\Bigr)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv\mu$, $f(x)=\dfrac{\gamma^2}{2}x^2$, and $\Phi(x)\equiv 1$.

**13.6.** Verify that

$$u(t,x)=\exp\Bigl(-\frac{\gamma x}{\theta}\bigl(1-e^{-\theta t}\bigr)+\frac{\gamma^2\sigma^2}{2\theta^2}\Bigl(2\theta t+1-\bigl(2-e^{-\theta t}\bigr)^2\Bigr)\Bigr)$$

is the solution of (13.14), (13.15) with $b(x)\equiv\sigma^2\theta$, $a(x)=-\theta x$, $\theta>0$, $f(x)=\gamma x$, and $\Phi(x)\equiv 1$.

**13.7.** Verify that

$$u(t,x)=\frac{\sqrt{\gamma}e^{\theta t/2}}{\sqrt{\operatorname{sh}(t\gamma\theta)+\gamma\operatorname{ch}(t\gamma\theta)}}\exp\Bigl(-\frac{x^2(\gamma^2-1)\operatorname{sh}(t\gamma\theta)}{4\sigma^2(\operatorname{sh}(t\gamma\theta)+\gamma\operatorname{ch}(t\gamma\theta))}\Bigr)$$

is the solution of (13.14), (13.15) with $b(x)\equiv\sigma^2\theta$, $a(x)=-\theta x$, $\theta>0$, $f(x)=\dfrac{(\gamma^2-1)\theta}{4\sigma^2}x^2$, $\gamma\ge 1$, and $\Phi(x)\equiv 1$.

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