On the spectrum of the hierarchical Schrödinger operator

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Abstract

The goal of this paper is the spectral analysis of the Schrödinger operator $H = L + V$, the perturbation of the Taibleson-Vladimirov multiplier $L = \mathcal{D}^\alpha$ by a potential V. Assuming that V belongs to a class of fast decreasing potentials we show that the discrete part of the spectrum of H may contain negative energies, it also appears in the spectral gaps of L. We will split the spectrum of H in two parts: high energy part containing eigenvalues which correspond to the eigenfunctions located on the support of the potential V, and low energy part which lies in the spectrum of certain bounded Schrödinger operator acting on the Dyson hierarchical lattice. The spectral asymptotics strictly depend on the transience versus recurrence properties of the underlying hierarchical random walk. In the transient case we will prove results in spirit of CLR theory, for the recurrent case we will provide Bargmann's type asymptotics.

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1 Introduction

The spectral theory of nested fractals similar to the Sierpinski gasket, i.e. the spectral theory of the corresponding Laplacians, is well understood. It has several important features: Cantor-like structure of the essential spectrum and, as result, the large number of spectral gaps, presence of infinite number of eigenvalues each of which has infinite multiplicity and compactly supported eigenstates, non-regularly varying at infinity heat kernels which contain oscilated in $\log t$ scale terms etc, see [14], [10] and [7].

The spectral properties mentioned above occure in the very precise form for the Taibleson-Vladimirov Laplacian $\mathcal{D}^\alpha$, the operator of fractional derivative of order α. This operator can be introduced in several different forms (say, as L^2-multiplier in the p-adic analysis setting, see [33]) but we select the geometric approach [11], [28], [27], [3], [4], [5] and [6].

The Dyson's hierarchical model   Let us fix an integer $p \geq 2$ and consider the family $\{\Pi_r : r \in \mathbb{Z}\}$ of partitions of $X = [0, +\infty[$ such that each Π_r consists of all p-adic intervals $[kp^r, (k+1)p^r[$. We call r the rank of the partition Π_r (respectively, the rank of the interval $I \in \Pi_r$). Each interval of rank r is the union of p disjoint intervals of rank $(r-1)$. Each point $x \in X$ belongs to a certain interval $I_r(x)$ of rank r, and intersection of all p-adic intervals $I_r(x)$ is $\{x\}$.

The hierarchical distance $d(x, y)$ is defined as follows:

$$d(x, y) = p^{\mathfrak{n}(x,y)}, \text{ where } \mathfrak{n}(x, y) = \inf\{r : y \in I_r(x)\}.$$

Since any two points x and y belong to a certain p-adic interval, $d(x, y) < \infty$. Clearly $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for arbitrary x, y and z in X, i.e. $d(x, y)$ is an *ultrametric.*

The set X equipped with the ultrametric $d(x, y)$ is *complete, separable* and *proper* metric space. In the metric space (X, d) the set of all open balls is countable, it coincides with the set of all p-adic intervals. The Borel σ-algebra $\mathcal{B}(X, d)$ coincides with the Borel σ-algebra $\mathcal{B}(X, \mathrm{d})$ corresponding to the Eucledian distance d.

Following [29] we introduce *the hierarchical Laplacian L* defined pointwise as convexe linear combination of "elementary Laplacians"

$$(Lf)(x) = \sum_{r=-\infty}^{+\infty} C(r) \left(f(x) - \frac{1}{m(I_r(x))} \int_{I_r(x)} f \, dm \right), \qquad (1.1)$$

where $C(r) = (1 - \kappa)\kappa^{r-1}$, and m is the Lebesgues measure. Here $\kappa \in (0, 1)$ is the second parameter of the model. Recall that the first parameter of the model is p, the integer which defines the family of partitions $\{\Pi_r\}$. The series in (1.1) diverges in general but it is finite and belongs to $L^2 = L^2(X, m)$ for all f which take constant values on the p-adic intervals of the rank r.

The operator L admits a complete system of compactly supported eigenfunctions. Indeed, let I be a p-adic interval of rank r, let $I_1, I_2, ..., I_p$ be its p-adic subintervals of rank $r - 1$. Let us consider p functions

$$\psi_{I_i} = \frac{1_{I_i}}{m(I_i)} - \frac{1_I}{m(I)}.$$

We have $L\psi_{I_i} = \kappa^{r-1}\psi_{I_i}$. Since $\sum_{i=1}^p \psi_{I_i} = 0$ the rank of the system $\{\psi_{I_i} : i = 1, 2, ..., p\}$ is $p - 1$. When I runs over the set of all p-adic intervals the system of eigenfunctions $\{\psi_{I_i}\}$ is complete in L^2 whence L is essentially self-adjoint operator having pure point spectrum

$$Spec(L) = \{0\} \cup \{\kappa^r : r \in \mathbb{Z}\}.$$

Each eigenvalue $\lambda_r = \kappa^{r-1}$ has infinite multiplicity. We will see below that writing $\kappa = p^{-\alpha}$, i.e. setting $\alpha = \ln \frac{1}{\kappa} / \ln p$, the operator L coinsides with the Taibleson-Vladimirov operator $\mathcal{D}^\alpha$, the operator of fractional derivative of order α. The constant $s_h = 2 \ln p / \ln \frac{1}{\kappa}$ is called *spectral dimension* of the triple (X, d, L). It gives the on-diagonal asymptotics of the transition density

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$p(t, x, x) \asymp t^{-s_h/2}$ of the Markov semigroup $(e^{-tL})_{t>0}$, see [29, Proposition 2.3] and [4].

There are already several publications on the spectrum of the hierarchical Laplacian acting on a general ultrametric measure space (X, d, m) [2], [1], [28], [27], [3], [4], [5], [6]. Accordingly, the hierarchical Schrödinger operator was studied in [12], [28], [29], [30], [9], [23], [24], [25] (the hierarchical lattice of Dyson) and in [35], [34], [19] (the field of p-adic numbers).

By the general theory developed in [3], [4] and [5], any hierarchical Laplacian L acts in $L^2(X, m)$, is essentially self-adjoint operator and can be represented in the form

$$Lf(x) = \int_X (f(x) - f(y)) J(x, y) dm(y). \tag{1.2}$$

It has a pure point spectrum, and its Markov semigroup $(e^{-tL})_{t>0}$ admits with respect to m a continuous transition density $p(t, x, y)$. It turns out that in terms of certain (intrinsically related to L) ultrametric d_*,

$$J(x, y) = \int_0^{1/d_*(x,y)} N(x, \tau) d\tau, \tag{1.3}$$

$$p(t, x, y) = t \int_0^{1/d_*(x,y)} N(x, \tau) \exp(-t\tau) d\tau, \tag{1.4}$$

and

$$p(t, x, x) = \int_0^\infty \exp(-t\tau) dN(x, \tau)$$

where $N(x, \tau)$ is the so called *spectral function* related to L. The analytic properties of the function $p(t, x, y)$ play essential role in the study of the Schrödinger operator $H = L + V$, see paper [30].

Notation For two positive functions f and g we write $f \asymp g$ if the ratio f/g is bounded from above and from below by positive constants for a specified range of variables. We write $f \sim g$ if the ratio f/g tends to identity.

A non-decreasing function $N : \mathbb{R}_+ \to \mathbb{R}_+$ is called *doubling* if the inequality

$$N(2r) \leq CN(r)$$

holds for all $r > 0$ and some $C > 1$. The doubling property implies that

$$\frac{N(R)}{N(r)} \leq C' \left(\frac{R}{r}\right)^{\tau}$$

for all $R > r > 0$ and some constants $\tau, C' > 0$.

A non-decreasing function $M : \mathbb{R}_+ \to \mathbb{R}_+$ is called *reverse doubling* if the inequality

$$\frac{M(R)}{M(r)} \geq C'' \left(\frac{R}{r}\right)^{\nu},$$

holds for all $R > r > 0$ and some constants $C'', \nu > 0$.

Outline Let us describe the main body of the paper. In Section 2 we introduce the notion of homogeneous hierarchical Laplacian L and list its basic properties such as: the set $Spec(L)$, the spectrum of the operator L, is pure point, all eigenvalues of L have infinite multiplicity and compactly supported eigenfunctions, the heat kernel $p(t, x, y)$ exists and is a continuous function having nice asymptotic properties etc.). As a special example we consider the case $X = \mathbb{Q}_p$, the ring $\mathbb{Q}_p$ of p-adic numbers, endowed with its standard ultrametric $d(x, y) = |x - y|_p$ and the normed Haar measure m. The hierarchical Laplacian L in our example coincides with the Taibleson-Vladimirov operator $\mathfrak{D}^{\alpha}$, the operator of fractional derivative of order α, see [33], [35], and [19]. The most complete sourse for the basic definitions and facts related to the p-adic analysis is [18] and [32].

In the next sections we consider the Schrödinger operator $H = L + V$ with a continuous descending at infinity potential of the form $V = \sum \sigma_i 1_{B_i}$, where B_i are balls. The main aim here is to study the set $Spec(H)$. Since $V(x) \to 0$ as $x \to \infty$ the set $Spec(H)$ is pure point with possibly countaly many limit points - the eigenvalues of the operator L. We splitt the set $Spec(H)$ in two disjoint parts: the first part is related to $Spec(L)$ and the second part is countably infinite set Ξ. In the case when $d(B_i, B_j), i \neq j$, become large enough we specify the structure of the set Ξ. We obtain in certain cases lower bounds on $Neg(H)$, the number of negative eigenvalues of the operator H counted with their multiplicity. Our lower bounds match the well-known upper bounds from CLR theory.

2 Preliminaries

2.1 Homogeneous ultrametric space

Let (X, d) be a locally compact and separable ultrametric space. Recall that a metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \tag{2.1}$$

that is stronger than the usual triangle inequality. The basic consequence of the ultrametric property is that each open ball is a closed set. Moreover, each point x of a ball B can be regarded as its center, any two balls A and B either do not intersect or one is a subset of another etc. See e.g. Section 1 in [5] and references therein. In this paper we assume that the ultrametric space (X, d) is not compact and that it is *proper*, i.e. each closed d-ball is a compact set.

Let $\mathcal{B}$ be the set of all open balls and $\mathcal{B}(x) \subset \mathcal{B}$ the set of all balls centred at x. Notice that the set $\mathcal{B}$ is atmost countable whereas X by itself may well be uncountable, e.g. $X = [0, +\infty[$ with $\mathcal{B}$ consisting of all p-adic intervals as explained in the introduction.

To any ultrametric space (X, d) one can associate in a standard fashion a tree $\mathcal{T}$. The vertices of the tree are metric balls, the boundary $\partial \mathcal{T}$ can be identified with the one-point compactification $X \cup \{\varpi\}$ of X. We refere to [5] for a treatment of the association between a ultrametric space and the tree of its metric balls.

A ultrametric measure space (X, d, m) is called *homogeneous* if the group of isometries of (X, d) acts transitively and preserves the measure. In particular, a homogeneous ultrametric measure space is eather discrete or perfect. In a homogeneous ultrametric measure space any two balls A and B having the same diameter satisfy $m(A) = m(B)$.

2.2 Homogeneous hierarchical Laplacian

Let (X, d, m) be a homogeneous ultrametric space. Let $C : \mathcal{B} \to (0, \infty)$ be a function satisfying the following two conditions: (i) $C(A) = C(B)$ for any two balls A and B of the same diameter, and (ii) for all non-singletone $B \in \mathcal{B}$,

$$\lambda(B) := \sum_{T \in \mathcal{B}:\ B \subseteq T} C(T) < \infty. \tag{2.2}$$

The class of functions $C(B)$ satisfying (i) and (ii) is reach enough, e.g. one can choose

$$C(B) = (1/m(B))^\alpha - (1/m(B'))^\alpha$$

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for any two closest neighboring balls $B \subset B'$. In this case

$$\lambda(B) = (1/m(B))^{\alpha}.$$

Let $\mathcal{D}$ be the set of all locally constant functions having compact support. The set $\mathcal{D}$ belongs to Banach spaces $C_0(X)$ and $L^p = L^p(X, m)$, $1 \le p < \infty$, and is a dence subset there. Given the data $(\mathcal{B}, C, m)$ we define (pointwise) *the homogeneous hierarchical Laplacian L* as follows

$$Lf(x) := \sum_{B \in \mathcal{B}(x)} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right). \tag{2.3}$$

The operator $(L, \mathcal{D})$ acts in L^2, is symmetric and admits a complete system of eigenfunctions

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}, \tag{2.4}$$

where the couple $B \subset B'$ runs over all nearest neighboring balls having positive measure. The eigenvalue corresponding to f_B is $\lambda(B')$ defined at (2.2),

$$Lf_B(x) = \lambda(B')f_B(x).$$

Since the system of eigenfunctions is complete, we conclude that $(L, \mathcal{D})$ is essentially self-adjoint operator.

The intrinsic ultrametric $d_*(x, y)$ is defined as follows

$$d_*(x, y) := \begin{cases} 0 & \text{when} \quad x = y \\ 1/\lambda(x \curlywedge y) & \text{when} \quad x \ne y \end{cases}, \tag{2.5}$$

where $x \curlywedge y$ be the minimal ball containing both x and y. In particular, for any ball B,

$$\lambda(B) = \frac{1}{\operatorname{diam}_*(B)}. \tag{2.6}$$

The spectral function $\tau \to N(\tau)$, see equation (1.3), is defined as a left-continuous step-function having jumps at the points $\lambda(B)$, and

$$N(\lambda(B)) = 1/m(B).$$

The volume function $V(r)$ is defined by setting $V(r) = m(B)$ where the ball B has d_*-radius r. It is easy to see that

$$N(\tau) = 1/V(1/\tau). \tag{2.7}$$

The Markov semigroup $P_t = e^{-tL}$, $t > 0$, admits a density $p(t, x, y)$ w.r.t. m, we call it *the heat kernel*. $p(t, x, y)$ is a continuous function which can be represented in the form

$$p(t, x, y) = t \int\limits_0^{1/d_*(x,y)} N(\tau) \exp(-t\tau) d\tau. \tag{2.8}$$

For $\lambda > 0$ the resolvent operator $R_\lambda = (\lambda + L)^{-1}$ admits a continuous strictly positive kernel $R(\lambda, x, y)$ with respect to the measure m. The operator R_λ is well defined for $\lambda = 0$, i.e. the Markov semigroup $(P_t)_{t>0}$ is transient, if and only if for some (equivalently, for all) $x \in X$ the function $\tau \to 1/V(\tau)$ is integrable at ∞. Its kernel $R(0, x, y)$, called also the Green function, is of the form

$$R(0, x, y) = \int\limits_{d_*(x,y)}^{+\infty} \frac{d\tau}{V(\tau)}. \tag{2.9}$$

Under certain reasonable conditions the equation from above takes the form

$$R(0, x, y) \asymp \frac{d_*(x, y)}{V(d_*(x, y))}.$$

2.3 An example

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. For any two nearest neighbouring balls $B \subset B'$ we define

$$C(B) = \Phi(1/m(B)) - \Phi(1/m(B')). \tag{2.10}$$

Then the following properties hold:

(i) $\lambda(B) = \Phi(1/m(B))$,

(ii) $d_*(x, y) = 1/\Phi(1/m(x \curlywedge y))$,

(iii) $V(r) \leq 1/\Phi^{-1}(1/r)$. Moreover, $V(r) \asymp 1/\Phi^{-1}(1/r)$ whenever both Φ and Φ^{-1} are doubling and $m(B') \leq cm(B)$ for some $c > 0$ and all neighboring balls $B \subset B'$. In turn, this yields

$$p(t, x, y) \asymp t \cdot \min \left\{ \frac{1}{t} \Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{m(x \curlywedge y)} \Phi\left(\frac{1}{m(x \curlywedge y)}\right) \right\}, \tag{2.11}$$

and

$$p(t, x, x) \asymp \Phi^{-1}\left(\frac{1}{t}\right) \tag{2.12}$$

for all $t > 0$ and $x, y \in X$.

2.4 L^2-multipliers

As a special case of the general construction consider $X = \mathbb{Q}_p$, the ring of p-adic numbers equipped with its standard ultrametric $d(x, y) = |x - y|_p$. Notice that the ultrametric spaces $(\mathbb{Q}_p, \mathrm{d})$ and $([0, \infty), d)$ with non-eucledian d, as explained in the introduction, are isometric.

Let m be the normed Haar measure on the Abelian group $\mathbb{Q}_p$, $L^2 = L^2(\mathbb{Q}_p, m)$ and $\mathcal{F} : f \to \hat{f}$ the Fourier transform of function $f \in L^2$. It is known, see [32], [35], [19], that $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is a bijection.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. The self-adjoint operator $\Phi(\mathfrak{D})$ we define as L^2-multiplier, that is,

$$\widehat{\Phi(\mathfrak{D})f}(\xi) = \Phi(|\xi|_p)\hat{f}(\xi), \quad \xi \in \mathbb{Q}_p.$$

By [4, Theorem 3.1], $\Phi(\mathfrak{D})$ is a homogeneous hierarchical Laplacian. The eigenvalues $\lambda(B)$ of the operator $\Phi(\mathfrak{D})$ are of the form

$$\lambda(B) = \Phi\left(\frac{p}{m(B)}\right). \tag{2.13}$$

Let $p(t, x, y)$ be the heat kernel associated with the operator $\Phi(\mathfrak{D})$. Assume that both Φ and Φ^{-1} are doubling, then equations (2.11) and (2.12) apply. Since $m(x \curlywedge y) = |x - y|_p$ we obtain

$$p(t, x, y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{|x - y|_p}\Phi\left(\frac{1}{|x - y|_p}\right)\right\}, \tag{2.14}$$

and

$$p(t, x, x) \asymp \Phi^{-1}\left(\frac{1}{t}\right). \tag{2.15}$$

The Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$ is L^2-multiplier. By what we said above, its heat kernel $p_\alpha(t, x, y)$ satisfy

$$p_\alpha(t, x, y) \asymp \frac{t}{(t^{1/\alpha} + |x - y|_p)^{1+\alpha}}, \tag{2.16}$$

$$p_\alpha(t, x, x) \asymp \frac{1}{t^{1/\alpha}}.$$

The Markov semigroup $(e^{-t\mathfrak{D}^\alpha})_{t>0}$ is transient if and only if $\alpha < 1$. In the transient case the Green function is of the form

$$R_\alpha(0, x, y) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}}\frac{1}{|x - y|_p^{1-\alpha}}. \tag{2.17}$$

For all facts listed above we refer the reader to [3], [4] and [5].

2.5 Schrödinger operator

Let L be a homogeneous hierarchical Laplacian acting on (X, d, m). Notice that thanks to homogenuity X can be identified with certain locally compact Abelian group equipped with translation invariant ultrametric d and the Haar measure m. This identification is not unique (!) One possible way to define such identification is to choose the sequence $\{a_n\}$ of forward degrees associated with the tree of balls $\Upsilon(X)$. This sequence is two-sided if X is non-compact and perfect, it is one-sided if X is compact and perfect, or if X is discrete. In the 1st case we identify X with Ω_a, the ring of a-adic numbers, in the 2nd case with $\Delta_a \subset \Omega_a$, the ring of a-adic integers, and in the 3rd case with the factor group Ω_a/Δ_a. We refere to [15, (10.1)-(10.11), (25.1), (25.2)] for the comprehensive treatment of special groups Ω_a, Δ_a and Ω_a/Δ_a.

By this identification $-L$ becomes a translation invariant isotropic Markov generator, it acts as L^2-multiplicator as explained in the preceding subsection. Yet, by (1.2), the operator $(-L, \mathcal{D})$ can be regarded as a symmetric Lévy generator

$$- Lf(x) = \int_X (f(x + y) - f(x)) J(y) dm(y) \qquad (2.18)$$

where the measure $J(y)dm(y)$ (the Lévy measure associated to $-L$) is finite on the complement O^c of any open neighbourhood O of the neutral element. Respectively, the semigroup $(e^{-tL})_{t>0}$ acts as a weakly continuous convolution semigroup of probability measures $(\mu_t)_{t>0}$. Each measure μ_t is absolutely continuous with respect to m and admits a continuous symmetric density $\mu_t(x)$. In particular, the transition density $p(t, x, y)$ can be expressed in the form $p(t, x, y) = \mu_t(x - y)$. The same of course true for the λ-Green function $R(\lambda, x, y)$, the density of the resolvent $(L + \lambda)^{-1}$ with respect to m.

Consider the Schrödinger operator $Hu(x) = Lu(x) + V(x)u(x)$ where $V(x)$ is a real locally bounded measurable function.

Theorem 2.1 *The following properties hold true:*

(i) The operator H is essentially self-adjoint.[1]

(ii) Assume that $V(x)$ tends to plus infinity at infinity. Then the operator H has a compact resolvent, so that its spectrum is discrete.

(iii) Assume that $V(x)$ tends to zero at infinity. Then the essential spectrum of H coincides with the essential spectrum of L. (Thus, $Spec(H)$ is

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[1]For the classical Schrödinger operator similar statement is known as Sears's theorem. It holds true if the potential admits certain low bound and may fail otherwise, see [8, Chapter II, Theorem 1.1 and Example 1.1]

pure point and the negative part of the spectrum of H consists of isolated eigenvalues of finite multiplicity).

(iv) Assume that $V(x)$ tends to zero at infinity, the Markov semigroup $(e^{-tL})_{t>0}$ is transient and that

$$p(t, x, x) \asymp \Phi^{-1}(1/t)$$

for some Φ as in (2.12) (see also (2.15)). Then the number $Neg(H)$ of negative eigenvalues counted with their multiplicity is finite and satisfies

$$Neg(H) \leq C \int_X \Phi^{-1} \circ V_-(x) dm(x) \qquad (2.19)$$

for some constant $C > 0$ and $V_-(x) = -\min\{V(x), 0\}$.

Proof. (i) We follow the argument of [19, Theorem 3.2]. Let us choose an open ball O which contains the neutral element and write equation (2.18) in the form

$$Lf(x) = \left(\int_O + \int_{O^c} \right) [f(x) - f(x+y)] J(y) dm(y)$$
$$= L_O f(x) + L_{O^c} f(x).$$

We have $Hf = L_O f + L_{O^c} f + Vf$, where the operator V is the operator of multiplication by the function $V(x)$. The operator $L_{O^c} f = J(O^c)(f - a * f)$, where $a(y) = J(y)1_{O^c}(y)/J(O^c)$, is bounded in $L^2(X, m)$ (as $f \to a * f$ is the operator of convolution with probability measure $a(y) dm(y)$) and thus does not influence self-adjointness. As L_O is minus Lévy generator it is essentially self-adjoint (one more way to make this conclusion is that the matrix of the operator L_O is diagonal in the basis $\{f_B\}$ of eigenfunctions of the operator L, see [20]).

For any ball B which belongs to the same horocycle $\mathcal{H}$ as O we denote $\mathfrak{H}_B$ the subspace of $L^2(X, m)$ which consists of all functions f having support in B. It is easy to see that $\mathfrak{H}_B$ is invariant with respect to symmetric operator $H_O = L_O + V$. Moreover, $\mathfrak{H}_B$ reduces H_O.

The ultrametric space X can be covered by a sequence of non-intersecting balls B_n (recall that due to the ultrametric property two balls of the same diameter either coincide or do not itersect). This leads to the orthogonal decomposition

$$L^2(X, m) = \bigoplus_n \mathfrak{H}_{B_n}$$

where each $\mathfrak{H}_{B_n}$ reduces H_O. The restriction of the essentially self-adjoint operator L_O to its invariant subspace $\mathfrak{H}_{B_n}$ is an essentially self-adjoint operator, while the restriction of the operator V is bounded. Thus H_O is essentially self-adjoint as orthogonal sum of essentially self-adjoint operators $H_{O,n}$, the restriction of H_O to $\mathfrak{H}_{B_n}$.

(ii) The proof is similar to the one for the Schrödinger operators given in [35, Theorem X.3]; the main tools are boundness from below of the operator H and the analogues of the Riesz and Rellich compactness criteria for subsets of $L^2(X, m)$.

(iii) Let us show that the operator V is $L-compact.$ Then, by [17, Theorem IV.5.35], the essential spectrums of the operators H and L coinside. Recall that $L-$compactness means that if a sequence $\{u_n\}$ is such that both $\{u_n\}$ and $\{Lu_n\}$ are bounded then there exists a subsequence $\{u'_n\} \subset \{u_n\}$ such that the sequence $\{Vu'_n\}$ converges.

1. Denote $v_n = Lu_n + u_n$. By assumption, the sequence $\{v_n\}$ is bounded, and $u_n = r_1 * v_n$. It follows that the quantity

$$\left(\int |u_n(x+h) - u_n(x)|^2 \, dm(x) \right)^{1/2} \leq \|v_n\|_{L^2} \int |r_1(z+h) - r_1(z)| \, dm(z)$$

tends to zero uniformly in n as h tends to the neutral element. Thus, the sequence $\{u_n\}$ consists of equicontinuous on the whole in $L^2(X, m)$ functions. The same is true for the sequence $\{Vu_n\}$. Indeed, for any ball B which contains the neutral element we write

$$\left(\int |Vu_n(x+h) - Vu_n(x)|^2 \, dm(x) \right)^{1/2} \leq I + II + III,$$

where

$$I = \|V\|_{L^\infty} \left(\int |u_n(x+h) - u_n(x)|^2 \, dm(x) \right)^{1/2},$$

$$II = \|u_n\|_{L^2} \left(\int_B |V(x+h) - V(x)|^2 \, dm(x) \right)^{1/2},$$

$$III = \|u_n\|_{L^2} \sup_{x \in B^c} |V(x+h) - V(x)|.$$

Clearly I, II and III tend to zero uniformly in n as h tends to the neutral element and $B \nearrow X$.

2. The sequence $\{Vu_n\}$ consists of functions with equicontinuous $L^2(X, m)$ integrals at infinity. Indeed, for any ball B which contains the neutral element we have

$$\int_{B^c} |Vu_n(x)|^2 \, dm(x) \leq \|u_n\|_{L^2} \sup_{x \in B^c} |V(x)| \to 0$$

uniformly in n as $B \nearrow X$.

Thus, the sequence $\{Vu_n\}$ is bounded in $L^2(X, m)$, consists of equicontinuous on whole in $L^2(X, m)$ functions with equicontinuous $L^2(X, m)$ integrals at infinity. By the Riesz-Kolmogorov criterion of compactness in $L^2(X, m)$, the set $\{Vu_n\}$ is compact, whence it contains a convergent subsequence $\{Vu'_n\}$, as claimed.

(iv) By [30, Theorem 2.1 and Remark 2.2], the number of negative eigenvalues counted with their multiplicity can be estimated as follows:

$$Neg(H) \le \frac{1}{c(\sigma)} \int_X V_-(x) \left(\int_{\frac{\sigma}{V_-(x)}}^{\infty} p(t, x, x) dt \right) dm(x) \qquad (2.20)$$

holds for any $\sigma > 0$ with $c(\sigma) = e^{-\sigma} \int_0^{\infty} z(z + \sigma)^{-1} e^{-z} dz$. Since we assume that Φ^{-1} is doubling,

$$\int_\tau^{\infty} p(t, x, x) dt \asymp \int_\tau^{\infty} \Phi^{-1}(1/t) dt \asymp \tau \Phi^{-1}(1/\tau) \text{ at } \infty.$$

Choosing σ big enough and applying the inequality (2.20) we obtain the inequality (2.19), as desired. ∎

3 Discrete ultrametric spaces

Recall that a hierarchical Laplacian L acting on a homogeneous ultrametric measure space (X, d, m) is called *homogeneous* if it is invariant under the action of the group of isometries. By the homogenuity property for any two balls A and B having the same diameter the eigenvalues $\lambda(A)$ and $\lambda(B)$ coinside. We denote by λ_k the common value of eigenvalues for balls which belong to the horocycle $\mathcal{H}_k$. In this section we assume that the ultrametric mesure space (X, d, m) is *countably infinite and homogeneous*.

3.1 Rank one perturbations

Let us consider the Schrödinger operator $H = L - V$ with potential $V = \sigma \delta_a$. The operator $V : f(x) \to V(x)f(x)$ can be written in the form

$$V f(x) = \sigma (f, \delta_a) \delta_a(x),$$

i.e. H can be regarded as rank one perturbation of the operator L.

Denote $\psi(x) = \mathcal{R}(\lambda, x, y)$, the solution of the equation

$$L\psi(x) - \lambda\psi(x) = \delta_y(x).$$

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Let $\psi_V(x) = \mathcal{R}_V(\lambda, x, y)$ be the solution of the equation

$$H\psi_V(x) - \lambda\psi_V(x) = \delta_y(x).$$

Notice that L and H are symmetric operators whence both $(x, y) \to \mathcal{R}(\lambda, x, y)$ and $(x, y) \to \mathcal{R}_V(\lambda, x, y)$ are symmetric functions.

Proposition 3.1 *In the notation from above*

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \frac{\sigma\mathcal{R}(\lambda, x, a)\mathcal{R}(\lambda, a, y)}{1 - \sigma\mathcal{R}(\lambda, a, a)}. \qquad (3.21)$$

In particular,

$$\mathcal{R}_V(\lambda, a, y) = \frac{\mathcal{R}(\lambda, a, y)}{1 - \sigma\mathcal{R}(\lambda, a, a)}. \qquad (3.22)$$

and

$$\mathcal{R}_V(\lambda, a, a) = \frac{\mathcal{R}(\lambda, a, a)}{1 - \sigma\mathcal{R}(\lambda, a, a)}. \qquad (3.23)$$

Proof. We have

$$L\psi_V(x) - \lambda\psi_V(x) = \delta_y(x) + \sigma\delta_a(x)\psi_V(x)$$
$$= \delta_y(x) + \sigma\delta_a(x)\psi_V(a).$$

It follows that

$$\psi_V(x) = \mathcal{R}(\lambda, x, y) + \sigma\psi_V(a)\mathcal{R}(\lambda, x, a). \qquad (3.24)$$

Setting $x = a$ in the above equation we obtain

$$\psi_V(a) = \mathcal{R}(\lambda, a, y) + \sigma\psi_V(a)\mathcal{R}(\lambda, a, a)$$

or

$$\psi_V(a)(1 - \sigma\mathcal{R}(\lambda, a, a)) = \mathcal{R}(\lambda, a, y).$$

Since $\psi_V(a) = \mathcal{R}_V(\lambda, a, y)$ we obtain equation (3.23). In turn, equations (3.23) and (3.24) imply (3.21) and (3.22). ∎

Let $\Upsilon(X)$ be the tree of balls associated with the ultrametric space (X, d). Consider in $\Upsilon(X)$ the infinite geodesic path from a to $\varpi : \{a\} = B_0 \subsetneq B_1 \subsetneq \ldots \subsetneq B_k \subsetneq \ldots$. The series below converges uniformly and in L^2,

$$\delta_a = \left(\frac{1_{B_0}}{m(B_0)} - \frac{1_{B_1}}{m(B_1)}\right) + \left(\frac{1_{B_1}}{m(B_1)} - \frac{1_{B_2}}{m(B_2)}\right) + \ldots = \sum_{k=0}^{\infty} f_{B_k}.$$

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Notice that all f_{B_k} are eigenfunctions of the operator L, to be more precise $Lf_{B_k} = \lambda(B_{k+1})f_{B_k} = \lambda_{k+1}f_{B_k}$. Observe that by definition $\mathcal{R}(\lambda, x, y) = (L - \lambda)^{-1}\delta_y(x)$ whence we obtain

$$\mathcal{R}(\lambda, a, a) = \frac{1}{\lambda_1 - \lambda}f_{B_0}(a) + \frac{1}{\lambda_2 - \lambda}f_{B_1}(a) + ...$$
$$= \frac{1}{\lambda_1 - \lambda}\left(\frac{1}{m(B_0)} - \frac{1}{m(B_1)}\right)$$
$$+ \frac{1}{\lambda_2 - \lambda}\left(\frac{1}{m(B_1)} - \frac{1}{m(B_2)}\right) + ... \,,$$

or in the final form

$$\mathcal{R}(\lambda, a, a) = \sum_{k=1}^{\infty}\frac{A_k}{\lambda_k - \lambda}, \quad A_k = \left(\frac{1}{m(B_{k-1})} - \frac{1}{m(B_k)}\right). \tag{3.25}$$

Since $\lambda \to \mathcal{R}(\lambda, a, a)$ is an increasing function, the equation

$$1 - \sigma\mathcal{R}(\lambda, a, a) = 0, \quad \sigma \neq 0, \tag{3.26}$$

has precisely one solution λ_k^{σ} lying in each open interval $]\lambda_{k+1}, \lambda_k[$,

$$\lambda_{k+1} < \lambda_k^{\sigma} < \lambda_k, \quad k = 1, 2, ... \,.$$

Claim 1 All numbers λ_k^{σ} are eigenvalues of the operator H. Indeed, the function $\psi(x) = \mathcal{R}(\lambda, x, a)$ with $\lambda = \lambda_k^{\sigma}$ satisfies the equation

$$H\psi(x) - \lambda\psi(x) = L\psi(x) - \lambda\psi(x) - \sigma\delta_a(x)\psi(x)$$
$$= L\psi(x) - \lambda\psi(x) - \sigma\delta_a(x)\psi(a)$$
$$= L\psi(x) - \lambda\psi(x) - \delta_a(x) = 0.$$

Claim 2 All numbers λ_k are eigenvalues of the operator H. Indeed, for any ball B which does not contain a but belongs to the horocycle as $\mathcal{H}_{k-1}$ we have

$$Hf_B = Lf_B = \lambda_k f_B.$$

When $\sigma > 0$ there may exist one more eigenvalue $\lambda_-^{\sigma} < 0$, a solution of the equation (3.26). Indeed, $\lambda \to \mathcal{R}(\lambda, a, a)$ is an increasing function, continuous on the interval $]-\infty, 0]$. Since $\mathcal{R}(\lambda, a, a) \to 0$ as $\lambda \to -\infty$ and $\mathcal{R}(\lambda, a, a) \to \mathcal{R}(0, a, a) \leq +\infty$ as $\lambda \to -0$, equation (3.26) has a unique solution $\lambda = \lambda_-^{\sigma} < 0$ in the following two cases:

(i) The semigroup $(e^{-tL})_{t>0}$ is recurrent, i.e. $\mathcal{R}(0, a, a) = +\infty$.

(ii) The semigroup $(e^{-tL})_{t>0}$ is transient, i.e. $\mathcal{R}(0, a, a) < +\infty$, and σ is such that $\mathcal{R}(0, a, a) > 1/\sigma$.

Summarizing all the above we obtain the following result

Proposition 3.2 *The operator $H = L - V$ with $V = \sigma\delta_a$ has at most one negative eigenvalue and countably many positive eigenvalues with accumulating point 0. The operator H has precisely one negative eigenvalue λ^σ_- if and only if $\sigma > 0$ and one of the conditions (i) and (ii) above holds. In this case the set $Spec(H)$ consists of points*

$$\lambda^\sigma_- < 0 < ... < \lambda_{k+1} < \lambda^\sigma_k < \lambda_k < ... < \lambda_2 < \lambda^\sigma_1 < \lambda_1.$$

Otherwise the set $Spec(H)$ consists of points

$$0 < ... < \lambda_{k+1} < \lambda^\sigma_k < \lambda_k < ... < \lambda_2 < \lambda^\sigma_1 < \lambda_1$$

The eigenvalues λ_k are at the same time eigenvalues of the operator L. All λ_k have infinite multiplicity and compactly supported eigenfunctions, the eigenfunctions of the operator L whose supports do not contain a. The eigenvalue λ^σ_k (resp. λ^σ_-) is the unique solution of the equation 3.26 in the interval $]\lambda_{k+1}, \lambda_k[$ (resp. in the interval $] -\infty, 0[$). λ^σ_k (resp. λ^σ_-) has multiplicity one and non-compactly supported eigenfunction $\psi_k(x) = \mathcal{R}(\lambda^\sigma_k, x, a)$ (resp. $\psi_-(x) = \mathcal{R}(\lambda^\sigma_-, x, a)$).

3.2 Finite rank perturbations

Let us consider the Schrödinger operator $H = L - V$ with potential $V = \sum_{i=1}^N \sigma_i \delta_{a_i}$. The operator $V : f(x) \to V(x)f(x)$ can be written in the form

$$Vf(x) = \sum_{i=1}^N \sigma_i(f, \delta_{a_i})\delta_{a_i}(x),$$

i.e. H can be regarded as rank N perturbation of the operator L.

Lemma 3.3 *Let A and B be two symmetric operators, $rank(B) = N$ and $H = A + B$. Let (a, b) be an interval lying in the complement of the set $Spec(A)$. The set $Spec(H) \cap (a, b)$ consists of at most N eigenvalues.*

Proof. Assume that $N = 1$. In this case $Bf = \sigma(f, f_1)f_1$. For $\lambda \in (a, b)$ any solution of the equation $Hf - \lambda f = 0$ can be written in the form $f = -\sigma(f, f_1)R_\lambda f_1$ where $R_\lambda = (A - \lambda)^{-1}$. Taking inner product in both parts

of this equation we get $(f, f_1) = -\sigma(f, f_1)(R_\lambda f_1, f_1)$ or $\sigma(R_\lambda f_1, f_1) + 1 = 0$. Since the function $\lambda \to (R_\lambda f_1, f_1)$ is strictly increasing the equation $\sigma(R_\lambda f_1, f_1) + 1 = 0$ has at most one solution in the interval (a, b). This solution is eigenvalue of the operator H.

Assume that the statement holds true for $N = k$, and let $\lambda_1 \leq ... \leq \lambda_k$ be the corresponding eigenvalues in the interval (a, b). The numbers λ_i split the interval (a, b) in at most $k + 1$ open intervals I_i each of which does not intersect the spectrum of the operator $A + \sum_{i=1}^{k} \sigma_i(f, f_i)f_i$. Let us consider the operator $H = A + \sum_{i=1}^{k+1} \sigma_i(f, f_i)f_i$ and write $H = A' + \sigma_{k+1}(f, f_{k+1})f_{k+1}$. By what we have proved above each of the $k + 1$ open intervals I_i contains at most one eigenvalue of the operator H, i.e. H contains in (a, b) at most $k + 1$ eigenvalues. The proof is finished. ∎

Example 3.4 *Consider the case: $\sigma_i = \sigma > 0$ and a_i fulfill the whole ball $B_0 \subset X$, i.e. $V(x) = \sigma 1_{B_0}(x)$. Assume that B_0 belongs to the horocycle $\mathcal{H}_{k_0}$. Let us select the following three Hilbert subspaces of $L^2(X, m)$:*

- $\mathcal{L}_+ = \mathrm{span}\{1_B : B \in \mathcal{H}_{k_0}\}$, the linear subspace of $L^2 = L^2(X, m)$ spanned by the indicators of balls which belong to the horocycle $\mathcal{H}_{k_0}$,

- $\mathcal{L}_- = L^2(X, m) \ominus \mathcal{L}_+$, the orthogonal complement of $\mathcal{L}_+$, and

- $\mathcal{L}_B = \mathrm{span}\{f_T : T \subsetneq B\}$, the linear space spanned by the eigenfunctions $f_T = 1_T/m(T) - 1_{T'}/m(T')$ of the operator L such that $T' \subseteq B$.

Let $\langle H | \mathcal{L}_+ \rangle$, $\langle H | \mathcal{L}_- \rangle$ and $\langle H | \mathcal{L}_B \rangle)$ be the restriction of the operator H to its invariant subspaces listed above.

Spectrum of the operator $\langle H | \mathcal{L}_- \rangle$: The system of eigenfunctions $\{f_T : T \in \mathcal{B}\}$ is complete in $L^2(X, m)$. Each eigenfunction f_T either belongs to $\mathcal{L}_+$ or to its orthogonal complement $\mathcal{L}_-$ whence

$$\mathcal{L}_- = \bigoplus_{B \in \mathcal{H}_{k_0}} \mathcal{L}_B = \mathcal{L}_{B_0} \bigoplus \mathcal{L}'.$$

For every ball $B \in \mathcal{H}_{k_0}$ we have

$$\langle H | \mathcal{L}_B \rangle = \begin{cases} \langle L | \mathcal{L}_{B_0} \rangle - \sigma, & B = B_0 \\ \langle L | \mathcal{L}_B \rangle, & B \neq B_0 \end{cases}$$

whence

$$\langle H | \mathcal{L}_- \rangle = (\langle L | \mathcal{L}_{B_0} \rangle - \sigma) \bigoplus \langle L | \mathcal{L}' \rangle$$

and

$$\langle L | \mathcal{L}' \rangle = \bigoplus_{B \neq B_0} \langle L | \mathcal{L}_B \rangle.$$

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It follows that

$$Spec(\langle H|\mathcal{L}_-\rangle) = \{\lambda_k - \sigma : 1 \le k \le k_0\} \cup \{\lambda_k : 1 \le k \le k_0\}. \qquad (3.27)$$

The number $Neg(\langle H|\mathcal{L}_-\rangle)$: The operator $\langle H|\mathcal{L}_-\rangle$ has finite number of negative eigenvalues. Given $\sigma > \lambda_{k_0}$ this set is not empty. Let us estimate $Neg(\langle H|\mathcal{L}_-\rangle)$, the number of negative eigenvalues counted with their multiplicity. Assuming that $\sigma > \lambda_{k_0}$ let us choose the integer $k_* \le k_0$ such that

$$\lambda_{k_*} < \sigma < \lambda_{k_*-1}. \qquad (3.28)$$

Let $\{n_k\}$ be the sequence of forward degrees associated with the tree of balls $\Upsilon(X)$. We have

$$Neg(\langle H|\mathcal{L}_-\rangle) = (n_{k_0} - 1) + n_{k_0}(n_{k_0-1} - 1) + ... + n_{k_0}n_{k_0-1}...n_{k_*+1}(n_{k_*} - 1)$$

$$= n_{k_0}\left(1 - \frac{1}{n_{k_0}}\right) + ... + n_{k_0}n_{k_0-1}...n_{k_*+1}n_{k_*}\left(1 - \frac{1}{n_{k_*}}\right).$$

Since all $n_k \ge 2$ we obtain

$$\frac{1}{2}\left(n_{k_0} + ... + n_{k_0}n_{k_0-1}...n_{k_*}\right) < Neg(\langle H|\mathcal{L}_-\rangle) < n_{k_0} + ... + n_{k_0}n_{k_0-1}...n_{k_*}$$

or by the same reasongs

$$\frac{1}{2}n_{k_0}n_{k_0-1}...n_{k_*} < Neg(\langle H|\mathcal{L}_-\rangle) < \frac{3}{2}n_{k_0}n_{k_0-1}...n_{k_*}. \qquad (3.29)$$

Let $B_* \subseteq B_0$ be a ball in $\mathcal{H}_{k_*}$, then

$$n_{k_0}n_{k_0-1}...n_{k_*} = \frac{m(B_0)}{m(B_*)}$$

whence

$$\frac{1}{2}\frac{m(B_0)}{m(B_*)} < Neg(\langle H|\mathcal{L}_-\rangle) < \frac{3}{2}\frac{m(B_0)}{m(B_*)}. \qquad (3.30)$$

Let us choose an increasing function Φ such that the eigenvalues of the operator L can be written in the form $\lambda(B) = \Phi(1/m(B))$, then equation (3.28) yields

$$\frac{1}{n_{k_*}}\int_X \Phi^{-1} \circ V(x)dm(x) < \frac{m(B_0)}{m(B_*)} < \int_X \Phi^{-1} \circ V(x)dm(x) \qquad (3.31)$$

Spectrum of the operator $\langle H|\mathcal{L}_+\rangle$: We say that $x \sim y$ if and only if x and y belong to the same ball $B \in \mathcal{H}_{k_0}$. Clearly this is an equivalence relation.

The set $[X]$ of all equivalence classes $[x]$ equipped with the induced metric and with the induced measure is a discrete homogeneous ultrametric measure space. All balls $B \in \mathcal{H}_{k_0}$ become singletones $[B]$ in $[X]$. Let $\tau : x \to [x]$ be the canonical mapping of X onto $[X]$. The mapping $\Upsilon : f \to f \circ \tau$ maps $L^2([X], [m])$ onto $\mathcal{L}_+$ isometrically. The operator $[L] = \Upsilon^{-1} \circ \langle L | \mathcal{L}_+ \rangle \circ \Upsilon$ acting in $L^2([X], [m])$ is a homogeneous hierarchical Laplacian. Since in the formula

$$\langle L | \mathcal{L}_+ \rangle f(x) = \sum_{B \in \mathcal{B}(x)} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right),$$

the sum can be taken over all balls $B \in \mathcal{B}(x)$ each of which belongs to the horocycle $\mathcal{H}_k$ with $k > k_0$, the complete list of eigenvalues of the operator $[L]$ is: $\lambda_{k_0+1} > \lambda_{k_0+2} > \lambda_{k_0+3} > ....$ The Markov semigroup $(e^{-t[L]})_{t>0}$ is transient (resp. recurrent) if and only if the Markov semigroup $(e^{-tL})_{t>0}$ is transient (resp. recurrent).

The operator $\Upsilon^{-1} \circ \langle H | \mathcal{L}_+ \rangle \circ \Upsilon$ acting in $L^2([X], [m])$ coincides with the Schrödinger operator $[H] = [L] - [V]$ where $[V] = \sigma \cdot \delta_{[B_0]}$. Thus $Spec(\langle H | \mathcal{L}_+ \rangle) = Spec([H])$ and the results of Proposition 3.2 apply.

Spectrum of the operator H : The operator $H = L - \sigma 1_{B_0}$ has purely point spectrum. The set of positive eigenvalues of the operator H can be splittet in three subsets Ξ_1, Ξ_2 and Ξ_3: Ξ_1 consists of eigenvalues λ_k of the operator L, $\Xi_2 = \{\lambda_k - \sigma : k \leq k_*\}$ where $k_* = \max\{k \leq k_0 : \lambda_k - \sigma > 0\}$ and Ξ_3 consists of eigenvalues λ_k^σ, $k \geq k_0$, each lying in the open interval $(\lambda_{k+1}, \lambda_k)$. The set Ξ_- of negative eigenvalues of the operator H consists of eigenvalues $\lambda_k - \sigma$, $k_* < k \leq k_0$, and the eigenvalue λ_-^σ described in Proposition 3.2. Each eigenvalue in Ξ_1 has infinite multiplicity, the eigenfunctions in Ξ_2, Ξ_3 and Ξ_- have finite multiplicity. The eigenfunctions corresponding to Ξ_1, Ξ_2 and $\Xi_- \backslash \{\lambda_-^\sigma\}$ are compactly supported, the eigenfunctions corresponding Ξ_3 and λ_-^σ have full support.

The number $Neg(H)$: The operator H has finite number of negative eigenvalues. Evidently this set is not empty if, for instance, $\sigma > \lambda_{k_0}$. Assuming that $\sigma > \lambda_{k_0}$ we estimate $Neg(H)$, the number of negative eigenvalues counted with their multiplicity:

$$Neg(H) > \frac{1}{2n_{k_*}} \int_X \Phi^{-1} \circ V(x) dm(x), \qquad (3.32)$$

where k_* is defined by equation (3.28), and

$$Neg(H) < 1 + \frac{3}{2} \int_X \Phi^{-1} \circ V(x) dm(x). \qquad (3.33)$$

Proposition 3.5 *Assume that $V = \sum_{i=1}^{N} \sigma_i \delta_{a_i}$ and that all $\sigma_i > 0$ are different. For any δ large enough there exists $k(\delta)$ such that $\min_{i \neq j} d(a_i, a_j) > \delta$ implies that the operator H has precisely N different eigenvalues in each open interval $]\lambda_{k+1}, \lambda_k[$: $1 \leq k \leq k(\delta)$. Moreover there exists precisely N negative eigenvalues in the following two cases:*

(i) The semigroup $(e^{-tL})_{t>0}$ is recurrent.

(ii) The semigroup $(e^{-tL})_{t>0}$ is transient and $\sigma_i > 1/\mathcal{R}(0, a, a)$.

Proof. Let $\psi(x) := \mathcal{R}(\lambda, x, y)$ be the solution of the equation $L\psi(x) - \lambda\psi(x) = \delta_y(x)$, and $\psi_V(x) := \mathcal{R}_V(\lambda, x, y)$ be the solution of the equation $H\psi_V(x) - \lambda\psi_V(x) = \delta_y(x)$. To find the perturbed resolvent $\mathcal{R}_V(\lambda, x, y)$ we write

$$L\psi_V(x) - \lambda\psi_V(x) = \delta_y(x) + \sum_{i=1}^{N} \sigma_j \delta_{a_j}(x)\psi_V(x)$$

$$= \delta_y(x) + \sum_{j=1}^{N} \sigma_j \psi_V(a_j)\delta_{a_j}(x),$$

or

$$\psi_V(x) = \mathcal{R}(\lambda, x, y) + \sum_{j=1}^{N} \sigma_j \psi_V(a_j)\mathcal{R}(\lambda, x, a_j) \tag{3.34}$$

Choosing in the above equation $x = a_1, x = a_2, ..., x = a_N$ we obtain system of N linear equations with N variables $\xi_i = \psi_V(a_i)$,

$$\xi_i = \mathcal{R}(\lambda, a_i, y) + \sum_{j=1}^{N} \sigma_j \mathcal{R}(\lambda, a_i, a_j)\xi_j, \quad i = 1, 2, ..., N,$$

or, in the vector form,

$$(\mathrm{E} - \mathfrak{R}(\lambda)\Theta)\xi = \mathfrak{R}(\lambda, y), \tag{3.35}$$

where we use the following notation $\xi = (\xi_i)_{i=1}^{N}$, $\mathfrak{R}(\lambda, y) = (\mathcal{R}(\lambda, a_i, y))_{i=1}^{N}$, $\mathfrak{R}(\lambda) = (\mathcal{R}(\lambda, a_i, a_j))_{i,j=1}^{N}$, $\mathrm{E} = (\delta_{ij})_{i,j=1}^{N}$ and $\Theta = \mathrm{diag}(\sigma_i)$.

Let us substitute $\xi_i = \mathcal{R}_V(\lambda, a_i, y)$ and $\mathfrak{R}_V(\lambda, y) := (\mathcal{R}_V(\lambda, a_i, y))_{i=1}^{N}$ in the above equation

$$(\mathrm{E} - \mathfrak{R}(\lambda)\Theta)\mathfrak{R}_V(\lambda, y) = \mathfrak{R}(\lambda, y). \tag{3.36}$$

Choosing in equation (3.36) $y = a_1, y = a_2, ..., y = a_N$ and setting $\mathfrak{R}_V(\lambda) = (\mathcal{R}_V(\lambda, a_i, a_j))_{i,j=1}^N$ we get the following matrix equation

$$(E - \mathfrak{R}(\lambda)\Theta)\mathfrak{R}_V(\lambda) = \mathfrak{R}(\lambda)$$

or equivalently

$$(E - \mathfrak{R}(\lambda)\Theta)\mathfrak{R}_V(\lambda)\Theta = \mathfrak{R}(\lambda)\Theta.$$

Similarly, since $H = L + \sum_{i=1}^N (-\sigma_i)\delta_{a_i}$, we get

$$(E + \mathfrak{R}_V(\lambda)(-\Theta))\mathfrak{R}(\lambda)(-\Theta) = \mathfrak{R}_V(\lambda)(-\Theta).$$

It follows that

(i) the matrices $\mathfrak{R}(\lambda)\Theta$ and $\mathfrak{R}_V(\lambda)\Theta$ commute, and

(ii) $\lambda \in Spec(H) \setminus Spec(L)$ if and only if λ satisfies the equation

$$\det(E - \mathfrak{R}(\lambda)\Theta) = 0. \tag{3.37}$$

Observe that the variable $z := \mathcal{R}(\lambda, a_i, a_i)$ does not depend on i, and its range is the whole interval $]-\infty, \infty[$ when λ takes values in each of open interval $]\lambda_{k+1}, \lambda_k[$. Equation (3.37) can be written as characteristic equation

$$\det(\mathfrak{A} - zE) = 0 \tag{3.38}$$

where $\mathfrak{A} = (\mathfrak{a}_{ij})_{i,j=1}^N$ is symmetric $N \times N$ matrix with entries $\mathfrak{a}_{ii} = 1/\sigma_i$ and $\mathfrak{a}_{ij} = -\mathcal{R}(\lambda, a_i, a_j)$ for $i \neq j$. Thus all solutions of equation (3.38) are eigenvalues of the matrix $\mathfrak{A}$.

Let us compute $\mathcal{R}(\lambda, a_i, a_j)$. For any two neighboring balls $B \subset B'$ let us denote

$$A(B) = \frac{1}{m(B)} - \frac{1}{m(B')}.$$

Let $a_i \curlywedge a_j$ be the minimal ball which contains both a_i and a_j. Following the same line of reasongs as in the proof of equation (3.25) we obtain

$$\mathcal{R}(\lambda, a_i, a_i) = \sum_{B: \, a_i \in B}^{\infty} \frac{A(B)}{\lambda(B) - \lambda}$$

and

$$\mathcal{R}(\lambda, a_i, a_j) = -\frac{m(a_i \curlywedge a_j)^{-1}}{\lambda(a_i \curlywedge a_j) - \lambda} + \sum_{B: \, a_i \curlywedge a_j \subset B} \frac{A(B)}{\lambda(B) - \lambda}.$$

Let us assume that $\min_{i \neq j} d(a_i, a_j) > \sigma >> 1$. Then $\max \lambda(a_i \curlywedge a_j) = \lambda_{k(\delta)+1}$ for some $k(\delta) >> 1$. For all $\lambda \geq \lambda_{k(\delta)}$ and all $i \neq j$ we get

$$|\mathcal{R}(\lambda, a_i, a_j)| < \frac{2m(a_i \curlywedge a_j)^{-1}}{\lambda_{k(\delta)+1} - \lambda_{k(\delta)}} \leq \frac{2\max\{m(a_i \curlywedge a_j)^{-1}\}}{\lambda_{k(\delta)+1} - \lambda_{k(\delta)}}.$$

Denote the left-hand side of the above inequality by $\varepsilon(\delta)/N$ and observe that this quantity tend to zero as $\delta \to \infty$. Let us choose δ big enough so that the intervals $\{s : |1/\sigma_i - s| \leq \varepsilon(\delta)\}$ do not intersect. By Gershgorin Circle Theorem the matrix $\mathfrak{A}$ admits N different eigenvalues $\mathfrak{a}_i$ each of which lies in the corresponding open interval $\{s : |1/\sigma_i - s| < \varepsilon(\delta)\}$. The eigenvalues $\mathfrak{a}_i$ are analytic functions of λ in each open interval $]\lambda_{k+1}, \lambda_k[$, $1 \leq k \leq k(\delta)$, see [31, Theorem XII.1]. Whence in each of these intervals the number of different solutions of the equation $\mathfrak{a}_i = \mathcal{R}(\lambda, a_i, a_i)$ is at least N. Lemma 3.3 says that the number of different solutions is at most N. Thus the number of different solutions is precisely N as desired. ∎

4   Perfect ultrametric spaces

Let us consider a homogeneous ultrametric measure space (X, d, m) which is also assumed to be *non-compact, perfect and proper*. Recall that the last two properties mean: X has no isolated points, and all closed balls are compact sets.

4.1   Potentials of finite rank

Let L be a homogeneous hierarchical Laplacian L. Let us fix a horocycle $\mathcal{H} = \mathcal{H}_k$ and define $V(x) = \sum_{B \in \mathcal{H}} \sigma(B) 1_B(x)$. We say that $V(x)$ is of *rank k function*.

Let $Hu(x) = Lu(x) - Vu(x)$ be the Schrödinger operator with potential $V(x)$. By Theorem 2.1(i), symmetric operator $(H, \mathcal{D})$ is essentially self-adjoint. Notice that (in contrary to the case when the phase space is discrete) the operator $(L, \mathcal{D})$ is indeed *unbounded* symmetric operator.

Among invariant subspaces of the operator $(H, \mathcal{D})$ we select the following three Hilbert spaces:

- $\mathcal{L}_+ = \mathrm{span}\{1_B : B \in \mathcal{H}\}$, the linear subspace of $L^2(X, m)$ spanned by the indicators of balls which belong to the horocycle $\mathcal{H}$,

- $\mathcal{L}_- = L^2 \ominus \mathcal{L}_+$, the orthogonal complement of $\mathcal{L}_+$, and

- $\mathcal{L}_B = \mathrm{span}\{f_T : T \subsetneq B\}$, the linear space spanned by the eigenfunctions $f_T = 1_T/m(T) - 1_{T'}/m(T')$ of the operator L such that $T' \subseteq B$. The space $\mathcal{L}_B$ is indeed the invariant subspace of H.

Each of the invariant subspaces spaces $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_B$ reduces the operators L and H. We denote $\langle L|\mathcal{L}_+\rangle$, $\langle H|\mathcal{L}_+\rangle$ etc. the restriction of the operators L and H to these invariant subspaces.

Spectrum of the operator $\langle H|\mathcal{L}_-\rangle$ The system of functions $\{f_T : T \in \mathcal{B}\}$ is complete in $L^2(X,m)$. Since each f_T either belongs to $\mathcal{L}_+$ or to its orthogonal complement $\mathcal{L}_-$, we write

$$\mathcal{L}_- = \bigoplus_{B \in \mathcal{H}} \mathcal{L}_B.$$

For any ball $B \in \mathcal{H}$ we have

$$\langle H|\mathcal{L}_B\rangle = \langle L|\mathcal{L}_B\rangle - \sigma(B),$$

whence

$$\langle H|\mathcal{L}_-\rangle = \bigoplus_{B \in \mathcal{H}} (\langle L|\mathcal{L}_B\rangle - \sigma(B)).$$

Let us set $\mathfrak{S}(\mathcal{H}) = Spec\,\langle L|\mathcal{L}_B\rangle$, then the above equation yield

$$Spec\,\langle H|\mathcal{L}_-\rangle = \bigcup_{B \in \mathcal{H}} \{\mathfrak{S}(\mathcal{H}) - \sigma(B)\}. \qquad (4.39)$$

Spectrum of the operator $\langle H|\mathcal{L}_+\rangle$ Consider the equivalence relation: $x \sim y$ if and only if x and y belong to the same ball $B \in \mathcal{H}$. The set $[X]$ of all equivalence classes equipped with the induced ultrametric $[d]$ and with the induced measure $[m]$ is a discrete homogeneous ultrametric measure space.Without loss of generality we may assume that $m(B) = 1$, then the measure of all singletones in $[X]$ is equal to 1. Let $\tau : x \to [x]$ be the canonical mapping of X onto $[X]$. The mapping $\Upsilon : f \to f \circ \tau$ maps $L^2([X],[m])$ onto $\mathcal{L}_+$ isometrically.

For any function $f(x)$ which takes constant values on the balls in $\mathcal{H}$ we have

$$Lf(x) == \sum_{[x] \subset B} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f\,dm \right)$$

where $[x]$ is the unique ball in $\mathcal{H}$ (the equivalence class) which contains x. The operator $[L] = \Upsilon^{-1} \circ \langle L|\mathcal{L}_+\rangle \circ \Upsilon$ is a homogeneous hierarchical Laplacian acting on the discrete homogeneous ultrametric measure space $([X],[d],[m])$. The Markov semigroup $(e^{-t[L]})_{t>0}$ is transient (resp. recurrent) if and only if the semigroup $(e^{-tL})_{t>0}$ is transient (resp. recurrent).

The operator $\Upsilon^{-1} \circ \langle H|\mathcal{L}_+\rangle \circ \Upsilon$ acting in $L^2([X],[m])$ coincides with the Schrödinger operator $[H] = [L] - [V]$ with potential $[V] = \sum \sigma(B) \cdot \delta_{[B]}$. In particular, we obtain the following result

$$Spec\,\langle H|\mathcal{L}_+\rangle = Spec[H].$$

Summing all the above we come to the following conclusion

Proposition 4.1 *Assume that $V(x) \to 0$ as $x \to \infty$. Then the spectrum of the operator $H = L - V$ is pure point with possibly accumulating points λ_k, the eigenvalues of the operator L.*

Assume that $\lambda(B) = \Phi(1/m(B))$ for some increasing homeomorphism $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, and for all balls B, then

$$Neg(H) \geq C_1 \int_{\{x:\, V(x)>\lambda_*\}} \Phi^{-1} \circ V(x) dm(x)$$

for some constant $C_1 > 0$, and for λ_*- the eigenvalue corresponding to balls in $\mathcal{H}$.

Assume that the Markov semigroup $(e^{-tL})_{t>0}$ is transient and that both Φ and Φ^{-1} are doubling, then

$$Neg(H) \leq C_2 \int_X \Phi^{-1} \circ V_+(x) dm(x)$$

for some constant C_2, and $V_+(x) = \max\{0, V(x)\}$.

Proof. Negative eigenvalues E_i of H lie below the essential spectrum of H. Hence to compute E_i the $\min - \max$ principle applies, see [31, Theorem XIII.1]. In particular, E_i depend monotonically on the potential $V(x)$, whence without loss of generality one can assume that $V(x) \geq 0$.

As $B \downarrow \{a\}$ the eigenvalue $\lambda(B) = 1/\text{diam}_*(B)$ tend to ∞. It follows that the set of negative eigenvalues of the operator $\langle H | \mathcal{L}_B \rangle$ is finite. In turn, since $V(x) \to 0$, the set of negative eigenvalues of the operator $\langle H | \mathcal{L}_- \rangle$ is finite. Hence the proof of the statement reduces to the discrete setting: $Spec \langle H | \mathcal{L}_+ \rangle = Spec[H]$. Thus, we apply Theorem 2.1(*ii*) to get the first part of our claim. The second part of the statement follows from equation (3.32). The Markov semigroup $(e^{-tL})_{t>0}$ is transient and admits a continuous transition density $p(t, x, y)$ such that $p(t, x, x) \asymp \Phi^{-1}(1/t)$. Since we assume that Φ^{-1} is doubling,

$$\int_\tau^\infty p(t, x, x) dt \asymp \tau \Phi^{-1}(1/\tau) \text{ at } \infty$$

and

$$\int_0^1 t^m p(t, x, x) dt < \infty \quad \text{for some } m > 0.$$

Thus, inequality (2.19) applies and we come to the desired conclusion

$$Neg(H) \leq C_2 \int_X \Phi^{-1} \circ V_+(x) dm(x).$$

The proof is finished. ∎

4.2 The operator $H = \mathfrak{D}^\alpha - \sigma 1_B$

As an example let us consider $X = \mathbb{Q}_p$, the ring of p-adic numbers and $\mathfrak{D}^\alpha$ the operator of fractional derivative of order α. Choose $\sigma > 0$ and $B = \mathbb{Z}_p$, the set of p-adic integers. Let $H = \mathfrak{D}^\alpha - V$ be the Schrödinger operator with potential $V = \sigma 1_B$.

The eigenvalues of the operator $\mathfrak{D}^\alpha$ are numbers

$$\lambda(B) = \left(\frac{p}{m(B)}\right)^\alpha = p^{-\alpha(k-1)}, \quad k \in \mathbb{Z},$$

therefore the operator $\langle H|\mathcal{L}_+\rangle$ has eigenvalues $\lambda_k = p^{-\alpha(k-1)}$, $k = 1, 2, ...$

Let $B_0 \subset B_1 \subset ...$ be the infinite geodesic path in the homogeneous (self-similar) tree $\Upsilon(\mathbb{Q}_p)$ starting at $B_0 = B$ and ending at ϖ. The operator $\langle H|\mathcal{L}_+\rangle$ we identify with operator $[H]$ acting on the discrete lattice $\mathbb{Q}_p/\mathbb{Z}_p$ which can be identified with the set $\mathbb{N}$ of integers equipped with the family of p-adic partitians (a discrete counterpart of the Dyson's model). Let us compute the resolvent $\mathcal{R}(\lambda, [B], [B])$ of the operator $[H]$ at $\lambda = 0$. Following our computations in Section I we obtain

$$\mathcal{R}(\lambda, [B], [B]) = \sum_{k \geq 1} \frac{A_k}{\lambda_k - \lambda} = (p-1)\sum_{k \geq 1} \frac{1}{p^k(\lambda_k - \lambda)}.$$

In particular, $\mathcal{R}(0, [B], [B]) = +\infty$ if $\alpha \geq 1$, otherwise

$$\mathcal{R}(0, [B], [B]) = \frac{p-1}{p}\sum_{k \geq 0} \frac{1}{p^{k(1-\alpha)}} = \frac{p-1}{p - p^\alpha}.$$

By Proposition 3.2, the operator $\langle H|\mathcal{L}_+\rangle$ has atmost one negative eigenvalue. It does have a negative eigenvalue if and only if either (i) $\alpha \geq 1$ or (ii) $0 < \alpha < 1$ and $\sigma > (p - p^\alpha)(p-1)^{-1}$.

By equation (4.39), the operator $\langle H|\mathcal{L}_-\rangle$ has atmost finite number of negative eigenvalues. Evidently this set is not empty if $\sigma > \lambda(B) = p^\alpha$. To estimate $Neg\,\langle H|\mathcal{L}_-\rangle$, the number of negative eigenvalues counted with their multiplicity, we choose the integer $k_* \geq 0$ such that

$$p^{k_*} \leq \frac{\sigma^{1/\alpha}}{p} < p^{k_*+1}. \tag{4.40}$$

Let $B_* \subseteq B$ be a ball such that $m(B_*) = p^{-k_*}$. According to our choice $\lambda(B_*)$ is the minimal eigenvalue satisfying $\lambda(B) \leq \lambda(T) < \sigma$. Equations (3.30) and (3.31) yield

$$\frac{1}{2}\frac{m(B)}{m(B_*)} < Neg\,\langle H|\mathcal{L}_-\rangle < \frac{3}{2}\frac{m(B)}{m(B_*)}$$

and

$$\frac{1}{p}\int V(x)^{1/\alpha}dm(x) < \frac{m(B)}{m(B_*)} < \int V(x)^{1/\alpha}dm(x).$$

Let us define three subsets of the set $\{(\alpha,\sigma): \alpha > 0, \sigma > 0\}:$

- $\Omega_1 = \{(\alpha,\sigma): \sigma \le (p-p^\alpha)(p-1)^{-1},$
- $\Omega_2 = \{(\alpha,\sigma): (p-p^\alpha)(p-1)^{-1} < \sigma \le p^\alpha\},$
- $\Omega_3 = \{(\alpha,\sigma): \sigma > p^\alpha\}.$

Let $Neg(H)$ be the number of negative eigenvalues of the operator H counted with their multiplicity. Summing all the above we conclude that

$$Neg(H) = \begin{cases} 0 & \text{if} \quad (\alpha,\sigma) \in \Omega_1 \\ 1 & \text{if} \quad (\alpha,\sigma) \in \Omega_2 \end{cases}$$

and, if $(\alpha,\sigma) \in \Omega_3$, then

$$\frac{1}{2p}\int V(x)^{1/\alpha}dm(x) \le Neg(H) \le \frac{3}{2}\int V(x)^{1/\alpha}dm(x).$$

5 The Dyson's dyadic model

Let us consider $X = \{0,1,2,...\}$ equipped with the counting measure m. The set $\{\Pi_r : r = 0,1,...\}$ of partitians of X each of which consists of dyadic intervals $I_r = \{l \in X : k2^r \le l < (k+1)2^r\}$ induces in the standard way the ultrametric structure - a discrete version of Dyson's model, as explained in the introduction. We call r the rank of the interval I_r. We denote $I_r(x)$ the interval of rank r which contains the point x. Recall that the set of balls in the metric space X coincides with the set of all dyadic intervals.

5.1 Potentials of infinite rank We consider the Schrödinger operator $H = L + V$ with bounded potential V of the form $V(x) = \sum \sigma_k 1_{B_k}(x)$, the sequence of balls B_k is chosen such that the rank of B_k tends to infinity. The choice of the potential V will allow us to conclude that $Spec_{sc}(H)$, the singular continuous part of the spectrum of H, is not empty set. Here L is the Dyson's hierarchical Laplacian

$$Lf(x) = \sum_{r=1}^{\infty}(1-\varkappa)\varkappa^r\left(f(x) - \frac{1}{m(I_r(x))}\int_{I_r(x)} f\,dm\right),$$

with $\varkappa \in]0,1[$ a fixed parameter. Writing $\varkappa = 2^{-\alpha}$ we see that L coinsides with the hierarchical Laplacian introduced in (2.10) with $\Phi(\tau) = \tau^\alpha$. In particular, all eigenvalues $\lambda(B)$ of the operator L are of the form

$$\lambda(B) = \varkappa^r = |B|^{-\alpha},$$

the natural number $r \geq 1$ is the rank of the ball B and $|B|$ its cardinality.

To define the potential $V(x)$ we choose a sequence of balls $B_0 = \{0,1\}$, $B_1 = \{2,3\}$, $B_2 = \{4,5,6,7\}$,..., $B_k = \{2^k,...,2^{k+1}-1\}$, ... and a sequence $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_k, ...$ of negative reals such that $\sigma_0 \neq \sigma_1$, and set

$$V(x) = \sum \sigma_k 1_{B_k}(x).$$

We select the folowing $H-$ invariant subspaces of $L^2(X,m)$

- $\mathcal{L}_+ = \mathrm{span}\{1_{B_k} : k = 0,1,2,...\}$,

- $\mathcal{L}_{B_k} = \mathrm{span}\{f_T : T \subsetneq B_k\}$, and

- $\mathcal{L}_- = L^2(X,m) \ominus \mathcal{L}_+ = \bigoplus \mathcal{L}_{B_k}$.

5.2 Spectrum of $\langle H|\mathcal{L}_-\rangle$ As in the previous section we conclude that

- $\langle H|\mathcal{L}_{B_k}\rangle = \langle L|\mathcal{L}_{B_k}\rangle + \sigma_k$,

- $\langle H|\mathcal{L}_-\rangle = \bigoplus(\langle L|\mathcal{L}_{B_k}\rangle + \sigma_k)$, and

- $Spec\,\langle H|\mathcal{L}_-\rangle = \overline{\bigcup\{\mathfrak{S}_k + \sigma_k\}}$ where $\mathfrak{S}_k = Spec\,\langle L|\mathcal{L}_{B_k}\rangle$.

Let B_k' be the closest neighbouring ball to B_k. Since the operator L is homogeneous, $\mathfrak{S}_k = Spec\,\langle L|\mathcal{L}_{B_k'}\rangle$. The sequence $\{B_k'\}$ monotone increase to X whence $\mathfrak{S}_k \uparrow Spec(L)$. In particular, the following statement holds true

Proposition 5.1 *If the sequence $\{\sigma_k\}$ forms a dense subset in some interval $\mathcal{I}$ then the operator $\langle H|\mathcal{L}_-\rangle$ has a pure point spectrum. It contains all intervals $\tau + \mathcal{I}$, where τ runs over the set $Spec(L)$. In particular, the set $Spec\,\langle H|\mathcal{L}_-\rangle$ consists of finite number of disjoint intervals.*

5.3 Spectrum of $<H|\mathcal{L}_+>$ As $f_{B_k} \perp \mathcal{L}_{B_k}$ the sequence $\{f_{B_k}\} \subset \mathcal{L}_+$. We claim that $\{f_{B_k} : k \geq 1\}$ is a *complete orthogonal system* in the Hilbert space $\mathcal{L}_+$. Indeed, when $k,l \geq 1$ and $k \neq l$ we have $f_{B_k} \perp f_{B_l}$ because $B_k' \cap B_l' = \oslash$. Let us show that $\{f_{B_k} : k \geq 1\}$ is complete. Assume that $\psi = \sum_{k\geq 0} \psi_k 1_{B_k}$ is orthogonal to all f_{B_k}, $k = 1,2,...$. Since $f_{B_0} + f_{B_1} = 0$,

ψ is orthogonal to f_{B_0} as well. Thus, we obtain an infinite system of linear equations $\sum_{l \geq 0} \psi_l(1_{B_l}, f_{B_k}) = 0$, $k = 0, 1, 2, ...$, or in equivalent form,

$$|B_k'|\psi_k = \sum_l \psi_l(1_{B_l}, 1_{B_k'})$$
$$= \sum_{l \leq k} \psi_l(1_{B_l}, 1_{B_k'}) = \sum_{l \leq k} |B_l|\psi_l.$$

Setting $\xi_l = |B_l|\psi_l$, we obtain an infinite system of linear equations $2\xi_k = \sum_{l \leq k} \xi_l$ which has unique solution $\xi_k = 0$, $k = 0, 1, 2, ...$, as claimed.

Thus, the system of functions $F_k = \sqrt{2|B_k|}f_{B_k}$, $k \geq 1$, is an orthnormal basis in $\mathcal{L}_+$. The matrix M_L of the operator $< L|\mathcal{L}_+ >$ in the basis $\{F_k\}$ is diagonal $M_L = \mathsf{diag}\{\varkappa^2, \varkappa^3, ...\}$. In particular, L belongs to the trace class, therefore by [17, Theorem IV.5.35 and Theorem X.4.4],

$$Spec_{ess} < H|\mathcal{L}_+ >= Spec_{ess} < V|\mathcal{L}_+ >$$

and

$$Spec_{ac} < H|\mathcal{L}_+ >= Spec_{ac} < V|\mathcal{L}_+ > .$$

The system of functions $E_{k+1} = |B_k|^{-1/2}I_{B_k}$, $k \geq 0$, is an orthnormal basis in $\mathcal{L}_+$. The matrix M_V of the operator $< V|\mathcal{L}_+ >$ in the basis $\{E_k\}$ is diagonal $M_V = \mathsf{diag}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, ...\}$. In particular, we conclude that:

- $Spec < V|\mathcal{L}_+ >= \overline{\{\sigma_n : n = 0, 1, ...\}}$,

- $Spec_{ac} < V|\mathcal{L}_+ >$ and $Spec_{sc} < V|\mathcal{L}_+ >$ are empty sets,

- $Spec_d < V|\mathcal{L}_+ >$ consists of isolated σ_k having finite multiplicity,

- $Spec_{ess} < V|\mathcal{L}_+ >$ consists of isolated σ_k having infinite multiplicity and of occumulating points of the sequence $\{\sigma_k\}$.

Summing up all fact from above we obtain

Proposition 5.2 *The set $Spec_{ess} < H|\mathcal{L}_+ >$ consists of isolated σ_k having infinite multiplicity and of occumulating points of the sequence $\{\sigma_k\}$, its subset $Spec_{ac} < H|\mathcal{L}_+ >= \varnothing$. In particular, if the sequence $\{\sigma_k\}$ forms a dense subset in some interval $\mathcal{I}$ then $Spec_{ess} \langle H|\mathcal{L}_+\rangle = \mathcal{I}$.*

5.4 Generalized eigenfunctions Thus, we are left to find $Spec_{pp} \langle H|\mathcal{L}_+\rangle$ and its subset $Spec_d \langle H|\mathcal{L}_+\rangle$. Let us consider the equation $H\psi = \lambda\psi$. We are

looking for a bounded solution ψ of the form $\psi = \sum_k \psi_k 1_{B_k}$ which satisfies the equation $H\psi = \lambda\psi$ in a weak sense, that is,

$$(H\phi - \lambda\phi, \psi) = 0 \text{ or } (L\phi, \psi) = ((\lambda - V)\phi, \psi),$$

for any test function $\phi \in \mathcal{D}$ of the form $\phi = \sum_k \phi_k 1_{B_k}$.

Let us choose $\phi = 1_{B_n}$. As $V = \sum \sigma_k 1_{B_k}$ and $L f_B = \lambda(B') f_B$, we obtain:

$$(\lambda - \sigma_n)\psi_n = (\psi, L(1_{B_n} / |B_n|)) = (\psi, L(f_{B_n} + f_{B_n'} + f_{B_n''} + ...)) \qquad (5.41)$$
$$= (\psi, \lambda_{n+1} f_{B_n} + \lambda_{n+2} f_{B_n'} + \lambda_{n+3} f_{B_n''} + ...).$$

Since $f_{B_n'} = -f_{B_{n+1}}$, $f_{B_n''} = -f_{B_{n+2}}$, etc equation (5.41) gives for $n \geq 1$,

$$(\lambda - \sigma_n)\psi_n = \lambda_{n+1}(\psi, f_{B_n}) - \lambda_{n+2}(\psi, f_{B_{n+1}}) - \lambda_{n+3}(\psi, f_{B_{n+2}}) + ..., \quad (5.42)$$

and

$$(\lambda - \sigma_0)\psi_0 = -\lambda_2(\psi, f_{B_1}) - \lambda_3(\psi, f_{B_2}) - \lambda_4(\psi, f_{B_3}) - ... \ .$$

It follows that for $n \geq 1$,

$$(\lambda - \sigma_n)\psi_n - (\lambda - \sigma_{n+1})\psi_{n+1} = \lambda_{n+1}(\psi, f_{B_n}) - 2\lambda_{n+2}(\psi, f_{B_{n+1}}), \qquad (5.43)$$

and

$$(\lambda - \sigma_0)\psi_0 - (\lambda - \sigma_1)\psi_1 = -2\lambda_2(\psi, f_{B_1}). \qquad (5.44)$$

Next we compute $\lambda_{n+1}(\psi, f_{B_n})$:

$$\lambda_2(\psi, f_{B_1}) = \frac{\lambda_2}{|B_1|}(\psi, 1_{B_1}) - \frac{\lambda_2}{|B_1'|}(\psi, 1_{B_1'})$$
$$= \frac{1}{2}\lambda_2(\psi_1 - \psi_0),$$

$$\lambda_3(\psi, f_{B_2}) = \frac{\lambda_3}{|B_2|}(\psi, 1_{B_2}) - \frac{\lambda_3}{|B_2'|}(\psi, 1_{B_2'})$$
$$= \frac{1}{2}\lambda_3\left(\psi_2 - \frac{1}{2}(\psi_1 + \psi_0)\right)$$

and, for $n \geq 3$,

$$\lambda_{n+1}(\psi, f_{B_n}) = \frac{\lambda_{n+1}}{|B_n|}(\psi, 1_{B_n}) - \frac{\lambda_{n+1}}{|B_n'|}(\psi, 1_{B_n'})$$
$$= \frac{1}{2}\lambda_{n+1}\left(\psi_n - \frac{1}{2}\psi_{n-1} - ... - \frac{1}{2^{n-2}}\psi_2 - \frac{1}{2^{n-1}}(\psi_1 + \psi_0)\right).$$

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Equations (5.44), (5.43) and computations from above yield

$$(\lambda - \sigma_0 - \lambda_2)\,\psi_0 = (\lambda - \sigma_1 - \lambda_2)\,\psi_1, \qquad (5.45)$$

$$\lambda_2(\psi, f_{B_1}) - 2\lambda_3(\psi, f_{B_2})$$
$$= \lambda_2\left(-\varkappa\psi_2 + \frac{1}{2}(1+\varkappa)\psi_1 - \frac{1}{2}(1-\varkappa)\psi_0\right),$$

and, for $n \geq 2$,

$$\lambda_{n+1}(\psi, f_{B_n}) - 2\lambda_{n+2}(\psi, f_{B_{n+1}}) \qquad (5.46)$$
$$= \lambda_{n+1}\left(-\varkappa\psi_{n+1} + \frac{1}{2}(1+\varkappa)\psi_n - \frac{1}{2^2}(1-\varkappa)\psi_{n-1} - \dots\right.$$
$$\left.-\frac{1}{2^n}(1-\varkappa)(\psi_1 + \psi_0)\right).$$

Thus, applying equations (5.43) and (5.46) we obtain

$$\frac{-\varkappa\lambda_{n+1} + (\lambda - \sigma_{n+1})}{\lambda_{n+1}}\psi_{n+1} + \frac{-2(\lambda - \sigma_n) + (1+\varkappa)\lambda_{n+1}}{2\lambda_{n+1}}\psi_n$$
$$= \frac{(1-\varkappa)}{2^2}\psi_{n-1} + \left(\frac{(1-\varkappa)}{2^3}\psi_{n-2} + \dots + \frac{(1-\varkappa)}{2^n}(\psi_1 + \psi_0)\right)$$

and

$$\frac{-\varkappa\lambda_n + (\lambda - \sigma_n)}{\lambda_n}\psi_n + \frac{-2(\lambda - \sigma_{n-1}) + (1+\varkappa)\lambda_n}{2\lambda_n}\psi_{n-1}$$
$$= \frac{(1-\varkappa)}{2^2}\psi_{n-2} + \frac{(1-\varkappa)}{2^3}\psi_{n-3} + \dots + \frac{(1-\varkappa)}{2^{n-1}}(\psi_1 + \psi_0).$$

Let us define variables

- $A_n = -(\lambda - \sigma_n) + \varkappa^{n+1}$, $n \geq 1$,

- $B_n = (1 + \frac{\varkappa}{2})(\lambda - \sigma_n) - (\frac{1}{2} + \varkappa)\varkappa^{n+1}$, $n \geq 1$, and
  $B_1 = (\lambda - \sigma_1) - \frac{\varkappa^2}{2}(1+\varkappa)$, $B_0 = (\lambda - \sigma_0) - \varkappa^2$,

- $C_n = -\frac{\varkappa}{2}(\lambda - \sigma_n) + \frac{1}{2}\varkappa^{n+2}$, $n \geq 1$, and $C_0 = -\frac{\varkappa^2}{2}(1+\varkappa)$, $C_{-1} = 0$.

The computations from above show that the sequence $\{\psi_n\}$ satisfies the following homogeneous second order difference equation

$$A_{n+1}\psi_{n+1} + B_n\psi_n + C_{n-1}\psi_{n-1} = 0,$$

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or equivalently, setting $(\lambda - \sigma_n)\psi_n = \theta_n$, we get

$$\theta_{n+1} = D_n\theta_n + E_{n-1}\theta_{n-1} \quad \text{and} \quad \theta_1 = D_0\theta_0. \tag{5.47}$$

The coefficients D_n and E_{n-1} satisfy

$$D_0 = 1 + O(1)\varkappa^2, \quad D_n = 1 + \frac{\varkappa}{2} + O(1)\varkappa^n, \quad E_{n-1} = -\frac{\varkappa}{2} + O(1)\varkappa^n,$$

whenever the following condition holds

$$\lambda \notin \overline{\{\sigma_n : n = 0, 1, ...\}}. \tag{5.48}$$

If this is the case, by the asymptotic theory of linear second order difference equations, see [16, Theorem 1.5] and [13, Theorem 8.25 and Corollary 8.27], there exist two fundamenthal solutions $\theta_{1,n}$ and $\theta_{2,n}$ of the equation

$$\theta_{n+1} = D_n\theta_n + E_n\theta_{n-1}$$

such that asymptotically as $n \to \infty$,

$$\theta_{1,n} = [1 + o(1)] \quad \text{and} \quad \theta_{1,n} = \left(\frac{\varkappa}{2}\right)^n [1 + o(1)].$$

Thus, general solution θ_n of the equation (5.47) asymptotically can be written in the form

$$\theta_n = \left(C_1 + C_2 \left(\frac{\varkappa}{2}\right)^n\right)[1 + o(1)], \tag{5.49}$$

where the constants C_1 and C_2 depend on θ_0 (remember, $\theta_1 = D_0\theta_0$).

On the other hand, by (5.42), we get

$$|\theta_n| \leq \lambda_{n+1}|(\psi, f_{B_n})| + \lambda_{n+2}|(\psi, f_{B_{n+1}})| + ... \tag{5.50}$$
$$\leq \lambda_{n+1}\|\psi\|_{L^\infty}\|f_{B_n}\|_{L^1} + \lambda_{n+2}\|\psi\|_{L^\infty}\|f_{B_{n+1}}\|_{L^1} + ...$$
$$= \|\psi\|_{L^\infty}\left(\varkappa^{n+1} + \varkappa^{n+2} + ...\right) = O(\varkappa^n),$$

in particular, the sequence θ_n tends to zero. Thus, comparing equations (5.49) and (5.50) we conclude that under condition (5.48),

$$\theta_n = C_2 \left(\frac{\varkappa}{2}\right)^n [1 + o(1)], \tag{5.51}$$

or equivalently,

$$\psi_n = C_3 \left(\frac{\varkappa}{2}\right)^n [1 + o(1)], \tag{5.52}$$

for some constant C_3 which depends on the distance of λ to the set $\overline{\{\sigma_n\}}$.

Proposition 5.3 $Spec < H|\mathcal{L}_+ > \subset \overline{\{\sigma_n\}}$. *In particular, if the sequence* $\{\sigma_k\}$ *forms a dense subset in some interval* $\mathcal{I}$, *then*

$$Spec < H|\mathcal{L}_+ >= Spec_{ess} < H|\mathcal{L}_+ >= \mathcal{I}.$$

Proof. Let us fix $\lambda \notin \mathcal{I}$. By Proposition 5.2 it is enought to show that $\lambda \notin Spec < H|\mathcal{L}_+ >$. Assume that in contrary the corresponding generalized λ-eigenfunction ψ is not eidentically zero. We already know, see equation (5.52), that $\psi \in L^2(X, m)$. We claim that for any $0 < \varkappa < 1$ there exists $0 < \varepsilon_* < 1$ such that for all $0 < \varepsilon < \varepsilon_*$ the function $|\psi|^{1-\varepsilon}$ belongs to $L^1(X, m)$. Indeed, let us choose $0 < \varepsilon < 1$ such that

$$\frac{\varepsilon}{1 - \varepsilon} < \log_2 \frac{1}{\varkappa}.$$

Then, by our choice, $2^\varepsilon \varkappa^{1-\varepsilon} < 1$. Thus applying equation (5.52) we get

$$\left\| |\psi|^{1-\varepsilon} \right\|_{L^1} \leq C_4 \sum 2^n \left(\frac{\varkappa}{2} \right)^{n(1-\varepsilon)} = C_4 \sum \left(2^\varepsilon \varkappa^{1-\varepsilon} \right)^n < \infty$$

as claimed. Next we apply equation (5.42),

$$
\begin{aligned}
|\theta_n| &\leq \varkappa^{n+1} \left[|(\psi, f_{B_n})| + \varkappa |(\psi, f_{B_n})| + ... \right] \\
&\leq \varkappa^{n+1} \left\| |\psi|^{1-\varepsilon} \right\|_{L^1} \left[\left\| |\psi|^\varepsilon f_{B_n} \right\|_{L^\infty} + \varkappa \left\| |\psi|^\varepsilon f_{B_{n+1}} \right\|_{L^\infty} + ... \right] \\
&\leq C_5 \varkappa^n \left[\left(\frac{\varkappa}{2} \right)^{n\varepsilon} \frac{1}{2^{n+1}} + \varkappa \left(\frac{\varkappa}{2} \right)^{(n+1)\varepsilon} \frac{1}{2^{n+2}} + ... \right] \\
&= C_6 \left(\frac{\varkappa}{2} \right)^n \left(\frac{\varkappa}{2} \right)^{n\varepsilon} \left[1 + \frac{\varkappa}{2} + \left(\frac{\varkappa}{2} \right)^2 + ... \right] \leq C_7 \left(\frac{\varkappa}{2} \right)^{(n+1)\varepsilon}.
\end{aligned}
$$

This evidently contradicts to (5.52). Thus, $\lambda \notin Spec < H|\mathcal{L}_+ >$. The proof is finished. ∎

Proposition 5.4 *Assume that* $\{\sigma_n\}$ *is a bounded sequence. The set* $Spec_{pp} \langle H|\mathcal{L}_+ \rangle$, *pure point part of the spectrum of* $\langle H|\mathcal{L}_+ \rangle$, *is an empty set (Thus, by Propositions 5.2 and 5.3,* $Spec \langle H|\mathcal{L}_+ \rangle$ *coinsides with its singular continuous part* $Spec_{sc} \langle H|\mathcal{L}_+ \rangle$).

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